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Positive solutions for singular nonlinear fractional differential equation with integral boundary conditions

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Abstract

In this article, we study the existence of positive solutions for a class of singular nonlinear fractional differential equations with Riemann-Stieltjes integral boundary conditions. Using the properties of the Green function and the fixed point theory in cones, we obtain some results on the existence of positive solutions. Our results extend and improve many known results including singular and nonsingular cases.

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1 Introduction

In this article, we consider the existence of solutions for the following class of singular fractional differential equations:

$$\begin{cases} D_{0^+}^\alpha u(t) + p(t)f(t, u(t)) + q(t)g(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \int_0^1 h(s)u(s)\,dA(s), \end{cases} \tag{1.1}$$

where $n - 1 < \alpha \le n$, $n \ge 2$, $p, q \in C(0, 1)$, $p(t)$ and $q(t)$ are allowed to be singular at $t = 0$ or $t = 1$, $f, g : [0, 1] \times (0, \infty) \to [0, \infty)$ are continuous and may be singular at $x = 0$, $h : (0, 1) \to [0, \infty)$ is continuous with $h \in L^1(0, 1)$, and $\int_0^1 h(s)u(s)\,dA(s)$ denotes the Riemann-Stieltjes integral with a signed measure, in which $A : [0, 1] \to \mathbb{R}$ is a function of bounded variation.

Fractional differential equations have attracted more and more attention from the research communities due to their numerous applications in many fields of science and engineering including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, probability, *etc.* For details, see [1–6] and the references therein. On the other hand, boundary value problems with integral boundary conditions for ordinary differential equations arise in many fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics. The existence and multiplicity of positive solutions for such problems have become an important area of investigation in recent years.

Cabada and Hamdi [7] studied the existence of positive solutions of the following nonlinear fractional differential equation with integral boundary value conditions:

$$
\begin{cases}
D_{0^+}^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\
u(0) = u'(0) = 0, & u(1) = \lambda \int_0^1 u(s)\, ds,
\end{cases}
$$

where $2 < \alpha \leq 3$, $0 < \lambda$, $\lambda \neq \alpha$, $D_{0^+}^\alpha$ is the Riemann-Liouville fractional derivative, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

By means of the monotone iteration method, Sun and Zhao [8] investigated the existence of positive solutions for the fractional differential equation with integral boundary conditions

$$
\begin{cases}
D_{0^+}^\alpha u(t) + q(t)f(t, u(t)) = 0, & 0 < t < 1, \\
u(0) = u'(0) = 0, & u(1) = \int_0^1 g(s)u(s)\, ds,
\end{cases}
$$

where $2 < \alpha \leq 3$, $D_{0^+}^\alpha$ is the standard Riemann-Liouville derivative of order α, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, and $g, q : (0,1) \to [0,\infty)$ are also continuous with $g, q \in L^1(0,1)$.

Zhang *et al.* [9] considered the following nonlinear fractional differential equation with integral boundary conditions:

$$
\begin{cases}
D_{0^+}^\alpha u(t) + h(t)f(t, u(t)) = 0, & 0 < t < 1, \\
u(0) = u'(0) = u''(0) = 0, & u(1) = \lambda \int_0^\eta u(s)\, ds,
\end{cases}
$$

where $3 < \alpha \leq 4$, $0 < \eta \leq 1$, $0 \leq \frac{\lambda \eta^\alpha}{\alpha} < 1$, $D_{0^+}^\alpha$ is the Riemann-Liouville fractional derivative, $h : (0,1) \to [0,\infty)$ is continuous, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is also continuous.

Motivated by the above-mentioned papers, the purpose of this article is to investigate the existence of positive solutions for the more general fractional differential equations BVP (1.1) under suitable conditions on f and g. Obviously, our discussion is different from that in [7–11]. The main new features presented in this paper are as follows. First, the boundary value problem has a more general form, in which p, q are allowed to be singular at $t = 0, 1$ and f, g may be singular at $x = 0$. Second, the existence and multiplicity of positive solutions of BVP (1.1) are obtained. Third, $\int_0^1 u(s)\, dA(s)$ denotes the Riemann-Stieltjes integral, where A is a function of bounded variation, and dA may be a signed measure. The work includes the multipoint boundary problems and integral boundary problems as special cases.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that are used to prove our main results. We also develop some properties of the Green function. In Section 3, we discuss the existence of a positive solution of BVP (1.1). In Section 4, we discuss the existence of multiple positive solutions of BVP (1.1). In Section 5, we give an example to demonstrate an application of our theoretical results.

2 Preliminaries and lemmas

In this section, for the convenience of reader, we present some notation and lemmas that will be used in the proof of our main results. They can be found in the literature; see [5, 6, 12, 13].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) \, ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t - s)^{\alpha-n+1}} \, ds,$$

where $n = [\alpha] + 1$ with $[\alpha]$ denoting the integer part of a number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 [6] *Let $\alpha > 0$. If $u \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad C_i \in \mathbb{R}, i = 1, 2, \ldots, N,$$

as the unique solution, where $N = [\alpha] + 1$.

From the definition of the Riemann-Liouville derivative we can obtain the following statement.

Lemma 2.2 [6] *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}$ $(i = 1, 2, \ldots, N)$, where $N = [\alpha] + 1$.

In the following, we present the Green function of the fractional differential equation boundary value problem.

Lemma 2.3 *Given $y \in C(0,1) \cap L^1(0,1)$, $n - 1 < \alpha \leq n$, the problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \int_0^1 h(s) u(s) \, dA(s) \end{cases} \tag{2.1}$$

is equivalent to

$$u(t) = \int_0^1 G(t, s) y(s) \, ds,$$

where

$$G(t,s) = G_0(t,s) + \frac{t^{\alpha-1}}{1-\Gamma}g_A(s) \tag{2.2}$$

with

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases}$$

$$\Gamma = \int_0^1 t^{\alpha-1}h(t)\,dA(t), \qquad g_A(s) = \int_0^1 G_0(t,s)h(t)\,dA(t).$$

Proof We may apply Lemma 2.2 to reduce (2.1) to the equivalent integral equation

$$u(t) = -I_{0^+}^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for some $C_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. Consequently, the general solution of (2.1) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)\,ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}. \tag{2.3}$$

Since $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we get that $C_2 = C_3 = \cdots = C_n = 0$ by (2.3). Then, we obtain

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)\,ds + C_1 t^{\alpha-1}. \tag{2.4}$$

On the other hand, we can combine (2.1) with

$$u(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}y(s)\,ds + C_1,$$

yielding

$$C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}y(s)\,ds + \int_0^1 h(s)u(s)\,dA(s). \tag{2.5}$$

So, by (2.4) and (2.5) we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)\,ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}y(s)\,ds \\ &\quad + t^{\alpha-1} \int_0^1 h(s)u(s)\,dA(s) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \left[t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} \right] y(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}y(s)\,ds + t^{\alpha-1} \int_0^1 h(s)u(s)\,dA(s). \end{aligned}$$

Let

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

Then

$$u(t) = \int_0^1 G_0(t,s)y(s)\,ds + t^{\alpha-1}\int_0^1 h(s)u(s)\,dA(s). \tag{2.6}$$

Multiplying both sides of (2.6) by $h(t)$ and then integrating from 0 to 1, we have

$$\int_0^1 h(t)u(t)\,dA(t) = \int_0^1 h(t)\int_0^1 G_0(t,s)y(s)\,ds\,dA(t)$$
$$+ \int_0^1 t^{\alpha-1}h(t)\int_0^1 h(s)u(s)\,dA(s)\,dA(t). \tag{2.7}$$

Then, by (2.7) we obtain

$$\left(1 - \int_0^1 t^{\alpha-1}h(t)\,dA(t)\right)\int_0^1 h(t)u(t)\,dA(t) = \int_0^1\int_0^1 h(t)G_0(t,s)\,dA(t)y(s)\,ds. \tag{2.8}$$

Let

$$\Gamma = \int_0^1 t^{\alpha-1}h(t)\,dA(t), \qquad g_A(s) = \int_0^1 G_0(t,s)h(t)\,dA(t). \tag{2.9}$$

From (2.8) and (2.9) we deduce that

$$\int_0^1 h(t)u(t)\,dA(t) = \frac{1}{1-\Gamma}\int_0^1 g_A(s)y(s)\,ds. \tag{2.10}$$

Substituting (2.10) into (2.6), we obtain

$$u(t) = \int_0^1 G_0(t,s)y(s)\,ds + \frac{t^{\alpha-1}}{1-\Gamma}\int_0^1 g_A(s)y(s)\,ds = \int_0^1 G(t,s)y(s)\,ds.$$

The proof is complete. $\qquad\square$

Lemma 2.4 *Let $0 \le \Gamma < 1$ and $g_A(s) \ge 0$ for $s \in [0,1]$. Then the Green function $G(t,s)$ defined by (2.2) satisfies*

(1) $G : [0,1] \times [0,1] \to [0,\infty)$ *is continuous.*
(2) *For any $t,s \in [0,1]$, we have $c_0(t)\phi(s) \le G(t,s) \le \phi(s)$, where*

$$\phi(s) = \phi_0(s) + \frac{g_A(s)}{1-\Gamma}, \qquad \phi_0(s) = \frac{\tau(s)^{\alpha-2}s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)},$$

$$\tau(s) = \frac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}}, \qquad c_0(t) = \min\left\{\frac{(\alpha-1)^{\alpha-1}t^{\alpha-2}(1-t)}{(\alpha-2)^{\alpha-2}}, t^{\alpha-1}\right\}.$$

(3) *Then, taking $\theta \in (0, \frac{1}{2})$, for any $t \in [\theta, 1-\theta]$ and $s \in [0,1]$, we have $G(t,s) \ge \gamma_0\phi(s)$, where $\gamma_0 = \min_{\theta \le t \le 1-\theta} c_0(t) \in (0,1)$.*

The proof of Lemma 2.4 is similar to that in [14], so we omit it.

Remark 2.1 By Lemma 2.4 we can obtain that $\frac{\alpha-2}{\alpha-1} \leq \tau(s) \leq 1$, $\phi_0(s) \leq \frac{1}{\alpha-1}$. That is, $\phi(s)$ is bounded, and there exists a constant $\overline{g} > 0$ such that $\phi(s) \leq \overline{g}$. It is easy to verify that $\max_{0 \leq t \leq 1} c_0(t) \leq 1$.

Let $E = C[0,1]$ and $\|u\| = \sup_{0 \leq t \leq 1} | u(t) |$. Then $(E, \| \cdot \|)$ is a Banach space. Let

$$P = \{u \in E : u(t) \geq 0, t \in [0,1]\}$$

and

$$K = \{u \in P : u(t) \geq c_0(t)\|u\|, t \in [0,1]; u(t) \geq \gamma_0\|u\|, t \in [\theta, 1-\theta]\},$$

and let $K_r = \{u \in K : \|u\| < r\}$. It is easy to see that K is a cone in E and $\overline{K_R} \setminus K_r \subset K \subset P$.

Throughout the paper, we need the following conditions:

(H$_1$) $A : [0,1] \to \mathbb{R}$ is a function of bounded variation, and $g_A(s) \geq 0$ for all $s \in [0,1]$;

(H$_2$) $h \in C(0,1) \cap L^1(0,1)$, and $0 \leq \Gamma = \int_0^1 t^{\alpha-1}h(s)\,dA(s) < 1$;

(H$_3$) $p, q : (0,1) \to [0,\infty)$ are continuous, $p(t) \not\equiv 0$, $q(t) \not\equiv 0$, $t \in [0,1]$, and

$$\int_0^1 \phi(s)p(s)\,ds < +\infty, \qquad \int_0^1 \phi(s)q(s)\,ds < +\infty;$$

(H$_4$) $f, g : [0,1] \times (0,\infty) \to [0,\infty)$ are continuous, and for any $0 < r < R < +\infty$, $\lim_{m \to \infty} \sup_{u \in \overline{K_R} \setminus K_r} \int_{H(m)} (p(s)f(s, u(s)) + q(s)g(s, u(s)))\,ds = 0$, where $H(m) = [0, \frac{1}{m}] \cup [\frac{m-1}{m}, 1]$.

In what follows, let us define the nonlinear operator $T : \overline{K_R} \setminus K_r \to K$ by

$$(Tu)(t) = \int_0^1 G(t,s)\big[p(s)f\big(s, u(s)\big) + q(s)g\big(s, u(s)\big)\big]\,ds, \quad t \in [0,1]. \tag{2.11}$$

Lemma 2.5 *Suppose that* (H$_1$)-(H$_4$) *hold. Then* $T : \overline{K_R} \setminus K_r \to K$ *is a completely continuous operator and the fixed point of* T *in* $\overline{K_R} \setminus K_r$ *is the positive solution to BVP* (1.1).

Proof It follows from (H$_4$) that there exists a natural number $m \geq 2$ such that

$$\sup_{u \in \overline{K_R} \setminus K_r} \int_{H(m)} p(s)f\big(s, u(s)\big) + q(s)g\big(s, u(s)\big)\,ds < 1.$$

It is easy to see that for any $u \in \overline{K_R} \setminus K_r$ and $t \in [\frac{1}{m}, \frac{m-1}{m}]$, we have $\lambda_0 r \leq u(t) \leq R$, where $\lambda_0 = \min\{c_0(t) : \frac{1}{m} \leq t \leq \frac{m-1}{m}\} > 0$. Let

$$M_1 = \max\left\{f(t,x) : \frac{1}{m} \leq t \leq \frac{m-1}{m}, \lambda_0 r \leq x \leq R\right\},$$

$$M_2 = \max\left\{g(t,x) : \frac{1}{m} \leq t \leq \frac{m-1}{m}, \lambda_0 r \leq x \leq R\right\}.$$

So, by (H$_3$), (H$_4$), and Remark 2.1 we have

$$\sup_{u \in \overline{K_R} \backslash K_r} \int_0^1 G(t,s) \big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big) \big]\, ds$$

$$\leq \sup_{u \in \overline{K_R} \backslash K_r} \int_0^1 \phi(s) \big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big) \big]\, ds$$

$$\leq \overline{g} \sup_{u \in \overline{K_R} \backslash K_r} \int_{H(m)} p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big)\, ds$$

$$+ \sup_{u \in \overline{K_R} \backslash K_r} \int_{\frac{1}{m}}^{\frac{m-1}{m}} \phi(s) \big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big) \big]\, ds$$

$$\leq \overline{g} + (M_1 + M_2) \int_{\frac{1}{m}}^{\frac{m-1}{m}} \phi(s)\big(p(s) + q(s)\big)\, ds$$

$$\leq \overline{g} + (M_1 + M_2) \int_0^1 \phi(s)\big(p(s) + q(s)\big)\, ds < +\infty.$$

This implies that the operator T defined by (2.11) is well defined.

Now, we show that $T : \overline{K_R} \backslash K_r \subset K$. For any $u \in \overline{K_R} \backslash K_r$, $t \in [0,1]$, we have

$$(Tu)(t) = \int_0^1 G(t,s) \big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big) \big]\, ds$$

$$\leq \int_0^1 \phi(s) \big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big) \big]\, ds.$$

Hence,

$$\| Tu \| \leq \int_0^1 \phi(s) \big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big) \big]\, ds.$$

On the other hand, by Lemma 2.4 we have

$$(Tu)(t) \geq c_0(t) \int_0^1 \phi(s) \big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big) \big]\, ds \geq c_0(t) \| Tu \|, \quad t \in [0,1],$$

and at the same time, we can get

$$(Tu)(t) \geq \gamma_0 \| Tu \|, \quad t \in [\theta, 1 - \theta].$$

Thus, $Tu \in K$. Consequently, $T : \overline{K_R} \backslash K_r \subset K$. Finally, we prove that T, which maps $\overline{K_R} \backslash K_r$ into K, is a completely continuous map. Let $D \subset \overline{K_R} \backslash K_r$ be an arbitrary bounded set. Then from the previous proof we know that $T(D)$ is uniformly bounded.

Next, we show $T(D)$ is equicontinuous. In fact, for any $\epsilon > 0$, there exists a natural number $m \geq 2$ such that

$$\sup_{u \in \overline{K_R} \backslash K_r} \int_{H(m)} p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big)\, ds < \frac{\epsilon}{4\overline{g}}. \tag{2.12}$$

Since $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$, for the above $\epsilon > 0$, there exists $\delta > 0$ such that for any $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, $s \in [\frac{1}{m}, \frac{m-1}{m}]$,

$$\left| G(t_1, s) - G(t_2, s) \right| < \frac{\epsilon}{2(\overline{p}M_1 + \overline{q}M_2)}.$$

Consequently, for any $u \in D$, $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, we have

$$
\begin{aligned}
&\left| (Tu)(t_1) - (Tu)(t_2) \right| \\
&= \left| \int_0^1 \left(G(t_1, s) - G(t_2, s) \right) \left(p(s)f(s, u(s)) + q(s)g(s, u(s)) \right) ds \right| \\
&\leq 2 \int_{H(m)} \phi(s) \left[p(s)f(s, u(s)) + q(s)g(s, u(s)) \right] ds \\
&\quad + \sup_{u \in D} \int_{\frac{1}{m}}^{\frac{m-1}{m}} \left| \left(G(t_1, s) - G(t_2, s) \right) \right| \left(p(s)f(s, u(s)) + q(s)g(s, u(s)) \right) ds \\
&< 2\overline{g} \frac{\epsilon}{4\overline{g}} + \frac{\epsilon}{2(\overline{p}M_1 + \overline{q}M_2)} (\overline{p}M_1 + \overline{q}M_2) \\
&= \epsilon,
\end{aligned}
$$

where

$$\overline{p} = \max \left\{ p(t) : \frac{1}{m} \leq t \leq \frac{m-1}{m} \right\}, \qquad \overline{q} = \max \left\{ q(t) : \frac{1}{m} \leq t \leq \frac{m-1}{m} \right\}.$$

This shows that $T(D)$ is equicontinuous. By the Arzela-Ascoli theorem, $T : D \to K$ is compact.

Finally, we prove that $T : \overline{K_R} \setminus K_r \to K$ is continuous. Assume that $u_0, u_n \in \overline{K_R} \setminus K_r$ ($n = 0, 1, 2, \dots$) and $\|u_n - u_0\| \to 0$ ($n \to \infty$). Then $r \leq \|u_n\| \leq R$ and $r \leq \|u_0\| \leq R$. For any $\epsilon > 0$, by (H_4) there exists a natural number $m \geq 2$ such that

$$\sup_{u \in \overline{K_R} \setminus K_r} \int_{H(m)} p(s)f(s, u(s)) + q(s)g(s, u(s)) \, ds < \frac{\epsilon}{\overline{g}}.$$

On the other hand, for any $t \in [\frac{1}{m}, \frac{m-1}{m}]$, we have $\gamma_0 r \leq u_n(t) \leq R$ ($n = 0, 1, 2, \dots$). Since $f(t, u), g(t, u)$ are uniformly continuous in $[\frac{1}{m}, \frac{m-1}{m}] \times [\gamma_0 r, R]$, we have that

$$\lim_{n \to \infty} \left| f(s, u_n(s)) - f(s, u_0(s)) \right| = 0,$$

$$\lim_{n \to \infty} \left| g(s, u_n(s)) - g(s, u_0(s)) \right| = 0$$

uniformly on $[\frac{1}{m}, \frac{m-1}{m}]$. Then the Lebesgue dominated convergence theorem yields that

$$\int_{\frac{1}{m}}^{\frac{m-1}{m}} \phi(s) \left[p(s) \left(f(s, u_n(s)) - f(s, u_0(s)) \right) + q(s) \left(g(s, u_n(s)) - g(s, u_0(s)) \right) \right] ds \to 0,$$

$$n \to \infty.$$

Thus, for the above $\epsilon > 0$, there exists a natural number N such that for $n > N$, we have

$$\int_{\frac{1}{m}}^{\frac{m-1}{m}} \phi(s)\big[p(s)\big(f\big(s, u_n(s)\big) - f\big(s, u_0(s)\big)\big) + q(s)\big(g\big(s, u_n(s)\big) - g\big(s, u_0(s)\big)\big)\big]\,ds < \frac{\epsilon}{2}. \quad (2.13)$$

It follows from (2.12), (2.13) that for $n > N$,

$$
\begin{aligned}
\|Tu_n - Tu_0\| &\leq \sup_{u \in \overline{K_R} \setminus K_r} \int_{H(m)} \phi(s)\big[p(s)\big(f\big(s, u_n(s)\big) - f\big(s, u_0(s)\big)\big) \\
&\quad + q(s)\big(g\big(s, u_n(s)\big) - g\big(s, u_0(s)\big)\big)\big]\,ds \\
&\quad + \sup_{u \in \overline{K_R} \setminus K_r} \int_{\frac{1}{m}}^{\frac{m-1}{m}} \phi(s)\big[p(s)\big(f\big(s, u_n(s)\big) - f\big(s, u_0(s)\big)\big) \\
&\quad + q(s)\big(g\big(s, u_n(s)\big) - g\big(s, u_0(s)\big)\big)\big]\,ds \\
&< \overline{g}\frac{\epsilon}{2\overline{g}} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

This implies that $T : \overline{K_R} \setminus K_r \to K$ is continuous. Thus, $T : \overline{K_R} \setminus K_r \to K$ is completely continuous. It is clear that if u is a fixed point of T in $\overline{K_R} \setminus K_r$, then u satisfies (1.1) and is a positive solution of BVP (1.1). $\qquad\square$

To prove the main results, we need the following well-known fixed point theorem.

Lemma 2.6 [15] *Let P be a positive cone in a Banach space E, Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$, $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator, where θ denotes the zero element of E. Suppose that one of the following two conditions holds:*

(i) $\|Au\| \leq \|u\|, \forall u \in P \cap \partial\Omega_1$; $\|Au\| \geq \|u\|, \forall u \in P \cap \partial\Omega_2$;

(ii) $\|Au\| \geq \|u\|, \forall u \in P \cap \partial\Omega_1$; $\|Au\| \leq \|u\|, \forall u \in P \cap \partial\Omega_2$.

Then A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3 Existence of a positive solution

In the following, for convenience, we set

$$\overline{f}_0 = \limsup_{x \to 0} \max_{t \in [0,1]} \frac{f(t,x)}{x}, \qquad \overline{g}_0 = \limsup_{x \to 0} \max_{t \in [0,1]} \frac{g(t,x)}{x},$$

$$\underline{f}_0 = \liminf_{x \to 0} \min_{t \in [\theta, 1-\theta]} \frac{f(t,x)}{x}, \qquad \underline{g}_0 = \liminf_{x \to 0} \min_{t \in [\theta, 1-\theta]} \frac{g(t,x)}{x},$$

$$\overline{f}_\infty = \limsup_{x \to \infty} \max_{t \in [0,1]} \frac{f(t,x)}{x}, \qquad \overline{g}_\infty = \limsup_{x \to \infty} \max_{t \in [0,1]} \frac{g(t,x)}{x},$$

$$\underline{f}_\infty = \liminf_{x \to \infty} \min_{t \in [\theta, 1-\theta]} \frac{f(t,x)}{x}, \qquad \underline{g}_\infty = \liminf_{x \to \infty} \min_{t \in [\theta, 1-\theta]} \frac{g(t,x)}{x},$$

$$m_0 = \left[\int_0^1 \phi(s)\big[p(s) + q(s)\big]\,ds\right]^{-1}, \qquad m_1 = \left[\gamma_0 \int_\theta^{1-\theta} \phi(s)p(s)\,ds\right]^{-1},$$

$$m_2 = \left[\gamma_0 \int_\theta^{1-\theta} \phi(s)q(s)\,ds\right]^{-1}.$$

Theorem 3.1 *Let $f, g : [0,1] \times [0, +\infty) \to [0, +\infty)$ be continuous. Assume that (H_1)-(H_3) and $\overline{f}_0 < m_0, \overline{g}_0 < m_0$ hold and one of the two conditions $\underline{f}_{-\infty} > m_1$ and $\underline{g}_{-\infty} > m_2$ is satisfied. Then BVP (1.1) has at least one positive solution.*

Proof First, we consider the case $\underline{f}_{-\infty} > m_1$. Since $\overline{f}_0 < m_0, \overline{g}_0 < m_0$, we can choose $\epsilon_0 > 0$ such that $m_0 - \epsilon_0 > 0$ and $\overline{f}_0 < m_0 - \epsilon_0, \overline{g}_0 < m_0 - \epsilon_0$, and also there exists $r_0 > 0$ such that for any $0 < x \leq r_0, 0 \leq t \leq 1$, we have

$$f(t,x) \leq (m_0 - \epsilon_0)x, \qquad g(t,x) \leq (m_0 - \epsilon_0)x. \tag{3.1}$$

Set $\Omega_0 = \{u \in E : \|u\| < r_0\}$. We can see that Ω_0 is a bounded open subset in E, and for any $u \in \partial\Omega_0 \cap K$, $0 < u(t) \leq r_0$, $t \in [0,1]$, from expression (3.1) we deduce the following inequalities:

$$\|Tu\| = \sup_{t \in [0,1]} \left| \int_0^1 G(t,s)\big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big)\big]\,ds \right|$$

$$\leq (m_0 - \epsilon_0) \sup_{t \in [0,1]} \left| \int_0^1 G(t,s)\big[p(s) + q(s)\big]u(s)\,ds \right|$$

$$\leq (m_0 - \epsilon_0) \sup_{t \in [0,1]} \left| \int_0^1 G(t,s)\big[p(s) + q(s)\big]\,ds \right| \|u\|$$

$$\leq m_0 \int_0^1 \phi(s)\big[p(s) + q(s)\big]\,ds\|u\| = \|u\|.$$

Consequently,

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_0 \cap K. \tag{3.2}$$

On the other hand, since $\underline{f}_{-\infty} > m_1$, we can choose $\epsilon_1 > 0$ such that $\underline{f}_{-\infty} > (m_1 + \epsilon_1)$, and also there exists $R_1 > r_0 > 0$ such that for any $x \geq R_1$, $t \in [\theta, 1 - \theta]$, we have

$$f(t,x) \geq (m_1 + \epsilon_1)x. \tag{3.3}$$

Let $\overline{R_1} = \frac{R_1}{\gamma_0}$ and $\Omega_1 = \{u \in E : \|u\| < \overline{R_1}\}$. It is easy to see that Ω_1 is a bounded open subset in E, and from the expression (3.3) we have that for any $u \in \partial\Omega_1 \cap K$, $u(t) \geq \gamma_0\|u\| = R_1$, $t \in [\theta, 1 - \theta]$,

$$\|Tu\| = \sup_{t \in [0,1]} \left| \int_0^1 G(t,s)\big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big)\big]\,ds \right|$$

$$\geq \int_\theta^{1-\theta} G(t,s)p(s)f\big(s,u(s)\big)\,ds$$

$$\geq \gamma_0(m_1 + \epsilon_1) \int_\theta^{1-\theta} \phi(s)p(s)\,ds\|u\|$$

$$\geq \gamma_0 m_1 \int_\theta^{1-\theta} \phi(s)p(s)\,ds\|u\| = \|u\|.$$

Consequently,

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_1 \cap K. \tag{3.4}$$

Applying Lemma 2.6 to (3.2) and (3.4) yields that T has a fixed point $u^* \in K \cap (\overline{\Omega_1} \setminus \Omega_0)$, and hence u^* is a positive solution of BVP (1.1).

In a similar way, when the case that $\underline{g}_\infty > m_2$ holds, we can prove that the conclusion of Theorem 3.1 also holds. The proof of Theorem 3.1 is completed. $\square$

Theorem 3.2 *Assume that* (H_1)-(H_4) *and* $\overline{f}_\infty < m_0, \overline{g}_\infty < m_0$ *hold. Then BVP* (1.1) *has at least one positive solution.*

The proof is similar to the previous one, so we omit it.

4 Existence of multiple positive solutions

In this section, we discuss the existence of multiple positive solutions of BVP (1.1). We obtain the following existence theorems.

Theorem 4.1 *Assume that* (H_1)-(H_4) *hold and* $\underline{f}_\infty > m_1$ *are satisfied. Further, assume that the following condition holds:*

$$(*) \quad \int_0^1 p(s)f\big(s, u(s)\big) + q(s)g\big(s, u(s)\big)\, ds < \frac{1}{\overline{g}}, \quad \forall u \in \partial\Omega \cap K,$$

where $\phi(s) \leq \overline{g}$, $\partial\Omega = \{u \in E : \|u\| = 1\}$. *Then BVP* (1.1) *has at least two positive solutions.*

Proof Since $\underline{g}_0 = +\infty > m_2$, there exist $\epsilon_2 > 0$, $0 < r_1 < 1$ such that for any $0 < x \leq r_1$, $\theta \leq t \leq 1 - \theta$, we have

$$g(t, x) \geq (m_2 + \epsilon_2)x. \tag{4.1}$$

Set $\Omega_2 = \{u \in E : \|u\| < r_1\}$. Then Ω_2 is a bounded open subset in E, and for any $u \in \partial\Omega_2 \cap K$ and $0 < u(t) \leq r_1$, by expression (4.1) we have

$$
\begin{aligned}
\|Tu\| &= \sup_{t \in [0,1]} \left| \int_0^1 G(t, s)\big[p(s)f\big(s, u(s)\big) + q(s)g\big(s, u(s)\big)\big]\, ds \right| \\
&\geq \int_\theta^{1-\theta} G(t, s)q(s)g\big(s, u(s)\big)\, ds \\
&\geq \gamma_0(m_2 + \epsilon_2) \int_\theta^{1-\theta} \phi(s)q(s)\, ds \|u\| \\
&\geq \gamma_0 m_2 \int_\theta^{1-\theta} \phi(s)q(s)\, ds \|u\| = \|u\|.
\end{aligned}
$$

Consequently,

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_2 \cap K. \tag{4.2}$$

Since $\underline{f}_\infty > m_1$, there exist $\epsilon_3 > 0$, $R_2 > 1$ such that for any $x \geq R_2$, $\theta \leq t \leq 1 - \theta$, we have

$$f(t, x) \geq (m_1 + \epsilon_3)x. \tag{4.3}$$

Let $R_3 = \frac{R_2}{\gamma_0}$ and $\Omega_3 = \{u \in E : \|u\| < R_3\}$. Then Ω_3 is a bounded open subset in E, and so for any $u \in \partial\Omega_3 \cap K$ and $u(t) \geq \gamma_0\|u\| = R_2$, $t \in [\theta, 1-\theta]$, by expression (4.3) we have

$$\|Tu\| = \sup_{t\in[0,1]} \left| \int_0^1 G(t,s)\big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big)\big]\,ds \right|$$

$$\geq \int_\theta^{1-\theta} G(t,s)p(s)f\big(s,u(s)\big)\,ds$$

$$\geq \gamma_0(m_1 + \epsilon_3) \int_\theta^{1-\theta} \phi(s)p(s)\,ds \|u\|$$

$$\geq \gamma_0 m_1 \int_\theta^{1-\theta} \phi(s)p(s)\,ds \|u\| = \|u\|.$$

Consequently,

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_3 \cap K. \tag{4.4}$$

On the other hand, by $(*)$ we have for any $u \in \partial\Omega \cap K$, $t \in [0,1]$,

$$\|Tu\| = \sup_{t\in[0,1]} \left| \int_0^1 G(t,s)\big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big)\big]\,ds \right|$$

$$\leq \int_0^1 \phi(s)\big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big)\big]\,ds$$

$$\leq \overline{g} \int_0^1 \big[p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big)\big]\,ds \leq \overline{g}\frac{1}{\overline{g}} = 1 = \|u\|.$$

Consequently,

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega \cap K. \tag{4.5}$$

Therefore, from (4.2), (4.5), and Lemma 2.6 it follows that BVP (1.1) has a positive solution $u_0 \in K$ with $r_1 \leq \|u\| < 1$. In the same way, from (4.4), (4.5), and Lemma 2.6 it follows that BVP (1.1) has a positive solution $u_1 \in K$ with $1 < \|u\| \leq R_3$. Therefore, BVP (1.1) has at least two positive solutions $u_0, u_1 \in K$. □

Theorem 4.2 *Assume that* (H_1)-(H_4) *hold and* $\underline{g}_\infty > m_2$ *are satisfied. Further, assume that the following condition holds*:

$$\int_0^1 p(s)f\big(s,u(s)\big) + q(s)g\big(s,u(s)\big)\,ds < \frac{1}{\overline{g}}, \quad \forall u \in \partial\Omega \cap K,$$

where $\phi(s) \leq \overline{g}$, $\partial\Omega = \{u \in E : \|u\| = 1\}$. *Then BVP* (1.1) *has at least two positive solutions*.

The proof is similar to the previous one, so we omit it.

5 Example

Example 5.1 We consider BVP (1.1) with $\alpha = \frac{7}{2}$, $p(t) = q(t) = \frac{1}{200\sqrt{1-t^2}}$, $h(t) = t^{-\frac{1}{2}}$, $f(t,x) = 5 - t + |\ln x| + x^{\frac{3}{2}}$, $g(t,x) = 2 - t + |\ln x|$, and

$$
A(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}), \\ 2, & t \in [\frac{1}{2}, 1]. \end{cases}
$$

Thus, $\Gamma = \int_0^1 t^{\frac{5}{2}} t^{-\frac{1}{2}} \, dA(t) = 2 \times (\frac{1}{2})^2 = \frac{1}{2}$. Obviously, $g_A(s) \le 1$ and $\phi(s) = \phi_0(s) + \frac{g_A(s)}{1-\Gamma} \le 4 = \overline{g}$. We define the cone

$$
K = \{ u \in C[0,1] : u(t) \ge c_0(t)\|u\|, t \in [0,1]; u(t) \ge \gamma_0 \|u\|, t \in [\theta, 1-\theta] \},
$$

in which

$$
c_0(t) = \min\left\{ \frac{(\frac{5}{2})^{\frac{5}{2}} t^{\frac{3}{2}} (1-t)}{\frac{3}{2}^{\frac{3}{2}}}, t^{\frac{5}{2}} \right\}.
$$

For any $0 < r < R < +\infty$ and $u \in \overline{K_R} \backslash K_r$, we have $u(t) \ge c_0(t)\|u\|, t \in [0,1]$. Then $0 \le rc_0(t) \le u(t) \le R, t \in [0,1]$. Since $|\ln x|$ is decreasing on $(0,1)$ and is increasing on $(1, +\infty)$, we have

$$
\left| \ln u(x) \right| \le \left| \ln rc_0(t) \right| + |\ln R| \le |\ln r| + |\ln R| + \left| \ln c_0(t) \right|. \tag{5.1}
$$

It is easy to verify that $\frac{|\ln c_0(t)|}{200\sqrt{1-t^2}} \in L^1(0,1)$. The absolute continuity of the integral yields that

$$
\lim_{m \to \infty} \int_{H(m)} \frac{|\ln c_0(s)|}{200\sqrt{1-s^2}} \, ds = 0. \tag{5.2}
$$

So, by (5.1) and (5.2) we get

$$
\sup_{u \in \overline{K_R} \backslash K_r} \int_{H(m)} \frac{1}{200\sqrt{1-s^2}} \left[2 - s + \left| \ln u(s) \right| + 5 - s + \left| \ln u(s) \right| + (u(s))^{\frac{3}{2}} \right] ds
$$

$$
\le \int_{H(m)} \frac{1}{200\sqrt{1-s^2}} \left[7 - 2s + 2|\ln r| + 2|\ln R| + 2\left| \ln c_0(s) \right| + R^{\frac{3}{2}} \right] ds
$$

$$
\le \left(8 + R^{\frac{3}{2}} + 4|\ln r| + 4|\ln R| \right) \int_{H(m)} \frac{1}{200\sqrt{1-s^2}} \, ds + \int_{H(m)} \frac{1}{100\sqrt{1-s^2}} \left| \ln c_0(s) \right| ds
$$

$$
= \frac{8 + R^{\frac{3}{2}} + 4|\ln r| + 4|\ln R|}{200} \left[\arcsin \frac{1}{m} + \arcsin 1 - \arcsin \left(1 - \frac{1}{m} \right) \right]
$$

$$
+ \int_{H(m)} \frac{1}{100\sqrt{1-s^2}} \left| \ln c_0(s) \right| ds. \tag{5.3}
$$

By (5.2) we know that, as $m \to \infty$, the limit of (5.3) is 0. On the other hand, by calculation we have

$$
\underline{g}_0 = \liminf_{x \to 0} \min_{t \in [\theta, 1-\theta]} \frac{g(t,x)}{x} = \liminf_{x \to 0} \frac{1 + \theta + |\ln x|}{x} = \infty > m_2,
$$

$$
\underline{f}_\infty = \liminf_{x \to \infty} \min_{t \in [\theta, 1-\theta]} \frac{f(t,x)}{x} = \liminf_{x \to \infty} \frac{4 + \theta + |\ln x|}{x} + x^{\frac{1}{2}} = \infty > m_1,
$$

where

$$m_1 = m_2 = \left[\gamma_0 \int_\theta^{1-\theta} \frac{\phi(s)}{200\sqrt{1-s^2}} \, ds \right]^{-1}$$

with $\gamma_0 = \min_{\theta \leq t \leq 1-\theta} c_0(t)$. On the other hand, for any $u \in \partial\Omega \cap K$, $t \in [0,1]$, we have

$$f(t, u(t)) + g(t, u(t)) = 7 - 2t + 2\left| \ln u(t) \right| + u(t)^{\frac{3}{2}}$$

$$\leq 8 + 2\left| \ln c_0(t) \right|.$$

So we obtain

$$\int_0^1 p(s)f(s, u(s)) + q(s)g(s, u(s)) \, ds = \int_0^1 \frac{1}{200\sqrt{1-s^2}} \left[f(s, u(s)) + g(s, u(s)) \right] ds$$

$$\leq \int_0^1 \frac{1}{200\sqrt{1-s^2}} \left[8 + 2\left| \ln c_0(s) \right| \right] ds$$

$$= \frac{\pi}{50} + \frac{1}{100} \int_0^1 \frac{\left| \ln c_0(s) \right|}{\sqrt{1-s^2}} \, ds$$

$$\leq \frac{\pi}{50} + \frac{1}{10}$$

$$< \frac{1}{4}.$$

Therefore, the assumptions of Theorem 4.1 are satisfied. Thus the problem considered possesses two positive solutions in K.

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