A Coordinate-Free Approach to Instantaneous Kinematics of Two Rigid Objects with Rolling Contact and Its Implications for Trajectory Planning

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***Abstract*—This paper adopts a coordinate-free approach to investigate the kinematics of rigid bodies with rolling contact. A new equation of angular velocity of the moving body is derived in terms of the magnitude of rolling velocity and two sets of geometric invariants belonging to the respective contact curves. This new formulation can be differentiated up to any order. Furthermore, qualitative information about trajectory planning can be deduced from this equation if the characteristics of rolling objects and the motion are taken into consideration.**

I. Introduction

When a multifingered hand is manipulating a grasped object, rolling and sliding often occur at the contact point between the fingertips and the grasped object. Since there are many benefits of rolling contact including reduction of abrasion wear, simplification of controller, and enlargement of reachable configurations, rolling without sliding contact is preferred when fine-tuning the grasped object. Kinematics of rigid objects with rolling contact is a problem of nonholonomic constraint, that is, the equation relating two bodies is expressed in terms of their velocities rather than in terms of their positions. Moreover, the equation becomes a function of the shapes of the two objects. The kinematics of rolling contact is essential to the subsequent development of dynamics, controllability, and motion planning of the mechanical systems.

Kinematics of rolling contact was sometimes included as a special case of kinematics of contact. One exception was Neimark and Fufaev [1], who studied this problem in a way similar to this paper. However, their equations depended on the frames along the directions of lines of curvature and therefore were not general. Cai and Roth [2]–[3] investigated kinematics of rigid bodies in point contact both in planar and spatial cases and concentrated on two special motions including sliding and pure-rolling motion. Montana [4] studied the kinematics of contact from a geometric point of view and derived the equations of contact. Sarkar, Kumar, and Yun [5] extended Montana's work to include acceleration terms. By using intrinsic geometric properties for the contact surfaces, they showed the explicit dependence on the Christoffel symbols and their time derivatives.

It is widely acknowledged that the curvature effects and rolling can be turned to play in advantage of the design of simpler and dexterous hands. Hence it is not a surprise to see that plenty of literature appearing in recent years related to dynamics, grasp, manipulation, and control of multi-fingered hands concerning kinematics of rolling contact between fingertips and object. Ghafoor, Dai, and Duffy [6]–[7] simplified kinematics of rolling contact to line contact to study stiffness modeling and fine motion control of the interaction between fingertips and the grasped object. Kerr and Roth [8] discussed how to compute the movement of the fingers in order to produce a given displacement of the object. Montana [9] studied the kinematics of multi-fingered manipulation based on his equations of contact. Salisbury [10] defined the grip Jacobian which calculated the velocity of an object in the grasp of fingers of a hand given the velocity of the joints of fingers. Nagashima, Seki, and Takano [11] analyzed the kinematic relationship between object and finger joint motions in the case when the contact motion was pure rolling, twist-rolling, and slide-rolling. Hwang and Sasaki [12] evaluated various finger-joint configurations for three types of contact motion, namely pure rolling, twist-rolling, and slide-twist-rolling. From the point of view of motion planning, Li and Canny [13] used Montana's contact equations to study whether an admissible path existed between two configurations in the case of pure-rolling, and if it existed, how to find it. Han and Trinkle [14]–[15] showed all systems variables (the finger joint, object, and contact velocities) needed to be included in the differential kinematic equation used for manipulation planning and further studied the relevant theories of contact kinematics, nonholonomic motion planning, coordinated object manipulation, grasp stability and finger gaiting to develop a general framework for dexterous manipulation planning. Kiss, Lévine, Lantos [16] addressed the motion planning problem (open-loop trajectory design) for manipulating rigid bodies with permanent rolling contact without slipping. Some authors studied this issue from the point of view of control. Cole, Hauser, and Sastry [17] were the first to study the control of multifingered hands with rolling contact. Sarkar, Yun, Kumar [18] addressed the problem of maintaining rolling contact between the robot arm and an external moving object. Bicchi and Marigo [19] derived analogous equations with Montana's contact equations, but with a different approach that allowed an analysis of admissibility of rolling contact. Harada and Kaneko [20] used Montana's equations of contact to study the rolling based manipulation under neighborhood equilibrium. Remond, Perdereau, and Drouin [21] acknowledged the need of rolling compensation for multifingered hand control structure and propose polar parameterization of the local shape around the point of contact to get a four independent relation system. From the point of view of grasp, Maekawa, Tanie, and Komoriya [22] represented the local shape of the fingertip and the object by a quadratic surface and developed the kinematic-static relation for the motion-force of the fingertip, contact location and the object.

However, the above mentioned mathematical derivations of kinematics of rolling contact suffer from some drawbacks. Firstly, they are local in nature. In other words, they depend on the parameterization of the contact surfaces. The formulations are not valid any more if the origin or orientation of the global coordinate changes. Secondly, the derivation of equations is generally difficult to obtian, as stated in [21], cannot be easily done in general case. Thirdly, the formulations can only be differentiated to a certain order, usually one or two.

The contribution of this paper is that a new equation of the angular velocity of the moving object is derived in terms of the magnitude of the rolling velocity and two sets of geometric invariants. The approach is based on moving frames and differential one-forms, which is coordinate-free in nature. Hence, the equations of moving frames can be formulated in a convenient local coordinate system and later be used in a global one. The effects of the relative curvature and torsion on rolling kinematics are explicitly presented in this equation. In addition, qualitative information about trajectory planning can be deduced from the equation if the characteristics of rolling objects and the motion are taken into consideration.

This paper is organized as follows. Section 2 introduces the concept of moving frames and Darboux frame. Section 3 obtains a new equation of the angular velocity of the rolling objects in terms of the rolling speed and two sets of geometric invariants. Section 4 discusses some implications of this new equation for trajectory planning. Finally, section 5 concludes this paper.

II. MOVING FRAME METHOD AND DARBOUX FRAME

A brief introduction concerning some basic concepts of differential geometry is given in this section. Details can be found in Carmo [23], Cartan [24], and Cartan [25].

A. Moving frame method

Consider an ordinary three-dimensional Euclidean space. Take a right-handed orthonormal frame T at an arbitrary point M of this space. The set of all orthogonal frames of the space depends on six parameters. The equations of an infinitesimal displacement of the frame T are

$$\begin{cases} d\boldsymbol{M} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3 \\ d\begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{bmatrix} \end{cases} \tag{1}$$

where $\{\omega_1, \omega_2, \omega_3, \omega_{12}, \omega_{13}, \omega_{23}\}$ are differential one-forms.

B. Darboux frame

Let L be an oriented curve traced on an oriented surface S in E^3. *Darboux frame* $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a right-handed orthogonal frame associated with each point of $M \in L$, where $\boldsymbol{e}_1$ is the unit tangent vector to L; $\boldsymbol{e}_3$ is the unit normal to S; $\boldsymbol{e}_2$ is tangential to S and $\boldsymbol{e}_2 = \boldsymbol{e}_3 \times \boldsymbol{e}_1$ as in Fig. 1.

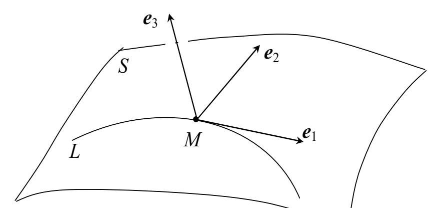

Fig. 1 A Darboux frame at point M

Thus the curve L traced on S defines a one parameter family of frames. The equations of motion of the frame are

$$\begin{cases} d\boldsymbol{M} = \omega_1 \boldsymbol{e}_1 \\ d\begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{bmatrix} \end{cases} \tag{2}$$

where ω_1, ω_{12}, ω_{13}, and ω_{23} are differential one-forms. If the curve L is parameterized by the arc length s, it follows that $\omega_1 = ds$, $\omega_{12} = a\,ds$, $\omega_{13} = b\,ds$, and $\omega_{23} = c\,ds$, where a, b, and c are functions on L. They are called geodesic curvature, normal curvature, and geodesic torsion, respectively. Traditionally they are written as

$$k_g = \frac{\omega_{12}}{\omega_1}, k_n = \frac{\omega_{13}}{\omega_1}, \tau_g = \frac{\omega_{23}}{\omega_1} \tag{3}$$

III. THE KINEMATICS OF ROLLING CONTACT BETWEEN TWO OBJECTS IN E^3 SPACE

Assume that Obj1 and Obj2 undergo rolling contact at every moment. Curve L is the contact curve on the surface S_1 of Obj1 and curve L' on the surface S_2 of Obj2. Since this paper is to study the relative motion between Obj1 and Obj2, Obj1 can be assumed to be fixed. Set up frame {1} fixed on Obj1 and frame {2} on Obj2. The subscripts {1} and {2} will be used to represent in which frame a vector is expressed. The moving frame attached to the contact point M of curve L is $\{\boldsymbol{e}_1^{\{1\}}, \boldsymbol{e}_2^{\{1\}}, \boldsymbol{e}_3^{\{1\}}\}$ and to the contact point M of L' is $\{\boldsymbol{e}_1^{\{2\}}, \boldsymbol{e}_2^{\{2\}}, \boldsymbol{e}_3^{\{2\}}\}$. Vector $\boldsymbol{e}_3^{\{1\}}$ is the unit normal of surface S_1 and $\boldsymbol{e}_3^{\{2\}}$ of S_2. Vector $\boldsymbol{e}_1^{\{1\}}$ is the unit tangent to the curve L and $\boldsymbol{e}_1^{\{2\}}$ to the curve L'. Both $\{\boldsymbol{e}_1^{\{1\}}, \boldsymbol{e}_2^{\{1\}}, \boldsymbol{e}_3^{\{1\}}\}$ and $\{\boldsymbol{e}_1^{\{2\}}, \boldsymbol{e}_2^{\{2\}}, \boldsymbol{e}_3^{\{2\}}\}$ are right-handed orthonormal frame. Due to the rolling constraints, vectors $\boldsymbol{e}_1^{\{1\}}$ and $\boldsymbol{e}_1^{\{2\}}$ are always collinear, so are $\boldsymbol{e}_3^{\{1\}}$ and $\boldsymbol{e}_3^{\{2\}}$. Hence, the two frames can always be made to coincide as in Fig. 2, where vector $\boldsymbol{e}_3^{\{1\}}$ is pointing outwards of the surface S_1 and $\boldsymbol{e}_3^{\{2\}}$ inwards of the surface S_2.

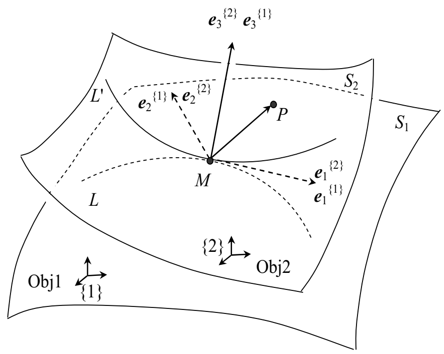

Fig. 2 Obj2 rolls on Obj1 along curve L' of Obj2 and curve L of Obj1

A. The kinematic geometry of rolling contact

Assume that both curve L and L' are parameterized by arc length. Let s denote the arc length of L and s' of L'. While vectors change following a certain rule under coordinate change, scalars remain invariant. In the language of tensor analysis, scalars are type of (0, 0) tensor and vectors are type of (1, 0) tensor. Let P be an arbitrary point of Obj2, then the vector of point P can be expressed in frame {2} as

$$\boldsymbol{P}^{\{2\}} = \boldsymbol{M}^{\{2\}} + u_1'\boldsymbol{e}_1^{\{2\}} + u_2'\boldsymbol{e}_2^{\{2\}} + u_3'\boldsymbol{e}_3^{\{2\}} \tag{4}$$

Differentiating (4) with respect to s' yields

$$\begin{aligned}\frac{d\boldsymbol{P}^{\{2\}}}{ds'} &= \left(1+\frac{du_1'}{ds'} - u_2'k_g' - u_3'k_n'\right)\boldsymbol{e}_1^{\{2\}} \\ &+ \left(\frac{du_2'}{ds'} + u_1'k_g' - u_3'\tau_g'\right)\boldsymbol{e}_2^{\{2\}} \\ &+ \left(\frac{du_3'}{ds'} + u_1'k_n' + u_2'\tau_g'\right)\boldsymbol{e}_3^{\{2\}}\end{aligned} \tag{5}$$

where k_g', k_n', τ_g' are the geodesic curvature, normal curvature, and geodesic torsion at point M of L', respectively. Since point P is a fixed point of Obj2, it follows that (5) equals to 0. Equaling the right side of (5) to 0 yields

$$\begin{cases}\dfrac{du_1'}{ds'} = u_2'k_g' + u_3'k' - 1 \\ \dfrac{du_2'}{ds'} = -u_1'k_g' + u_3'\tau_g' \\ \dfrac{du_3'}{ds'} = -u_1'k_n' - u_2'\tau_g'\end{cases} \tag{6}$$

On the other hand, point P can also be expressed in frame {1} as

$$\boldsymbol{P}^{\{1\}} = \boldsymbol{M}^{\{1\}} + u_1\boldsymbol{e}_1^{\{1\}} + u_2\boldsymbol{e}_2^{\{1\}} + u_3\boldsymbol{e}_3^{\{1\}} \tag{7}$$

Differentiating (7) with respect to s yields

$$\begin{aligned}\frac{d\boldsymbol{P}^{\{1\}}}{ds} &= \left(1+\frac{du_1}{ds} - u_2k_g - u_3k_n\right)\boldsymbol{e}_1^{\{1\}} \\ &+ \left(\frac{du_2}{ds} + u_1k_g - u_3\tau_g\right)\boldsymbol{e}_2^{\{1\}} \\ &+ \left(\frac{du_3}{ds} + u_1k_n + u_2\tau_g\right)\boldsymbol{e}_3^{\{1\}}\end{aligned} \tag{8}$$

where k_g, k_n and τ_g are the geodesic curvature, normal curvature, and geodesic torsion at point M of L, respectively. The rolling constraints require that the velocities of the two contact curves be equal, which results in the arc length covered by the two contact points in the same time period being the same. That is, $s = s'$. Since moving frame $\{\boldsymbol{e}_1^{\{1\}}, \boldsymbol{e}_2^{\{1\}}, \boldsymbol{e}_3^{\{1\}}\}$ and $\{\boldsymbol{e}_1^{\{2\}}, \boldsymbol{e}_2^{\{2\}}, \boldsymbol{e}_3^{\{2\}}\}$ are made to coincide at any moment, it follows that

$$u_1 = u_1', u_2 = u_2', u_3 = u_3' \tag{9}$$

and consequently

$$\frac{du_1}{ds} = \frac{du_1'}{ds'}, \frac{du_2}{ds} = \frac{du_2'}{ds'}, \frac{du_3}{ds} = \frac{du_3'}{ds'} \tag{10}$$

From now on let s denote both s and s' and let u_i denote both u_i and u_i'. Differentiation with respect to s will be denoted by "dot". Substituting (6) and (10) into (8) yields

$$\begin{aligned}\dot{\boldsymbol{P}}^{\{1\}} &= \left(u_2k_g^* + u_3k_n^*\right)\boldsymbol{e}_1^{\{1\}} + \left(-u_1k_g^* + u_3\tau_g^*\right)\boldsymbol{e}_2^{\{1\}} \\ &+ \left(-u_1k_n^* - u_2\tau_g^*\right)\boldsymbol{e}_3^{\{1\}}\end{aligned} \tag{11}$$

where

$$k_g^* = k_g' - k_g, k_n^* = k_n' - k_n, \tau_g^* = \tau_g' - \tau_g \tag{12}$$

are called induced geodesic curvature, normal curvature, and geodesic torsion, respectively.

B. The kinematic meaning of the induced geodesic curvature, normal curvature, and geodesic torsion

In this subsection the kinematic meaning of induced geodesic curvature, normal curvature, and geodesic torsion will be discussed. To simplify the notation, the superscript {1} will be omitted. It is understood that all the vectors are expressed in frame {1}.

From (11) the velocity of an arbitrary point P on Obj2 can be obtained as

$$\boldsymbol{v}_P = \frac{d\boldsymbol{P}}{ds}\frac{ds}{dt} = \sigma\begin{pmatrix}\left(u_2k_g^* + u_3k_n^*\right)\boldsymbol{e}_1 + \left(-u_1k_g^* + u_3\tau_g^*\right)\boldsymbol{e}_2 \\ + \left(-u_1k_n^* - u_2\tau_g^*\right)\boldsymbol{e}_3\end{pmatrix} \tag{13}$$

where $\sigma = ds/dt$ is the magnitude of rolling velocity. On the other hand, suppose that the angular velocity of Obj2 with respect to Obj1 is $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2 + \omega_3\boldsymbol{e}_3$, the velocity of point P can also be obtained as

$$\begin{aligned}\boldsymbol{v}_P = \boldsymbol{\omega}\times\boldsymbol{r}_{MP} &= \left(-u_2\omega_3 + u_3\omega_2\right)\boldsymbol{e}_1 + \left(u_1\omega_3 - u_3\omega_1\right)\boldsymbol{e}_2 \\ &+ \left(-u_1\omega_2 + u_2\omega_1\right)\boldsymbol{e}_3\end{aligned} \tag{14}$$

where $\boldsymbol{r}_{MP} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3$. Comparing (13) with (14) gives

$$\omega_1 = -\sigma\tau_g^*, \omega_2 = \sigma k_n^*, \omega_3 = -\sigma k_g^* \tag{15}$$

Thus the angular velocity of Obj2 is

$$\boldsymbol{\omega} = \sigma\left(-\tau_g^*\boldsymbol{e}_1 + k_n^*\boldsymbol{e}_2 - k_g^*\boldsymbol{e}_3\right) \tag{16}$$

From (16), higher order differentiation of $\boldsymbol{\omega}$ and $\boldsymbol{v}_p$ can be obtained with respect to time t.

C. Examples

Two examples are given in this subsection to show how to apply the proposed approach.

Example 1. Consider the classical example of a disk of radius R rolling on the plane as in Fig. 3. The disk always remains exactly upright. The contact curves are the circle L' of the disk and the curve L in the plane. In this case, the disk can be considered as a degenerated surface or a ball with a big circle as the contact curve.

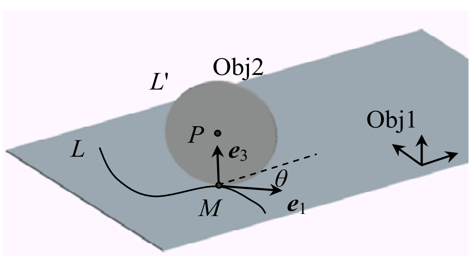

Fig. 3 A disk rolls on the plane

Suppose the curvature of L is k and $\boldsymbol{e}_3$ is in the outward direction of the plane. The Darboux equations of L are

$$\frac{d}{ds}\begin{bmatrix}\boldsymbol{e}_1^{\{1\}}\\ \boldsymbol{e}_2^{\{1\}}\\ \boldsymbol{e}_3^{\{1\}}\end{bmatrix} = \begin{bmatrix}0 & k & 0\\ -k & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_1^{\{1\}}\\ \boldsymbol{e}_2^{\{1\}}\\ \boldsymbol{e}_3^{\{1\}}\end{bmatrix} \tag{17}$$

Let the unit normal $\boldsymbol{e}_3$ of the disk be inward, the Darboux equations of L' are

$$\frac{d}{ds}\begin{bmatrix}\boldsymbol{e}_1^{\{2\}}\\ \boldsymbol{e}_2^{\{2\}}\\ \boldsymbol{e}_3^{\{2\}}\end{bmatrix} = \begin{bmatrix}0 & 0 & \frac{1}{R}\\ 0 & 0 & 0\\ -\frac{1}{R} & 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_1^{\{2\}}\\ \boldsymbol{e}_2^{\{2\}}\\ \boldsymbol{e}_3^{\{2\}}\end{bmatrix} \tag{18}$$

From (16) it follows that the angular velocity of the disk is

$$\boldsymbol{\omega}^{\{1\}} = \sigma\left(\frac{1}{R}\boldsymbol{e}_2^{\{1\}} - k\boldsymbol{e}_3^{\{1\}}\right) \tag{19}$$

Remark 1: In textbooks about dynamics, such as [26], four general coordinates (x, y, θ, φ) are taken to study this mechanical system. If the following replacement is made

$$\begin{aligned}&\boldsymbol{e}_1^{\{1\}} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \boldsymbol{e}_2^{\{1\}} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j},\\ &\sigma = R\frac{d\varphi}{dt} = \frac{ds}{dt}, k = \frac{d\theta}{ds}\end{aligned} \tag{20}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the usual Euclidean coordinates in the x and y directions. It is easily checked (19) is the same as those appearing in the textbooks. □

Example 2. A ball of radius R_2 (Obj2) rolls without sliding on a cylindrical fingertip of radius R_1 (Obj1). The small circle L' is the contact curve on Obj2 and the helix L is the contact curve on Obj1 as in Fig. 4.

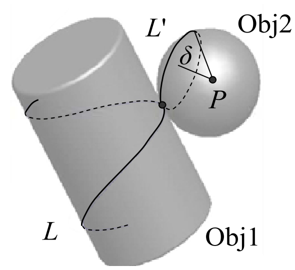

Fig. 4 Obj2 rolls on Obj1 along curves L' of Obj2 and L of Obj1

Suppose the pitch of curve L is h and the unit normal $\boldsymbol{e}_3$ of Obj1 is outward, then the Darboux equations of L are

$$\frac{d}{ds}\begin{bmatrix}\boldsymbol{e}_1^{\{1\}}\\ \boldsymbol{e}_2^{\{1\}}\\ \boldsymbol{e}_3^{\{1\}}\end{bmatrix} = \begin{bmatrix}0 & 0 & -\frac{R_1}{R_1^2+h^2}\\ 0 & 0 & \frac{h}{R_1^2+h^2}\\ \frac{R_1}{R_1^2+h^2} & -\frac{h}{R_1^2+h^2} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_1^{\{1\}}\\ \boldsymbol{e}_2^{\{1\}}\\ \boldsymbol{e}_3^{\{1\}}\end{bmatrix} \tag{21}$$

Let the unit normal $\boldsymbol{e}_3$ of Obj2 be inward, then the Darboux equations of L' are

$$\frac{d}{ds}\begin{bmatrix} \boldsymbol{e}_1^{\{2\}} \\ \boldsymbol{e}_2^{\{2\}} \\ \boldsymbol{e}_3^{\{2\}} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\cot\delta}{R_2} & \frac{1}{R_2} \\ \frac{\cot\delta}{R_2} & 0 & 0 \\ -\frac{1}{R_2} & 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{e}_1^{\{2\}} \\ \boldsymbol{e}_2^{\{2\}} \\ \boldsymbol{e}_3^{\{2\}} \end{bmatrix} \tag{22}$$

where δ is the half cone angle. From (16) the angular velocity of Obj2 can be obtained as

$$\boldsymbol{\omega}^{\{1\}} = \sigma\left(\frac{h}{R_1^2+h^2}\boldsymbol{e}_1^{\{1\}} + \left(\frac{1}{R_2}+\frac{R_1}{R_1^2+h^2}\right)\boldsymbol{e}_2^{\{1\}} + \frac{\cot\delta}{R_2}\delta\boldsymbol{e}_3^{\{1\}}\right) \tag{23}$$

IV. SOME IMPLICATIONS FOR TRAJECTORY PLANNING

The properties of a surface are revealed through studying curves on it. Some curves have special continuous geometric characteristics. For instant, a line of curvature is a regular connected curve such that the geodesic torsion τ_g equals to zero. A geodesic is a regular connected curve such that the geodesic curvature k_g equals to zero. Previous literature on rolling motion only paid attention to the properties at a single point of the surfaces in question and neglected these characteristic curves. It turns out that qualitative information about the necessary contact trajectory curves can be deduced from (16) if some constraints, such as motion or shapes of objects, are taken into consideration. In this section, pure-rolling motion and twist-rolling motion with specific geometric constraints are investigated to illustrate this point.

A. Pure Rolling motion

If the angular velocity in the direction of $\boldsymbol{e}_3$ is 0, the rolling object is defined to undergo a pure-rolling motion. It can be seen from (16) that if there only exists velocity constraint, the moving object will undergo twist-rolling motion. For an object to undergo pure rolling motion, there need additional physical constraints. Furthermore, It is obvious that if two rigid objects undergo pure-rolling motion, that is, $\omega_3 = 0$, the values of the geodesic curvature of the two contact curves have to be the same. Thus the following conclusion can be reached.

Corollary 1: If two objects undergo pure-rolling motion, the values of the geodesic curvature of the two corresponding contact curves have to be the same. □

Remark 2: From the above corollary, it can be seen that if the contact curves are two geodesics of the two bodies, the condition that the values of geodesic curvature of the contact curves have the same value can be naturally satisfied. The other benefit is that the distance between two configurations is the shortest if the surfaces are complete, which is often true for objects under consideration. □

It is often true that the shapes of the objects are known in advance and a certain motion of the moving object is preferred. In this case, qualitative information about the contact curve on the fixed object can be deduced. Next two examples are given to illustrate this point.

Example 3. Again, the classical example of a disk of radius R rolling on the plane is used. This time the disk is preferred to undergo pure-rolling motion. From (19), it can be seen that the curvature of the curve on the plane has to be zero, which means that the curve is a straight line on the plane.

Example 4. A sphere with radius R_2 undergoes pure-rolling motion on a plane along a small circle L' on the sphere and a curve L on the plane as in Fig. 5.

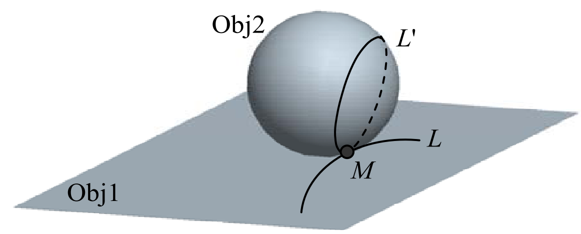

Fig. 5 A sphere rolls on a plane along a small circle of the sphere

From (16), (17), and (22), it follows that the angular velocity of the twist-rolling motion is

$$\boldsymbol{\omega}^{\{1\}} = \sigma\left(\frac{1}{R_2}\boldsymbol{e}_2^{\{1\}} - \left(\frac{\cot\delta}{R_2}+k\right)\boldsymbol{e}_3^{\{1\}}\right) \tag{24}$$

Since pure-rolling motion is preferred, it follows that

$$k = -\frac{\cot\delta}{R_2} \tag{25}$$

which means that curve C has to be a circle with radius $R_2/\cot\delta$.

B. Twist-rolling motions considering characteristics of the shapes

Even if an object undergoes a more general twist-rolling motion, qualitative information about the contact curves can be deduced. Such information is valuable when motion planning task is performed. One example is given to show this point.

Example 5. A disk of radius R rolls without sliding on a surface and remains upright at all time as in Fig. 6. The constraint of remaining upright means that the normal vector of the surface always points to the center of the disk. It can also be imagined as a ball rolls on a smooth surface along one big circle.

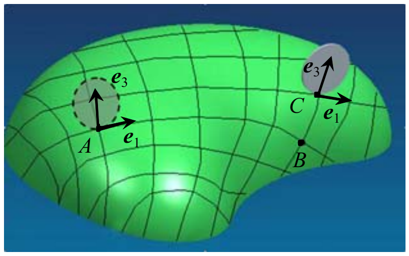

Fig. 6 A disk rolls on a surface

From (2), (16), and (17), it follows that the angular velocity of the disk is

$$\boldsymbol{\omega}^{\{1\}} = \sigma\left(\tau_g \boldsymbol{e}_1^{\{1\}} + \left(\frac{1}{R} - k_n\right)\boldsymbol{e}_2^{\{1\}} + k_g \boldsymbol{e}_3^{\{1\}}\right) \tag{26}$$

The disk remaining upright means that there is no angular velocity in the direction of $\boldsymbol{e}_1$. Hence τ_g has to be zero. From this it follows that the contact curve on the surface has to be one of the lines of curvature.

The black curves in Fig. 6 mark the lines of curvature of the surface. If the disk is to roll uprightly from position *A* to position *C*, it can follow segment *AB* of one line of curvature and then along segment *BC* of other line of curvature. Of course, there are infinitely many paths the disk can follow as long as the paths are lines of curvature.

V. Conclusions

This paper adopted the moving frame method to study the instantaneous kinematics of rigid bodies with rolling contact. The proposed approach had the benefit that it did not depend on the origin and orientation of the global coordinate system. The rolling constraints required that the velocities of the two contact curves be equal, which resulted in the arc length covered by the two contact points in the same period of time being the same. This greatly simplified the geometric kinematics analysis. After time being taken into consideration, it was found that the rolling speed and two sets of geometric invariants belonging to the respective contact surfaces, totally determined the instantaneous kinematics of moving object. The result was expressed in terms of geometric invariants which could be readily generalized to arbitrary parametric surfaces and contact curves. The effects of the relative curvature and torsions on rolling kinematics were explicitly represented. This paper then showed that qualitative information about the contact curves could be deduced from the derived equation and discussed two cases including pure-rolling motion and twist-rolling motion with additional geometric constraints. It was hoped that new light could be shed on dexterous manipulations and motion planning.

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