Pareto optimality solution of the Gauss-Helmert model

B.Paláncz
Department of Photogrammetry and Geoinformatics
Budapest University of Technology and Economy, H-1521, Hungary
e-mail: palancz@epito.bme.hu

J. L. Awange
Department of Spatial Sciences
Curtin University, Australia
e-mail: j.awange@curtin.edu.au

L. Völgyesi
Department of Geodesy and Surveying
Budapest University of Technology and Economy, H-1521, Hungary
e-mail: volgyesi@eik.bme.hu

Abstract

The Pareto optimality method is applied to the parameter estimation of the Gauss-Helmert weighted 2D similarity transformation assuming that there are measurement errors and/or modeling inconsistencies.

In some cases of parametric modeling, the residuals to be minimized can be expressed in different forms resulting in different parameter values for the estimated parameters. Sometimes these objectives may compete in the Pareto sense, namely a small change in the parameters can result in an increase in one of the objectives on the one hand, and a decrease of the other objective in the other hand. In this study, Pareto optimality approach was employed to find the optimal trade-off solution between the conflicting objectives and the results compared to those from ordinary least squares (OLS), total least squares (TLS) techniques and the least geometric mean deviation (LGMD) approach.

The results indicate that Pareto optimality can be considered as their g*eneralization* since the Pareto optimal solution produces a set of optimal parameters represented by the Pareto-set containing the solutions of these techniques (error models). From the Pareto-set, a single optimal solution can be selected on the basis of the decision maker's criteria. The application of Pareto optimality needs nonlinear multi-objective optimization, which can be easily achieved in parallel via hybrid genetic algorithms built-in engineering software systems such as *Matlab*. A real-word problem is investigated to illustrate the effectiveness of this approach.

Keywords
Pareto optimality, Gauss-Helmert transformation, parameter estimation, measurement and modeling errors, least squares approach, genetic algorithm.

1. Introduction

Parameter estimation of coordinate transformations is a central problem in geodesy, and requires the handling of overdetermined systems of nonlinear equations containing more equations (n) than unknown parameters (m)

$$y_i = g_i(x, p) + e_{y_i}, \quad i = 1, 2, ..., n, \tag{1}$$

which includes a parameter vector p of dimension m, as unknown, and observation vectors x and y. In this model, the underlying assumption is that the observation vector x is error-free, and only y has measurement errors e_y. This traditional error model can be solved in the least squares sense by using the ordinary least squares method (OLS).

However, this model is often invalid as many physical systems encounter errors in both the observation vectors, arising from, e.g., imprecise measuring instruments, human errors, etc. In such cases, the deviation between the model and the measured variables can be expressed by the compensation in the input variables, too

$$y_i = g_i(x - e_x, p) + e_{y_i}, \quad i = 1, 2, ..., n. \tag{2}$$

This model representing errors in both input as well as in output variables is called *errors-in-all-variables* (EIV) model whose parameter estimation problem leads to a minimization

$$\min_{p, e}(e^T \cdot e) \tag{3}$$

under the constraint of Eq. (2), where $e = \begin{pmatrix} e_y \\ e_x \end{pmatrix}$. This is a nonlinear problem even if the model (g) is linear, since now the errors (e) are also unknowns besides the parameters (p).

One of the frequently employed techniques to solve this parameter estimation problem is the so called total least squares method (TLS), see e.g. Golub and Van Loan (1980). Since this technique is basically designed for linear cases, one needs to linearize the original problem. For the *linear* Gauss-Helmert (GH) transformation, the EIV problem has been solved e.g., by Felus and Schaffrin (2005) employing the TLS technique. For weighted coordinates, namely

$$\min_{p, e}(e_y, e_x) \begin{pmatrix} w_y & 0 \\ 0 & w_x \end{pmatrix} \begin{pmatrix} e_y \\ e_x \end{pmatrix}, \tag{4}$$

Neitzel (2010) discussed the solution suggesting an *iteratively linearized* TLS (IL - TLS) method.

The EIV model assumes that there are no modeling inconsistencies, namely the $y = g(x, p)$ model would be adequate, error-free if there were no measurement errors. This assumption is represented by Eq. (2).

In order to avoid this supposition and taking into account possible modeling errors, Tofallis (2002) and (2003) introduced a different approach to solve the EIV problem. He considered a simple line fitting, assuming that $x, y \in R^1$ and $p \in R^2$. Then Eq (1) becomes

$$y_i - e_{y_i} = x_i p_1 + p_2, \quad i = 1, 2, ..., n. \tag{5}$$

Now the deviation between the measurements and the model can be expressed by the compensation of the input too, namely

$$y_i - e_{y_i} = (x_i - e_{x_i})p_1 + p_2, \quad i = 1, 2, ..., n .$$ (6)

He then proposed to minimize the sum of the squares of the geometric means of the errors,

$$\min_{p_1, p_2} \sum_{i=1}^{n} \left(\sqrt{|e_{x_i} e_{y_i}|} \right)^2 .$$ (7)

This technique is referred as *least geometric mean deviation* (LGMD) method.

To find another solution technique for the EIV problem of the GH model, let us express the squares of the errors considering Eqs (5) and (6) as,

$$f_1(p) = e_y^T e_y$$ (8)

and

$$f_2(p) = e_x^T e_x .$$ (9)

Realizing that these functions are competing such that one increases at the expense of the other, Pareto optimality can be employed as an alternative method.

In this study, an extension of this approach for the GH model with weighted coordinates will be illustrated and the results compared to those of the IL-TLS, as well as, to those of the LGMD method.

The study is organized as follows: Section 2 provides a brief outline of the Pareto optimality approach while Section 3 considers, the parameter estimation problem of the 2D similarity transformation (GH) as a multiobjective minimization problem. In Section 4, a numerical example illustrates the suggested approach. Section 5 compares the different parameter estimation techniques and Section 6 concludes the features of the Pareto optimality solution by contributing to the solution of a real-world 2D datum transformation problem of Hungarian geodetic stations covering an area of 70×70 km.

2. Pareto optimality

In many real-life situations, there are *multi-objective optimality problems*, which means that there is more than one objective to be minimized or maximized. In cases where all of the objective functions increase or decrease, there exists no optimum. However, in regions where these objective functions are competing or conflicting with each other, meaning that a small change in the independent variables will result in an increase of one objective function, an optimum can exist. We call such regions *feasible regions for optimal solutions*.

A solution in such region is said to be a *Pareto optimal solution* if it is *not dominated* by any other solution in that region. Pareto optimality is defined as follows, see e.g. Marler and Arora (2004):

Definition: A point, $x^* \in X$, is Pareto optimal if there exists no other point, $x \in X$, such that $f(x) \leq f(x^*)$ and $f_i(x) < f_i(x^*)$ for at least one function.

From the above definition, the Pareto optimal solution is therefore a set of solutions, rather than a single one. The independent variables representing these solutions in the variable space form a *Pareto-set*, and the corresponding values of the objective functions are labeled as the *Pareto-front*. In our case the objective functions are convex therefore the Pareto front is also convex and connected.

The selection of a single optimum from the Pareto set needs a trade-off strategy to be implemented by the user (*decision maker*). A possible selection can be a point of the Pareto-front which is closest to the *ideal point* represented by ($f_i = 0$) for all *i*.

3. The multi-objective problem

Let us consider the Gauss-Helmert model of a weighted 2D similarity transformation problem,

$$\begin{pmatrix} X \\ Y \end{pmatrix} = F(x,y) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \tag{10}$$

where α is the rotation angle, β is the scale factor and γ, δ are the coordinates of the translation. Considering X, Y and x, y as Cartesian coordinates of the transformed and original systems, respectively, the problem is now concerned with determining the 4 unknown parameters (α, β, γ, δ) from the corresponding measured data pairs {(X_i, Y_i), (x_i,y_i)} having the weights {(WX_i, WY_i) and (wx_i, wy_i)}. It has been assumed that x_i and y_i are considered as independent. The following multi-objective problems can be based on the sum of squares of the local residuals:

a) In case of the transformation $(x,y) \rightarrow (X,Y)$ given by

$$f_1(\alpha,\beta,\gamma,\delta) = \sum_{i=1}^{n} f_{1i}(\alpha,\beta,\gamma,\delta). \tag{11}$$

where

$$\begin{aligned} f_{1i}(\alpha,\beta,\gamma,\delta) = &WX_i(X_i - \beta(\cos(\alpha)x_i - \sin(\alpha)y_i) + \gamma)^2 \\ &+ WY_i(Y_i - \beta(\sin(\alpha)x_i + \cos(\alpha)y_i) + \delta)^2. \end{aligned} \tag{12}$$

b) The inverse transformation $(X,Y) \rightarrow (x,y)$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = F^{-1}(X,Y) = \begin{pmatrix} 1/\beta & 0 \\ 0 & 1/\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} X-\gamma \\ Y-\delta \end{pmatrix} \tag{13}$$

and a similar objective,

$$f_2(\alpha,\beta,\gamma,\delta) = \sum_{i=1}^{n} f_{2i}(\alpha,\beta,\gamma,\delta) \tag{14}$$

where

$$\begin{aligned} f_{2i}(\alpha,\beta,\gamma,\delta) = &wx_i\left(x_i - \frac{\cos(\alpha)(X-\gamma) + \sin(\alpha)(Y-\delta)}{\beta}\right)^2 \\ &+ wy_i\left(y_i - \frac{\cos(\alpha)(Y-\gamma) - \sin(\alpha)(X-\gamma)}{\beta}\right)^2. \end{aligned} \tag{15}$$

The multiobjective problem is to find the minimum of the competing objectives f_1 and f_2 in the design space – in our case $(\alpha,\beta,\gamma,\delta) \in R^4$ – in the sense of Pareto optimality.

To solve the multiobjective optimization problem, the multiobjective genetic algorithm (GA) implemented as a built-in function in *Matlab* as (*gamultiobj*) is applied.

In order to improve the efficiency of the procedure a hybrid scheme is used to find an optimal Pareto front for our multiobjective problem. The function *gamultiobj* can reach the region near an optimal Pareto front relatively quickly, but it can take many function evaluations to achieve convergence. A commonly used technique is to run *gamultiobj* for a small number of generations to get near an optimum front. Then the solution from *gamultiobj* is used as an initial point for another optimization solver that is faster and more efficient for a local search. Here *fgoalattain* is employed as the hybrid solver with *gamultiobj*.

One of the main advantages of this genetic hybrid algorithm is the parallelization, – the parallel evaluation of the genetic algorithm – especially in our days, when multicore processors are easily available.

4. Numerical Example

For the numerical computation the data from *Akyilmaz* (2007) is employed. Corresponding coordinates of the two systems can be seen in Table I and the weights are given in Table II. Fig. 1 shows the Pareto-front of our multiobjective minimization problem.

To ensure reliable results the following parameters were used in the *gamultiobj* function, error limit *TolFun* $= 10^{-12}$, limit for the number of generations *StallGenLimit* $= 1000$ and the number of the individuals of the population *PopulationSize* $= 300$. The computed Pareto-front consisted of 127 points. A single optimum can be selected by considering

$$\min_{\alpha,\beta,\gamma,\delta} \sqrt{f_1(\alpha,\beta,\gamma,\delta) + f_2(\alpha,\beta,\gamma,\delta)}, \qquad (16)$$

where f_1 and f_2 are the coordinates of a point of the Pareto-front. Practically, this optimal point is the closest point of the Pareto-front to the ideal point (0,0) in the sense of the L_1 norm. The corresponding point of the Pareto-set $(\alpha,\beta,\gamma,\delta)$ can be considered as the *selected Pareto solution*. The selected optimum point of the Pareto-front as the minimum of f_1+f_2 can be seen on Fig. 2.

The parameters belonging to the selected point of the Pareto-front and the coordinates of the corresponding point of the Pareto-set are presented in Table III.

5. Comparing Pareto optimality to other approaches

Let us compare our results to those of the other approaches into the solution of the EIV problems.

5.1 Ordinary least squares solutions

a) In case of the transformation $(x, y) \rightarrow (X, Y)$ minimizing Eq. (11), we get the parameters, displayed in Table IV.

b) The inverse transformation $(X, Y) \rightarrow (x, y)$ minimizing Eq. (14), leads to the parameters, shown in Table V.

5.2 Total least squares solution

The TLS approach leads to a constrained minimization problem. The objective function is the sum of the weighted squares of the adjusted values of the measured variables,

$$F = \sum_{i=1}^{n} WX_i \Delta X_i^2 + \sum_{i=1}^{n} WY_i \Delta Y_i^2 + \sum_{i=1}^{n} wx_i \Delta x_i^2 + \sum_{i=1}^{n} wy_i \Delta y_i^2 , \qquad (17)$$

under the constraints of the model equations,

$$g_{1i}(\alpha, \beta, \gamma, \Delta X_i, \Delta x_i, \Delta y_i) = 0, \quad i = 1, 2, ..., n , \qquad (18)$$

$$g_{2i}(\alpha, \beta, \delta, \Delta Y_i, \Delta x_i, \Delta y_i) = 0, \quad i = 1, 2, ..., n , \qquad (19)$$

where

$$g_{1i}(\alpha, \beta, \gamma, \Delta X_i, \Delta x_i, \Delta y_i) = X_i + \Delta X_i - \beta\big(\cos(\alpha)(x_i + \Delta x_i) - \sin(\alpha)(y_i + \Delta y_i)\big) + \gamma , \quad (20)$$

$$g_{2i}(\alpha, \beta, \delta, \Delta Y_i, \Delta x_i, \Delta y_i) = Y_i + \Delta Y_i - \beta\big(\sin(\alpha)(x_i + \Delta x_i) + \cos(\alpha)(y_i + \Delta y_i)\big) + \delta . \quad (21)$$

Here $\Delta X_i, \Delta Y_i, \Delta x_i$ and Δy_i, the adjusted values of the output as well as the input variables are also unknowns besides the four model parameters $(\alpha, \beta, \gamma, \delta)$.

Neitzel (2010) has solved this problem by employing the iteratively linearized least squares technique. The result can be seen in Table VI.

Since the model is linear, ΔX_i and ΔY_i can be expressed from Eqs (20) and (21) and can be substituted into Eq. (17). In this way the constrained optimization problem can be transformed into an unconstrained one and the number of unknowns can be reduced from $4n+4$ down to $2n+4$.

5.3 Least geometric mean deviation approach

Employing the LGMD technique suggested by *Tofallis* (2002) for a single variable problem, one needs to consider

$$G(\alpha, \beta, \gamma, \delta) = \sum_{i=1}^{n} \sqrt{f_{1i}(\alpha, \beta, \gamma, \delta) \, f_{2i}(\alpha, \beta, \gamma, \delta)} , \qquad (22)$$

which corresponds to Eq (7), see Eqs. (12) and (15). The results of this minimization can be found in Table VII. The resulted values of the parameters are very close to the values given by the IL - TLS as well as Paretro optimum method.

5.4 Comparison

The results of these techniques are compared to the Pareto optimality solution in Fig. 3. According to our illustrative example, the LGMD solution is Pareto optimal, while

the IL-TLS approach is very near to the Pareto front, therefore practically the Pareto-set can represent all of these solutions. Consequently the Pareto optimality solution can be considered as a generalization of these solutions, see Fig. (3).

The selected Pareto solution is the point of the Pareto-front whose distance from the ideal point (0,0) is minimum. The Pareto solution and the LGMD solution have practically the same value. The difference of the residuals representing the measure of the symmetry of the errors in the two different coordinate systems is also minimal in the case of the selected Pareto solution, see the third row in Table VIII.

Consequently, the selected Pareto solution can be considered as a single Pareto optimal solution, which has a mostly balanced minimum sum of squares of residuals. However, the user (decision maker) can independently define his/her specified trade-off between the errors of the direct and inverse transformations by selecting another point on the Pareto-front.

6. A real-word problem

Let us consider the Budapest stereographic and the EOV coordinates (Unified National Projection System in Hungary (Völgyesi et al, 1996)) of 13 geodetic stations covering an area of 70×70 km around Budapest, in Hungary. The coordinates of the same physical points are given in the systems (x,y) and (X,Y).

In this case no weights were applied. The different approaches were employed to estimate the parameters of the Gauss-Helmert transformation model. The minimization of the total residual was carried out with the genetic algorithm as global optimization method. The Pareto solution has been computed via a hybrid genetic multiobjective solver. The results can be seen in Table IX.

These results indicate very small deviations in the parameters estimated by the different methods although the Pareto solution produced the minimal total residual. The small differences of the residuals of the two coordinate systems $f_1 - f_2$ shows that their errors are well balanced. Figs. (4) and (5) illustrate the distribution of the local error (norm of the local error vector) computed by the Pareto method in the system (X,Y) and (x,y), respectively.

These two figures are nearly identical indicating that the distribution of the local errors is also well balanced between the two systems. The relatively large local errors of all of the methods indicate that the Gauss-Helmert model is not properly suitable for the present geodetic transformation problem. However, this example shows that neither of the considered technique is able to cure the problem of the GH transformation in case of large distortions. To reduce the local errors considerably, a nonlinear transformation should be employed, see Völgyesi et al. (1996).

Fig. 6 illustrates the adjusted vectors of the TLS solution. The arrows-vectors represent the corrections (adjustments) of the measured coordinates. This figure shows that the local error of the Pareto approach and the adjustments of the TLS model are in good harmony, since where the local error of the selected Pareto solution is large, there the TLS solution provides large adjusted values.

In this sense, the Pareto approach – as a generalization of the error models discussed here – can be a good candidate for the parameter estimation of geodetic datum transformations.

7. Conclusions

According to the results there is no significant difference between the selected Pareto solution and the extended LGMD method, although the latter requires less computation power than the Pareto solution. We should stress that none of these methods strictly satisfies the model equations, therefore they are able to take modeling inconsistencies into account to a certain extent. If the model is proved to be perfectly adequate then the TLS solution taking only measurement errors into account is more reasonable, since it satisfies the model equations and provides the necessary adjustments of the coordinates in both systems. However, the in case of real world problems, to use the TLS method requires global minimization with $4n+m$ variables. In our case, the number of the variables was $4\times13+4=56$, which could be reduced by utilizing the linearity of the transformation down to $2\times13+4=30$.

Acknowledgements

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Table Captions

Table I: Corresponding coordinates of the two systems

Table II: Weights of the measured coordinates of the two systems

Table III: Results of the Pareto optimality computation

Table IV: Results of the Ordinary Least Squares solution in case $(x,y) \rightarrow (X,Y)$

Table V: Results of the Ordinary Least Squares solution in case $(X,Y) \rightarrow (x,y)$

Table VI: Iteratively Linearized Total Least Squares solution, Neitzel (2010)

Table VII: Results of the Least Geometric Mean Deviation approach

Table VIII: Norm and the difference of the objective functions

Table IX: Resulting parameters of the Pareto solution

Table I. Corresponding coordinates of the two systems

X [m]	Y[m]	x [m]	y[m]
4540134.278	382379.896	4540124.094	382385.998
4539937.389	382629.787	4539927.225	382635.869
4539979.739	381951.478	4539969.567	381957.570
4540326.461	381895.009	4540316.294	381901.093
4539216.387	382184.435	4539206.211	382190.528

Table II. Weights of the measured coordinates of the two systems

WX	WY	wx	wy
10.0000	14.2857	5.8824	12.5000
0.8929	1.4286	0.9009	1.7241
7.1429	10.0000	7.6923	16.6667
2.2222	3.2259	4.1667	6.6667
7.6923	11.1111	8.3333	16.6667

Table III. Results of the Pareto optimality computation

	Selected Pareto optimum
α [rad]	-0.0000039722
β	0.9999951685
γ [m]	30.593
δ [m]	13.786
f_1 [m^2]	0.002200
f_2 [m^2]	0.002332

Table IV. Results of the Ordinary Least Square solution in case $(x,y) \rightarrow (X,Y)$

	OLS $(x,y) \rightarrow (X,Y)$
α [rad]	-0.0000058613
β	0.9999970178
γ [m]	21.475
δ [m]	21.654
f_1 [m^2]	0.002113
f_2 [m^2]	0.002610

Table V. Results of the Ordinary Least Square solution in case $(X,Y) \rightarrow (x, y)$

	OLS $(X,Y) \rightarrow (x, y)$
α [rad]	-0.0000024465
β	0.9999937289
γ [m]	37.709
δ [m]	7.410
f_1 [m^2]	0.002419
f_2 [m^2]	0.002256

Table VI. Iteratively Linearized Total Least Square solution, Neitzel (2010)

	IL - TLS
α [rad]	-0.0000042050
β	0.9999953579
γ [m]	29.643
δ [m]	14.769
f_1 [m^2]	0.002174
f_2 [m^2]	0.002401

Table VII. Results of the Least Geometric Mean Deviation approach

	LGMD
α [rad]	-0.0000041015
β	0.9999952510
γ [m]	30.168
δ [m]	14.341
f_1 [m^2]	0.002193
f_2 [m^2]	0.002340

Table VIII. Norm and the difference of the objective functions

	$\sqrt{f_1 + f_2} \times 10^3$	$|f_1 - f_2| \times 10^4$
OLS $(x, y) \rightarrow (X, Y)$	3.3581	4.97
OLS $(X, Y) \rightarrow (x, y)$	3.3077	1.63
IL-TLS	3.2390	2.27
LGMD	3.2070	1.47
Selected Pareto optimum	3.2060	1.32

Table IX. Resulting parameters of the Pareto solution

α [rad]	0.000063492
β	0.999931677
γ [m]	649999.949
δ [m]	238104.852
f_1 [m^2]	3.2329
f_2 [m^2]	3.2333
$\sqrt{f_1 + f_2}$ [m]	4.5723
$|f_1 - f_2|$ [m^2]	0.0004

Figure Captions

Fig. 1:. The Pareto-front

Fig. 2: The selected optimum point of the Pareto-front as the minimum of f_1+f_2

Fig. 3: Results of the different approaches: TLS(▲), LGMD (■) and the selected Pareto (●) solution

Fig. 4: Contour plot of the local error in the X-Y system in case of the Pareto solution.

Fig. 5: Contour plot of the local error in the x-y system in case of the Pareto solution.

Fig. 6: Contour plot of the local error in the X-Y system in case of the Pareto solution. The vector field illustrates the adjusted vectors (ΔX, ΔY) of the TLS solution
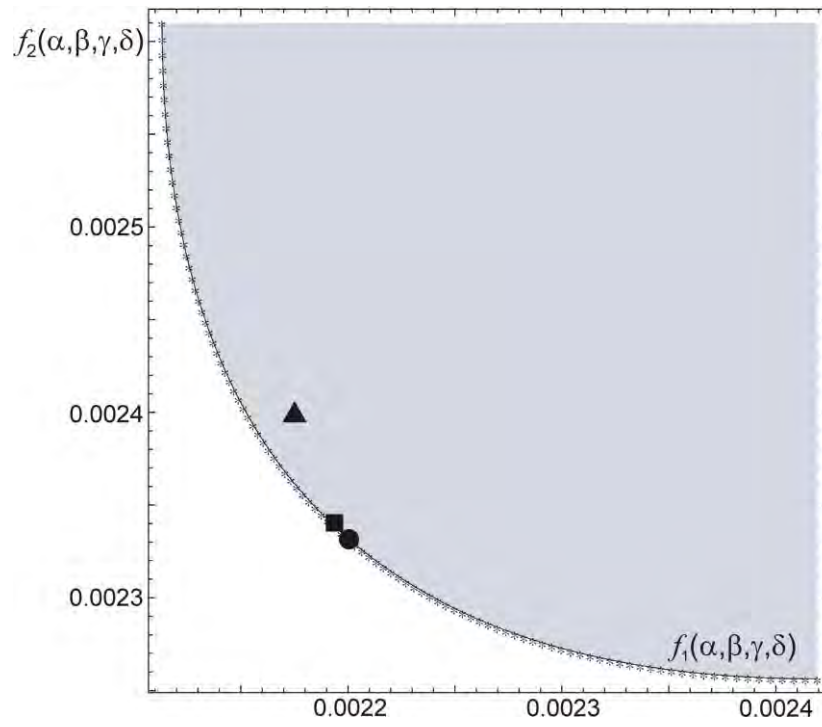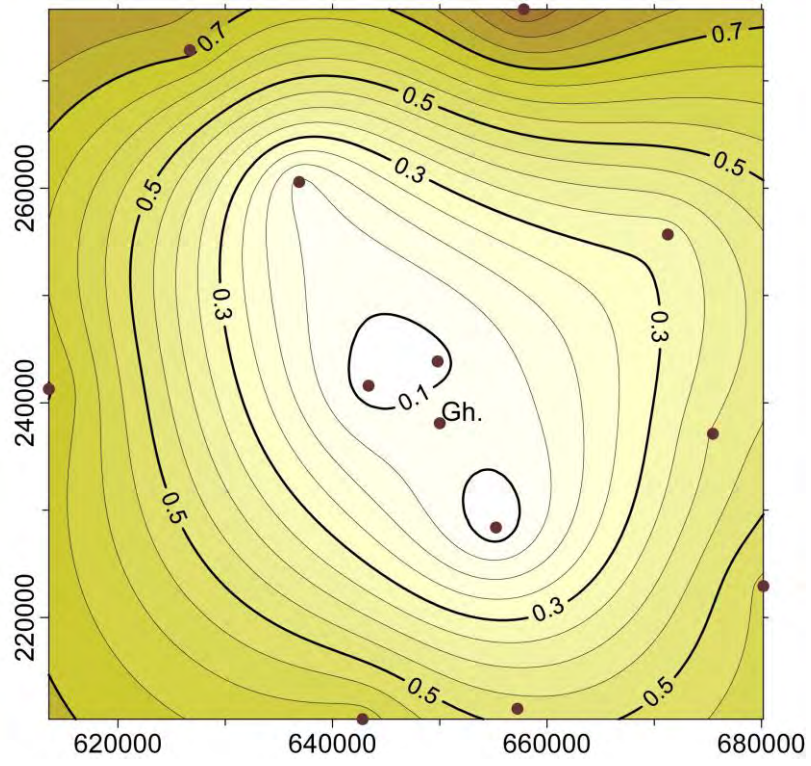
Fig. 1.

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6