White Noise: A Time Domain Basis

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Abstract--**Time domain definitions for finite bandwidth white noise, and filtered white noise, are detailed and are of pedagogical value. The associated power spectral density and autocorrelation functions are given. The potential Gaussianity of white noise is noted.**

Keywords--***white noise; Gaussian white noise; power spectral density; autocorrelation***.

I. INTRODUCTION

In Engineering, it is usual to define a white noise random process mathematically in terms of a specified autocorrelation and/or power spectral density function with the underlying physical random process being left undefined. Specifically, white noise is usually defined, e.g. [1], [2], as a noise phenomena which is uncorrelated at all arbitrarily chosen pairs of time instants, with a time averaged autocorrelation function, $R(\tau)$, which is an impulse and with a power spectral density function, $G(f)$, which is constant over all frequencies, i.e.

$$R(\tau) = k\delta(\tau) \qquad G(f) = \eta/2 \tag{1}$$

Here δ is the Dirac delta. In communication theory it is usual to define white noise by its power spectral density according to $G(f) = \eta/2$. The autocorrelation function and power spectral density functions are assumed to satisfy the Wiener-Khintchine theorem, e.g. [2], i.e.

$$R(\tau) = \int_{-\infty}^{\infty} G(f)e^{j2\pi f\tau}df \qquad G(f) = \int_{-\infty}^{\infty} R(\tau)e^{-j2\pi f\tau}d\tau \tag{2}$$

There are technicalities associated with such an approach, definitions and the statement of the Wiener-Khintchine theorem, e.g. [3]. These arise from the result of the implicit assumption of an infinite power random process and to define white noise without reference to a physical time domain signal. It is of interest, and of pedagogical value, to be able to define white noise signals in the time domain and then to establish their characteristics such as their power spectral density and autocorrelation functions and in a manner that avoids undue technicalities [3].

The approach taken is to start with a physically realizeable finite bandwidth white noise random process. An analytical expression for its power spectral density is established and from this the associated autocorrelation function can be established in a straight forward manner and without undue technicalities. For an arbitrarily small correlation time, or, equivalently, an arbitrarily large frequency range, a physically realizable white noise random process can be defined.

II. DEFINING FINITE BANDWIDTH WHITE NOISE

A general approach to defining a finite bandwidth white noise random process is to use frequencies randomly chosen, at an average rate, from an interval $[0, f_{\max}]$ (see [4] for a slightly less general approach). With a nominal minimum frequency of f_o, and an average rate of $\lambda = 1/f_o$, there is, on average, $N = f_{\max}/f_o$, frequency components for each signal. Such a white noise random process is underpinned by the following experiment:

1. Take the n values in the interval $[0, f_{\max}]$, which define a vector $(f_1, \ldots, f_n)$, and which arise from a trial of a point experiment: an experiment which yields points with a uniform density of $\lambda = 1/f_o$ points per unit interval.
2. A sub-experiment of taking a number at random from the interval $[0, 2\pi)$ is repeated n times to create the vector $(\phi_1, \ldots, \phi_n)$. The sample space of experimental outcomes is

$$S = \left\{ \begin{array}{l} ((\phi_1, f_1), \ldots, (\phi_n, f_n)): \\ \qquad (f_1, \ldots, f_n) \text{ from point experiment} \\ \qquad \phi_k \in [0, 2\pi), k \in \{1, \ldots, n\} \end{array} \right\} \tag{3}$$

A white noise random process, X, based on this experiment, can be defined according to

$$X(((\phi_1, f_1), \ldots, (\phi_n, f_n)), t) = \sum_{k=1}^{n} \frac{\sqrt{2}A_o}{\sqrt{n}} \sin(2\pi f_k t + \phi_k)u(t) \tag{4}$$

where $((\phi_1, f_1), \ldots, (\phi_n, f_n)) \in S$ and u is the unit step function. The random process is for the interval $[0, kT_o]$, $k \in \boldsymbol{Z}^+$, with a fundamental frequency of f_o, $f_o = 1/T_o$ and with an average power of A_o^2.

A. Example

Consider a white noise random process, defined by (4), on the interval $[0, 10]$ with $A_o = 1$, $f_{max} = 10$ and, on average, comprising of 100 sinusoids with frequencies chosen consistent with a Poisson point process with a rate of $\lambda = 10$. One signal from this random process is shown in Fig. 1. The power spectral density of this random process, obtained by averaging the individual power spectral densities of 100 signals, is shown in Fig. 2.

B. Gaussian White Noise

As the number of sinusoids increases, it follows, from the central limit theorem, that the random variable defined by the white noise random process at any set time will have, approximately, a Gaussian probability density function. Consider the white noise random process, defined by (4), and the parameters previously defined. The probability density function of the amplitude, $f_{X(t)}(a)$, estimated from 10000 signals, is shown in Fig. 3 along with a Gaussian approximation based on the sample mean and sample variance.

III. FILTERED WHITE NOISE

In many contexts, the observed noise is filtered by a linear filter and the model shown in Fig. 4 is appropriate. The filter is assumed to be causal with an impulse response h and

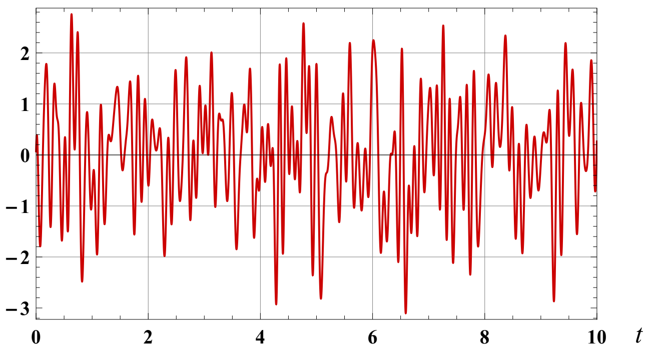

Fig. 1. Graph of one signal from a white noise random process defined on the interval $[0, 10]$ and comprising, on average, of 100 sinusoids.

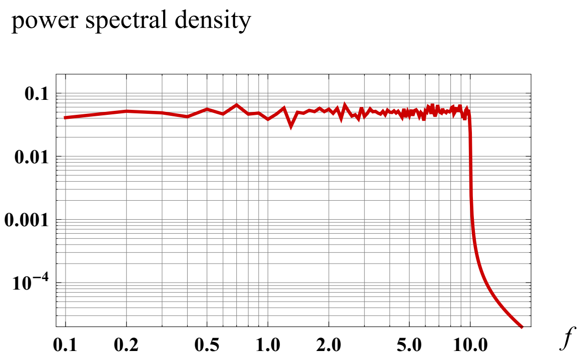

Fig. 2. Graph of the power spectral density of the defined white noise random process with $f_{max} = 10$ and obtained by averaging the individual power spectral densities (based on the FFT) of 100 signals.

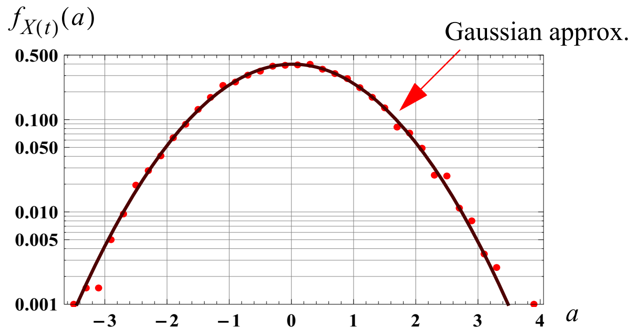

Fig. 3. Histogram approximation to the probability density function of the amplitude, at $t = 5$, and based on 10000 signals.

system transfer function H. As is well known, in steady state the response of such a system to a sinusoid:

$$A\sin(2\pi f_C t)u(t) \tag{5}$$

is

$$y_{SS}(t) = A|H(j2\pi f_C)|\sin[2\pi f_C t + \varphi(2\pi f_C)] \tag{6}$$

where $|H(j2\pi f_C)|$ and $\varphi(2\pi f_C)$, respectively, are the magnitude and argument of $H(s)$ for $s = 2\pi f_C$.

A. Model for Filtered White Noise

The following experiment underpins the definition of a filtered white noise random process. Assume the filter bandwidth is f_{BW}. Consider the experiment, where $f_{max} \gg f_{BW}$, defined as follows:

1. Take n values, from the interval $[0, f_{max}]$, to define a vector $(f_1, \ldots, f_n)$ and which arise from a trial of an sub-experiment yielding points, placed at random, and with a uniform density of $\lambda = 1/f_o$ points per unit interval.
2. A sub-experiment, of taking a number at random from the interval $[0, 2\pi)$, is repeated n times to create the vector $(\phi_1, \ldots, \phi_n)$.

A filtered white noise random process X for the interval $[0, kT_o]$, $k \in \boldsymbol{Z}^+$, with a fundamental frequency of f_o, $f_o = 1/T_o$, with frequencies chosen at random from the interval $[0, f_{max}]$ with a rate of $\lambda = 1/f_o$, and with an average input power of A_o^2, is defined according to: $Y: S \to S_X$.

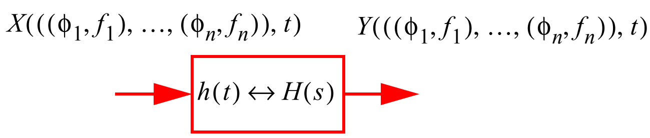

Fig. 4. Linear filtering by a filter with impulse response h.

$$Y(((\phi_1,f_1),\ldots,(\phi_n,f_n)),t) = \sum_{k=1}^{n} \frac{\sqrt{2}A_o A(f_k)}{\sqrt{n}} \sin(2\pi f_k t + \varphi(f_k) + \phi_k) \quad (7)$$

where $((\phi_1,f_1),\ldots,(\phi_n,f_n)) \in S$ and

$$A(f_k) = |H(j2\pi f_k)| \qquad \varphi(f_k) = \arg(H(j2\pi f_k)) \quad (8)$$

B. Example

Consider a white noise random process, defined by (4) and with linear filtering by a filter with a transfer function

$$H(s) = \left(1 + \frac{s}{2\pi f_p}\right)^{-2} \Leftrightarrow h(t) = \frac{t}{\tau^2} \cdot e^{-t/\tau} u(t) \quad (9)$$

where $\tau = 1/2\pi f_p$. Assume $f_p = 1$, $f_{max} = 10$, $A_o = 1$ and a time interval $[0, 10]$, which implies a fundamental frequency of $f_o = 1/10Hz$ and, on average, 100 sinusoidal components for each signal with frequencies chosen consistent with a point process with a rate of $\lambda = 10$. One signal from a random process with these parameters is shown in Fig. 5. The power spectral density of the random process, obtained by averaging the individual power spectral densities of 100 signals, is illustrated in Fig. 6.

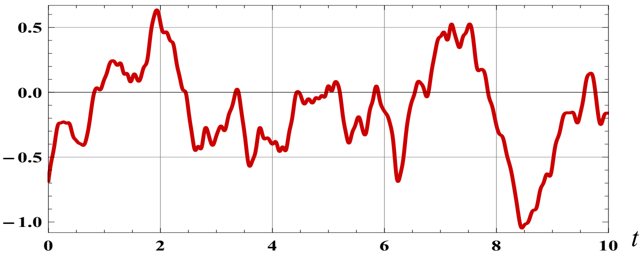

Fig. 5. Graph of one signal from a filtered white noise random process defined on the interval [0, 10] and comprising, on average, of 100 sinusoids.

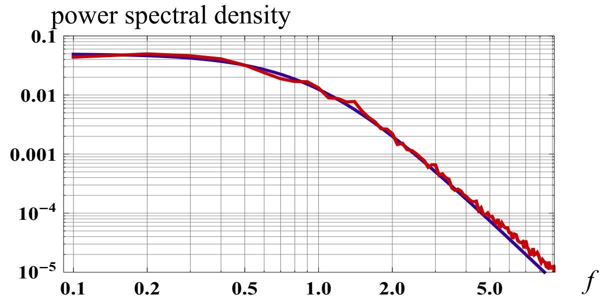

Fig. 6. Graph of the power spectral density of the defined filtered white noise random process with $f_{max} = 10$ and obtained by averaging the individual power spectral densities of 100 signals.

IV. POWER SPECTRAL DENSITY

The following theorem states the power spectral density for a finite bandwidth white noise random process:

Theorem 1: Power Spectral Density. The power spectral density of the white noise random process defined by (4) is

$$G_X(T,f) = \frac{A_o^2}{2\pi f_{max}} \cdot \left[\frac{\sin(\pi(f-f_{max})T)^2}{\pi(f-f_{max})T} - \frac{\sin(\pi(f+f_{max})T)^2}{\pi(f+f_{max})T}\right] + \frac{A_o^2}{2\pi f_{max}} \cdot [\mathrm{Si}[2\pi(f+f_{max})T] - \mathrm{Si}[2\pi(f-f_{max})T]] \quad (10)$$

$$G_X(\infty,f) = \lim_{T\to\infty} G_X(T,f) = \begin{cases} \dfrac{A_o^2}{2f_{max}} & -f_{max} < f < f_{max} \\ \dfrac{A_o^2}{4f_{max}} & f = \pm f_{max} \\ 0 & \text{elsewhere} \end{cases} \quad (11)$$

Here Si is the sine integral function:

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(\lambda)}{\lambda} d\lambda \qquad \begin{aligned} \mathrm{Si}(-x) &= -\mathrm{Si}(x) \\ \mathrm{Si}(\pm\infty) &= (\pm\pi)/2 \end{aligned} \quad (12)$$

With the definition of $\eta = A_o^2/f_{max}$, the power spectral density, for the infinite interval, can be written as

$$G_X(\infty,f) = \eta/2 \qquad -f_{max} < f < f_{max} \quad (13)$$

and with $G_X(\infty,f) = \eta/4$ for $f = \pm f_{max}$ and $G_X(\infty,f) = 0$ elsewhere.

Proof

The proof is detailed in Appendix 1.

A. Results

The power spectral density is shown in Fig. 7 for the cases of $f_{max} = 1$, $A_o = 1$ and for $T = 1, 2, 10, 100$.

V. AUTOCORRELATION FUNCTION

The time averaged autocorrelation function is given by

$$\bar{R}(T,\tau) = \int_{-\infty}^{\infty} G(T,f) e^{j2\pi f\tau} df \quad (14)$$

It then follows, as $T \to \infty$, that the power spectral density - time averaged autocorrelation function relationship is

$$G_X(\infty, f) = \frac{\eta}{2}[u(f+f_{max}) - u(f-f_{max})] \quad \Leftrightarrow \quad \bar{R}(\infty, \tau) = \eta f_{max}\mathrm{sinc}(2\tau f_{max}) \tag{15}$$

where $\eta = A_o^2/f_{max}$. See Fig. 8. Note that the average power, as expected, is:

$$\bar{P} = \bar{R}(\infty, 0) = \int_{-\infty}^{\infty} G(T, f)df = \eta f_{max} = A_o^2 \tag{16}$$

VI. CONCLUSION

The basis of defining white noise on a finite interval, and with finite power, avoids the complications associated with infinite power random processes and, accordingly, has pedagogical value. Consistent with (15), it follows, for an arbitrarily small correlation time and a flat power spectral density approximation over a finite, but arbitrarily large, frequency range, that a physically realizable white noise random process can be defined which is consistent with the Wiener-Khintchine relationships.

APPENDIX 1: Proof of Theorem 1

The Fourier transform of

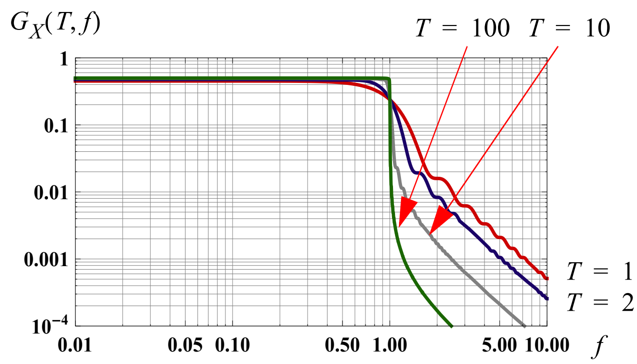

Fig. 7. Graph of the power spectral density of a white noise random process for the case of $f_{max} = 1$ and $A_o = 1$.

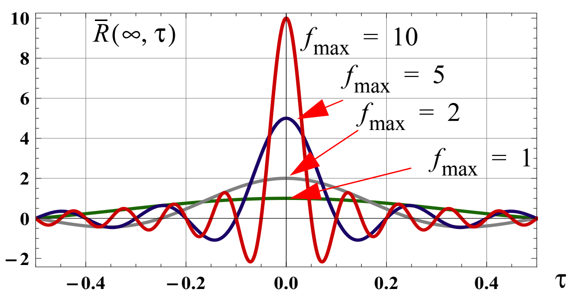

Fig. 8. Time averaged autocorrelation function for the infinite interval case when $\eta = 1$.

$$\begin{aligned} x_k(t) &= \sin(2\pi f_k t + \phi_k) \\ &= \cos(\phi_k)\sin(2\pi f_k t) + \sin(\phi_k)\cos(2\pi f_k t) \end{aligned} \tag{17}$$

evaluated on $[0, T]$ is

$$X_k(f) = \frac{-jTe^{-j\pi fT}}{2}\Big[e^{j\pi f_k T} e^{j\phi_k}\mathrm{sinc}[(f-f_k)T] - e^{-j\pi f_k T} e^{-j\phi_k}\mathrm{sinc}[(f+f_k)T] \Big] \tag{18}$$

By definition, the power spectral density evaluated on the interval $[0, T]$ is

$$\begin{aligned} G_X(T, f) = \frac{1}{T}\int_0^{f_{max}} \ldots \int_0^{f_{max}} \int_0^{2\pi} \ldots \int_0^{2\pi} \\ |X(((\phi_1, f_1), \ldots, (\phi_N, f_N)), T, f)|^2 f_{\Phi_1}(\phi_1) \cdot \ldots \cdot f_{\Phi_N}(\phi_N) \cdot \\ f_{F_1}(f_1) \ldots f_{F_N}(f_N) \cdot d\phi_1 \ldots d\phi_N \cdot df_1 \ldots df_N \end{aligned} \tag{19}$$

where

$$f_{\Phi_k}(\phi_k) = \frac{1}{2\pi} \qquad f_{F_k}(f_k) = \frac{1}{f_{max}} \qquad k \in \{1, \ldots, N\} \tag{20}$$

It then follows that

$$\begin{aligned} G_X(T, f) = \frac{A_o^2}{2\pi f_{max}}\Bigg[\\ \frac{\sin(\pi(f-f_{max})T)^2}{\pi(f-f_{max})T} - \frac{\sin(\pi(f+f_{max})T)^2}{\pi(f+f_{max})T} + \\ \mathrm{Si}[2\pi(f+f_{max})T] - \mathrm{Si}[2\pi(f-f_{max})T] \Bigg] \end{aligned} \tag{21}$$

For the infinite interval, the result $\mathrm{Si}(\pm\infty) = \pm\pi/2$ yields

$$G_X(\infty, f) = \frac{A_o^2}{2f_{max}} \qquad -f_{max} < f < f_{max} \tag{22}$$

and with $G_X(\infty, f) = A_o^2/4f_{max}$ for $f = \pm f_{max}$ and $G_X(\infty, f) = 0$ elsewhere.

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