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The Well-posedness and Solutions of Boussinesq-type Equations

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Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

Signature:

Date: 13 March 2009

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Abstract

We develop well-posedness theory and analytical and numerical solution techniques for Boussinesq-type equations. Firstly, we consider the Cauchy problem for a generalized Boussinesq equation. We show that under suitable conditions, a global solution for this problem exists. In addition, we derive sufficient conditions for solution blow-up in finite time.

Secondly, a generalized Jacobi/exponential expansion method for finding exact solutions of non-linear partial differential equations is discussed. We use the proposed expansion method to construct many new, previously undiscovered exact solutions for the Boussinesq and modified Korteweg-de Vries equations. We also apply it to the shallow water long wave approximate equations. New solutions are deduced for this system of partial differential equations.

Finally, we develop and validate a numerical procedure for solving a class of initial boundary value problems for the improved Boussinesq equation. The finite element method with linear B-spline basis functions is used to discretize the equation in space and derive a second order system involving only ordinary derivatives. It is shown that the coefficient matrix for the second order term in this system is invertible. Consequently, for the first time, the initial boundary value problem can be reduced to an explicit initial value problem, which can be solved using many accurate numerical methods. Various examples are presented to validate this technique and demonstrate its capacity to simulate wave splitting, wave interaction and blow-up behavior.

List of Publications Related to This Thesis

1. Qun Lin, Yong Hong Wu, Ryan Loxton and Shaoyong Lai, Linear B-spline finite element method for the improved Boussinesq equation, Journal of Computational and Applied Mathematics, 224 (2009) 658-667.

2. Qun Lin, Yong Hong Wu and Ryan Loxton, On the Cauchy problem for a generalized Boussinesq equation, Journal of Mathematical Analysis and Applications, 353 (2009) 186-195.

3. Qun Lin, Yong Hong Wu and Ryan Loxton, A generalized expansion method for non-linear wave equations, Journal of Physics A: Mathematical and Theoretical, accepted for publication.

4. Qun Lin, Yong Hong Wu and Shaoyong Lai, On global solution of an initial boundary value problem for a class of damped nonlinear equations, Nonlinear Analysis: Theory, Methods and Applications, 69 (2008) 4340-4351.

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Chapter 1

Introduction

1.1 Background

The propagation of surface waves is of fundamental and practical importance in oceanography and marine engineering. Boussinesq-type equations are capable of providing accurate description of water evolution in coastal regions (see [27, 87]). The earliest original Boussinesq equation was derived by Boussinesq in 1870s which takes into account the effects of weak dispersion due to finite depth and weak nonlinearity due to finite amplitude. Boussinesq-type equations also can be applied to many other areas of mathematical physics dealing with wave phenomena. Applications to waves in one-dimensional anharmonic lattices, ion acoustic waves in plasmas, and acoustical waves on circular elastic rods are described in references [41, 90, 91]. In addition, according to [38] (as quoted by Makhankov [73]), Boussinesq equation is also closely connected with the so-called Fermi-Pasta-Ulam problem.

The general form of the $1 + 1$ dimensional Boussinesq equation is

$$u_{tt} + \alpha_1 u_{xx} + \alpha_2 u_{xxxx} + \alpha_3 \left(u^2\right)_{xx} = 0, \tag{1.1}$$

where $u := u(x, t)$ represents the wave height from the free surface in the case of shallow water wave propagation, α_j, $j = 1, 2, 3$, are known constants, and the subscripts denote partial differentiation. In the literature, the Boussinesq equation (1.1) with $\alpha_j = -1$ ($j = 1, 2, 3$) is typically referred to as the "bad" or ill-posed Boussinesq equation. The "bad" Boussinesq equation describes unrealistic instability at short wavelengths (see [15, 73]) and so it can not be solved by using a sufficiently fine grid along the x-axis. Note that the dispersion relation of the "bad" Boussinesq equation is given as follows:

$$\omega^2 = k^2(1 - k^2), \tag{1.2}$$

where k is the wave number and ω is the wave circular frequency. When $k > 1$, the "bad" Boussinesq equation gives rise to an unrealistic instability. This is the physical reason why Cauchy problems for the "bad" Boussinesq equation become incorrect for $k > 1$. Choosing $\alpha_2 = 1$ and $\alpha_1 = \alpha_3 = -1$, the Boussinesq equation (1.1) is known as the "good" or well-posed Boussinesq equation whose dispersion relation is given by

$$\omega^2 = k^2(1 + k^2).$$

An improved Boussinesq equation is as follows:

$$u_{tt} - u_{xx} - u_{xxtt} - \left(u^2\right)_{xx} = 0, \tag{1.3}$$

in which $u := u(x, t)$ stands for the plasma density in the case of ion-sound wave propagation. Note that the dispersion relation of equation (1.3) is defined by

$$\omega^2 = \frac{k^2}{1 + k^2}. \tag{1.4}$$

The "improved" term means that, in comparison with the "bad" Boussinesq equation, equation (1.3) does not admit such a kind of instability for $k > 1$. Hence, the improved Boussinesq equation (1.3) is more suitable for computer simulation. Moreover, the improved Boussinesq equation (1.3) and its dispersion relation (1.4) approach the "bad" Boussinesq equation and its dispersion relation (1.2) when k is much smaller than 1.

The generalized Boussinesq equation has the form

$$u_{tt} + \alpha_1 u_{xx} + \alpha_2 u_{xxxx} + \left[f(u)\right]_{xx} = 0, \tag{1.5}$$

where constants α_j, $j = 1, 2$, and function $f : \mathbb{R} \to \mathbb{R}$ are given. Equation (1.5) arises in the study of one-dimensional anharmonic lattice waves (see [90]). Note that, if constants α_1 and α_2 are negative, then equation (1.5) is referred to as the generalized "bad" Boussinesq equation.

Equations (1.1) and (1.5) are certain perturbations of the wave equations which take into account the effects of small non-linearity and dispersion. It has also been established that, in many practical scenarios, the effect of damping is at least as significant as non-linearity and dispersion, if not more so. Hence, Varlamov [97] introduced the following damped Boussinesq equation:

$$u_{tt} - 2\alpha_1 u_{txx} + \alpha_2 u_{xxxx} - u_{xx} + \alpha_3 \big(u^2\big)_{xx} = 0, \tag{1.6}$$

where α_j, $j = 1, 2, 3$, denote constants satisfying α_1, $\alpha_2 > 0$ and the mixed derivative term is responsible for strong dissipation.

Next, we will introduce some well-known systems of Boussinesq-type equations which have been studied in the scientific literature on water waves.

As waves propagate toward shore or around marine structures, the wave field is transformed due to the effects of shoaling, refraction, diffraction and reflection. Boussinesq-type equations have been shown to be capable of simulating wave diffraction in shallow waters. The classical Boussinesq equations derived by Peregrine [87] are as follows:

$$\left.\begin{array}{l} \eta_t + \nabla \cdot \big[(h+\eta)\boldsymbol{u}\big] = 0, \\ \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + g\nabla\eta - \frac{1}{2}h\nabla\big[\nabla\cdot(h\boldsymbol{u}_t)\big] + \frac{1}{6}h^2\nabla\big(\nabla\cdot\boldsymbol{u}_t\big) = 0, \end{array}\right\} \tag{1.7}$$

where $\boldsymbol{u} := \boldsymbol{u}(x, y, t) = \big(u(x, y, t), v(x, y, t)\big)$ is the two-dimensional depth-averaged velocity vector, $\eta := \eta(x, y, t)$ is the wave amplitude, $h := h(x, y)$ is the varying water depth as measured from the still water level, constant g is the gravitational acceleration, and ∇ is the two-dimensional horizontal gradient operator. The dispersion relation of equations (1.7) is given as follows:

$$\omega^2 = \frac{gh\boldsymbol{k}^2}{1 + h^2\boldsymbol{k}^2/3},$$

where $\boldsymbol{k}^2 = k_1^2 + k_2^2$ and k_1, k_2 denote the components of the wave number vector $\boldsymbol{k}$ in the $x-$ and $y-$directions, respectively. Equations (1.7) can be used to describe the propagation of long waves in water of varying depth. However, this set of equations is not suitable for deep water.

To extend the applicability of the classical Boussinesq equations in deep water, many efforts have been made to improve the dispersion property of the equations. By rearranging the dispersion terms, Beji and Nadaoka [12] introduced the following improved Boussinesq equations:

$$\left.\begin{aligned}&\eta_t + \nabla\cdot\big[(h+\eta)\boldsymbol{u}\big] = 0,\\&\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + g\nabla\eta\\&\qquad -\tfrac{1}{2}h(1+\beta)\nabla\big[\nabla\cdot(h\boldsymbol{u}_t)\big] - \tfrac{1}{2}\beta g h\nabla\big[\nabla\cdot(h\nabla\eta)\big]\\&\qquad\qquad +\tfrac{1}{6}(1+\beta)h^2\nabla\big(\nabla\cdot\boldsymbol{u}_t\big) + \tfrac{1}{6}\beta g h^2\nabla\big(\Delta\eta\big) = 0,\end{aligned}\right\}\tag{1.8}$$

with the improved dispersion relation

$$\frac{\omega^2}{g\boldsymbol{k}} = \frac{\boldsymbol{k}h(1+\beta\boldsymbol{k}^2h^2/3)}{1+(1+\beta)\boldsymbol{k}^2h^2/3},\tag{1.9}$$

where the constant β is determined to yield a better dispersion characteristics. In [12], it has been shown that $\beta = 1/5$ is the best choice.

Equations (1.7) and (1.8) are derived by using the depth-averaged velocity. Instead, Nwogu [83] obtained the following extended Boussinesq equations using the velocity $\boldsymbol{u} := \boldsymbol{u}(x,y,t)$ at an arbitrary elevation $z := z(x,y)$:

$$\left.\begin{aligned}&\eta_t + \nabla\cdot\big[(h+\eta)\boldsymbol{u}\big] + \nabla\cdot\Big[\left(\tfrac{1}{2}z^2 - \tfrac{1}{6}h^2\right)h\nabla(\nabla\cdot\boldsymbol{u})\\&\qquad\qquad\qquad + \left(z+\tfrac{1}{2}h\right)h\nabla\big[\nabla\cdot(h\boldsymbol{u})\big]\Big] = 0,\\&\boldsymbol{u}_t + g\nabla\eta + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \tfrac{1}{2}z^2\nabla\big(\nabla\cdot\boldsymbol{u}_t\big) + z\nabla\Big[\nabla\cdot\big(h\boldsymbol{u}_t\big)\Big] = 0.\end{aligned}\right\}\tag{1.10}$$

This set of equations can describe the horizontal propagation of irregular, multi-directional waves in water of varying depth. It is noted that the dispersion relation of (1.10) is the same as (1.9) if z is set to $-(1+\beta)/3$.

In [119], Zhao et al. introduced a variable ϕ and derived the following generalized Boussinesq equations:

$$\left.\begin{aligned}&\eta_t + \nabla\cdot\big[(h+\eta)\nabla\phi\big] - \tfrac{1}{2}\nabla\cdot\big(h^2\nabla\eta_t\big)\\&\qquad\qquad +\tfrac{1}{6}h^2\Delta\eta_t - \tfrac{1}{15}\nabla\cdot\big[h\nabla(h\eta_t)\big] = 0,\\&\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + g\eta - \tfrac{1}{15}gh\nabla\cdot(h\nabla\eta) = 0.\end{aligned}\right\}\tag{1.11}$$

The dispersion relation of (1.11) is the same as (1.9) with $\beta = 1/5$. However, equations (1.11) are more efficient for calculations and can be easily implemented by any numerical methods since there are no spatial derivatives with an order higher than 2.

The following equations are referred to as the variant Boussinesq equation [88]:

$$\left.\begin{array}{l} H_t + (Hu)_x + u_{xxx} = 0, \\ u_t + H_x + uu_x = 0, \end{array}\right\} \tag{1.12}$$

where $u := u(x,t)$ is the velocity and $H := H(x,t)$ is the total depth of wave. Compared with other systems of Boussinesq-type equations, equations (1.12) are much more simple. Traveling wave solutions for equations (1.12) have been derived in the literature [11, 37, 63, 116, 117, 118].

1.2 Objectives

Although a significant advance in the study of Boussinesq-type equations and their associated initial or initial boundary value problems has been made, there are still many problems which require further investigation. In this thesis, we will study Boussinesq-type equations from three aspects. Firstly, we will consider a Cauchy problem governed by the generalized Boussinesq equation (1.5) and derive conditions for the existence of a global solution, as well as conditions for the solution blow-up in finite time. Secondly, we develop a generalized expansion method to construct exact solutions for non-linear partial differential equations and derive traveling wave solutions for Boussinesq-type equations. Finally, using the finite element method, we will propose a numerical scheme to solve an initial boundary value problem for the improved Boussinesq equation (1.3). The specific objectives are detailed below.

(I) Study the existence and blow-up of the solution for a Cauchy problem for the generalized Boussinesq equation

Consider the Cauchy problem for the following generalized Boussinesq equation

$$u_{tt} - \alpha u_{xx} + u_{xxxx} + \big[f(u)\big]_{xx} = 0, \tag{1.13}$$

subject to the initial conditions

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \tag{1.14}$$

where positive constant α and functions f, u^0, $u^1 : \mathbb{R} \to \mathbb{R}$ are given. The objective of this work is to establish conditions that ensure the existence of a global solution for the Cauchy problem (1.13)-(1.14). We will also establish conditions that guarantee solution blow-up in finite time.

(II) Construct new traveling wave solutions for Boussinesq-type equations

In this work, we aim to develop a generalized expansion method for finding traveling wave solutions of the Boussinesq equation (1.1). Furthermore, to demonstrate the flexibility and power of the proposed expansion method, we will apply it to study the modified Korteweg-de Vries equation and the shallow water long wave approximate equations.

(III) Develop a numerical method for solving initial boundary value problems for the improved Boussinesq equation

Consider the initial boundary value problem defined by the improved Boussinesq equation

$$u_{tt} = u_{xx} + u_{xxtt} + \left(u^2\right)_{xx}, \quad x \in (a,b), \quad t > 0, \tag{1.15}$$

the initial conditions

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x \in (a,b), \tag{1.16}$$

and the boundary conditions

$$u(a,t) = 0, \quad u(b,t) = 0, \quad t > 0, \tag{1.17}$$

where u^0 and u^1 are given functions. In this work, we aim to stimulate complex wave phenomena governed by the improved Boussinesq equation. To do this, we will propose an efficient and practical finite element scheme to solve the initial boundary value problem (1.15)-(1.17).

1.3 Outline of the thesis

In this thesis, we develop the theoretical results for the generalized Boussinesq equation, construct exact solutions for some well-known partial differential equations and investigate the numerical solutions for the improved Boussinesq equation. The thesis is organized as follows:

- In Chapter 1, we describe the background of Boussinesq-type equations and the objectives of the research project.

- In Chapter 2, we review previous research results relevant to Boussinesq-type equations.

- In Chapter 3, we construct sufficient conditions for the existence and nonexistence of a global solution for the Cauchy problem (1.13)-(1.14).

- In Chapter 4, we propose a generalized expansion method to derive exact solutions for non-linear partial differential equations.

- In Chapter 5, we present a numerical scheme to solve various initial boundary value problems for the improved Boussinesq equation.

- In Chapter 6, we conclude the research project and discuss some problems for further research.

Chapter 2

Review

2.1 An overview

The Boussinesq's theory is the first to give a satisfactory, scientific explanation of the phenomena of solitary waves, which are of permanent form and localized within a region, and can emerge from the collision with other solitary waves unchanged, except for a phase shift. However, the mathematical theory for Boussinesq-type equations is not so complete as the case for Korteweg-de Vries-type equations. Part of the reason for relative paucity of results about Boussinesq-type equations may be the fact that Cauchy problems for Boussinesq-type equations are not always globally well posed.

How to utilize modern mathematical techniques to study Boussinesq-type equations has been a major concern to mathematicians and physicists. We will review Boussinesq-type equations from the following three perspectives: (I) well-posedness theory for Boussinesq-type equations; (II) exact solutions of Boussinesq-type equations; (III) numerical methods for Boussinesq-type equations.

2.2 Well-posedness theory

As mentioned before, Cauchy problems for Boussinesq-type equations are not always globally well posed. Even if the initial wave and velocity profiles are smooth, the corresponding solution might lose regularity in finite time. Hence, a time evolution of an arbitrary initial wave packet is one of the most important problems related to Boussinesq-type equations.

In [77], the solitary-wave interaction mechanism for the “good” Boussinesq equation is investigated. It has been shown that when small amplitude solitons of the “good” Boussinesq equation collide, they emerge from the non-linear interaction with no change in shape or velocity. However, the large amplitude solitons change to the so-called antisolitons as they come out from the interaction. This difference in behavior is linked to a potential well of the “good” Boussinesq equation. Moreover, sufficient conditions on the initial data have been established for the existence and nonexistence of a global solution for the “good” Boussinesq equation.

Using the Faedo-Galerkin method, Pani and Saranga [85] have shown that there exists a unique weak solution to the initial boundary value problem for the “good” Boussinesq equation. The weak solution is also called a generalized solution, namely, a solution for which the derivatives appearing in the equation may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. An optimal rate of convergence in L^2-norm has been derived and priori error estimates for the fully descrete scheme in time have been established.

Turitsyn [96] considered the Boussinesq equation (1.1) with $\alpha_1 = -1$ and $\alpha_2 = \alpha_3 = 1$ for the case of periodic boundary conditions. Sufficient conditions have been determined for the corresponding solution to the Cauchy problem to blow up in finite time.

The generalized Boussinesq equation (1.13) with $\alpha = 1$ has been studied in references [16, 60, 65, 66, 67, 68, 69] through its equivalent system

$$\left.\begin{array}{l} u_t = v_x, \\ v_t = \big[u - u_{xx} - f(u)\big]_x. \end{array}\right\} \tag{2.1}$$

In [16, 60, 65], local existence for Cauchy problems for system (2.1) has been investigated. Using Kato’s abstract theory of quasi-linear evolution equation [52, 53], Bona and Sachs [16] have shown that the Cauchy problem is always locally well posed if f is an infinitely differentiable function satisfying $f(0) = 0$. Applying the contraction principle, Linares [60] has established the local well-posedness theory for system (2.1). Applying the semi-group theory [86], Liu [65] has shown that, for any initial data from space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, if f is a continuously differentiable function satisfying $f(0) = 0$, then the corresponding Cauchy problem possesses a uniquely weak solution. Moreover, the interval of existence can be extended to a maximal interval for which either the solution exists globally, or it blows up in finite time (see [65]).

It is well-known that, system (2.1) with $f(s) = |s|^{p-1}s$ for some real number $p > 1$ admits the following solitary wave solutions for all speeds c satisfying $c^2 < 1$:

$$\left.\begin{aligned} u(x,t) &= A\text{sech}^{2/(p-1)}\big(B(x-ct)\big), \\ v(x,t) &= -cA\text{sech}^{2/(p-1)}\big(B(x-ct)\big), \end{aligned}\right\} \tag{2.2}$$

where $A = [(p+1)(1-c^2)/2]^{1/(p-1)}$ and $B = (p-1)\sqrt{1-c^2}/2$. Bona and Sachs [16] verified that the solitary wave solutions (2.2) are stable in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$-norm if $1 < p < 5$ and $(p-1)/4 < c^2 < 1$. Combining the stability with the local existence result [16], one can conclude that the solutions emanating from the initial data lying relatively close to the stable solitary wave solutions exist globally. In contrast to the stability, Liu [65] complemented the work of Bona and Sachs and obtained instability of solitary wave solutions (2.2) when either $1 < p < 5$, $c^2 < (p-1)/4$ or $p \geq 5$, $c^2 < 1$.

In [66], Liu investigated conditions for the existence and nonexistence of global solutions to the generalized Boussinesq equation (1.13) with $\alpha = 1$. Sufficient conditions on the initial data and function f have been established for the blow-up of the corresponding solution in finite time. In particular, when $f(s) = |s|^{p-1}s$ ($p > 1$), two invariant sets have been constructed in terms of the energy of the function

$$\phi(x) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \text{sech}^{\frac{2}{p-1}} \left(\frac{(p-1)x}{2}\right).$$

Liu proved that, under some conditions, the solution exists globally if the initial wave belongs to one of the variant sets, while the solution blows up in finite time if the initial wave belongs to the other variant set. Note that the blow-up result for the special case of f is referred to as an improved blow-up theorem in which the energy could be larger. Furthermore, Liu obtained the strong instability of $\phi(x)$. More precisely, some solutions with initial waves arbitrarily close to $\phi(x)$ blow up in finite time. In [68], Liu investigated the strong instability of the solitary wave solutions

$$\phi_c(x) = \left[\frac{(p+1)(1-c^2)}{2}\right]^{\frac{1}{p-1}} \text{sech}^{\frac{2}{p-1}} \left(\frac{\sqrt{1-c^2}(p-1)(x-ct)}{2}\right)$$

with $0 < c^2 < 1$ for the generalized Boussinesq equation (1.13) with $f(s) = |s|^{p-1}s$ ($p > 1$).

In [69], Liu and Xu investigated the existence and nonexistence of global solutions to the generalized Boussinesq equation (1.13) with $\alpha = 1$ and $f(s) = \pm|s|^p$ or $\pm|s|^{p-1}s$ ($p > 1$). A family of potential wells and the corresponding family of outside sets have been introduced. Based on these sets, Liu and Xu obtained two invariant sets, vacuum isolating of solutions, and some threshold results of the existence and nonexistence of global solutions.

In [67], Liu studied the long-time behavior of small solutions for the Cauchy problem involving system (2.1) and obtained a lower bound for the degrees of non-linearity to establish a non-linear scattering result for small perturbations.

The generalized "bad" Boussinesq equation has been studied in references [109, 110]. In [109], Yang introduced a series of isometrically isomorphic Hilbert spaces. By virtue of the topological invariance of these spaces and the Galerkin approximation, it has been proved that, under rather mild conditions on the function f and initial data, the initial boundary value problems admit local weak solutions. Furthermore, if the function f is concave, then sufficient conditions on initial data and f have been determined such that the corresponding solution for the initial boundary value problem blows up in finite time. Yang and Wang [110] continued the work of [109] and derived some blow-up results according to the energy method and the Fourier transform method.

In Chapter 3, we will consider a Cauchy problem for equation (1.13). Note that, in [66], Liu only considered the existence of a global solution for the generalized Boussinesq equation (1.13) for a special case, i.e., $f(s) = |s|^{p-1}s$ ($p > 1$) and $\alpha = 1$. In this thesis, we will generalize the global existence theorem of [66] and derive sufficient conditions for the existence of a global solution for equation (1.13) when f is in a more general form and α is an arbitrary constant. In addition, we will derive a similar but improved blow-up theorem of [66] allowing f to be in a general form.

2.3 Exact solutions

Finding analytical solutions for non-linear partial differential equations is a difficult and challenging task. By employing a computer algebra software such as Maple or Mathematica, the large amounts of tedious working required to verify candidate solutions can be avoided. The capability and power of these softwares has increased dramatically over the past decade. Hence, a direct search for exact solutions is now much more viable. In this section, we will first introduce some popular methods which have been employed to derive exact solutions for non-linear partial differential equations. Then, we will review previous results on exact solutions to Boussinesq-type equations.

Generally, for direct search methods, certain transformation is required to reduce the partial differential equation under consideration to an ordinary differential equation. To simplify the presentation, let ξ denote the variable of the reduced ordinary differential equation. We can use the transformation $\xi = k(x - \nu t)$ if the partial differential equation is $1 + 1$ dimensional. Then, the solution of the reduced ordinary differential equation is represented in terms of a given function with some parameters to be determined later. For instance, the following expression has been used in several direct search methods:

$$\sum_{j=0}^{n} c_j \left[\Phi(\xi)\right]^j, \tag{2.3}$$

where n is an integer determined by balancing the highest order derivative term with the highest order non-linear term in the reduced ordinary differential equation, Φ is a given function and c_j, $j = 0, \ldots, n$, are constants to be determined later.

Using different function Φ in (2.3) yields different expansion method, such as the tanh method in which $\Phi(\cdot) = \tanh(\cdot)$, sine/cosine method where $\Phi(\cdot) = \sin(\cdot)$ or $\cos(\cdot)$, and Jacobi elliptic function expansion method where $\Phi(\cdot) = \mathrm{sn}(\cdot, m)$, cn$(\cdot, m)$ or dn$(\cdot, m)$, and $m \in (0, 1)$ is the modulus of the Jacobi elliptic functions. Moreover, Φ can be in a more general form. The generalized Jacobi elliptic function expansion method presented in [22, 30] chooses Φ satisfying the following ordinary differential equation:

$$\big[\Phi'(\xi)\big]^2 = q_4\big[\Phi(\xi)\big]^4 + q_3\big[\Phi(\xi)\big]^3 + q_2\big[\Phi(\xi)\big]^2 + q_1\Phi(\xi) + q_0, \tag{2.4}$$

where $'$ denotes differentation with respect to ξ and q_j, $j = 0, \ldots, 4$, are constants. The improved tanh function method proposed in [32] sets Φ to be a solution of the following Riccati equation:

$$\Phi'(\xi) = p_2\big[\Phi(\xi)\big]^2 + p_1\Phi(\xi) + p_0, \tag{2.5}$$

where p_j, $j = 0, 1, 2$, are constants.

It is noted that $\tanh(\xi)$ is a solution of equation (2.5) with $p_2 = -1$, $p_1 = 0$ and $p_0 = 1$. Hence, the tanh method is a subcase of the improved tanh function method [32]. It is also noted that the solutions of equation (2.5) also satisfy equation (2.4) with $q_4 = p_2^2$, $q_3 = 2p_1p_2$, $q_2 = 2p_0p_2 + p_1^2$, $q_1 = 2p_0p_1$ and $q_0 = p_0^2$. However, the improved tanh function method has an advantage. Letting Φ denote a solution of (2.5) and substituting expression (2.3) into the reduced ordinary different equation, we can obtain an equation in terms of Φ. If Φ is a solution of (2.4), we might end up with an equation in terms of Φ and Φ'.

Note that each solution of equation (2.4) generates a corresponding solution to the partial differential equation. However, different solutions of (2.4) sometimes create the same solution for the partial differential equation. Generally, the more solutions of (2.4) you can find, the more solutions of the partial differential equation you can generate. Many solutions of equation (2.4), including the Jacobi elliptic function solutions and the Weierstrass elliptic function solutions, have been reported in references [23, 30, 82, 116, 118].

On the other hand, the expression (2.3) also can be generalized. To simplify the presentation, let Φ denote a solution of equation (2.4), n be an integer determined by balancing the highest order derivative term with the highest order non-linear term in the equation, c_j, C_j, μ and μ_j are constants to be determined later.

The following generalized expression has been used in [22]:

$$c_0 + \sum_{j=1}^{n} \frac{c_j \left[\Phi(\xi)\right]^j + C_j \left[\Phi(\xi)\right]^{j-1} \Phi'(\xi)}{\left[\mu \Phi(\xi) + 1\right]^j}. \tag{2.6}$$

Note that expression (2.3) is a special case of (2.6). In addition, when $q_1 = q_3 = 0$, the following expressions have been studied in references [7, 30, 42]:

$$\sum_{j=-n}^{n} c_j \left[\Phi(\xi)\right]^j, \tag{2.7}$$

$$\sum_{j=-n}^{n} c_j \left[\Phi(\xi)\right]^j + \frac{\Phi'(\xi)}{\left[\Phi(\xi)\right]^2} \left(\sum_{j=-n}^{n} C_j \left[\Phi(\xi)\right]^j \right), \tag{2.8}$$

$$\sum_{j=-n}^{n} c_j \left[\Phi(\xi)\right]^j + \frac{\Phi'(\xi)}{\left[\Phi(\xi)\right]^2} \left(\sum_{j=-n}^{n+1} C_j \left[\Phi(\xi)\right]^j \right). \tag{2.9}$$

The expression (2.7) is a special case of the expressions (2.8) and (2.9). The expression (2.9) seems more general than the expression (2.8). Indeed, they are the same as the constant C_{n+1} in (2.9) will be equal to zero. For details, see Section 4.3 of Chapter 4. In [113, 114], the special expression

$$c_0 + \sum_{j=1}^{n} \frac{c_j \left[\text{sn}(\xi)\right]^j + C_j \left[\text{sn}(\xi)\right]^{j-1} \text{cn}(\xi)}{\left[\mu_1 \text{sn}(\xi) + \mu_2 \text{cn}(\xi) + 1\right]^j} \tag{2.10}$$

has been used to derive the Jacobi elliptic function solutions for the generalized Hirota-Satsuma coupled KdV equations, asymmetric Nizhnik-Novikov-Veselov equations and Davey-Stewartson equations.

The Exp-function method [45, 103] assumes that the solutions can be expressed in the form

$$\frac{\sum_{j=-n_1}^{n_2} c_j e^{j\xi}}{\sum_{l=-n_3}^{n_4} C_l e^{l\xi}},$$

where the positive integers n_j, $j = 1, \ldots, 4$ will be determined later. Note that this method includes the sine/cosine method and the ones in which the solution can be expressed in terms of exponential functions, such as the tanh method, cosh/sinh ansatz I-III method (see [100]) and those reported in [11, 104, 105, 116]. However, the method can not derive Jacobi elliptic function solutions or Weierstrass elliptic function solutions for non-linear partial differential equations.

In [107], an interesting transformation

$$u(x,t) = 2\frac{\partial}{\partial x}\Big[\text{arctan}\big(\phi(x,t)\big)\Big] = \frac{2\phi_x(x,t)}{1+\phi^2(x,t)} \tag{2.11}$$

has been applied to convert the modified KdV$^+$ equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{2.12}$$

into another partial differential equation

$$(1-\phi^2)(\phi_t + \phi_{xxx}) + 6\phi_x(\phi_x^2 - \phi\phi_{xx}) = 0. \tag{2.13}$$

Hence, combining the existed solutions of (2.13) with the transformation (2.11), one can obtain binary traveling wave periodic solutions for the modified KdV$^+$ equation (2.12). A different transformation is used to solve the modified KdV$^-$ equation in [107]. The method tells us that we can use a transformation to convert a partial differential equation into a new partial differential equation. By solving the new partial differential equation, we can obtain exact solutions for the original one. It should be addressed here that the exact solutions obtained by this way are different from the ones constructed by direct search methods.

Note that all the methods mentioned above are used to solve non-linear partial differential equations without boundary conditions. In [75], the tanh method has been modified to solve partial differential equations with boundary conditions. To satisfy the boundary conditions, the expression of the solution has to be modified. For example, if the solution must vanish as $\xi \to +\infty$, then the solution can be represented by

$$\big[1-\tanh(\xi)\big]^{n_1} \sum_{j=0}^{n-n_1} c_j \big[\tanh(\xi)\big]^j, \tag{2.14}$$

where $n_1 \in \{1, \ldots, n\}$ can be determined later.

Now, let us turn to Boussinesq-type equations. Based on the direct search methods introduced above, some Boussinesq-type equations have been solved in references [11, 37, 49, 63, 81, 100, 116, 117, 118].

In [100], some new hyperbolic schemes have been introduced to solve the Boussinesq equation (1.1). As mentioned before, these kinds of methods are subcases of the Exp-function method.

The generalized tanh function method developed in [49] assumes that the solution is represented by

$$\sum_{j=0}^{n} c_j \left[\Phi(\xi)\right]^j,$$

where $c_j = c_j(x,t)$ and $\xi := \xi(x,t) = \alpha x + q(t)$, and Φ is a solution of the Riccati equation (2.5). The method looks like more general than the improved tanh function method [32]. However, applying these two methods to the Boussinesq equation (1.1), you can obtain the same results.

The Boussinesq equation (1.1) and variant Boussinesq equations (1.12) have been solved by the Jacobi elliptic function expansion method [63] and extended Jacobi elliptic function expansion method [117]. Some Jacobi elliptic function solutions to equations (1.1) and (1.12) have been reported there. Note that the extended Jacobi elliptic function expansion method [117] includes the Jacobi elliptic function expansion method [63]. In [116, 118], by seeking new exact solutions of equation (2.4), new exact solutions to equations (1.1) and (1.12) have been obtained.

The hyperbola function method [11], in which the solution is represented by

$$\sum_{j=0}^{n} c_j \left[\text{csch}(\xi)\right]^j + \sum_{j=1}^{n} C_j \left[\text{csch}(\xi)\right]^{j-1} \text{coth}(\xi),$$

has been applied to solve the variant Boussinesq equation (1.12). Note that the method is also included in the Exp-function method.

In[37], a new algebraic method has been proposed to solve the variant Boussinesq equations (1.12). It is noted that in [37] only the case $r = 4$ has been used to solve equations (1.12). Hence, the new algebraic method applied to equations (1.12) is technically the same as the generalized Jacobi elliptic function expansion method

[22, 30]. Compared with the methods presented in [11, 63, 116, 117, 118], the proposed method [37] gives new and more general solutions due to more solutions to equation (2.4) available.

In addition, Natsis [81] derived a class of solitary wave solutions for the $1+1$ dimensional improved Boussinesq equations (1.8) with $\beta = 0$ by choosing expression (2.3) and $\Phi(\cdot) = \text{sech}(\cdot)$.

In Chapter 4, we will develop a generalized expansion method to construct exact solutions for non-linear partial differential equations without considering boundary conditions. Many new solutions of ordinary differential equation (2.4) will be reported. These new solutions together with expression (2.3) ensure that the proposed expansion method can yield many new solutions for non-linear partial differential equations. To demonstrate the proposed expansion method, we apply it to the Boussinesq equation (1.1), the improved Boussinesq equation (1.3) and the modified KdV equation. Clearly, we can obtain many additional solutions using the new solutions of (2.4) together with expression (2.6). We will apply expression (2.6) to the shallow water long wave approximate equations.

2.4 Numerical methods

Up to now, Boussinesq-type equations are mostly solved by the finite difference method (see [12, 17, 18, 19, 20, 26, 27, 33, 34, 50, 83, 84, 87, 91, 98, 111]). The finite difference method is easy to implement calculations. Derivatives in the equation under consideration are approximated by finite difference approximations, such as forward difference, back difference, and central difference. However, all such approximations cause truncation errors. Hence, it is important to study the stability and convergence of the finite difference method. Fourier method and matrix method are two well-known methods for determining stability criteria.

In [84], Cauchy problems for the "good" Boussinesq equation have been investigated by the finite difference method. Some simple finite difference schemes have been developed and their non-linear stability and convergence have been analyzed. In addition, the numerical schemes have been tested in the long-time integration of solitary waves and collision of solitary waves.

In [34], based on linearization and the finite-difference technique, an implicit scheme has been proposed for solving the initial boundary value problem involving the “good” Boussinesq equation. By using Fourier’s stability method, it has been proved that this numerical scheme is unconditionally stable. Complex wave phenomena, such as wave splitting and wave interaction, have been simulated by using the proposed numerical scheme. The numerical results confirmed the theoretical results reported in [77].

In [17, 19], Bratsos considered initial boundary value problems for both “good” and “bad” Boussinesq equations. Using finite difference formulation, the original problem has been converted to a Cauchy problem for a system of ordinary differential equations. Numerical methods have been developed by replacing the matrix-exponential term in a recurrence relation by rational approximations. In [17], Bratsos developed a seven-point three-level explicit and fifteen-point three-level implicit schemes. The later gives rise to a non-linear algebraic system and is solved by the Gauss-Seidel method. The local truncation and stability for the schemes have been analyzed. In [19], Bratsos developed a predictor-corrector scheme and a modified predictor-corrector scheme. Both numerical schemes are based on the explicit and implicit methods developed in [17]. The exact solutions have been used to test the proposed numerical schemes. Numerical experiments show that the numerical schemes proposed in [19] are able to give a satisfactory approximation.

Initial boundary value problems for the improve Boussinesq equation (1.3) have been solved numerically in [18, 20, 33, 50]. In [50], using a linearization technique and finite difference approximations, a three-level iterative scheme with second-order local truncation error was derived to solve the problem numerically. The scheme was used to investigate head-on collisions between solitary waves. In [33], an improved scheme with a Crank-Nicolson modification has been developed. A solitary wave solution of the equation has been used to test the accuracy and efficiency of the developed scheme. Numerical experiments show that the scheme is able to simulate complex wave phenomena, such as wave breaking-up and head-on collision. In [18], Bratsos applied finite difference approximations to reduce the improved Boussinesq equation to a system of ordinary differential equations and employed a Padé approximation to derive a three-level implicit time-step scheme. In addition, Bratsos [20] applied an implicit finite difference method associated with a predictor-corrector scheme to solve the problem. The efficiency of the proposed method [20] has been tested by various wave packets and the numerical results have been compared with the relevant ones given in [15, 33, 50].

In [26], a class of initial boundary value problems for the damped Boussinesq equation (1.6) have been studied by the finite difference method. The temporal and spatial derivatives have been approximated by finite difference formulae. Choo and Chung applied the Fourier transform and perturbation technique to derive the stability of the proposed numerical scheme. In addition, error estimate for the scheme has been given.

In [62], a second-order accurate numerical scheme has been presented to solve the extended Boussinesq equations (1.10). Finite difference formulae have been used to approximate spatial derivatives of various orders. Then, the equations are matched in time by a predictor-corrector scheme, in which the predictor and corrector steps are implemented by the explicit third-order Adams-Bashforth and forth-order Adams-Moulton methods respectively. The predictor-corrector scheme has been iterated until certain accuracy requirement on the error between two successive results has been satisfied. The stability of the presented numerical scheme has been analyzed by a Von Neumann stability analysis and the stability condition for the scheme has been given. Compared with available theory, other numerical results from a Navier-Stokes equations solver [61] and experimental data, the numerical experiments show that the proposed numerical scheme has very good properties for mass and energy conservation and that equations (1.10) are able to describe a wide range of water wave problems.

Although the finite difference method is easy to implement calculations, fine grid will be required to increase the accuracy of the numerical solutions. Moreover, if the computational domain is irregular, then the finite element method, in which the governing equations can be discretized in an unstructured mesh system, is more preferable than the finite difference method. In [35, 59, 76, 85, 98, 119], Boussinesq-type equations have been solved by using the finite element method.

In [76], using a Petro-Galerkin method with linear “hat” trial functions and cubic B-spline test functions, the Cauchy problem governed by the “good” Boussinesq equation has been converted to a system of ordinary differential equations. Using central differential approximation of the second order derivatives, a predictor-corrector scheme has been developed to solve the ordinary differential equations. Numerical experiments have been given to demonstrate its capability in simulating complex wave phenomena, such as wave splitting and wave interaction. In addition, an analytical formula for the two-soliton solution for the “good” Boussinesq equation has been given and numerical experiments confirm the theoretical result for the two-soliton solution.

In [35], spectral$/hp$ discontinuous Galerkin methods for the classical Boussinesq equations (1.7) have been developed on unstructured triangular meshes. Two different numerical schemes have been proposed to solve the equations. It has been shown that these two schemes are equivalent and give identical results in terms of the accuracy, convergence and restriction on the time-step.

In [119], the finite element method has been used to discretize the generalized Boussinesq equations (1.11) in space. A fourth-order predictor-corrector scheme which is similar to the predictor-corrector scheme presented in [62] has been used in the time integration. A damping layer has been applied to the open boundary for absorbing the outgoing waves. In comparison with experimental data and other numerical results available in literature [51, 57, 58, 72, 93, 95, 102], the numerical results demonstrate that equations (1.11) are capable of simulating wave transformation from relative deep water to shallow water.

In [98], using a linear element spatial discretization method coupled with a sophisticated adaptive time integration package, a numerical scheme for Nwogu’s one-dimensional extended Boussinesq equations (1.10) has been developed. Numerical experiments with wave propagating in variable water depth are compared with theoretical and experimental data [28, 29, 115]. The comparison confirms the accuracy of the numerical results and shows that the proposed numerical scheme competes well with the existing finite difference methods.

In [59], the improved Boussinesq system (1.8) has been studied by finite element method. Based on quadrilateral elements with linear interpolating functions, spatial derivatives have been discretized. Then the problem was reduced to a system of ordinary differential equations, which was solved by the Adams-Bashforth-Moulton predictor-corrector method which is similar to the one used in [62, 119]. The numerical results are in good agreement with the experimental results [102, 112]. Their numerical results show that the proposed scheme is capable of providing satisfactory results in engineering applications.

The Adomian's decomposition method [9] and its modification [1] have also been applied to solve non-linear partial differential equations. In the method, the solution is expressed as a series, where the terms are determined recursively. An approximation is obtained by truncating the series after a sufficient number of terms. However, it is difficult to prove the convergence of the series. Some convergence results have been given in references [3, 2, 4, 24, 25, 78].

In [10, 31, 47, 54, 55, 56, 99], Boussinesq-type equations have been solved by the Adomian's decomposition method. To demonstrate the efficiency and accuracy of the method, the numerical results have been compared with exact solutions. Moreover, some exact solutions can be derived by the Adomian's decomposition method (see [54, 99]).

In [1], the Adomian decomposition-Padé technique has been used to solve Cauchy problems for the "good" Boussinesq equation. Note that the convergence region in time for the Adomian's decomposition method is generally limited. Using Padé's technique, the region can be extended. Numerical examples show that the method can give approximate solutions with faster convergence rate and higher accuracy than using Adomian's decomposition method alone. However, the disadvantage of Adomian's decomposition method still remains, that is, the error increases rapidly as t increases.

In addition, the variational iteration method introduced by He [43, 44] is capable of solving Boussinesq-type equations (see [48]). In the method, a sequence can be derived from a correction functional. Tatari and Dehghann [94] established sufficient conditions for the convergence of this sequence. Extensive numerical experiences indicate that the variational iteration method is efficient for a large class of non-linear partial differential equations (see [5, 6, 39, 43, 44, 46, 48, 70, 79, 80, 94, 101]). Numerical examples show that the solutions obtained by the variational iteration method converge to their exact solutions faster than those obtained by the Adomian's decomposition method (see [44, 80, 101]).

In Chapter 5, a numerical scheme for solving the initial boundary value problem (1.15)-(1.17) will be developed. The finite element method with linear B-spline basis functions is used to discretize the non-linear partial differential equation in space. Consequently, the original problem is converted into an ordinary differential system. Thus, many accurate numerical methods are readily applicable. Various examples are presented to validate this technique and demonstrate its capacity to simulate wave splitting, wave interaction and blow-up behavior.

Chapter 3

On the Cauchy problem for a generalized Boussinesq equation

3.1 Introductory remarks

Over the past two decades, a great deal of work has been carried out worldwide to study the properties and solutions of the generalized Boussinesq equation (1.13) (see [16, 60, 65, 66, 67, 68, 69]). In this chapter, we study the following Cauchy problem:

$$u_{tt} - \alpha u_{xx} + u_{xxxx} + [f(u)]_{xx} = 0, \tag{3.1}$$

and

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \tag{3.2}$$

where $u := u(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $\alpha > 0$ is a constant, f, u^0, $u^1 : \mathbb{R} \to \mathbb{R}$ are given functions and the subscripts denote partial differentiation.

Problem (3.1)-(3.2) with $\alpha = 1$ has been previously considered in [16, 66]. More specially, the authors in [16] used Kato's theory developed in [52, 53] to show that the Cauchy problem (3.1)-(3.2) is locally well posed. The solitary wave solutions of equation (3.1) were also investigated and it was found that within a certain range of phase speeds, those solutions are non-linearly stable. In [66], based on the ground state of a corresponding non-linear Euclidean scalar field equation (see Section 3.2 for a definition), sufficient conditions for solution blow-up were established. In addition, when $f(s) = |s|^{p-1}s$ for some $p > 1$ in (3.1), conditions guaranteeing the existence of a global solution for problem (3.1)-(3.2) were derived.

One of the aims of this chapter is to construct sufficient conditions for the existence of a global solution for problem (3.1)-(3.2) when f is in a more general form and α is an arbitrary constant. To do this, we first generalize Theorem 2.6 of [66]. As the method of proof employed in [66] is not suitable for the generalized problem considered here, we use a different approach to establish this result. Based on the new result, sufficient conditions for the existence of a global solution are established. The other aim is to derive conditions for the blow-up of the solution to problem (3.1)-(3.2) for some more general cases of f. For this purpose, we propose a different approach to derive a necessary inequality and consequently establish the blow-up results. It should be addressed here that our blow-up results extend those reported in [66] which is for the case $f(s) = |s|^{p-1}s$ ($p > 1$).

3.2 Preliminary results

Before proving our main results relating to problem (3.1)-(3.2), we first need to establish some preliminary lemmas involving a corresponding non-linear Euclidean scalar field equation. Although the space domain of (3.1) is $\mathbb{R}$, we will study this corresponding equation in the more general setting $\mathbb{R}^N$.

The non-linear Euclidean scalar field equation that we will consider is

$$-\Delta\phi + \alpha\phi = f(\phi), \tag{3.3}$$

where $\phi \in H^1(\mathbb{R}^N)\backslash\{0\}$, $\alpha > 0$ is a constant and f is a given function. The function f is required to satisfy some conditions. More specifically, we consider the following two cases:

Case 1. $f(s) = |s|^{p-1}s - |s|^{q-1}s$ for some real numbers p and q satisfying $1 < q < p < \kappa$, where

$$\kappa = \begin{cases} \frac{N+2}{N-2}, & N \geq 3, \\ +\infty, & N = 1, 2. \end{cases}$$

Case 2. f satisfies the following hypotheses:

(H_1). $f \in C^1(\mathbb{R})$; f is odd; $f'(0) = 0$ and $f(s) \geq 0$ for all $s \geq 0$.

(H$_2$). If $N \geq 3$, then $\lim\limits_{s\to+\infty} \dfrac{f(s)}{s^\ell} = 0$ and $\limsup\limits_{s\to+\infty} \dfrac{f'(s)}{s^{\ell-1}} < +\infty$, where $\ell = \frac{N+2}{N-2}$; otherwise, there exists an $\ell \in (1, \infty)$ such that

$$\lim_{s\to+\infty} \frac{f(s)}{s^\ell} = 0 \text{ and } \limsup_{s\to+\infty} \frac{f'(s)}{s^{\ell-1}} < +\infty.$$

(H$_3$). There exists a real number $\theta \in \left(0, \frac{1}{2}\right)$ such that

$$F(s) := \int_0^s f(\tau)d\tau \leq \theta s f(s)$$

for all $s \geq 0$.

(H$_4$). The function $\dfrac{f(s)}{s}$ is strictly increasing on $(0, +\infty)$.

Remark 3.1. For both Cases 1 and 2, f satisfies (H$_2$) and (H$_3$). Note that if $f(s) = |s|^{p-1}s - |s|^{q-1}s$, then f satisfies (H$_2$) and (H$_3$) by choosing $\theta = 1/(q+1)$ and $\ell = p+1$ if $N = 1, 2$.

For both Cases 1 and 2, f is an odd function satisfying $\lim\limits_{s\to 0} \frac{f(s)}{s} = 0$ and (H$_2$). Hence, there exists a positive constant C such that, for each $s \in \mathbb{R}$,

$$sf(s) \leq C|s|^{\ell+1} + \frac{\alpha}{2}s^2, \tag{3.4}$$

where ℓ is defined as in (H$_2$) (according to Remark 3.1, $\ell = p+1$ for Case 1 if $N = 1, 2$).

In this chapter, $|\cdot|_l$ denotes the norm of $L^l(\mathbb{R}^N)$, while $\|\cdot\|_{H^1(\mathbb{R}^N)}$ denotes the norm of $H^1(\mathbb{R}^N)$. According to [14], if f is a continuously differentiable function satisfying (H$_2$) and $f(0) = f'(0) = 0$, then the functionals

$$S(\psi; f, \alpha) := \int_{\mathbb{R}^N} \left[\frac{1}{2}\left|\nabla\psi(\boldsymbol{x})\right|^2 + \frac{\alpha}{2}\left|\psi(\boldsymbol{x})\right|^2 - F\big(\psi(\boldsymbol{x})\big)\right] d\boldsymbol{x}$$

and

$$R(\psi; f, \alpha) := \int_{\mathbb{R}^N} \left[\left|\nabla\psi(\boldsymbol{x})\right|^2 + \alpha\left|\psi(\boldsymbol{x})\right|^2 - \psi(\boldsymbol{x}) f\big(\psi(\boldsymbol{x})\big)\right] d\boldsymbol{x}$$

are well-defined on $H^1(\mathbb{R}^N)$. Normally, we will omit f and α when referring to those functions if the dependence is obvious.

Recall that a function $\varphi \in H^1(\mathbb{R}^N)\backslash\{0\}$ is called a ground state of equation (3.3) if

(i) φ is a solution of (3.3); and

(ii) $S(\varphi; f, \alpha) \leq S(\psi; f, \alpha)$ whenever ψ is a solution of (3.3).

In other words, φ minimizes S over the class of solutions of (3.3). For Case 2, it has been shown in reference [13] that such a ground state exists. This result is extended further in the following two lemmas.

Lemma 3.1. Suppose that f satisfies the conditions listed in either Case 1 or Case 2, and that $\alpha > 0$ and $\psi \in H^1(\mathbb{R}^N)\backslash\{0\}$. Then, there exists a unique $\lambda^* \in (0, +\infty)$ such that

$$R(\lambda\psi; f, \alpha) \begin{cases} > 0, & \text{if } 0 < \lambda < \lambda^*, \\ = 0, & \text{if } \lambda = \lambda^*, \\ < 0, & \text{if } \lambda > \lambda^*. \end{cases}$$

In addition, $S(\lambda^*\psi; f, \alpha) > S(\lambda\psi; f, \alpha)$ whenever $\lambda \neq \lambda^*$.

Proof. From the definitions of S and R, we see that, for each $\lambda \in [0, \infty)$,

$$S(\lambda\psi) = \int_{\mathbb{R}^N} \left[\frac{1}{2}\lambda^2|\nabla\psi(\boldsymbol{x})|^2 + \frac{\alpha}{2}\lambda^2|\psi(\boldsymbol{x})|^2 - F\big(\lambda\psi(\boldsymbol{x})\big)\right] d\boldsymbol{x}$$

and

$$R(\lambda\psi) = \int_{\mathbb{R}^N} \left[\lambda^2|\nabla\psi(\boldsymbol{x})|^2 + \alpha\lambda^2|\psi(\boldsymbol{x})|^2 - \lambda\psi(\boldsymbol{x}) f\big(\lambda\psi(\boldsymbol{x})\big)\right] d\boldsymbol{x}.$$

A straightforward calculation shows that

$$\frac{dS(\lambda\psi)}{d\lambda} = \frac{R(\lambda\psi)}{\lambda}. \tag{3.5}$$

Now, we prove that there exists a unique real number $\lambda^* \in (0, \infty)$ such that $R(\lambda^*\psi) = 0$, $R(\lambda\psi) > 0$ for $0 < \lambda < \lambda^*$ and $R(\lambda\psi) < 0$ for $\lambda > \lambda^*$. For Case 1, let

$$g(\lambda) := \lambda^{p-1} - a\lambda^{q-1} - b,$$

where $a = \dfrac{|\psi|_{q+1}^{q+1}}{|\psi|_{p+1}^{p+1}}$ and $b = \dfrac{\alpha|\psi|_2^2 + |\nabla\psi|_2^2}{|\psi|_{p+1}^{p+1}}$. Then,

$$g'(\lambda) = (p-1)\lambda^{p-2} - a(q-1)\lambda^{q-2} = (p-1)\lambda^{q-2}\left[\lambda^{p-q} - \frac{a(q-1)}{p-1}\right]. \tag{3.6}$$

Set $\lambda_0 := \left[\frac{(q-1)|\psi|_{q+1}^{q+1}}{(p-1)|\psi|_{p+1}^{p+1}}\right]^{\frac{1}{p-q}} > 0$. It is clear from (3.6) that

$$g'(\lambda) \begin{cases} < 0, & \text{if } \lambda \in (0, \lambda_0), \\ = 0, & \text{if } \lambda = \lambda_0, \\ > 0, & \text{if } \lambda \in (\lambda_0, +\infty). \end{cases}$$

Consequently, $g(\lambda)$ is strictly decreasing on $[0, \lambda_0]$ and strictly increasing on $(\lambda_0, +\infty)$. Since $g(0) < 0$ and $\lim\limits_{\lambda \to +\infty} g(\lambda) = +\infty$, there exists a unique $\lambda^* \in (\lambda_0, +\infty)$ such that

$$g(\lambda) \begin{cases} < 0, & \text{if } \lambda \in (0, \lambda^*), \\ = 0, & \text{if } \lambda = \lambda^*, \\ > 0, & \text{if } \lambda \in (\lambda^*, +\infty). \end{cases}$$

As $R(\lambda\psi) = -\lambda^2 |\psi|_{p+1}^{p+1} g(\lambda)$, we derive that $R(\lambda^*\psi) = 0$, $R(\lambda\psi) > 0$ for $0 < \lambda < \lambda^*$, and $R(\lambda\psi) < 0$ for $\lambda > \lambda^*$. For Case 2, the odd function f implies that

$$R(\lambda\psi) = \lambda^2 \left[|\nabla\psi|_2^2 + \alpha|\psi|_2^2 - \int_{\mathbb{R}^N} |\psi(\boldsymbol{x})|^2 \frac{f\big(\lambda|\psi(\boldsymbol{x})|\big)}{\lambda|\psi(\boldsymbol{x})|} d\boldsymbol{x} \right].$$

Note that f satisfies (H_4) and $\lim\limits_{s \to 0} \frac{f(s)}{s} = 0$. Hence, there exists a unique $\lambda^* \in (0, \infty)$ such that $R(\lambda^*\psi) = 0$, $R(\lambda\psi) > 0$ for $0 < \lambda < \lambda^*$ and $R(\lambda\psi) < 0$ for $\lambda > \lambda^*$.

In addition, from (3.5), we have

$$\frac{dS(\lambda\psi)}{d\lambda} \begin{cases} > 0, & \text{if } \lambda \in (0, \lambda^*), \\ = 0, & \text{if } \lambda = \lambda^*, \\ < 0, & \text{if } \lambda \in (\lambda^*, +\infty). \end{cases}$$

Hence, it follows that $S(\lambda^*\psi) > S(\lambda\psi)$ whenever $\lambda \neq \lambda^*$. ■

Lemma 3.2. Let $M := \{\psi \in H^1(\mathbb{R}^N) \backslash \{0\} : R(\psi; f, \alpha) = 0\}$, $\alpha > 0$ and suppose that f satisfies the conditions listed in either Case 1 or Case 2. Then, there exists a solution ψ to the following problem:

$$\min_{\psi \in M} S(\psi; f, \alpha). \tag{3.7}$$

Moreover, the set of solutions of problem (3.7) coincides with the set of ground states of equation (3.3).

Proof. Multiplying both sides of (3.3) by ϕ, integrating over $\mathbb{R}^N$ and using Green's formula, we see that any solution of (3.3) belongs to M. Since f satisfies (H_3), we have that

$$S(\psi) = \frac{1}{2}|\nabla\psi|_2^2 + \frac{\alpha}{2}|\psi|_2^2 - \int_{\mathbb{R}^N} F\big(\psi(\boldsymbol{x})\big)d\boldsymbol{x}$$
$$> \frac{1}{2}|\nabla\psi|_2^2 + \frac{\alpha}{2}|\psi|_2^2 - \theta\int_{\mathbb{R}^N} \psi(\boldsymbol{x})f\big(\psi(\boldsymbol{x})\big)d\boldsymbol{x}. \tag{3.8}$$

If $\psi \in M$, then it follows from (3.8) that

$$S(\psi) > \left(\frac{1}{2} - \theta\right)\left(|\nabla\psi|_2^2 + \alpha|\psi|_2^2\right). \tag{3.9}$$

Note that $\theta < 1/2$. Hence, S is bounded below on M. Accordingly, let $\{v_n\} \subset M$ be a minimizing sequence such that $\lim\limits_{n\to+\infty} S(v_n) = \inf\limits_{\psi\in M} S(\psi)$.

Let ψ^* denote the Schwarz spherical rearrangement of a function $|\psi|$. From [13], ψ^* is the spherically symmetric non-increasing (with respect to $|\boldsymbol{x}|$) function having the same distribution function as $|\psi|$ such that

$$\int_{\mathbb{R}^N} |\nabla\psi^*(\boldsymbol{x})|^2 d\boldsymbol{x} \le \int_{\mathbb{R}^N} |\nabla\psi(\boldsymbol{x})|^2 d\boldsymbol{x}$$

and

$$\int_{\mathbb{R}^N} G\big(\psi^*(\boldsymbol{x})\big)d\boldsymbol{x} = \int_{\mathbb{R}^N} G\big(\psi(\boldsymbol{x})\big)d\boldsymbol{x}$$

for any function $G : \mathbb{R} \to \mathbb{R}$. Therefore,

$$S(\psi^*) \le S(\psi) \tag{3.10}$$

for each $\psi \in H^1(\mathbb{R}^N)$. In addition, it is easy to check that, for each real number $\gamma > 0$, $(\gamma\psi)^* = \gamma\psi^*$.

For a given n, it follows from Lemma 3.1 that there exists a unique real number $\nu_n > 0$ such that $R(\nu_n(v_n^*)) = 0$. Let $u_n = \nu_n(v_n)^* = (\nu_n(v_n))^*$. Then, according to (3.10) and Lemma 3.1, we get

$$S(u_n) = S\Big(\big(\nu_n(v_n)\big)^*\Big) \le S\big(\nu_n(v_n)\big) \le S\big(v_n\big).$$

Therefore, the spherically symmetric non-increasing sequence $\{u_n\}$ is a minimizing sequence in M as well.

By virtue of (3.9), we have $S(u_n) > \left(\frac{1}{2} - \theta\right)(|\nabla u_n|_2^2 + \alpha |u_n|_2^2)$. Hence, the boundness of sequence $\{S(u_n)\}$ implies that sequence $\{u_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Applying the compactness lemma of Strauss [92] (see also [14]), there exists a subsequence of $\{u_n\}$, relabeled by $\{u_n\}$ for notational convenience, such that

$$\begin{aligned} &u_n \rightharpoonup u_\infty \text{ weakly in } H^1(\mathbb{R}^N), \\ &u_n \to u_\infty \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.11}$$

Arguing by contradiction, we can conclude that $u_\infty \neq 0$. Suppose that $u_\infty = 0$. Noting that u_n converges almost everywhere to 0 as $n \to \infty$, it is clear from $R(u_n) = 0$ that $\lim\limits_{n\to\infty} \|u_n\|_{H^1(\mathbb{R}^N)} = 0$. Thus, u_n strongly converges to 0 in $H^1(\mathbb{R}^N)$ as $n \to +\infty$. On the other hand, it follows from $R(u_n) = 0$ and (3.4) that

$$\begin{aligned} |\nabla u_n|_2^2 + \alpha |u_n|_2^2 &= \int_{\mathbb{R}^N} u_n(\boldsymbol{x}) f\big(u_n(\boldsymbol{x})\big) d\boldsymbol{x} \\ &\leq C|u_n|_{\ell+1}^{\ell+1} + \frac{\alpha}{2}|u_n|_2^2, \end{aligned}$$

where constants C and ℓ are defined as in (3.4). Hence,

$$\min\{1, \frac{\alpha}{2}\}\|u_n\|_{H^1(\mathbb{R}^N)}^2 \leq C|u_n|_{\ell+1}^{\ell+1}.$$

According to the definition of ℓ, we have the following Sobolev inequality

$$|u_n|_{\ell+1} \leq C_{\ell+1}^* \|u_n\|_{H^1(\mathbb{R}^N)},$$

where the positive constant $C_{\ell+1}^*$ is independent of u_n. Hence, we obtain that there exists a positive constant c satisfying

$$c \leq \|u_n\|_{H^1(\mathbb{R}^N)}.$$

This leads to a contradiction.

According to Lemma 3.1, there is a unique real number $\mu > 0$ such that $R(\mu u_\infty) = 0$. Let $\phi := \mu u_\infty$. In view of (3.11), we have

$$\begin{aligned} &\mu u_n \rightharpoonup \phi \text{ weakly in } H^1(\mathbb{R}^N), \\ &\mu u_n \to \phi \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.12}$$

As $R(u_n) = 0$, it follows from Lemma 3.1 that $S(\mu u_n) \leq S(u_n)$. Noticing that S is weakly sequential lower semi-continuous on $H^1(\mathbb{R}^N)$, we have

$$S(\phi) \leq \liminf_{n\to+\infty} S(\mu u_n) \leq \lim_{n\to+\infty} S(u_n) = \inf_{\psi \in M} S(\psi).$$

Note that $\phi \in M$. Hence ϕ is a solution of problem (3.7).

Now, we will prove that ϕ satisfies (3.3). Since ϕ solves problem (3.7), there exists a Lagrange multiplier Λ such that

$$S'(\phi) = \Lambda R'(\phi). \tag{3.13}$$

We claim that $\Lambda = 0$, which implies that ϕ is a solution of (3.3). Indeed, it follows from [14] that S and R are continuously Frechet-differentiable and

$$< S'(\phi), \phi > = |\nabla\phi|_2^2 + \alpha|\phi|_2^2 - \int_{\mathbb{R}^N} \phi(\boldsymbol{x}) f\big(\phi(\boldsymbol{x})\big) d\boldsymbol{x} = R(\phi) = 0,$$
$$< R'(\phi), \phi > = 2|\nabla\phi|_2^2 + 2\alpha|\phi|_2^2 - \int_{\mathbb{R}^N} \Big[\phi(\boldsymbol{x}) f\big(\phi(\boldsymbol{x})\big) + \phi^2(\boldsymbol{x}) f'\big(\phi(\boldsymbol{x})\big)\Big] d\boldsymbol{x},$$

where $< \cdot, \cdot > = < \cdot, \cdot >_{(H^{-1}(\mathbb{R}^N), H^1(\mathbb{R}^N))}$. If $< R'(\phi), \phi >$ is negative, then it follows from (3.13) that $\Lambda = 0$. For Case 1, we have that

$$\begin{aligned} < R'(\phi), \phi > &= 2|\nabla\phi|_2^2 + 2\alpha|\phi|_2^2 - (p+1)|\phi|_{p+1}^{p+1} + (q+1)|\phi|_{q+1}^{q+1} \\ &< 2|\nabla\phi|_2^2 + 2\alpha|\phi|_2^2 - (p+1)|\phi|_{p+1}^{p+1} + (p+1)|\phi|_{q+1}^{q+1} \\ &= (1-p)(|\nabla\phi|_2^2 + \alpha|\phi|_2^2) \\ &< 0. \end{aligned}$$

For Case 2, it is clear that f' is an even function as f is odd. Thus, from $\phi \in M$, we have that

$$\begin{aligned} < R'(\phi), \phi > &= \int_{\mathbb{R}^N} \Big[\phi(\boldsymbol{x}) f\big(\phi(\boldsymbol{x})\big) - \phi^2(\boldsymbol{x}) f'\big(\phi(\boldsymbol{x})\big)\Big] d\boldsymbol{x} \\ &= \int_{\mathbb{R}^N} \Big[|\phi(\boldsymbol{x})| f\big(|\phi(\boldsymbol{x})|\big) - |\phi(\boldsymbol{x})|^2 f'\big(|\phi(\boldsymbol{x})|\big)\Big] d\boldsymbol{x}. \end{aligned}$$

In addition, condition (H_4) implies that $sf'(s) - f(s) > 0$ for each $s > 0$. Thus, for Case 2, $< R'(\phi), \phi >$ is negative as well. Therefore, the solutions of problem (3.7) are also ground states of (3.3). Recalling that each solution of (3.3) belongs to M, we can conclude that the set of ground states of (3.3) coincides with the set of solutions of problem (3.7) . ∎

In view of Lemma 3.2, we see that equation (3.3) has a ground state if $\alpha > 0$ and f satisfies the conditions listed in either Case 1 or Case 2. Accordingly, set

$$d := \min_{\psi \in M} S(\psi). \tag{3.14}$$

Next we will prove a preliminary result that will be used in derivation of the conditions for the blow-up of the solution to problem (3.1)-(3.2). To do this, the following additional condition is required for Case 2:

(H_4') There exists a real number $\beta > 1$ such that the function $\frac{f(s)}{s^\beta}$ is increasing on $(0, \infty)$.

Note that the condition (H_4') is stronger than the condition (H_4). If f satisfies the hypotheses (H_1), (H_2), (H_3) and (H_4'), we refer to it as Case 2^+. Hence, Case 2^+ is included in Case 2. It is also noted that if $f(s) = |s|^{p-1}s$ for some real number $p > 1$, then f satisfies all the conditions listed in Case 2^+.

Lemma 3.3. Suppose that $\alpha > 0$ and f satisfies the conditions listed in either Case 1 or Case 2^+. If $\psi \in H^1(\mathbb{R}^N)\backslash\{0\}$ satisfying $R(\psi) < 0$, then, $R(\psi) < (\rho+1)\big[S(\psi) - d\big]$, where $\rho = q$ for Case 1 and $\rho = \beta$ for Case 2^+.

Proof. As $R(\psi) < 0$, it follows from Lemma 3.1 that there exists a unique number $\lambda^* \in (0, 1)$ such that $R(\lambda^*\psi) = 0$. Let

$$G(\lambda) := (\rho+1)S(\lambda\psi) - R(\lambda\psi).$$

Now, we are in the position to prove that $G(\lambda)$ is strictly increasing on $(0, \infty)$. Noting that the function f is odd, we have

$$\begin{aligned} G(\lambda) =& \frac{\rho-1}{2}\lambda^2\big[\alpha|\psi(\boldsymbol{x})|_2^2 + |\nabla\psi(\boldsymbol{x})|_2^2\big] \\ &+ \int_{\mathbb{R}^N}\Big[\lambda|\psi(\boldsymbol{x})|f\big(\lambda|\psi(\boldsymbol{x})|\big) - (\rho+1)F\big(\lambda|\psi(\boldsymbol{x})|\big)\Big]d\boldsymbol{x} \end{aligned}$$

and

$$G'(\lambda) = \lambda(\rho-1)\big[\alpha|\psi(\boldsymbol{x})|_2^2 + |\nabla\psi(\boldsymbol{x})|_2^2\big] + \lambda\int_{\mathbb{R}^N} |\psi(\boldsymbol{x})|^2 \left[f'\big(\lambda|\psi(\boldsymbol{x})|\big) - \rho\frac{f\big(\lambda|\psi(\boldsymbol{x})|\big)}{\lambda|\psi(\boldsymbol{x})|}\right] d\boldsymbol{x}.$$

Note that, for both Case 1 and Case 2^+, the function $f(s)/s^\rho$ is increasing on $(0,\infty)$. Thus, $f'(s) - \rho f(s)/s \geq 0$ for each $s > 0$. Hence, $G'(\lambda) > 0$ for each $\lambda > 0$. Consequently, we have that $G(1) > G(\lambda^*)$. That is,

$$(\rho+1)S(\psi) - R(\psi) > (\rho+1)S(\lambda^*\psi) - R(\lambda^*\psi).$$

Using the fact that $R(\lambda^*\psi) = 0$ and $S(\lambda^*\psi) \geq d$, we can obtain that

$$(\rho+1)\big[S(\psi) - d\big] > R(\psi).$$

■

3.3 Main results

In this section, we first introduce an equivalent form of problem (3.1)-(3.2). Then, on the basis of an existing local existence theorem, we construct conditions for the existence of global solution for problem (3.1)-(3.2) under Case 1 and Case 2, and then establish the sufficient conditions for the blow-up of the solution to problem (3.1)-(3.2) under Case 1 and Case 2^+.

Now, we consider the following problem which is equivalent to problem (3.1)-(3.2):

$$\left.\begin{array}{l} u_t = v_x, \\ v_t = \alpha u_x - u_{xxx} - [f(u)]_x\,, \end{array}\right\} \tag{3.15}$$

subject to the initial conditions

$$u(x,0) = u^0(x), \quad v(x,0) = v^0(x). \tag{3.16}$$

Note that $u^1(x)$ in problem (3.1)-(3.2) and $v^0(x)$ in problem (3.15)-(3.16) satisfy $u^1(x) = \big[v^0(x)\big]'$.

Set

$$E(u,v) := \int_{-\infty}^{+\infty} \left[\frac{\alpha}{2} u^2(x,t) + \frac{1}{2} u_x^2(x,t) + \frac{1}{2} v^2(x,t) - F\big(u(x,t)\big) \right] dx,$$
$$V(u,v) := \int_{-\infty}^{+\infty} u(x,t)v(x,t)dx,$$
$$I_1(u,v) := \int_{-\infty}^{+\infty} u(x,t)dx,$$
$$I_2(u,v) := \int_{-\infty}^{+\infty} v(x,t)dx.$$

According to [65, 66], it can be easily established that problem (3.15)-(3.16) is always locally well posed, and the above four functionals are invariant.

Theorem 3.1. (Local existence) [65, 66] If f is a continuously differentiable function such that $f(0) = 0$ and $(u^0, v^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, then problem (3.15)-(3.16) possesses a unique weak solution (u, v) in $C\big([0,T); H^1(\mathbb{R}) \times L^2(\mathbb{R})\big)$ such that $E(u,v) = E(u^0, v^0)$, $V(u,v) = V(u^0, v^0)$, $I_1(u,v) = I_1(u^0, v^0)$ and $I_2(u,v) = I_2(u^0, v^0)$. Moreover, the interval of existence $[0,T)$ can be extended to a maximal interval $[0, T_{\max})$ such that either

(i) $T_{\max} = +\infty$; or

(ii) $T_{\max} < +\infty$, $\lim\limits_{t \to T_{\max}^-} \|(u,v)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} = +\infty$,

where $\|(u,v)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} = \|u\|_{H^1(\mathbb{R})} + |v|_2$ denotes the norm of $H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

Remark 3.2. Note that Theorem 3.1 is slightly different from the ones reported in [65, 66] where $\alpha = 1$. Let $g(s) := f(s) - \alpha s + s$ for each $s \in \mathbb{R}$. If f satisfies the conditions listed in Theorem 3.1, then g is continuously differentiable and $g(0) = 0$.

Now, we define two subsets of $H^1(\mathbb{R})$ which will be proved to be invariant under the flow generated by problem (3.15)-(3.16) for Cases 1 and 2. Let

$$K_1 := \{\psi \in H^1(\mathbb{R}) : \quad S(\psi) < d, R(\psi) > 0\} \cup \{0\}$$

and

$$K_2 := \{\psi \in H^1(\mathbb{R}) : \quad S(\psi) < d, R(\psi) < 0\},$$

where d is defined by (3.14). Suppose that $(u^0, v^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ are such that $E(u^0, v^0) < d$. We will show that if $\alpha > 0$ and f satisfies the conditions listed in either Case 1 or Case 2 and $u^0 \in K_1$, then the corresponding solution exists globally. Furthermore, if, in addition to satisfying the conditions listed in either Case 1 or Case 2$^+$, $\alpha > 0$ and $u^0 \in K_2$, then the corresponding solution blows up in finite time. All these results are furnished precisely in the following theorems.

To simplify the presentation, for the remainder of this section we will use the following notation:

$$u(t) := u(x,t),$$
$$u_x(t) := u_x(x,t),$$
$$v(t) := v(x,t).$$

Lemma 3.4. Suppose that $\alpha > 0$ and f satisfies the conditions listed in either Case 1 or Case 2. If $\psi \in H^1(\mathbb{R})$ satisfying $R(\psi) < 0$, then, there exists a positive constant c which is independent of ψ such that $\|\psi\|_{H^1(\mathbb{R})} > c$.

Proof. Since $R(\psi) < 0$, it follows from inequality (3.4) that

$$\alpha|\psi|_2^2 + |\psi_x|_2^2 < C|\psi|_{\ell+1}^{\ell+1} + \frac{\alpha}{2}|\psi|_2^2,$$

where constants C and ℓ are defined as in (3.4). Applying the Sobolev inequality, we obtain that there exists a positive constant $C^*_{\ell+1}$ depending on ℓ such that

$$\min\{\frac{\alpha}{2}, 1\}\|\psi\|_{H^1(\mathbb{R})}^2 < C(C^*_{\ell+1})^{\ell+1}\|\psi\|_{H^1(\mathbb{R})}^{\ell+1}. \tag{3.17}$$

Note that both C and $C^*_{\ell+1}$ are independent of ψ. Inequality (3.17) shows that there is a positive constant c which is independent of ψ satisfying $\|\psi\|_{H^1(\mathbb{R})} > c$. ■

Theorem 3.2. (Invariant sets) Suppose that $\alpha > 0$ and f satisfies the conditions listed in either Case 1 or Case 2, and that $(u^0, v^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfying

$E(u^0, v^0) < d$. Let $(u, v) \in C\big([0, T_{\max}); H^1(\mathbb{R}) \times L^2(\mathbb{R})\big)$ be the weak solution of problem (3.15)-(3.16). If, for each $j \in \{1, 2\}$, $u^0 \in K_j$, then $u(t) \in K_j$ for $0 \leq t < T_{\max}$.

Proof. By virtue of Theorem 3.1, we have that $E\big(u(t), v(t)\big) = E(u^0, v^0) < d$ for each $t \in [0, T_{\max})$, which implies that $S\big(u(t)\big) < d$. Now we claim that if $R\big(u(t^*)\big) = 0$ where $t^* \in (0, T_{\max})$, then $u(t^*) = 0$. Indeed, if $u(t^*) \neq 0$, then, it follows from Lemma 3.2 that $S\big(u(t^*)\big) \geq d$. This contradicts $S\big(u(t^*)\big) < d$.

Now, let us show that $u(t) \in K_2$ for each $t \in [0, T_{\max})$ if $u^0 \in K_2$. Note that $R(u^0) < 0$ and $R\big(u(t)\big)$ is continuous on $[0, T_{\max})$. If there exists a $\bar{t} \in [0, T_{\max})$ such that $u(\bar{t}) \notin K_2$, i.e., $R\big(u(\bar{t})\big) \geq 0$, then, there is a $t^* \in (0, \bar{t}]$ such that $R\big(u(t^*)\big) = 0$ and $R\big(u(t)\big) < 0$ whenever $t \in [0, t^*)$. From $R\big(u(t^*)\big) = 0$, we know that $u(t^*) = 0$. On the other hand, according to Lemma 3.4, we have that, for each $t \in [0, t^*)$, there exists a positive constant c such that $\|u(t)\|_{H^1(\mathbb{R})} > c$. Noting that $\|u(t)\|_{H^1(\mathbb{R})}$ is continuous on $[0, T_{\max})$, we obtain that $\|u(t^*)\|_{H^1(\mathbb{R})} > c$, which contradicts $u(t^*) = 0$.

Similarly, we can verify that if $u^0 \in K_1$, then $u(t) \in K_1$ for $t \in [0, T_{\max})$. Suppose that there is a $\bar{t} \in (0, T_{\max})$ such that $u(\bar{t}) \notin K_1$. Note that if $R(u(\bar{t})) = 0$, then $u(\bar{t}) = 0$, that is, $u(\bar{t}) \in K_1$. Thus, $R(u(\bar{t})) < 0$ and $u(\bar{t}) \neq 0$. Since $R(u^0) > 0$. according to the continuity of $R\big(u(t)\big)$, there is a $t^* \in (0, \bar{t})$ such that $R\big(u(t^*)\big) = 0$, which implies that $u(t^*) = 0$, and $R\big(u(t)\big) < 0$ whenever $t \in (t^*, \bar{t}]$. In view of Lemma 3.4, we can obtain that there is a positive constant c satisfying $\|u(t^*)\|_{H^1(\mathbb{R})} > c$. This contradicts $u(t^*) = 0$. ■

Theorem 3.3. (Global existence in K_1) Suppose that $\alpha > 0$ and f satisfies the conditions listed in either Case 1 or Case 2. Then, if $u^0 \in K_1$ and $v^0 \in L^2(\mathbb{R})$ such that $E(u^0, v^0) < d$, problem (3.15)-(3.16) possesses a unique weak solution $(u, v) \in C\big([0, +\infty); H^1(\mathbb{R}) \times L^2(\mathbb{R})\big)$.

Proof. As stated by Theorem 3.1, it suffices to prove that $\|u(t)\|_{H^1(\mathbb{R})} + |v(t)|_2$

is bounded for $0 \le t < T_{\max}$. Since f satisfies (H_3), we have

$$\begin{aligned} S\big(u(t)\big) \geq & \frac{1}{2}\int_{-\infty}^{+\infty}\left[|u_x(t,x)|^2+\alpha\,|u(t,x)|^2\right]dx-\theta\int_{-\infty}^{+\infty}u(t,x)f\big(u(t,x)\big)dx \\ &= \left(\frac{1}{2}-\theta\right)\int_{-\infty}^{+\infty}\left[|u_x(t,x)|^2+\alpha\,|u(t,x)|^2\right]dx+\theta R\big(u(t)\big) \\ &\geq \left(\frac{1}{2}-\theta\right)\min\{1,\alpha\}\|u(t)\|_{H^1(\mathbb{R})}^2+\theta R\big(u(t)\big). \end{aligned}$$

Applying Theorem 3.2 yields $u(t) \in K_1$, i.e. $S\big(u(t)\big) < d$ and $R\big(u(t)\big) \geq 0$ for $0 \le t < T_{\max}$. Thus, $\|u(t)\|_{H^1(\mathbb{R})}$ is bounded on $[0, T_{\max})$ and $S\big(u(t)\big) > 0$. On the other hand, combining $E\big(u(t), v(t)\big) < d$ and $S\big(u(t)\big) > 0$, it is easily verified that $|v(t)|_2^2 < 2d$ for $0 \le t < T_{\max}$. ■

Theorem 3.4. (Solution blow-up in K_2) Let $\alpha > 0$ and f satisfy the conditions listed in either Case 1 or Case 2^+. Suppose that $u^0 \in K_2$ and $v^0 \in L^2(\mathbb{R})$ such that $E(u^0, v^0) < d$ and $\xi^{-1}\widehat{u^0} \in L^2(\mathbb{R})$, where $\widehat{u^0}$ denotes the Fourier transform of u^0. Let $(u, v) \in C\big([0, T_{\max}); H^1(\mathbb{R}) \times L^2(\mathbb{R})\big)$ be the weak solution of problem (3.15)-(3.16). Then $T_{\max} < +\infty$ and

$$\lim_{t\to T_{\max}^-}\big(\|u(t)\|_{H^1(\mathbb{R})}+|v(t)|_2\big)=+\infty.$$

Proof. Here we use proof by contradiction. Suppose that $T_{\max} = +\infty$. According to [66], it follows from $\xi^{-1}\widehat{u^0} \in L^2(\mathbb{R})$ that

$$\xi^{-1}\hat{u} \in C^1\left([0,\infty); L^2(\mathbb{R})\right).$$

Let

$$I(t) := |\xi^{-1}\hat{u}(t,\xi)|_2^2,\ t \in [0,\infty).$$

Then,

$$I'(t) = 2\big(\xi^{-1}\hat{u}(t,\xi), \xi^{-1}\hat{u}_t(t,\xi)\big) \tag{3.18}$$

and

$$I''(t) = 2|v(t)|_2^2 - 2R(u(t)), \tag{3.19}$$

where $\left(\xi^{-1}\hat{u}(t,\xi),\xi^{-1}\hat{u}_t(t,\xi)\right)=\int_{-\infty}^{+\infty}\xi^{-1}\hat{u}(t,\xi)\overline{\xi^{-1}\hat{u}_t(t,\xi)}d\xi$. Using the Cauchy-Schwarz inequality, it follows from (3.18) that $[I'(t)]^2\le 4I(t)|v(t)|_2^2$ for $t\in[0,\infty)$. Let $\rho=q$ for Case 1 and $\rho=\beta$ for Case 2^+. We have for each $t\in[0,\infty)$ that

$$\begin{aligned}I''(t)I(t)-\frac{\rho+3}{4}\left[I'(t)\right]^2&\\ &\ge -I(t)\Big[(\rho+1)|v(t)|_2^2+2R\big(u(t)\big)\Big]\\ &=-I(t)\Big\{2(\rho+1)\Big[E(u_0,v_0)-S\big(u(t)\big)\Big]+2R\big(u(t)\big)\Big\}.\end{aligned}$$

Noting that $E(u_0,v_0)<d$, we have from the above inequality that

$$\begin{aligned}I''(t)I(t)-\frac{\rho+3}{4}\left[I'(t)\right]^2&\\ &\ge -I(t)\Big\{2(\rho+1)\Big[d-S\big(u(t)\big)\Big]+2R\big(u(t)\big)\Big\}.\end{aligned}$$

It follows from Theorem 3.2 that $R\big(u(t)\big)<0$. Thus, using Lemma 3.3, we can obtain that $I''(t)I(t)-\frac{\rho+3}{4}\left[I'(t)\right]^2>0$. Define $J(t):=[I(t)]^{-\frac{\rho-1}{4}}$, then $J''(t)<0$ for each $t\ge 0$.

Now, we will prove that there exists a $t^*>0$ such that $I'(t^*)>0$. If not, then, for all $t\ge 0$, $I'(t)\le 0$. From (3.19) and $R\big(u(t)\big)<0$, it follows that $I''(t)>0$ for all $t\ge 0$. Note that

$$\lim_{t\to\infty}I'(t)=I'(0)+\int_0^\infty I''(s)ds$$

exists. Hence, there is a sequence $\{t_n\}$ such that

$$\lim_{n\to\infty}I''(t_n)=0.$$

Combining (3.19) and $R\big(u(t)\big)<0$, we get

$$\lim_{n\to\infty}R\big(u(t_n)\big)=0. \tag{3.20}$$

Using Lemma 3.3 again yields that

$$(\rho+1)\big[E(u^0,v^0)-d\big]\ge(\rho+1)\big[S\big(u(t_n)\big)-d\big]>R\big(u(t_n)\big).$$

By virtue of (3.20), we have $E(u^0,v^0)\ge d$, which leads to a contradiction.

For such a t^*, $J(t^*) > 0$ and $J'(t^*) < 0$. Noting that $J''(t) < 0$ for $t \geq 0$, there exists a $\hat{t} \in \left(0, -\frac{J(t^*)}{J'(t^*)}\right]$ such that $J(\hat{t}) = 0$. Hence,

$$\lim_{t \to \hat{t}^-} I(t) = +\infty. \tag{3.21}$$

Combining (3.18) and the Cauchy-Schwarz inequality, we see that, for each $t \in [0, \hat{t})$,

$$\frac{d\,[I(t)]^{\frac{1}{2}}}{dt} = \frac{1}{2}\,[I(t)]^{-\frac{1}{2}}\,I'(t) \leq \frac{1}{2}\,[I(t)]^{-\frac{1}{2}}\,2\,[I(t)]^{\frac{1}{2}}\,|v(t)|_2 = |v(t)|_2,$$

from which we obtain that, for each $t \in [0, \hat{t})$,

$$[I(t)]^{\frac{1}{2}} < [I(0)]^{\frac{1}{2}} + \int_0^t |v(\tau)|_2 d\tau.$$

Thus, in view of (3.21), we obtain

$$\int_0^{\hat{t}} |v(\tau)|_2 d\tau = +\infty,$$

which implies that there exists a sequence $\{\tau_n\}$ such that $0 < \tau_n < \hat{t}$, $\lim_{n \to \infty} \tau_n = \hat{t}$ and

$$\lim_{n \to +\infty} |v(\tau_n)|_2 = +\infty.$$

This contradicts $T_{\max} = +\infty$. Therefore, $T_{\max} < +\infty$ and

$$\lim_{t \to T_{\max}^-} \left(\|u(t)\|_{H^1(\mathbb{R})} + |v(t)|_2\right) = +\infty.$$

■

3.4 Concluding remarks

In this chapter, we have studied the solution to the Cauchy problem for a generalized Boussinesq equation. Based on the ground state of a corresponding non-linear Euclidean scalar field equation, we constructed two invariant sets. We have then established the sufficient conditions under which a unique solution exists globally if the initial function u^0 belongs to the first invariant set, while the solution blows up if u^0 belongs to the second invariant set.

Chapter 4

A generalized expansion method for non-linear wave equations

4.1 Introductory remarks

Solutions of partial differential equations have attracted significant interest in the literature. Exact traveling wave solutions, in particular, are useful both in practice and for verifying the accuracy and stability of popular numerical schemes such as the finite difference and finite element methods. The capability and power of computer algebra softwares such as Maple or Mathematica has increased dramatically over the past decade. Hence, the large amounts of tedious calculations required to verify candidate traveling wave solutions can be avoided.

Several effective direct search methods have been proposed in the literature. These include the tanh method [74, 75], the Exp-function method [45, 103], the Jacobi elliptic function method [63, 89], the Weierstrass elliptic function method [82], and the cosh/sinh ansatz I-III method [100].

In this Chapter, we extend the generalized expansion method developed in references [22, 30]. More specifically, we obtain some new Jacobi elliptic and exponential solution classes for the same auxiliary ordinary differential equation considered in these papers. The solutions of the ordinary differential equation are then used to construct candidate traveling wave solutions. Our new results ensure that, when applied to the classical Boussinesq and modified KdV equations, this generalized expansion method not only recovers all of the solutions reported in [45, 63, 89, 100, 117], but also discovers many new ones. Furthermore, this approach is flexible as well as powerful — it is easily adapted in Section 4.6 to handle the system of shallow water long wave approximate equations.

4.2 Preliminary results

The Jacobi elliptic functions are discussed thoroughly in [8, 40]. Since these special functions play an important role in the sequel, we will briefly introduce them here. We will also discuss some preliminary results that form the basis for our work in Sections 4.3-4.6. Note that we will follow the usual convention and let i denote the complex number satisfying $\mathrm{i}^2 = -1$. Moreover, for the remainder of this chapter, $m \in (0, 1)$ is arbitrary.

To begin, consider the integral

$$\zeta = \int_0^{\rho} \frac{d\eta}{\sqrt{1 - m^2 \sin^2(\eta)}}.$$

Here, the constant m is referred to as the modulus and the upper limit ρ is called the amplitude of ζ, which we denote as

$$\rho = \mathrm{am}(\zeta).$$

On this basis, the first three Jacobi elliptic functions are defined as

$$\mathrm{sn}(\zeta) := \sin[\mathrm{am}(\zeta)] = \sin(\rho),$$
$$\mathrm{cn}(\zeta) := \cos[\mathrm{am}(\zeta)] = \cos(\rho),$$

and

$$\mathrm{dn}(\zeta) := \sqrt{1 - m^2 \sin^2[\mathrm{am}(\zeta)]} = \sqrt{1 - m^2 \sin^2(\rho)}.$$

As $m \to 1$, we have

$$\mathrm{sn}\,(\zeta) \to \tanh\,(\zeta)\,, \qquad \mathrm{cn}\,(\zeta) \to \mathrm{sech}\,(\zeta)\,, \qquad \mathrm{dn}\,(\zeta) \to \mathrm{sech}\,(\zeta)\,.$$

Similarly, as $m \to 0$,

$$\mathrm{sn}\,(\zeta) \to \sin\,(\zeta)\,, \qquad \mathrm{cn}\,(\zeta) \to \cos\,(\zeta)\,, \qquad \mathrm{dn}\,(\zeta) \to 1.$$

Table 4.1: Definition of the constants $p_{j,l}(\gamma)$, $j = 1, \ldots, 12$, $l = 0, \ldots, 4$.

j	$p_{j,0}(\gamma)$	$p_{j,1}(\gamma)$	$p_{j,2}(\gamma)$	$p_{j,3}(\gamma)$	$p_{j,4}(\gamma)$
1	$m^2 - 1$	$4\gamma(1 - m^2)$	$2 - 6\gamma^2 + 6\gamma^2 m^2 - m^2$	$2\gamma(2\gamma^2 - 2 + m^2 - 2\gamma^2 m^2)$	$\gamma^4 m^2 + 2\gamma^2 - \gamma^4 - 1 - \gamma^2 m^2$
2	1	-4γ	$6\gamma^2 - 1 - m^2$	$2\gamma(1 + m^2 - 2\gamma^2)$	$\gamma^4 + m^2 - \gamma^2 - \gamma^2 m^2$
3	$1 - m^2$	$4\gamma(m^2 - 1)$	$2m^2 - 6\gamma^2 m^2 + 6\gamma^2 - 1$	$2\gamma(2\gamma^2 m^2 - 2\gamma^2 + 1 - 2m^2)$	$2\gamma^2 m^2 + \gamma^4 - m^2 - \gamma^4 m^2 - \gamma^2$
4	$-\frac{1}{4}$	γ	$\frac{-3\gamma^2+1-2m^2}{2}$	$\gamma(2m^2 + \gamma^2 - 1)$	$\frac{-\gamma^4-1-4\gamma^2 m^2+2\gamma^2}{4}$
5	$-\frac{1}{4}$	γ	$\frac{1-3\gamma^2+m^2}{2}$	$\gamma(\gamma^2 - 1 - m^2)$	$\frac{2\gamma^2+2m^2-\gamma^4-1+2\gamma^2 m^2-m^4}{4}$
6	$-\frac{m^2}{4}$	γm^2	$\frac{m^2-3\gamma^2 m^2-2}{2}$	$\gamma(\gamma^2 m^2 - m^2 + 2)$	$\frac{2\gamma^2 m^2-\gamma^4 m^2-m^2-4\gamma^2}{4}$
7	0	$m^2 - 1$	$3\gamma + 2 - 3\gamma m^2 - m^2$	$3\gamma^2 m^2 + 2\gamma m^2 - 3\gamma^2 - 4\gamma - 1$	$\gamma(\gamma + 1)(\gamma + 1 - \gamma m^2)$
8	0	$-2\sqrt{1 - m^2}$	$6\sqrt{1 - m^2}\gamma - 4m^2 + 5$	$(8m^2 - 10)\gamma - (6\gamma^2 + 4)\sqrt{1 - m^2}$	$(4\gamma + 2\gamma^3)\sqrt{1 - m^2} + 1 + (5 - 4m^2)\gamma^2$
9	$\frac{1}{4}$	0	$-m^2 + \frac{1}{2}$	0	$\frac{1}{4}$
10	$\frac{m^2}{4(1-m^2)}$	$\frac{\sqrt{m^4-m^2+1}}{m^2-1}$	$\frac{2m^4-3m^2+4}{2(1-m^2)}$	$\frac{\sqrt{m^4-m^2+1}}{m^2-1}$	$\frac{m^2}{4(1-m^2)}$
11	$\frac{1-m^2}{4}$	0	$\frac{1+m^2}{2}$	0	$\frac{1-m^2}{4}$
12	$\frac{m^2(2-m^2)}{4(1-m^2)}$	$\frac{\sqrt{1-m^4+m^2}}{m^2-1}$	$\frac{m^4-4}{2(m^2-1)}$	$\frac{\sqrt{1-m^4+m^2}}{m^2-1}$	$\frac{m^2(2-m^2)}{4(1-m^2)}$

Nine additional Jacobi elliptic functions can be defined in terms of these first three — see references [8, 40] for details.

In [22, 30], the following auxiliary ordinary differential equation was introduced:

$$[\Phi'(\xi)]^2 = q_0 + q_1\Phi(\xi) + q_2 [\Phi(\xi)]^2 + q_3 [\Phi(\xi)]^3 + q_4 [\Phi(\xi)]^4 , \qquad (4.1)$$

where q_j, $j = 0, \ldots, 4$, are given coefficients. Various solutions of the ordinary differential equation (4.1) were constructed using the Jacobi elliptic functions, and these results were exploited in the design of a systematic procedure for generating solutions of non-linear partial differential equations. We will follow a similar approach. In our work, the ordinary differential equation (4.1) will be considered assuming $q_4 \neq 0$. We will need to determine more general solution classes of the equation than those reported in [22, 30]. This is the motivation behind the preliminary results that follow.

Recall that m is an arbitrary real number satisfying $0 < m < 1$. With this in mind, for any (possibly complex) number γ, define the constants $p_{j,l}(\gamma)$, $j = 1, \ldots, 12$, $l = 0, \ldots, 4$, according to Table 4.1. Furthermore, let the functions $\varphi_{j,l}(\cdot, \gamma)$, $j = 1, \ldots, 12$, $l = 1, \ldots, 4$, be defined as follows:

$$\varphi_{1,1}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma \mathrm{dn}(\xi)+1},$$

$$\varphi_{1,2}(\xi,\gamma) = \frac{\sqrt{1-m^2}}{\gamma\sqrt{1-m^2}+\mathrm{dn}(\xi)},$$

$$\varphi_{1,3}(\xi,\gamma) = \frac{\sqrt{m^2-1}\mathrm{sn}(\xi)}{\gamma\sqrt{m^2-1}\mathrm{sn}(\xi)+\mathrm{cn}(\xi)},$$

$$\varphi_{1,4}(\xi,\gamma) = \frac{\mathrm{cn}(\xi)}{\gamma \mathrm{cn}(\xi)+\mathrm{isn}(\xi)},$$

$$\varphi_{2,1}(\xi,\gamma) = \frac{\mathrm{sn}(\xi)}{\gamma \mathrm{sn}(\xi)+1},$$

$$\varphi_{2,2}(\xi,\gamma) = \frac{1}{\gamma+m\mathrm{sn}(\xi)},$$

$$\varphi_{2,3}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma \mathrm{dn}(\xi)+m\mathrm{cn}(\xi)},$$

$$\varphi_{2,4}(\xi,\gamma) = \frac{\mathrm{cn}(\xi)}{\gamma \mathrm{cn}(\xi)+\mathrm{dn}(\xi)},$$

$$\varphi_{3,1}(\xi,\gamma) = \frac{\mathrm{cn}(\xi)}{\gamma\, \mathrm{cn}(\xi)+1},$$

$$\varphi_{3,2}(\xi,\gamma) = \frac{\sqrt{m^2-1}}{\gamma\sqrt{m^2-1}+m\mathrm{cn}(\xi)},$$

$$\varphi_{3,3}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma \mathrm{dn}(\xi)+\mathrm{i}m\mathrm{sn}(\xi)},$$

$$\varphi_{3,4}(\xi,\gamma) = \frac{\sqrt{1-m^2}\,\mathrm{sn}(\xi)}{\gamma\sqrt{1-m^2}\mathrm{sn}(\xi)+\mathrm{dn}(\xi)},$$

$$\varphi_{4,1}(\xi,\gamma) = \frac{1}{\gamma+\mathrm{i}m\,\mathrm{sn}(\xi)+\mathrm{dn}(\xi)},$$

$$\varphi_{4,2}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma\,\mathrm{dn}(\xi)+\mathrm{i}m\,\mathrm{cn}(\xi)+\sqrt{1-m^2}},$$

$$\varphi_{4,3}(\xi,\gamma) = \frac{\mathrm{sn}(\xi)}{\gamma\,\mathrm{sn}(\xi)+\mathrm{i}+\mathrm{i}\,\mathrm{cn}(\xi)},$$

$$\varphi_{4,4}(\xi,\gamma) = \frac{\mathrm{cn}(\xi)}{\gamma\,\mathrm{cn}(\xi)+\mathrm{i}\,\mathrm{dn}(\xi)+\sqrt{m^2-1}\,\mathrm{sn}(\xi)},$$

$$\varphi_{5,1}(\xi,\gamma) = \frac{1}{\gamma+m\,\mathrm{cn}(\xi)+\mathrm{dn}(\xi)},$$

$$\varphi_{5,2}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma\,\mathrm{dn}(\xi)+m\sqrt{1-m^2}\,\mathrm{sn}(\xi)+\sqrt{1-m^2}},$$

$$\varphi_{5,3}(\xi,\gamma) = \frac{\mathrm{sn}(\xi)}{\gamma\,\mathrm{sn}(\xi)+\mathrm{i}\,\mathrm{dn}(\xi)+\mathrm{i}\,\mathrm{cn}(\xi)},$$

$$\varphi_{5,4}(\xi,\gamma) = \frac{\mathrm{cn}(\xi)}{\gamma\,\mathrm{cn}(\xi)+\sqrt{m^2-1}+\sqrt{m^2-1}\,\mathrm{sn}(\xi)},$$

$$\varphi_{6,1}(\xi,\gamma) = \frac{1}{\gamma+\mathrm{i}\,\mathrm{sn}(\xi)+\mathrm{cn}(\xi)},$$

$$\varphi_{6,2}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma\,\mathrm{dn}(\xi)+\mathrm{i}\,\mathrm{cn}(\xi)+\sqrt{1-m^2}\,\mathrm{sn}(\xi)},$$

$$\varphi_{6,3}(\xi,\gamma) = \frac{m\,\mathrm{sn}(\xi)}{\gamma m\,\mathrm{sn}(\xi)+\mathrm{i}+\mathrm{i}\,\mathrm{dn}(\xi)},$$

$$\varphi_{6,4}(\xi,\gamma) = \frac{\mathrm{i}m\,\mathrm{cn}(\xi)}{\mathrm{i}\gamma m\,\mathrm{cn}(\xi)+\mathrm{dn}(\xi)+\sqrt{1-m^2}},$$

$$\varphi_{7,1}(\xi,\gamma) = \frac{\sqrt{1-m^2}[1+\mathrm{sn}(\xi)]}{\gamma\sqrt{1-m^2}+\sqrt{1-m^2}(\gamma+1)\,\mathrm{sn}(\xi)+\mathrm{dn}(\xi)},$$

$$\varphi_{7,2}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)+\mathrm{cn}(\xi)}{\gamma\,\mathrm{dn}(\xi)+(\gamma+1)\,\mathrm{cn}(\xi)+1},$$

$$\varphi_{7,3}(\xi,\gamma)= \frac{\sqrt{1-m^2}[1+m\,\mathrm{sn}(\xi)]}{\gamma m\sqrt{1-m^2}\,\mathrm{sn}(\xi)+\sqrt{1-m^2}(\gamma+1)+m\mathrm{i}\,\mathrm{cn}(\xi)},$$

$$\varphi_{7,4}(\xi,\gamma)= \frac{\mathrm{dn}(\xi)+m\,\mathrm{cn}(\xi)}{m\gamma\,\mathrm{cn}(\xi)+(\gamma+1)\,\mathrm{dn}(\xi)+\mathrm{i}m\,\mathrm{sn}(\xi)},$$

$$\varphi_{8,1}(\xi,\gamma)= \frac{\mathrm{dn}(\xi)+\sqrt{1-m^2}\,\mathrm{sn}(\xi)}{(1+\sqrt{1-m^2}\gamma)\,\mathrm{sn}(\xi)+\gamma\,\mathrm{dn}(\xi)},$$

$$\varphi_{8,2}(\xi,\gamma)= \frac{\sqrt{1-m^2}[\,\mathrm{cn}(\xi)+1]}{\sqrt{1-m^2}\gamma+\sqrt{1-m^2}\gamma\,\mathrm{cn}(\xi)+\,\mathrm{cn}(\xi)},$$

$$\varphi_{8,3}(\xi,\gamma)= \frac{\sqrt{1-m^2}+\mathrm{i}m\,\mathrm{cn}(\xi)}{1+\sqrt{1-m^2}\gamma+\mathrm{i}\gamma m\,\mathrm{cn}(\xi)},$$

$$\varphi_{8,4}(\xi,\gamma)= \frac{\sqrt{1-m^2}\,\mathrm{dn}(\xi)+m\sqrt{m^2-1}\,\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)+\gamma\sqrt{1-m^2}\,\mathrm{dn}(\xi)+m\sqrt{m^2-1}\gamma\,\mathrm{sn}(\xi)},$$

$$\varphi_{9,1}(\xi,\gamma)= \frac{\mathrm{sn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}{m\sqrt{2-m^2}+\sqrt{-m^4+m^2+1}\,\mathrm{cn}(\xi)},$$

$$\varphi_{9,2}(\xi,\gamma)= \frac{\mathrm{cn}(\xi)-1+m^2}{m\sqrt{2-m^2}\,\mathrm{dn}(\xi)+\sqrt{(-m^4+m^2+1)(1-m^2)}\,\mathrm{sn}(\xi)},$$

$$\varphi_{9,3}(\xi,\gamma)= \frac{\mathrm{dn}(\xi)+\mathrm{i}m(1-m^2)\,\mathrm{sn}(\xi)}{m^2\sqrt{2-m^2}\,\mathrm{cn}(\xi)+\sqrt{(-m^4+m^2+1)(m^2-1)}},$$

$$\varphi_{9,4}(\xi,\gamma)= \frac{1+m\sqrt{m^2-1}\,\mathrm{cn}(\xi)}{m^2\sqrt{2-m^2}\,\mathrm{sn}(\xi)+\sqrt{m^4-m^2-1}\,\mathrm{dn}(\xi)},$$

$$\varphi_{10,1}(\xi,\gamma)= \frac{\mathrm{cn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}{m^2-1+\sqrt{m^4-m^2+1}\,\mathrm{cn}(\xi)},$$

$$\varphi_{10,2}(\xi,\gamma)= \frac{\mathrm{sn}(\xi)+\sqrt{1-m^2}}{\sqrt{1-m^2}\,\mathrm{dn}(\xi)+\sqrt{m^4-m^2+1}\,\mathrm{sn}(\xi)},$$

$$\varphi_{10,3}(\xi,\gamma)= \frac{1+m\sqrt{1-m^2}\,\mathrm{sn}(\xi)}{m\sqrt{m^2-1}\,\mathrm{cn}(\xi)+\sqrt{m^4-m^2+1}},$$

$$\varphi_{10,4}(\xi,\gamma)= \frac{\mathrm{dn}(\xi)+m\sqrt{1-m^2}\,\mathrm{cn}(\xi)}{\mathrm{i}(m^3-m)\,\mathrm{sn}(\xi)+\sqrt{m^4-m^2+1}\,\mathrm{dn}(\xi)},$$

$$\varphi_{11,1}(\xi,\gamma)= \frac{\mathrm{cn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}{m+\sqrt{m^4-m^2+1}\,\mathrm{sn}(\xi)},$$

$$\varphi_{11,2}(\xi,\gamma)= \frac{\sqrt{1-m^2}\,\mathrm{sn}(\xi)-1+m^2}{m\,\mathrm{dn}(\xi)+\sqrt{m^4-m^2+1}\,\mathrm{cn}(\xi)},$$

$$\varphi_{11,3}(\xi,\gamma)= \frac{\mathrm{i}[\,\mathrm{dn}(\xi)+m\sqrt{1-m^2}\,\mathrm{cn}(\xi)]}{m^2\,\mathrm{sn}(\xi)+\sqrt{m^4-m^2+1}},$$

$$\varphi_{11,4}(\xi,\gamma)= \frac{\sqrt{m^2-1}[1+m\sqrt{1-m^2}\,\mathrm{sn}(\xi)]}{m^2\,\mathrm{cn}(\xi)+\sqrt{m^4-m^2+1}\,\mathrm{dn}(\xi)},$$

$$\varphi_{12,1}(\xi,\gamma)= \frac{\mathrm{cn}(\xi)+1-m^2}{\sqrt{1-m^2}\,\mathrm{dn}(\xi)+\sqrt{1-m^4+m^2}\,\mathrm{cn}(\xi)},$$

$$\varphi_{12,2}(\xi,\gamma)= \frac{\mathrm{sn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}{\sqrt{1-m^2}+\sqrt{1-m^4+m^2}\,\mathrm{sn}(\xi)},$$

$$\varphi_{12,3}(\xi,\gamma)= \frac{1+m\sqrt{m^2-1}\,\mathrm{cn}(\xi)}{m\sqrt{1-m^2}\,\mathrm{sn}(\xi)+\sqrt{1-m^4+m^2}},$$

$$\varphi_{12,4}(\xi,\gamma)= \frac{\mathrm{dn}(\xi)+\mathrm{i}(m^3-m)\,\mathrm{sn}(\xi)}{m\sqrt{1-m^2}\,\mathrm{cn}(\xi)+\sqrt{1-m^4+m^2}\,\mathrm{dn}(\xi)}.$$

Through lengthy calculation, we can readily verify the following result. Note that Maple can be used to help us for the calculation.

Theorem 4.1. Let γ be arbitrary. Then, for each $j = 1, \ldots, 12$, the ordinary differential equation (4.1) with coefficients $q_l = p_{j,l}(\gamma)$, $l = 0, \ldots, 4$, has solutions $\varphi_{j,l}(\cdot, \gamma)$, $l = 1, \ldots, 4$.

Remark 4.1. Theorem 4.1 can be generalized further. In fact, it remains valid even if $\text{cn}(\xi)$, $\text{sn}(\xi)$ and $\text{dn}(\xi)$ are replaced, respectively, by $\pm\text{cn}(\xi)$, $\pm\text{sn}(\xi)$ and $\pm\text{dn}(\xi)$ in the expressions for $\varphi_{j,l}$ given above.

In some cases, the solutions of the ordinary differential equation (4.1) can be used to generate additional solutions. This observation is furnished precisely in Theorem 4.2 and Theorem 4.3 below. Again, Maple can be used to conveniently verify these results.

Theorem 4.2. Suppose that φ is a solution of the ordinary differential equation (4.1) with coefficients $q_l = \hat{q}_l$, $l = 0, \ldots, 4$, where $\hat{q}_1 = \hat{q}_3 = 0$, and $\hat{q}_0$, $\hat{q}_2$ and $\hat{q}_4$ are given constants such that $\hat{q}_0 \neq 0$. Then,

$$\pm\sqrt{\frac{\hat{q}_4}{\hat{q}_0}}\varphi + \frac{1}{\varphi}$$

is a solution of the ordinary differential equation (4.1) with coefficients

$$q_0 = 8\hat{q}_4 \mp 4\hat{q}_2\sqrt{\frac{\hat{q}_4}{\hat{q}_0}}, \quad q_1 = 0, \quad q_2 = \hat{q}_2 \mp 6\hat{q}_0\sqrt{\frac{\hat{q}_4}{\hat{q}_0}}, \quad q_3 = 0, \quad q_4 = \hat{q}_0.$$

Theorem 4.3. Suppose that φ is a solution of the ordinary differential equation (4.1) with coefficients $q_l = \hat{q}_l$, $l = 0, \ldots, 4$, where $\hat{q}_l$, $l = 0, \ldots, 4$, are given constants such that $\hat{q}_1 \neq 0$ and $\hat{q}_4 = \frac{\hat{q}_0\hat{q}_3^2}{\hat{q}_1^2}$. Then,

$$\frac{\hat{q}_3}{\hat{q}_1}\varphi + \frac{1}{\varphi}$$

is a solution of the ordinary differential equation (4.1) with coefficients

$$q_0 = \frac{4\hat{q}_3(2\hat{q}_0\hat{q}_3 - \hat{q}_1\hat{q}_2)}{\hat{q}_1^2}, \quad q_1 = -4\hat{q}_3, \quad q_2 = \hat{q}_2 - \frac{6\hat{q}_0\hat{q}_3}{\hat{q}_1}, \quad q_3 = \hat{q}_1, \quad q_4 = \hat{q}_0.$$

Remark 4.2. From Table 4.1 and Theorem 4.1, the reader will notice that, for any γ, Theorem 4.3 can be invoked with $\varphi_{j,l}(\cdot,\gamma)$, $j \in \{10, 12\}$, $l = 1, \ldots, 4$.

We also seek for non-Jacobi elliptic solutions of the ordinary differential equation (4.1). As such, to conclude this section, we present the following two results. Both can be proved easily via direct substitution.

Theorem 4.4. Let a_{-1}, a_0, a_1 and b_0 be given constants such that $a_{-1} \neq 0$ and $a_0 \neq a_{-1}b_0$. Then,

$$\frac{a_{-1}\mathrm{e}^{-\xi} + a_0 + a_1\mathrm{e}^{\xi}}{\mathrm{e}^{-\xi} + b_0 + \frac{a_1}{a_{-1}}\mathrm{e}^{\xi}}$$

is a solution of the ordinary differential equation (4.1) with coefficients

$$\begin{aligned}
q_0 &= -\frac{\left(4a_{-1}a_1 - a_0^2\right)a_{-1}^2}{\left(a_{-1}b_0 - a_0\right)^2},\\
q_1 &= \frac{2a_{-1}\left(-a_0a_{-1}b_0 + 8a_{-1}a_1 - a_0^2\right)}{\left(a_{-1}b_0 - a_0\right)^2},\\
q_2 &= \frac{a_{-1}^2b_0^2 + 4a_{-1}a_0b_0 - 24a_{-1}a_1 + a_0^2}{\left(a_{-1}b_0 - a_0\right)^2},\\
q_3 &= \frac{2(8a_1 - a_{-1}b_0^2 - a_0b_0)}{\left(a_{-1}b_0 - a_0\right)^2},\\
q_4 &= \frac{a_{-1}b_0^2 - 4a_1}{a_{-1}\left(a_{-1}b_0 - a_0\right)^2}.
\end{aligned}$$

Theorem 4.5. Let a_{-1}, a_1, b_0 and b_1 be given constants such that $a_1 \neq b_1a_{-1}$ and $a_0 = \frac{b_0(a_{-1}b_1+a_1)\pm(a_{-1}b_1-a_1)\sqrt{b_0^2-4b_1}}{2b_1}$. Then,

$$\frac{a_{-1}\mathrm{e}^{-\xi} + a_0 + a_1\mathrm{e}^{\xi}}{\mathrm{e}^{-\xi} + b_0 + b_1\mathrm{e}^{\xi}}$$

is a solution of the ordinary differential equation (4.1) with

$$\begin{aligned}
q_0 &= \frac{a_{-1}^2a_1^2}{\left(b_1a_{-1} - a_1\right)^2},\\
q_1 &= \frac{-2a_{-1}a_1^2 - 2b_1a_{-1}^2a_1}{\left(b_1a_{-1} - a_1\right)^2},\\
q_2 &= \frac{a_1^2 + 4a_{-1}b_1a_1 + a_{-1}^2b_1^2}{\left(b_1a_{-1} - a_1\right)^2},\\
q_3 &= \frac{-2a_1b_1 - 2a_{-1}b_1^2}{\left(b_1a_{-1} - a_1\right)^2},\\
q_4 &= \frac{b_1^2}{\left(b_1a_{-1} - a_1\right)^2}.
\end{aligned}$$

Note that additional solutions of the ordinary differential equation (4.1) can be constructed using Weierstrass' elliptic function. The reader is directed to [82] for more details.

4.3 A generalized expansion method

We will briefly outline a generalized expansion method for constructing traveling wave solutions. Similar procedures have been developed in references [22, 30]. However, the new results given in the previous section ensure that our method yields many new solutions when applied to some classical partial differential equations. This will be clearly demonstrated in Sections 4.4-4.6.

We consider the following non-linear wave equation:

$$H\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots\right) = 0, \tag{4.2}$$

where $u := u(x,t)$ is a real or complex-valued function, H is a given function involving powers of its arguments and the subscripts denote differentiation. We will consider candidate traveling wave solutions that take the form

$$u(x,t) = \tilde{u}\left(\xi\right) = \sum_{j=0}^{n} c_j \left[\Phi\left(\xi\right)\right]^j, \tag{4.3}$$

where $\xi = k\left(x - \nu t\right)$, $k > 0$ is the wave number, ν is the traveling wave velocity, n is an integer, Φ is a non-trival solution of the ordinary differential equation (4.1) with coefficients q_l, $l = 0, \ldots, 4$, and c_j, $j = 0, \ldots, n$, are constants with $c_n \neq 0$. Depending on the form of H, k and ν will be determined or remain as free parameters.

Note that $\tilde{u}$ given by (4.3) is a polynomial function of Φ. Hence, it is readily seen that, for each integer $\kappa \geq 1$, $\tilde{u}^\kappa$ is also a polynomial in Φ. In this case, we use the degree notation $O(\cdot)$ to denote the index of the highest power of Φ. Thus,

$$O(\tilde{u}^\kappa) = n\kappa, \quad \kappa \geq 1. \tag{4.4}$$

The derivatives of Φ can be obtained by repeatedly differentiating both sides of (4.1). For example,

$$\left\{\begin{array}{lcl} \Phi'' & = & \frac{q_1}{2} + q_2\Phi + \frac{3q_3}{2}\Phi^2 + 2q_4\Phi^3, \\ \Phi''' & = & \left(q_2 + 3q_3\Phi + 6q_4\Phi^2\right)\Phi', \\ \Phi'''' & = & \left(3q_0q_3 + \frac{1}{2}q_1q_2\right) + \left(q_2^2 + \frac{9}{2}q_1q_3 + 12q_0q_4\right)\Phi \\ & & +15\left(\frac{1}{2}q_2q_3 + q_1q_4\right)\Phi^2 + \left(20q_2q_4 + \frac{15}{2}q_3^2\right)\Phi^3 \\ & & \qquad +30q_3q_4\Phi^4 + 24q_4^2\Phi^5. \end{array}\right. \tag{4.5}$$

It is not difficult to show that only the even derivatives are polynomials in Φ. The odd derivatives also contain terms of the form $\Phi^j(\Phi')$, where j is a non-negative integer. In this case, we define $O(\Phi') = 2$ and so

$$O(\Phi^j(\Phi')) = j + 2, \qquad j \geq 0.$$

By differentiating (4.3), we can also deduce the derivatives of $\tilde{u}$. For example,

$$\left\{\begin{array}{lcl} \tilde{u}' & = & (c_1 + \ldots + nc_n\Phi^{n-1})\Phi', \\ \tilde{u}'' & = & (c_1 + \ldots + nc_n\Phi^{n-1})\Phi'' + [2c_2 + \ldots + n(n-1)c_n\Phi^{n-2}](\Phi')^2, \\ \tilde{u}''' & = & (c_1 + \ldots + nc_n\Phi^{n-1})\Phi''' + 3[2c_2 + \ldots + n(n-1)c_n\Phi^{n-2}]\Phi'\Phi'' \\ & & \quad +[6c_3 + \ldots + n(n-1)(n-2)c_n\Phi^{n-3}](\Phi')^3, \\ \tilde{u}'''' & = & (c_1 + \ldots + nc_n\Phi^{n-1})\Phi'''' \\ & & +4[2c_2 + \ldots + n(n-1)c_n\Phi^{n-2}]\Phi'\Phi''' \\ & & +3[2c_2 + \ldots + n(n-1)c_n\Phi^{n-2}](\Phi'')^2 \\ & & +6[6c_3 + \ldots + n(n-1)(n-2)c_n\Phi^{n-3}](\Phi')^2\Phi'' \\ & & +[24c_4 + \ldots + n(n-1)(n-2)(n-3)c_n\Phi^{n-4}](\Phi')^4, \end{array}\right. \tag{4.6}$$

where the derivatives of Φ are given in (4.1) and (4.5). Higher order derivatives can be obtained similarly. Again, only the even derivatives of $\tilde{u}$ are polynomials in Φ. It is readily seen that

$$O\left(\frac{d^\kappa\tilde{u}}{d\xi^\kappa}\right) = n + \kappa, \quad \kappa \geq 1. \tag{4.7}$$

When $\tilde{u}$ is substituted into (4.2), the original partial differential equation in x and t is reduced to a non-linear ordinary differential equation in ξ. We will normally choose n so that the degrees of the highest order derivative term and the highest order non-linear term in this reduced ordinary differential equation are balanced. However, this does not always result in an integral value for n. In this case, it is sometimes possible to proceed by letting $\tilde{u} = v^{\frac{1}{\tau}}$, where τ is the denominator of the fractional value of n (assuming the denominator and numerator have no common factors), and solving the resulting equation for v. This is illustrated in the following example.

Example 4.1. Consider the following Boussinesq-type equation:

$$u_{tt} - u_{xx} + u_{xxxx} + (u^5 - u^3)_{xx} = 0.$$

By letting $u(x,t) = \tilde{u}(k(x - \nu t))$, the above partial differential equation is reduced to the following ordinary differential equation:

$$\nu^2 \tilde{u}'' - \tilde{u}'' + k^2 \tilde{u}'''' + (\tilde{u}^5 - \tilde{u}^3)'' = 0.$$

Integrating twice yields

$$\nu^2 \tilde{u} - \tilde{u} + k^2 \tilde{u}'' + \tilde{u}^5 - \tilde{u}^3 = 0. \tag{4.8}$$

Here, the highest order non-linear term is $\tilde{u}^5$, and the highest order derivative term is $\tilde{u}''$. Balancing these two terms using (4.4) and (4.7) gives $5n = n + 2$, or $n = \frac{1}{2}$. Setting $\tilde{u} = v^{\frac{1}{2}}$, (4.8) becomes

$$(\nu^2 - 1)v^2 + \frac{k^2}{4}[2vv'' - (v')^2] + v^4 - v^3 = 0. \tag{4.9}$$

Now, balancing $(v')^2$ and v^4 gives $n = 1$. Hence, we can search for traveling wave solutions of (4.9) which take the form $v\,(k(x - \nu t)) = c_0 + c_1 \Phi\,(k(x - \nu t))$, for constants c_0 and c_1. If such a v can be determined, then it is easy to derive $\tilde{u}$. ■

It is noted in Example 4.1 that substituting $\tilde{u}$ into (4.2) yields a non-linear ordinary differential equation in ξ. When the derivatives of $\tilde{u}$ are substituted into this reduced ordinary differential equation, we will obtain a linear combination of $\Phi^j(\Phi')^l$, where $j \geq 0$ is an integer and $l \in \{0, 1\}$. If ν, k, and c_j, $j = 0, \ldots, n$, and q_j, $j = 0, \ldots, 4$, can be chosen to make each coefficient in this linear combination zero, then the resulting $\tilde{u}$ will satisfy the original partial differential equation (4.2). However, in this procedure, we sometimes end up with $c_j = 0$, $j = 0, \ldots, n$ (we encounter this in Section 4.6). In this case, we can use the following alternative solution form proposed in [22]:

$$\tilde{u}(\xi) = c_0 + \sum_{j=1}^{n} \frac{c_j[\Phi(\xi)]^j + C_j[\Phi(\xi)]^{j-1}\Phi'(\xi)}{[\mu\Phi(\xi) + 1]^j}, \tag{4.10}$$

where c_j $(j = 0, \ldots, n)$, C_j $(j = 1, \ldots, n)$ and μ are constants.

In [7, 30], the solutions to the reduced ordinary differential equation are represented by

$$\tilde{u}(\xi) = \sum_{j=-n}^{n} c_j \big[\Phi(\xi)\big]^j + \frac{\Phi'(\xi)}{\big[\Phi(\xi)\big]^2} \left(\sum_{j=-n}^{n+1} C_j \big[\Phi(\xi)\big]^j \right). \tag{4.11}$$

However, the degree of $\tilde{u}$ given in (4.11) is $n + 1$. Hence, $C_{n+1} = 0$. Thus, it becomes

$$\tilde{u}(\xi) = \sum_{j=-n}^{n} c_j \big[\Phi(\xi)\big]^j + \frac{\Phi'(\xi)}{\big[\Phi(\xi)\big]^2} \left(\sum_{j=-n}^{n} C_j \big[\Phi(\xi)\big]^j \right). \tag{4.12}$$

In (4.12), there are $4n + 2$ variables which need to be determined, while there are only $2n + 2$ variables (c_j, $j = 0, \ldots, n$, C_j, $j = 1, \ldots, n$, and μ) in (4.10). Note that, in (4.12), $q_1 = q_3 = 0$ is required. Hence, combining Theorem 4.2 and expression (4.10) with $\mu = 0$, we can derive the same result as the ones derived from (4.12). But, (4.10) is much more easy than (4.12) for calculations.

Notice that each of the Jacobi elliptic solutions of the ordinary differential equation (4.1) reported in [22, 64, 71] can be written as a scalar multiple of some $\varphi_{j,l}(\cdot, 0)$, $j \in \{1, \ldots, 6\}$, $l \in \{1, \ldots, 4\}$. Hence, by applying our expansion method with (4.3) and Theorem 4.1 to a non-linear partial differential equation, we can replicate every Jacobi elliptic solution obtained using the methods presented in [64, 71]. Applying our expansion method with (4.10) and Theorem 4.1 to a non-linear partial differential equation, we can obtain all Jacobi elliptic solutions obtained using the method presented in [22]. Similarly, each Jacobi elliptic solution of the ordinary differential equation (4.1) reported in [30, 36] with $\omega = 1$ can be written as a scalar multiple of some $\varphi_{j,l}(\cdot, 0)$, $j \in \{1, \ldots, 6\}$, $l \in \{1, \ldots, 4\}$. It is also evident that, for the special case $\mu = 0$, using our expansion method with (4.10) and Theorem 4.1 and Theorem 4.2, we can recover every Jacobi elliptic solution obtained using the method of [30, 106]. Hence, by virtue of the new results in Section 4.2, our method is a significant generalization of the work reported in [7, 22, 30, 36, 42, 64, 71, 106].

4.4 Traveling wave solutions for the Boussinesq equation

Consider the Boussinesq equation

$$u_{tt} + \alpha_1 u_{xx} + \alpha_2 u_{xxxx} + \alpha_3 \left(u^2\right)_{xx} = 0, \tag{4.13}$$

where $u := u\,(x,t)$ is a real-valued function. Letting $u(x,t) = \tilde{u}(\xi)$, where ξ is as defined in Section 4.3, (4.13) becomes an ordinary differential equation

$$\nu^2 \tilde{u}'' + \alpha_1 \tilde{u}'' + \alpha_2 k^2 \tilde{u}'''' + \alpha_3 \left(\tilde{u}^2\right)'' = 0. \tag{4.14}$$

Now, let us solve the following ordinary differential equation:

$$\beta_1 \Psi''(\xi) + \beta_2 \Psi''''(\xi) + \beta_3 \left[\Psi^2(\xi)\right]'' = 0, \tag{4.15}$$

where constants β_j, $j = 1, 2, 3$, are non-zero. Balancing $[\Psi^2(\xi)]''$ and $\Psi''''(\xi)$ gives $2n+2 = n+4$, or $n = 2$. Hence, we will search for candidate solutions of the form

$$\Psi(\xi) = c_0 + c_1 \Phi(\xi) + c_2 \left[\Phi(\xi)\right]^2, \tag{4.16}$$

where $c_2 \neq 0$ and Φ satisfies the ordinary differential equation (4.1) with coefficients q_j, $j = 0, \ldots, 4$. Substituting (4.16) into (4.15) and using (4.1) and (4.5)-(4.6), we obtain the following sufficient conditions for Ψ to satisfy (4.15):

$$\begin{cases} c_0 = \dfrac{3\beta_2 q_3^2 - 16\beta_2 q_2 q_4 - 4\beta_1 q_4}{8\beta_3 q_4}, \\ c_1 = -\dfrac{3\beta_2 q_3}{\beta_3}, \\ c_2 = -\dfrac{6\beta_2 q_4}{\beta_3}, \\ q_1 = \dfrac{q_3\left(4q_2q_4 - q_3^2\right)}{8q_4^2}. \end{cases} \tag{4.17}$$

That is, if a solution Φ of the ordinary differential equation (4.1) with coefficients satisfying $q_1 = \frac{q_3\left(4q_2q_4 - q_3^2\right)}{8q_4^2}$ and $q_4 \neq 0$ can be found, then

$$\Psi(\xi) = c_0 + c_1 \Phi(\xi) + c_2 \left[\Phi(\xi)\right]^2, \tag{4.18}$$

where c_0, c_1, c_2 are as defined in (4.17), is a solution of the ordinary differential equation (4.15). Now, we generalize this solution form further. Note that, if $q_1 = q_3 = 0$, then (4.18) reduces to

$$\Psi(\xi) = -\tfrac{4\beta_2 q_2+\beta_1}{2\beta_3} - \tfrac{6\beta_2 q_4}{\beta_3}\left[\Phi(\xi)\right]^2. \tag{4.19}$$

If $q_0 \neq 0$, then using Theorem 4.2 with (4.19) gives the following solution form for equation (4.15):

$$\Psi(\xi) = -\frac{4\beta_2 q_2 + \beta_1}{2\beta_3} - \frac{6\beta_2 q_4}{\beta_3}\left[\Phi(\xi)\right]^2 - \frac{6\beta_2 q_0}{\beta_3\left[\Phi(\xi)\right]^2}. \tag{4.20}$$

Furthermore, note that (4.18) can be rewritten as

$$\Psi(\xi) = \frac{3\beta_2 q_3^2 - 8\beta_2 q_2 q_4 - 2\beta_1 q_4}{4\beta_3 q_4} - \frac{6\beta_2 q_4}{\beta_3}\left[\Phi(\xi) + \frac{q_3}{4q_4}\right]^2. \tag{4.21}$$

The solution forms (4.20) and (4.21) provide motivation for the following more general candidate solution:

$$\Psi(\xi) = \frac{3\beta_2 q_3^2 - 8\beta_2 q_2 q_4 - 2\beta_1 q_4}{4\beta_3 q_4} - \frac{6\beta_2 q_4}{\beta_3}\left[\Phi(\xi) + \frac{q_3}{4q_4}\right]^2 + \frac{d}{\left[\Phi(\xi) + \frac{q_3}{4q_4}\right]^2}, \tag{4.22}$$

where d is a constant. By substituting (4.22) into (4.15), the value of d can be determined. We summarize our results in the form of the following theorem.

Theorem 4.6. For each $j = 1, 2$, let $\varepsilon_j \in \{0, 1\}$. Suppose that Φ is a solution of the ordinary differential equation (4.1) with coefficients q_j, $j = 0, \ldots, 4$, satisfying $q_4 \neq 0$ and $q_1 = \frac{q_3\left(4q_2q_4 - q_3^2\right)}{8q_4^2}$. Then,

$$\begin{aligned}\Psi_1(\xi, \beta_1, \beta_2, \beta_3) = &\frac{3\beta_2 q_3^2 - 8\beta_2 q_2 q_4 - 2\beta_1 q_4}{4\beta_3 q_4} - \varepsilon_1 \frac{6\beta_2 q_4}{\beta_3}\left[\Phi(\xi) + \frac{q_3}{4q_4}\right]^2 \\ &+ \varepsilon_2 \frac{3\beta_2(16q_3^2 q_2 q_4 - 5q_3^4 - 256q_4^3 q_0)}{128\beta_3 q_4^3\left[\Phi(\xi) + \frac{q_3}{4q_4}\right]^2}\end{aligned}$$

is a solution of the ordinary differential equation (4.15).

Remark 4.3. Ψ_1 includes both (4.18) and (4.20) as special cases.

Remark 4.4. When $q_1 = q_3 = 0$, using the method proposed in [42], where

$\sum\limits_{j=-n}^{n} c_j \left[\Phi(\xi)\right]^j$ has been used, the same results will be derived as Ψ_1 with $q_1 = q_3 = 0$.

Applying Theorem 4.6 and using the solutions of (4.1) given in Theorems 4.1, 4.4 and 4.5, we can deduce many solutions for equation (4.15). Let the functions $\psi_j(\xi, \beta_1, \beta_2, \beta_3)$, $j = 1, \ldots, 26$, be defined as follows:

$$\begin{aligned}
\psi_1(\xi,\beta_1,\beta_2,\beta_3) =& \frac{2\beta_2-\beta_1}{2\beta_3} - \frac{3\beta_2}{2\beta_3}\Bigg[\varepsilon_1\left(\frac{-\mathrm{e}^{-\xi} \pm \sqrt{\lambda_1^2-4\lambda_2}+\lambda_2\mathrm{e}^{\xi}}{\mathrm{e}^{-\xi}+\lambda_1+\lambda_2\mathrm{e}^{\xi}}\right)^2 \\
&+\varepsilon_2\left(\frac{\mathrm{e}^{-\xi}+\lambda_1+\lambda_2\mathrm{e}^{\xi}}{-\mathrm{e}^{-\xi} \pm \sqrt{\lambda_1^2-4\lambda_2}+\lambda_2\mathrm{e}^{\xi}}\right)^2\Bigg],\\
\psi_2(\xi,\beta_1,\beta_2,\beta_3) =& -\frac{2\beta_2+\beta_1}{2\beta_3} - \frac{3\beta_2}{2\beta_3}\left[\varepsilon_1\left(\frac{\sin(\xi)\pm\sqrt{1-\lambda_3^2}}{\cos(\xi)+\lambda_3}\right)^2+\varepsilon_2\left(\frac{\cos(\xi)+\lambda_3}{\sin(\xi)\pm\sqrt{1-\lambda_3^2}}\right)^2\right],\\
\psi_3(\xi,\beta_1,\beta_2,\beta_3) =& \frac{4\beta_2\left(m^2+1\right)-\beta_1}{2\beta_3} - \frac{6\beta_2}{\beta_3}\left[\varepsilon_1 m^2\mathrm{sn}^2(\xi)+\varepsilon_2\mathrm{sn}^{-2}(\xi)\right],\\
\psi_4(\xi,\beta_1,\beta_2,\beta_3) =& \frac{4\beta_2\left(m^2-2\right)-\beta_1}{2\beta_3} + \frac{6\beta_2}{\beta_3}\left[\varepsilon_1\mathrm{dn}^2(\xi)+\frac{1-m^2}{\mathrm{dn}^2(\xi)}\right],\\
\psi_5(\xi,\beta_1,\beta_2,\beta_3) =& \frac{4\beta_2\left(m^2-2\right)-\beta_1}{2\beta_3} - \frac{6\beta_2}{\beta_3}\left[\varepsilon_1\frac{\mathrm{cn}^2(\xi)}{\mathrm{sn}^2(\xi)}+\frac{(1-m^2)\mathrm{sn}^2(\xi)}{\mathrm{cn}^2(\xi)}\right],\\
\psi_6(\xi,\beta_1,\beta_2,\beta_3) =& \frac{4\beta_2\left(m^2+1\right)-\beta_1}{2\beta_3} - \frac{6\beta_2}{\beta_3}\left[m^2\frac{\mathrm{cn}^2(\xi)}{\mathrm{dn}^2(\xi)}+\frac{\mathrm{dn}^2(\xi)}{\mathrm{cn}^2(\xi)}\right],\\
\psi_7(\xi,\beta_1,\beta_2,\beta_3) =& -\frac{\beta_1+4\beta_2\left(2m^2-1\right)}{2\beta_3} - \frac{6\beta_2}{\beta_3}\left[-m^2\mathrm{cn}^2(\xi)+\frac{1-m^2}{\mathrm{cn}^2(\xi)}\right],\\
\psi_8(\xi,\beta_1,\beta_2,\beta_3) =& -\frac{\beta_1+4\beta_2\left(2m^2-1\right)}{2\beta_3} - \frac{6\beta_2}{\beta_3}\left[\frac{\mathrm{dn}^2(\xi)}{\mathrm{sn}^2(\xi)}+\frac{m^2(m^2-1)\mathrm{sn}^2(\xi)}{\mathrm{dn}^2(\xi)}\right],\\
\psi_9(\xi,\beta_1,\beta_2,\beta_3) =& \frac{2\beta_2\left(2m^2-1\right)-\beta_1}{2\beta_3} - \frac{3\beta_2[1-\mathrm{cn}(\xi)]}{2\beta_3[1+\mathrm{cn}(\xi)]},\\
\psi_{10}(\xi,\beta_1,\beta_2,\beta_3) =& \frac{2\beta_2\left(2m^2-1\right)-\beta_1}{2\beta_3} - \frac{3\beta_2[\mathrm{dn}(\xi)-\sqrt{1-m^2}\mathrm{sn}(\xi)]}{2\beta_3[\mathrm{dn}(\xi)+\sqrt{1-m^2}\mathrm{sn}(\xi)]},\\
\psi_{11}(\xi,\beta_1,\beta_2,\beta_3) =& -\frac{\beta_1+2\beta_2\left(m^2+1\right)}{2\beta_3} + \frac{3\beta_2(1-m^2)[1-m\mathrm{sn}(\xi)]}{2\beta_3[1+m\mathrm{sn}(\xi)]},\\
\psi_{12}(\xi,\beta_1,\beta_2,\beta_3) =& -\frac{\beta_1+2\beta_2\left(m^2+1\right)}{2\beta_3} - \frac{3\beta_2(1-m^2)[1-\mathrm{sn}(\xi)]}{2\beta_3[1+\mathrm{sn}(\xi)]},\\
\psi_{13}(\xi,\beta_1,\beta_2,\beta_3) =& -\frac{\beta_1+2\beta_2\left(m^2+1\right)}{2\beta_3} - \frac{3\beta_2(1-m^2)[\mathrm{dn}(\xi)-\mathrm{cn}(\xi)]}{2\beta_3[\mathrm{dn}(\xi)+\mathrm{cn}(\xi)]},\\
\psi_{14}(\xi,\beta_1,\beta_2,\beta_3) =& -\frac{\beta_1+2\beta_2 2\left(m^2+1\right)}{2\beta_3} + \frac{3\beta_2(m^2-1)[m\mathrm{cn}(\xi)-\mathrm{dn}(\xi)]}{2\beta_3[m\mathrm{cn}(\xi)+\mathrm{dn}(\xi)]},
\end{aligned}$$

$$\psi_{15}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+2\beta_2\left(m^2-2\right)}{2\beta_3}-\frac{3\beta_2 m^2[\mathrm{dn}(\xi)-\sqrt{1-m^2}]}{2\beta_3[\mathrm{dn}(\xi)+\sqrt{1-m^2}]},$$

$$\psi_{16}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+2\beta_2\left(m^2-2\right)}{2\beta_3}-\frac{3\beta_2 m^2[1-\mathrm{dn}(\xi)]}{2\beta_3[1+\mathrm{dn}(\xi)]},$$

$$\psi_{17}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+\beta_2\left(2m^2+12m+2\right)}{2\beta_3}+\frac{3\beta_2(m-1)^2}{2\beta_3}\left\{\left[\frac{1-\sqrt{m}\mathrm{sn}(\xi)}{1+\sqrt{m}\mathrm{sn}(\xi)}\right]^2+\varepsilon_1\left[\frac{1+\sqrt{m}\mathrm{sn}(\xi)}{1-\sqrt{m}\mathrm{sn}(\xi)}\right]^2\right\},$$

$$\psi_{18}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+\beta_2\left(2m^2+12m+2\right)}{2\beta_3}+\frac{3\beta_2(m-1)^2}{2\beta_3}\left\{\left[\frac{\mathrm{dn}(\xi)-\sqrt{m}\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)+\sqrt{m}\mathrm{cn}(\xi)}\right]^2+\varepsilon_1\left[\frac{\mathrm{dn}(\xi)+\sqrt{m}\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)-\sqrt{m}\mathrm{cn}(\xi)}\right]^2\right\},$$

$$\psi_{19}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+\beta_2(2m^2-4+12\sqrt{1-m^2})}{2\beta_3}-\frac{3\beta_2(1+\sqrt{1-m^2})^2}{2\beta_3}\left\{\left[\frac{\mathrm{cn}(\xi)-\sqrt[4]{1-m^2}\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)+\sqrt[4]{1-m^2}\mathrm{sn}(\xi)}\right]^2+\varepsilon_1\left[\frac{\mathrm{cn}(\xi)+\sqrt[4]{1-m^2}\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)-\sqrt[4]{1-m^2}\mathrm{sn}(\xi)}\right]^2\right\},$$

$$\psi_{20}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+\beta_2(2m^2-4-12\sqrt{1-m^2})}{2\beta_3}-\frac{3\beta_2(1-\sqrt{1-m^2})^2}{2\beta_3}\left\{\left[\frac{\mathrm{dn}(\xi)-\sqrt[4]{1-m^2}}{\mathrm{dn}(\xi)+\sqrt[4]{1-m^2}}\right]^2+\varepsilon_1\left[\frac{\mathrm{dn}(\xi)+\sqrt[4]{1-m^2}}{\mathrm{dn}(\xi)-\sqrt[4]{1-m^2}}\right]^2\right\},$$

$$\psi_{21}(\xi,\beta_1,\beta_2,\beta_3)=\frac{2\beta_2\left(2m^2-1\right)-\beta_1}{2\beta_3}-\frac{3\beta_2}{2\beta_3}\left[\frac{\mathrm{sn}(\xi)+\sqrt{1-m^2}\mathrm{dn}(\xi)}{m\sqrt{2-m^2}+\sqrt{-m^4+m^2+1}\mathrm{cn}(\xi)}\right]^2-\frac{3\beta_2}{2\beta_3}\varepsilon_1\left[\frac{m\sqrt{2-m^2}+\sqrt{-m^4+m^2+1}\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)+\sqrt{1-m^2}\mathrm{dn}(\xi)}\right]^2,$$

$$\psi_{22}(\xi,\beta_1,\beta_2,\beta_3)=\frac{2\beta_2\left(2m^2-1\right)-\beta_1}{2\beta_3}-\frac{3\beta_2}{2\beta_3}\left[\frac{\mathrm{cn}(\xi)+(m^2-1)}{m\sqrt{2-m^2}\mathrm{dn}(\xi)+\sqrt{(-m^4+m^2+1)(1-m^2)}\mathrm{sn}(\xi)}\right]^2-\varepsilon_1\frac{3\beta_2}{2\beta_3}\left[\frac{m\sqrt{2-m^2}\mathrm{dn}(\xi)+\sqrt{(-m^4+m^2+1)(1-m^2)}\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)+(m^2-1)}\right]^2,$$

$$\psi_{23}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+2\beta_2\left(m^2+1\right)}{2\beta_3}+\frac{3\beta_2(m^2-1)}{2\beta_3}\left[\frac{\mathrm{cn}(\xi)+\sqrt{1-m^2}\mathrm{dn}(\xi)}{m+\sqrt{m^4-m^2+1}\mathrm{sn}(\xi)}\right]^2+\varepsilon_1\frac{3\beta_2(m^2-1)}{2\beta_3}\left[\frac{m+\sqrt{m^4-m^2+1}\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)+\sqrt{1-m^2}\mathrm{dn}(\xi)}\right]^2,$$

$$\psi_{24}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+2\beta_2\left(m^2+1\right)}{2\beta_3}-\frac{3\beta_2(1-m^2)^2}{2\beta_3}\left[\frac{\mathrm{sn}(\xi)+\sqrt{1-m^2}}{m\mathrm{dn}(\xi)+\sqrt{m^4-m^2+1}\mathrm{cn}(\xi)}\right]^2-\varepsilon_1\frac{3\beta_2}{2\beta_3}\left[\frac{m\mathrm{dn}(\xi)+\sqrt{m^4-m^2+1}\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)+\sqrt{1-m^2}}\right]^2,$$

$$\psi_{25}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+2\beta_2\left(m^2+1\right)}{2\beta_3}-\frac{3\beta_2(m^2-1)}{2\beta_3}\left[\frac{\mathrm{dn}(\xi)+\sqrt{1-m^2}\mathrm{cn}(\xi)}{m^2\mathrm{sn}(\xi)+\sqrt{m^4-m^2+1}}\right]^2-\varepsilon_1\frac{3\beta_2(m^2-1)}{2\beta_3}\left[\frac{m^2\mathrm{sn}(\xi)+\sqrt{m^4-m^2+1}}{\mathrm{dn}(\xi)+\sqrt{1-m^2}\mathrm{cn}(\xi)}\right]^2,$$

$$\psi_{26}(\xi,\beta_1,\beta_2,\beta_3)=-\frac{\beta_1+2\beta_2\left(m^2+1\right)}{2\beta_3}+\frac{3\beta_2(1-m^2)^2}{2\beta_3}\left[\frac{1+m\sqrt{1-m^2}\mathrm{sn}(\xi)}{m^2\mathrm{cn}(\xi)+\sqrt{m^4-m^2+1}\mathrm{dn}(\xi)}\right]^2+\varepsilon_1\frac{3\beta_2}{2\beta_3}\left[\frac{m^2\mathrm{cn}(\xi)+\sqrt{m^4-m^2+1}\mathrm{dn}(\xi)}{1+m\sqrt{1-m^2}\mathrm{sn}(\xi)}\right]^2,$$

where ε_1, $\varepsilon_2 \in \{0,1\}$, $m \in (0,1)$ is the modulus of the Jacobi elliptic functions, and

λ_j, $j = 1, 2, 3$, are arbitrary real constants satisfying $\lambda_1^2 \leq 4\lambda_2$ and $-1 \leq \lambda_3 \leq 1$.

Through direct substitution, we can prove the following result.

Theorem 4.7. For each $j = 1, \ldots, 26$, the function ψ_j is a solution of the ordinary differential equation (4.15).

Remark 4.5. It follows from Remark 4.1 that ψ_j, $j = 3, \ldots, 26$, still satisfy equation (4.15) even if cn(ξ), sn(ξ) and dn(ξ) are replaced, respectively, by $\pm$cn(ξ), $\pm$sn(ξ) and $\pm$dn(ξ).

Remark 4.6. It is interesting to note that, for each $j \in \{3, \ldots, 26\}$, the solution ψ_j becomes a special case of ψ_1 as $m \to 1$. Similarly, as $m \to 0$, ψ_j becomes a special case of ψ_2.

Next, we will make use of the solutions ψ_j, $j = 1, \ldots, 26$, to derive traveling wave solutions for the Boussinesq equation (4.13).

For each $j = 1, \ldots, 26$, let

$$u_j(x, t) := \psi_j\big(k(x - \nu t), \alpha_1 + \nu^2, \alpha_2 k^2, \alpha_3\big),$$

where k and ν are arbitrary real constants. According to Theorem 4.6 and (4.14), we know that, for each $j = 1, \ldots, 26$, the function u_j is a solution of the Boussinesq equation (4.13).

Note that, for some cases, the denominators in the expression of u_1 can be equal to zero at certain points, and thus, such a solution is unbounded. For example, u_1 with $\varepsilon_1 = \varepsilon_2 = 1$ and $\lambda_2 \neq 0$ is unbounded. It is also noted that, for some cases, the solution u_1 is bounded. For instance, u_1 with $\varepsilon_1 = 1$, $\varepsilon_2 = 0$, $0 \leq \lambda_2 \leq \lambda_1^2/4$ and $\lambda_1 \geq 0$ is bounded. For the bounded case, clearly, the solution u_1 gives a single wave that moves in the x-direction with velocity ν and $u_1(x, t) \to (2\alpha_2 k^2 - \alpha_1 - \nu^2)/(2\alpha_3) - 3\alpha_2 k^2(\varepsilon_1 + \varepsilon_2)/(2\alpha_3)$ as $k(x - \nu t) \to \pm\infty$.

To show the physical insight of these solutions, here we choose $\alpha_1 = -1$,

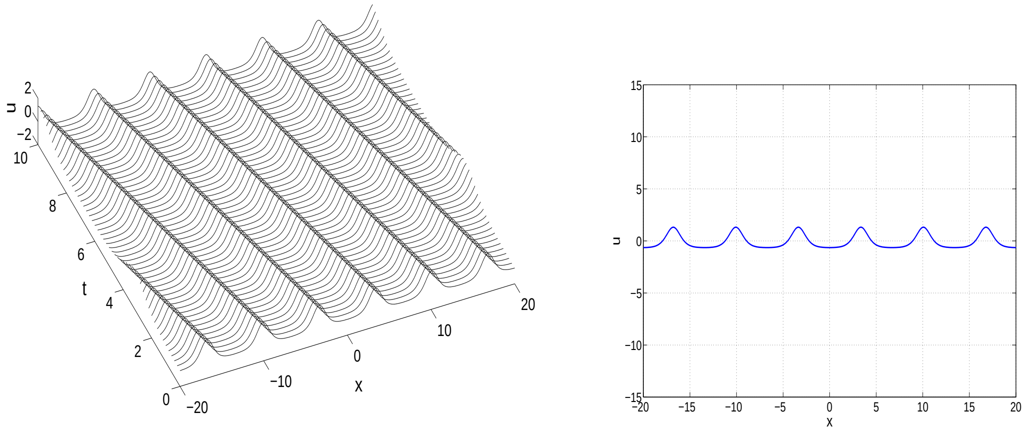

Figure 4.1: The plot of the solution u_4 to the Boussinesq equation (4.13) with $\alpha_1 = -1$, $\alpha_2 = -1$, $\alpha_3 = -3$, $m = 0.99$, $k = 1$ and $\nu = -1$ and the initial status of u_4.

$\alpha_2 = -1$, $\alpha_3 = -3$ and take u_4, u_7 as examples. Figure 4.1 shows the wave profile of the solution u_4 with $m = 0.99$, $k = 1$ and $\nu = -1$. Clearly, the solution is a periodic function describing the traveling of waves in the negative x-direction. Figure 4.2 shows the graph of the solution u_7 for $m = 0.9$, $k = 1$ and $\nu = -2$. Note that u_7 becomes infinity when $\text{cn}(k(x - \nu t), m) = 0$, that is, $k(x - \nu t) = (2j + 1)K$, where $K = \int_0^{\pi/2} \left(1 - m^2 \sin^2(s)\right)^{-1/2} ds$ and $j = 0$, ± 1, ... For instance, in Figure 4.2, u_7 becomes negative infinity when the point (x, t) is close to the lines $x + 2t = 2.280549138(2j + 1)$, where $j = 0$, ± 1, ... It is also noted from the expression of the solutions u_3 with $\varepsilon_2 = 1$ and u_j, $j \in \{5, 6, 7, 8, 9, 12, 13, 19, 20, 21, 22, 23, 24\}$, that these solutions are unbounded, since the denominator in the expression can be zero at certain points.

To show the power of the proposed expansion method, we compare our results with the solutions reported in [49, 63, 100, 116, 117, 118]. In [49], the multiple soliton-like solutions and triangular periodic solutions derived by Inan and Kaya are included in the solutions u_1 and u_2, respectively. In [100], the solutions of (4.13) were obtained using the sinh/cosh ansatz I-II method, the sinh-cosh ansatz III method, the tanh method and the sine-cosine method. Each of these solutions is a special case of u_1 or u_2. In [116], two solutions of the Boussinesq equation (4.13)

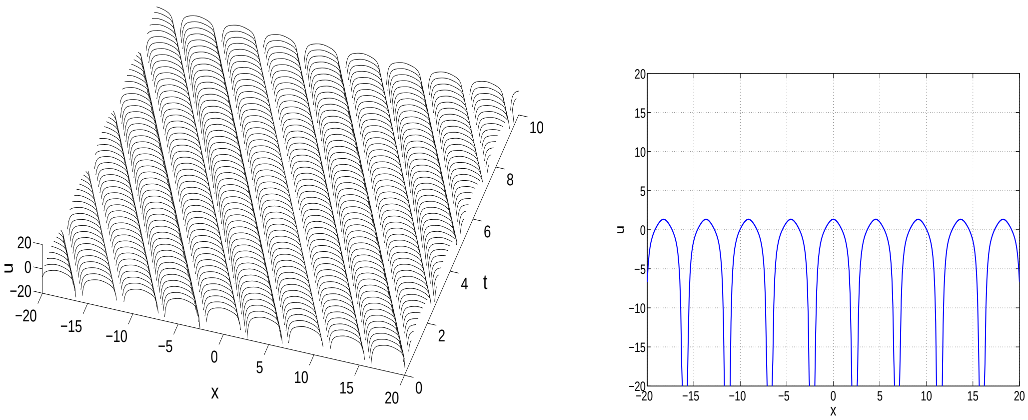

Figure 4.2: The plot of the solution u_7 to the Boussinesq equation (4.13) with $\alpha_1 = -1$, $\alpha_2 = -1$, $\alpha_3 = -3$, $m = 0.9$, $k = 1$ and $\nu = -2$ and the initial status of u_7.

are special cases of the solution u_1. The solution u_3 is identical to the solutions reported in [63, 117], and the solution u_9 is the same as the solution reported in [118]. However, all of the other Jacobi elliptic function solutions are new solutions.

Furthermore, if the candidate traveling wave solution of the forms (2.7) and (4.10) are considered and our new results in Section 4.2 are applied, then many additional solutions can be obtained.

As mentioned in Remark 4.4, Ψ_1 includes the results derived by (2.7) for the case $q_1 = q_3 = 0$. Hence, we will consider (2.7) under the case $q_3 \neq 0$. Substituting

$$\Psi(\xi) = \frac{c_{-2}}{\big[\Phi(\xi)\big]^2} + \frac{c_{-1}}{\Phi(\xi)} + c_0 + c_1\Phi(\xi) + c_2\big[\Phi(\xi)\big]^2$$

into (4.15), we can determine the coefficients c_j, $j = -2, \ldots, 2$.

Theorem 4.8. Let Φ be a solution of the ordinary differential equation (4.1) with coefficients q_j ($j = 0, \ldots, 4$). If $q_0 = \frac{16q_4^2q_2^2 - 8q_3^2q_2q_4 + q_3^4}{64q_4^3}$ and $q_1 = \frac{q_3(q_3^2 - 4q_2q_4)}{8q_4^2}$ ($q_4 \neq 0$), then a solution of the equation (4.15) is

$$\Psi_2(\xi, \beta_1, \beta_2, \beta_3) = c_0 + c_1\Phi(\xi) + c_2\big[\Phi(\xi)\big]^2 - \frac{3\beta_2 q_1}{\beta_3\Phi(\xi)} - \frac{6\beta_2 q_0}{\beta_3\big[\Phi(\xi)\big]^2},$$

where c_0, c_1 and c_2 are as defined in (4.17).

Note that the form (4.10) is more general than (4.16). Consequently, (4.10) can generate additional solutions for (4.15). Suppose that the solutions of (4.15) are represented by

$$\Psi(\xi) = c_0 + \frac{c_1\Phi(\xi) + C_1\Phi'(\xi)}{\mu\Phi(\xi)+1} + \frac{c_2\big[\Phi(\xi)\big]^2 + C_2\Phi(\xi)\Phi'(\xi)}{\big[\mu\Phi(\xi)+1\big]^2}, \tag{4.23}$$

where the constants μ, c_j $(j = 0, 1, 2)$ and C_j $(j = 1, 2)$ will be determined later.

If $C_1 = C_2 = 0$ and $\mu \neq 0$, then (4.23) becomes

$$\Psi(\xi) = c_0 + \frac{c_1\Phi(\xi)}{\mu\Phi(\xi)+1} + \frac{c_2\big[\Phi(\xi)\big]^2}{\big[\mu\Phi(\xi)+1\big]^2}. \tag{4.24}$$

The form (4.24) is not an ideal choice for solving exact solutions for non-linear partial differential equations. As far as we know, no one uses this form in the literature. In the sequel, we will show why it is not a good choice.

Note that if we want to derive exponential function solutions, then the form (4.18) is more preferable than the form (4.24), since they can derive the same results. According to Table 4.1, there are no sets of coefficients q_j $(j = 0, \ldots, 4)$ satisfying $q_0 = q_1 = 0$. Hence, to obtain Jacobi elliptic function solutions, we need to delete results satisfying $q_0 = q_1 = 0$. Substituting (4.24) into (4.15) and using Maple, we can only obtain the following three sets of results when $q_0 c_1 = 0$:

$$\begin{cases} q_0 = 0, \\ q_4 = \mu(q_3 - q_2\mu + q_1\mu^2), \\ c_0 = \frac{3q_1\beta_2\mu - q_2\beta_2 - \beta_1}{2\beta_3}, \\ c_1 = -\frac{3\beta_2(q_3 + 3\mu^2 q_1 - 2q_2\mu)}{2\beta_3}, \\ c_2 = 0, \end{cases} \tag{4.25}$$

$$\begin{cases} q_0 = 0, \\ q_2 = \frac{6q_1\beta_2 c_2^2 - \beta_3 c_1^3 + 18q_1\beta_2 c_1 c_2\mu}{6\beta_2 c_1 c_2}, \\ q_3 = -\frac{\beta_3 c_1^2 c_2 - 6q_1\beta_2 c_2^2\mu - 9q_1\beta_2 c_1 c_2\mu^2 + \beta_3 c_1^3\mu}{3\beta_2 c_1 c_2}, \\ q_4 = -\frac{\beta_3 c_1 c_2^2 - 6q_1\beta_2 c_2^2\mu^2 + 2\beta_3 c_1^2 c_2\mu + \beta_3 c_1^3\mu^2 - 6\beta_2 q_1 c_1 c_2\mu^3}{6\beta_2 c_1 c_2}, \\ c_0 = \frac{\beta_3 c_1^3 - 24q_1\beta_2 c_2^2 - 6\beta_1 c_1 c_2}{12\beta_3 c_1 c_2}, \end{cases} \tag{4.26}$$

and

$$\begin{cases} q_3 = \frac{q_1(4q_2q_0 - q_1^2)}{8q_0^2}, \\ \mu = \frac{q_1}{4q_0}, \\ c_0 = \frac{3\beta_2 q_1^2 - 8\beta_2 q_2 q_0 - 2\beta_1 q_0}{4\beta_3 q_0}, \\ c_1 = 0, \\ c_2 = \frac{3\beta_2(16q_1^2 q_2 q_0 - 5q_1^4 - 256q_0^3 q_4)}{128\beta_3 q_0^3}. \end{cases} \tag{4.27}$$

Note that if φ is a solution of equation (4.1) with $q_j = \hat{q}_j$, $j = 0, \ldots, 4$, then $1/\varphi$ is a solution of (4.1) with $q_j = \hat{q}_{4-j}$, $j = 0, \ldots, 4$. Hence, using the solution form Ψ_1 with $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$, we can derive (4.27). Actually, there exists another set of results in which $q_0 c_1 \neq 0$. However, it is too difficult to solve the non-linear equations, even if $q_1 = q_3 = 0$. Note that Ψ_1 with $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$ satisfies $q_0 c_1 \neq 0$. This means that such Ψ_1 gives a solution for (4.24) under the case $q_0 c_1 \neq 0$. However, it is impossible to derive Ψ_1 with $\varepsilon_1 = 1$ and $\varepsilon_2 = 1$ using (4.24).

In addition, we can derive solutions in the form (4.24) from (4.16). Rewriting (4.24), it follows

$$\Psi(\xi) = c_0 + c_1 \left[\frac{\Phi(\xi)}{\mu\Phi(\xi)+1}\right] + c_2 \left[\frac{\Phi(\xi)}{\mu\Phi(\xi)+1}\right]^2. \tag{4.28}$$

Note that if φ is a solution of the equation (4.1) with coefficients $q_j = \hat{q}_j$, $j = 0, \ldots, 4$, then $\varphi/(\mu\varphi + 1)$ is a solution of (4.1) with

$$\begin{cases} q_0 = \hat{q}_0, \\ q_1 = \hat{q}_1 - 4\hat{q}_0\mu, \\ q_2 = \hat{q}_2 - 3\hat{q}_1\mu + 6\hat{q}_0\mu^2, \\ q_3 = \hat{q}_3 - 2\hat{q}_2\mu + 3\hat{q}_1\mu^2 - 4\hat{q}_0\mu^3, \\ q_4 = \hat{q}_4 - \hat{q}_3\mu + \hat{q}_2\mu^2 - \hat{q}_1\mu^3 + \hat{q}_0\mu^4. \end{cases} \tag{4.29}$$

As we know, (4.16) is a solution of (4.15) if the coefficients q_j ($j = 0, \ldots, 4$) of (4.1) satisfy $q_1 = \dfrac{q_3\,(4q_2q_4 - q_3^2)}{8q_4^2}$. Combining this condition with (4.29), we have to solve a sixth order polynomial equation in μ. It is well-known that such equation is very difficult to solve. This is the reason why Maple can not solve the case $q_0 c_1 \neq 0$ for (4.24).

Now, let us consider (4.23) with $C_1 C_2 \neq 0$. Substituting (4.23) into (4.15), we obtain that

$$\begin{cases} c_0 = -\dfrac{6\beta_2\mu^2 q_0 - 3\beta_2\mu q_1 + \beta_2 q_2 + \beta_1}{2\beta_3}, \\ c_1 = \dfrac{3\beta_2(2\mu q_2 - 3\mu^2 q_1 - q_3 + 4\mu^3 q_0)}{2\beta_3}, \\ c_2 = -\dfrac{3\beta_2(q_4 + \mu^4 q_0 - \mu q_3 - \mu^3 q_1 + \mu^2 q_2)}{\beta_3}, \\ C_1 = \pm\dfrac{3\beta_2\sqrt{q_4 + \mu^4 q_0 - \mu q_3 - \mu^3 q_1 + \mu^2 q_2}}{\beta_3}, \\ C_2 = \mp\dfrac{3\beta_2\mu\sqrt{q_4 + \mu^4 q_0 - \mu q_3 - \mu^3 q_1 + \mu^2 q_2}}{\beta_3}. \end{cases} \tag{4.30}$$

It is noted that there is no requirement on q_j ($j = 0, \ldots, 4$) as in Theorem 4.6. It is also noted that $C_2 = -\mu C_1$. Thus, we can simplify the expression of (4.23). In addition, if $\mu = 0$, we can apply Theorems 4.2 and 4.3 to generate new solutions for equation (4.15) from (4.23). The results are summarized in the following theorem.

Theorem 4.9. Suppose that Φ is a solution of the ordinary differential equation (4.1) with coefficients q_j ($j = 0, \ldots, 4$). Let

$$\Psi_3(\xi, \beta_1, \beta_2, \beta_3) = c_0 + \frac{c_1\Phi(\xi)}{\mu\Phi(\xi) + 1} + \frac{c_2\big[\Phi(\xi)\big]^2 + C_1\Phi'(\xi)}{\big[\mu\Phi(\xi) + 1\big]^2},$$

where c_j ($j = 0, 1, 2$) and C_1 are defined as in (4.30). For each μ, if $q_4 + \mu^4 q_0 - \mu q_3 - \mu^3 q_1 + \mu^2 q_2 \geq 0$, then the function Ψ_3 satisfies the ordinary differential equation (4.15). Moreover, if $q_3^2 q_0 = q_1^2 q_4$, $q_0 > 0$ and $q_4 > 0$, then

$$\begin{aligned}\Psi_4(\xi, \beta_1, \beta_2, \beta_3) = -\frac{1}{2\beta_3}\Bigg[&\beta_1 + \beta_2(q_2 + 6\epsilon_1\sqrt{q_0 q_4}) + 6\beta_2 q_4\big[\Phi(\xi)\big]^2 + \frac{6\beta_2 q_0}{\big[\Phi(\xi)\big]^2} \\ &\pm 6\beta_2\left(\epsilon_1\sqrt{q_4}\Phi'(\xi) - \frac{\sqrt{q_0}\Phi'(\xi)}{\big[\Phi(\xi)\big]^2}\right)\Bigg],\end{aligned}$$

where $\epsilon_1 \in \{1, -1\}$, is a solution of equation (4.15).

Remark 4.7. Without the condition $q_1 = q_3 = 0$ as the authors assumed in [7, 30], applying (4.12) to (4.15), we can get exactly the same results as Ψ_1, Ψ_2, Ψ_3 with $\mu = 0$ and Ψ_4. However, there are too many variables involved in (4.12).

Using the solutions of equation (4.15), we also can derive traveling wave solutions for the improved Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxtt} - \left(u^2\right)_{xx} = 0.$$

Letting $\xi = k(x - \nu t)$, we have the following ordinary differential equation

$$(1 - \nu^2)\tilde{u}'' + k^2\nu^2\tilde{u}'''' + \left(\tilde{u}^2\right)'' = 0.$$

Thus, according to Theorem 4.7, for each $j \in \{1, \ldots, 26\}$ and any real numbers k and ν,

$$\psi_j\big(k(x-\nu t), 1-\nu^2, k^2\nu^2, 1\big),$$

is a solution of the improved Boussinesq equation. Moreover, using Ψ_2 given in Theorem 4.8 and Ψ_3, Ψ_4 given in Theorem 4.9, many additional solutions can be derived.

4.5 Traveling wave solutions for the modified KdV equation

We consider the following modified KdV equation:

$$u_t + u^2 u_x + u_{xxx} = 0, \tag{4.31}$$

where $u := u\,(x, t)$ is a complex-valued function. Letting $u(x,t) = \tilde{u}(\xi)$, where ξ is as defined in Section 4.3, (4.31) is reduced to the ordinary differential equation

$$-\nu\tilde{u}' + \tilde{u}^2\tilde{u}' + k^2\tilde{u}''' = 0. \tag{4.32}$$

Balancing $\tilde{u}^2\tilde{u}'$ and $\tilde{u}'''$ yields $n = 1$. Thus, we now consider candidate traveling wave solutions of the form

$$\tilde{u}(\xi) = c_0 + c_1\Phi(\xi),$$

where $c_1 \neq 0$, and Φ satisfies the ordinary differential equation (4.1) with coefficients q_j, $j = 0, \ldots, 4$. Substituting $\tilde{u}$ into (4.32), we obtain the following sufficient conditions for $\tilde{u}$ to satisfy (4.32):

$$\begin{cases} c_1^2 + 6k^2 q_4 = 0, \\ 2c_0c_1 + 3k^2 q_3 = 0, \\ -\nu + c_0^2 + k^2 q_2 = 0. \end{cases} \tag{4.33}$$

According to (4.33),

$$u(x,t) = \pm k\left[\frac{3q_3}{2\sqrt{-6q_4}} - \sqrt{-6q_4}\,\Phi\big(k(x-\nu_1 t)\big)\right], \tag{4.34}$$

is a solution of (4.31) in which $\nu_1 = k^2\left(q_2 - \frac{3q_3^2}{8q_4}\right)$ and k is an arbitrary constant. Now, if $q_1 = q_3 = 0$ and $q_0 \neq 0$, then Theorem 4.2 can be applied with (4.34) to give the following solution form of (4.31):

$$u(x,t) = \epsilon_1 k\sqrt{-6q_0}\left[\epsilon_2\sqrt{\frac{q_4}{q_0}}\Phi\big(k(x-\nu_2 t)\big) + \frac{1}{\Phi\big(k(x-\nu_2 t)\big)}\right], \tag{4.35}$$

where $\epsilon_j = \pm 1$, $j = 1, 2$, $\nu_2 = k^2\left(q_2 - \epsilon_2 6q_0\sqrt{\frac{q_4}{q_0}}\right)$ and k is an arbitrary constant. In addition, if $q_4 = \frac{q_0 q_3^2}{q_1^2}$ and $q_0 q_1 \neq 0$, then Theorem 4.3 can be applied with (4.34) to yield another solution form of (4.31):

$$u(x,t) = \pm k\left(\frac{3q_1}{2\sqrt{-6q_0}} - \sqrt{-6q_0}\left[\frac{q_3}{q_1}\Phi\big(k(x-\nu_3 t)\big) + \frac{1}{\Phi\big(k(x-\nu_3 t)\big)}\right]\right), \tag{4.36}$$

where $\nu_3 = k^2\left(q_2 - \frac{6q_0 q_3}{q_1} - \frac{3q_1^2}{8q_0}\right)$ and k is an arbitrary constant.

We can apply Theorem 4.4 with (4.34) to obtain the following class of traveling wave solutions of (4.31):

$$u_1(x,t) = \lambda + \frac{\frac{3\vartheta k^2}{\lambda}}{\mathrm{e}^{-k(x-(k^2+\lambda^2)t)} + \vartheta + \frac{\vartheta^2(2\lambda^2+3k^2)}{8\lambda^2}\mathrm{e}^{k(x-(k^2+\lambda^2)t)}},$$

where λ, ϑ and k are arbitrary parameters such that $\lambda \neq 0$. It is noted that, if λ, ϑ and k are all real constants satisfying $\lambda\vartheta k \neq 0$, then u_1 describes a single wave traveling in the x-direction and $u_1(x,t) \to \lambda$, as $k(x-(k^2+\lambda^2)t) \to \pm\infty$.

We can also apply Theorem 4.5 with (4.36) to obtain another class of solutions of (4.31):

$$\begin{aligned} u_2(x,t) =& \epsilon_1\frac{\sqrt{-6}k}{\sigma-1}\left[\frac{\sigma+1}{2} - \frac{\sigma\mathrm{e}^{-k(x-\nu_4 t)} + \frac{1}{2}\lambda(\sigma+1) + \frac{1}{2}\epsilon_2\sqrt{\lambda^2-4\vartheta}(\sigma-1) + \vartheta\mathrm{e}^{k(x-\nu_4 t)}}{\mathrm{e}^{-k(x-\nu_4 t)} + \lambda + \vartheta\mathrm{e}^{k(x-\nu_4 t)}}\right] \\ & -\epsilon_1\frac{\sqrt{-6}k\sigma}{\sigma-1}\left[\frac{\mathrm{e}^{-k(x-\nu_4 t)} + \lambda + \vartheta\mathrm{e}^{k(x-\nu_4 t)}}{\sigma\mathrm{e}^{-k(x-\nu_4 t)} + \frac{1}{2}\lambda(\sigma+1) + \frac{1}{2}\epsilon_2\sqrt{\lambda^2-4\vartheta}(\sigma-1) + \vartheta\mathrm{e}^{k(x-\nu_4 t)}}\right], \end{aligned}$$

where $\epsilon_j = \pm 1$, $j = 1, 2$, $\nu_4 = \frac{k^2(\sigma^2+10\sigma+1)}{2(\sigma-1)^2}$, and λ, ϑ and σ are arbitrary constants such that $\sigma \neq 1$. Note that u_1 is the same as solution (18) in [45], obtained using the Exp-function method. However, u_2 is a new solution.

We also can obtain Jacobi elliptic solutions to the modified KdV equation (4.31) by combining Theorem 4.1 with (4.34)-(4.36).

1. For $l \in \{1, \ldots, 4\}$, $j \in \{1, \ldots, 12\}$ and γ arbitrary, (4.34) with $\Phi = \varphi_{j,l}(\cdot, \gamma)$ and $q_n = p_{j,n}(\gamma)$, $n = 0, \ldots, 4$, is a solution of (4.31).

2. For $l \in \{1, \ldots, 4\}$ and $j \in \{1, 2, 3, 4, 5, 6, 9, 11\}$, (4.35) with $\Phi = \varphi_{j,l}(\cdot, 0)$ and $q_n = p_{j,n}(0)$, $n = 0, \ldots, 4$, is a solution of (4.31).

3. For $l \in \{1, \ldots, 4\}$ and $j \in \{10, 12\}$, (4.36) with $\Phi = \varphi_{j,l}(\cdot, 0)$ and $q_n = p_{j,n}(0)$, $n = 0, \ldots, 4$, is a solution of (4.31).

Thus, we can obtain many Jacobi elliptic solutions of (4.31). To keep the details to minimum, we will not list them all here. Instead, we just select some of them to compare our results with those reported in [89, 117]. Note that our method can also be applied to the modified KdV equation considered in [89, 117].

Let γ be such that $\gamma \neq \pm 1$ and $\gamma \neq \pm m$. Choosing $q_j = p_{2,j}(\gamma)$, $j = 0, \ldots, 4$, from (4.34), it follows that

$$
\begin{aligned}
u_3(x,t) &= k\left\{\frac{3\gamma(1+m^2-2\gamma^2)}{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}\ \text{sn}\,[k(x-\nu_5 t)]}{\gamma\ \text{sn}\,[k(x-\nu_5 t)]+1}\right\},\\
u_4(x,t) &= k\left\{\frac{3\gamma(1+m^2-2\gamma^2)}{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}}{\gamma + m\ \text{sn}\,[k(x-\nu_5 t)]}\right\},\\
u_5(x,t) &= k\left\{\frac{3\gamma(1+m^2-2\gamma^2)}{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}\ \text{dn}\,[k(x-\nu_5 t)]}{\gamma\ \text{dn}\,[k(x-\nu_5 t)] + m\ \text{cn}\,[k(x-\nu_5 t)]}\right\},\\
u_6(x,t) &= k\left\{\frac{3\gamma(1+m^2-2\gamma^2)}{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}\ \text{cn}\,[k(x-\nu_5 t)]}{\gamma\ \text{cn}\,[k(x-\nu_5 t)] + \text{dn}\,[k(x-\nu_5 t)]}\right\},
\end{aligned}
$$

are solutions of (4.31) in which $\nu_5 = k^2\left[6\gamma^2 - 1 - m^2 - \frac{3\gamma^2(1+m^2-2\gamma^2)^2}{2(m^2-\gamma^2)(1-\gamma^2)}\right]$ and k is an arbitrary constant. If γ is any real number such that $m < |\gamma| < 1$, then u_j, $j = 3, \ldots, 6$, are real and bounded. Moreover, if $\gamma = 0$, then according to (4.35), we can obtain the following two unbounded solutions:

$$
\begin{aligned}
u_7(x,t) &= \sqrt{-6}k\left\{\pm m\ \text{sn}\,[k(x-\nu_6 t)] + \frac{1}{\text{sn}\,[k(x-\nu_6 t)]}\right\},\\
u_8(x,t) &= \sqrt{-6}k\left\{\pm\frac{\text{dn}\,[k(x-\nu_6 t)]}{\text{cn}\,[k(x-\nu_6 t)]} + \frac{m\ \text{cn}\,[k(x-\nu_6 t)]}{\text{dn}\,[k(x-\nu_6 t)]}\right\},
\end{aligned}
$$

where $\nu_6 = -k^2(1 \pm 6m + m^2)$ and k is an arbitrary constant.

Similarly, if $q_j = p_{3,j}(\gamma)$, $j = 0, \dots, 4$, where γ is an arbitrary constant such that $\gamma \neq \pm 1$ and $\gamma \neq \pm \mathrm{i}\frac{m}{\sqrt{1-m^2}}$, then we get solutions of (4.31) as follows:

$$\begin{aligned} u_9(x,t) =& k\Big\{ \frac{3\gamma(1-2m^2-2\gamma^2+2\gamma^2m^2)}{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}} \\ & - \frac{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}\ \mathrm{cn}\,[k(x-\nu_7 t)]}{\gamma\ \mathrm{cn}\,[k(x-\nu_7 t)]+1} \Big\}, \\ u_{10}(x,t) =& k\Big\{ \frac{3\gamma(1-2m^2-2\gamma^2+2\gamma^2m^2)}{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}} \\ & - \frac{\sqrt{6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)(1-m^2)}}{\gamma\sqrt{m^2-1}+m\ \mathrm{cn}\,[k(x-\nu_7 t)]} \Big\}, \\ u_{11}(x,t) =& k\Big\{ \frac{3\gamma(1-2m^2-2\gamma^2+2\gamma^2m^2)}{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}} \\ & - \frac{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}\ \mathrm{dn}\,[k(x-\nu_7 t)]}{\gamma\ \mathrm{dn}\,[k(x-\nu_7 t)]+\mathrm{i}m\ \mathrm{sn}\,[k(x-\nu_7 t)]} \Big\}, \\ u_{12}(x,t) =& k\Big\{ \frac{3\gamma(1-2m^2-2\gamma^2+2\gamma^2m^2)}{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}} \\ & - \frac{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)(1-m^2)}\ \mathrm{sn}\,[k(x-\nu_7 t)]}{\gamma\sqrt{1-m^2}\ \mathrm{sn}\,[k(x-\nu_7 t)]+\ \mathrm{dn}\,[k(x-\nu_7 t)]} \Big\}, \end{aligned}$$

where $\nu_7 = k^2\left[2m^2-6\gamma^2m^2+6\gamma^2-1-\frac{3\gamma^2(1-2m^2-2\gamma^2+2\gamma^2m^2)^2}{2(m^2\gamma^2-m^2-\gamma^2)(1-\gamma^2)}\right]$ and k is an arbitrary constant. Moreover, if $\gamma = 0$, then we have the unbounded solutions

$$\begin{aligned} u_{13}(x,t) =& k\sqrt{6(m^2-1)}\left\{\pm\frac{m}{\sqrt{m^2-1}}\ \mathrm{cn}\,[k(x-\nu_8 t)]+\frac{1}{\mathrm{cn}\,[k(x-\nu_8 t)]}\right\}, \\ u_{14}(x,t) =& k\sqrt{6(1-m^2)}\left\{\mp\frac{1}{\sqrt{m^2-1}}\frac{\mathrm{dn}\,[k(x-\nu_8 t)]}{\mathrm{sn}\,[k(x-\nu_8 t)]}+m\frac{\mathrm{sn}\,[k(x-\nu_8 t)]}{\mathrm{dn}\,[k(x-\nu_8 t)]}\right\}, \end{aligned}$$

where $\nu_8 = k^2(2m^2-1\pm 6m\sqrt{m^2-1})$ and k is an arbitrary constant.

If $q_j = p_{6,j}(\gamma)$, $j = 0, \dots, 4$, where γ is an arbitrary constant such that

$m^2\gamma^4 + m^2 + 4\gamma^2 - 2m^2\gamma^2 \neq 0$, then we can obtain another four solutions of (4.31)

$$
\begin{aligned}
u_{15}(x,t) =& k\Big\{\frac{3\gamma(\gamma^2 m^2 - m^2 + 2)}{\sqrt{6(\gamma^4 m^2 + m^2 + 4\gamma^2 - 2\gamma^2 m^2)}} \\
& - \frac{\sqrt{6(\gamma^4 m^2 + m^2 + 4\gamma^2 - 2\gamma^2 m^2)}}{2\gamma + \mathrm{i}2\ \mathrm{sn}\,[k(x-\nu_9 t)] + 2\ \mathrm{cn}\,[k(x-\nu_9 t)]}\Big\}, \\
u_{16}(x,t) =& k\Big\{\frac{3\gamma(\gamma^2 m^2 - m^2 + 2)}{\sqrt{6(\gamma^4 m^2 + m^2 + 4\gamma^2 - 2\gamma^2 m^2)}} \\
& - \frac{\sqrt{6(\gamma^4 m^2 + m^2 + 4\gamma^2 - 2\gamma^2 m^2)}\ \mathrm{dn}\,[k(x-\nu_9 t)]}{2\gamma\ \mathrm{dn}\,[k(x-\nu_9 t)] + \mathrm{i}2\ \mathrm{cn}\,[k(x-\nu_9 t)] + 2\sqrt{1-m^2}\ \mathrm{sn}\,[k(x-\nu_9 t)]}\Big\}, \\
u_{17}(x,t) =& k\Big\{\frac{3\gamma(\gamma^2 m^2 - m^2 + 2)}{\sqrt{6(\gamma^4 m^2 + m^2 + 4\gamma^2 - 2\gamma^2 m^2)}} \\
& - \frac{\sqrt{6(\gamma^4 m^2 + m^2 + 4\gamma^2 - 2\gamma^2 m^2)} m\ \mathrm{sn}\,[k(x-\nu_9 t)]}{2\gamma m\ \mathrm{sn}\,[k(x-\nu_9 t)] + \mathrm{i}2 + \mathrm{i}2\ \mathrm{dn}\,[k(x-\nu_9 t)]}\Big\}, \\
u_{18}(x,t) =& k\Big\{\frac{3\gamma(\gamma^2 m^2 - m^2 + 2)}{\sqrt{6(\gamma^4 m^2 + m^2 + 4\gamma^2 - 2\gamma^2 m^2)}} \\
& - \frac{\sqrt{6(\gamma^4 m^2 + m^2 + 4\gamma^2 - 2\gamma^2 m^2)}\mathrm{i}m\ \mathrm{cn}\,[k(x-\nu_9 t)]}{\mathrm{i}2\gamma m\ \mathrm{cn}\,[k(x-\nu_9 t)] + 2\ \mathrm{dn}\,[k(x-\nu_9 t)] + 2\sqrt{1-m^2}}\Big\},
\end{aligned}
$$

where $\nu_9 = k^2\left[\frac{m^2-3\gamma^2 m^2-2}{2} + \frac{3\gamma^2(\gamma^2 m^2 - m^2+2)^2}{2(m^2\gamma^4+m^2+4\gamma^2-2m^2\gamma^2)}\right]$ and k is an arbitrary constant. Furthermore, choosing $\gamma = 0$ yields that, for any k,

$$
\begin{aligned}
u_{19}(x,t) =& \sqrt{-6}km\ \mathrm{sn}[kx + k^3(m^2+1)t], \\
u_{20}(x,t) =& \sqrt{-6}k\frac{1}{\mathrm{sn}[kx + k^3(m^2+1)t]}, \\
u_{21}(x,t) =& \sqrt{-6}km\frac{\mathrm{cn}[kx + k^3(m^2+1)t]}{\mathrm{dn}[kx + k^3(m^2+1)t]}, \\
u_{22}(x,t) =& \sqrt{-6}k\frac{\mathrm{dn}[kx + k^3(m^2+1)t]}{\mathrm{cn}[kx + k^3(m^2+1)t]}, \\
u_{23}(x,t) =& \sqrt{6}km\ \mathrm{cn}[kx - k^3(2m^2-1)t], \\
u_{24}(x,t) =& k\sqrt{6(m^2-1)}\frac{1}{\mathrm{cn}[kx - k^3(2m^2-1)t]}, \\
u_{25}(x,t) =& k\sqrt{-6}\frac{\mathrm{dn}[kx - k^3(2m^2-1)t]}{\mathrm{sn}[kx - k^3(2m^2-1)t]}, \\
u_{26}(x,t) =& km\sqrt{6(1-m^2)}\frac{\mathrm{sn}[kx - k^3(2m^2-1)t]}{\mathrm{dn}[kx - k^3(2m^2-1)t]},
\end{aligned}
$$

are solutions of (4.31).

Remark 4.8. It follows from Remark 4.1 that u_j, $j = 3, \ldots, 26$, still satisfy (4.31) even if cn$(\cdot)$, sn$(\cdot)$ and dn$(\cdot)$ are replaced, respectively, by $\pm$cn$(\cdot)$, $\pm$sn$(\cdot)$ and $\pm$dn$(\cdot)$.

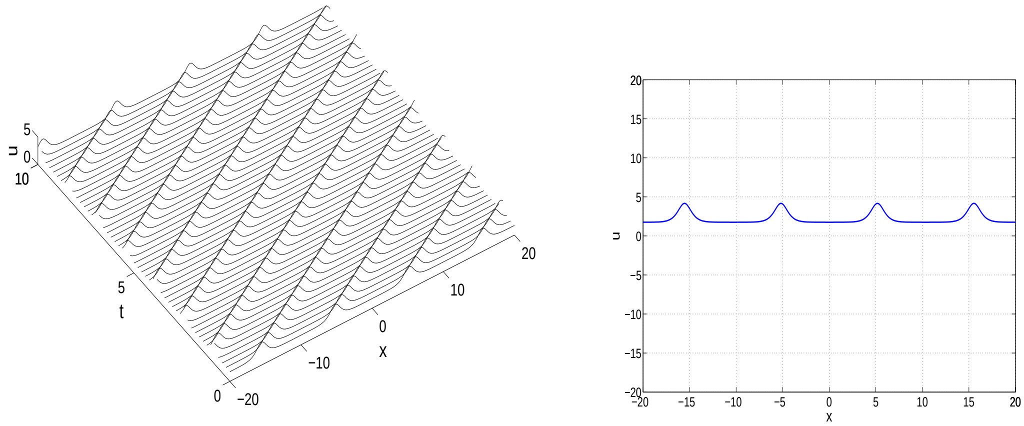

Figure 4.3: The plot of the solution u_6 to the modified KdV equation (4.31) with $m = 0.95$, $k = 1$ and $\gamma = 0.96$ and the initial status of u_6.

Remark 4.9. If $\gamma = 0$, then u_3, u_9 and u_{15} are the same as the solutions reported in [89] (with $a = 1$ and $b = 1$), and u_3, u_4 and u_7 are the same as those reported in [117] (for $\alpha = 1$ and $\beta = 1$). However, all of the other Jacobi elliptic solutions are new. More new solutions can be obtained if solution form (4.10) is used.

To demonstrate the physical insight of the new solutions, we take u_6 as an example. By choosing $m = 0.95$ and $k = 1$, the wave profiles of the solution u_6 for two different values of γ, $\gamma = 0.96$ and $\gamma = -0.96$, are displayed in Figures 4.3 and 4.4, respectively. Clearly, in both cases, the solutions describe the traveling of waves in the x-direction. Different values of γ yield different wave shapes.

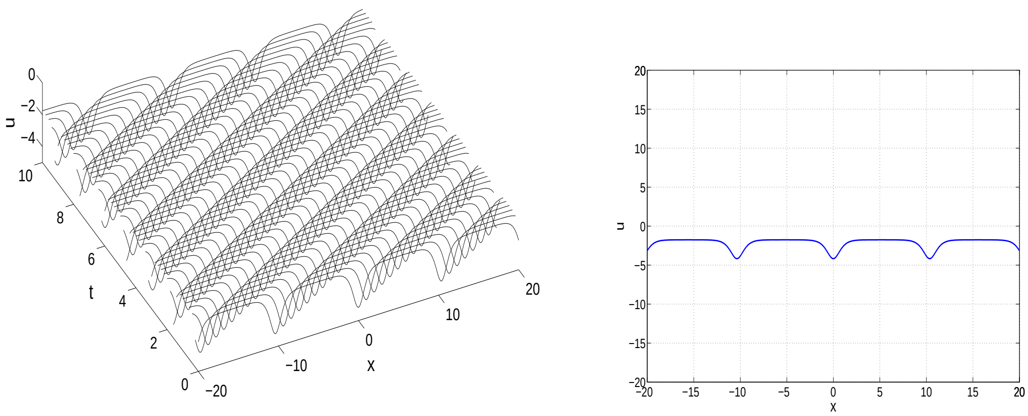

Figure 4.4: The plot of the solution u_6 to the modified KdV equation (4.31) with $m = 0.95$, $k = 1$ and $\gamma = -0.96$ and the initial status of u_6.

4.6 Traveling wave solutions for the shallow water long wave approximate equations

In this section, we will apply the method discussed in Section 4.3 to a system of partial differential equations. Consider the shallow water long wave approximate equations

$$\begin{cases} u_t - uu_x - v_x + \frac{1}{2}u_{xx} = 0, \\ v_t - vu_x - uv_x - \frac{1}{2}v_{xx} = 0, \end{cases} \tag{4.37}$$

where $u := u(x,t)$ is the horizontal velocity of water and $v := v(x,t)$ is the height that deviates from the equilibrium position of the water. Substituting $u(x,t) = \tilde{u}(\xi)$ and $v(x,t) = \tilde{v}(\xi)$, where ξ is as defined previously, into (4.37) and balancing the highest order derivative and non-linear terms, we obtain $n_u = 1$ and $n_v = 2$. If candidate traveling wave solutions are chosen according to (4.3), then all of the coefficients are required to be zero. Accordingly, we will use the more general form (4.10) and consider candidate solutions

$$\begin{cases} \tilde{u}(\xi) = \hat{c}_0 + \frac{\hat{c}_1\Phi(\xi)+\hat{C}_1\Phi'(\xi)}{\mu\Phi(\xi)+1}, \\ \tilde{v}(\xi) = \tilde{c}_0 + \frac{\tilde{c}_1\Phi(\xi)+\tilde{C}_1\Phi'(\xi)}{\mu\Phi(\xi)+1} + \frac{\tilde{c}_2\Phi^2(\xi)+\tilde{C}_2\Phi(\xi)\Phi'(\xi)}{(\mu\Phi(\xi)+1)^2}, \end{cases} \tag{4.38}$$

where Φ satisfies the ordinary differential equation (4.1) with coefficients q_j, $j = 0, \ldots, 4$. By substituting (4.38) into (4.37), we can ascertain the following sufficient conditions for $\tilde{u}$ and $\tilde{v}$ to satisfy the shallow water long wave approximate equations (4.37):

$$\begin{aligned}
k =&\pm\frac{\hat{c}_1}{\alpha}\\
\hat{c}_0 =&-\nu+\frac{-4q_0\hat{c}_1\mu^3+3q_1\hat{c}_1\mu^2-2q_2\hat{c}_1\mu+q_3\hat{c}_1}{4\alpha^2},\\
\hat{C}_1 =&0,\\
\tilde{c}_0 =&\frac{\hat{c}_1^2}{16\alpha^4}\big[12q_0q_1\mu^5-8q_0^2\mu^6-(12q_0q_2+3q_1^2)\mu^4+(16q_0q_3+4q_1q_2)\mu^3\\
&\qquad\qquad -(24q_0q_4+6q_1q_3)\mu^2+12q_1q_4\mu+q_3^2-4q_2q_4\big],\\
\tilde{c}_1 =&\frac{\hat{c}_1^2(4\mu^3q_0-3\mu^2q_1+2\mu q_2-q_3)}{4\alpha^2},\\
\tilde{c}_2 =&-\frac{\hat{c}_1^2}{2},\\
\tilde{C}_1 =&\pm\frac{\hat{c}_1^2}{2\alpha},\\
\tilde{C}_2 =&\mp\frac{\hat{c}_1^2\mu}{2\alpha},
\end{aligned}$$

where $\alpha=\sqrt{q_0\mu^4-q_1\mu^3+q_2\mu^2-q_3\mu+q_4}$ and μ, ν, $\hat{c}_1$ are arbitrary constants. Note that these requirements are the same as those reported in [22]. Note also that there are no conditions restricting the choice of coefficients q_j, $j=0,\dots,4$, of the ordinary differential equation (4.1). Using $\varphi_{j,l}(\cdot,0)$, $j=1,\dots,6$, $l=1,\dots,4$, from Theorem 4.1, we can reproduce the same Jacobi elliptic solutions of (4.37) reported in [22]. By applying Theorems 4.1-3, we also can deduce many new solutions. These solutions cannot be obtained using the results in [22]. For example, choosing $\mu=0$ and $q_j=p_{7,j}(\gamma)$, $j=0,\dots,4$, we can obtain the following solutions for the shallow water long wave approximate equations (4.37):

$$\begin{aligned}
u_j(x,t) =&-\nu+\frac{\vartheta\beta}{4\alpha^2}+\vartheta\varphi_{7,j}\big(k(x-\nu t)\big),\quad j=1\dots,4,\\
v_j(x,t) =&-\vartheta^2\Big\{\frac{\eta}{16\alpha^4}+\frac{\beta}{4\alpha^2}\varphi_{7,j}\big(k(x-\nu t)\big)-\frac{1}{2\alpha}\varphi_{7,j}'\big(k(x-\nu t)\big)\\
&+\frac{1}{2}\varphi_{7,j}^2\big(k(x-\nu t)\big)\Big\},\quad j=1\dots,4,
\end{aligned}$$

where $\varphi_{7,j}$, $j=1,\dots,4$, are as defined in Section 4.2, $k=\vartheta/\alpha$, $\alpha=\big[\gamma^3(1-m^2)+\gamma^2(2-m^2)+\gamma\big]^{1/2}$, $\beta=\gamma^2(3m^2-3)+\gamma(2m^2-4)-1$, $\eta=\gamma^4(3m^4-6m^2+3)+\gamma^3(4m^4-12m^2+8)+\gamma^2(6-6m^2)-1$, and ν, γ, ϑ, m are arbitrary. For the other solutions, we leave it to the reader.

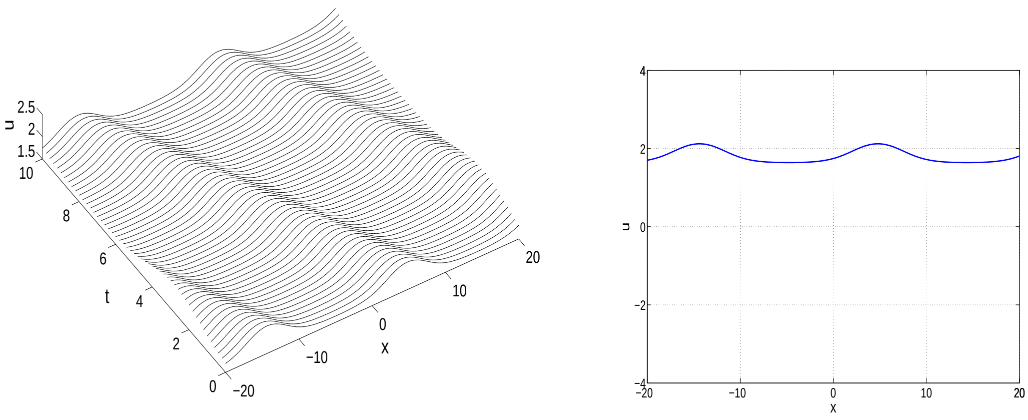

Figure 4.5: The plot of the solution u_1 of the shallow water long wave approximate equations (4.37) with $m = 0.99$, $\nu = -2$ and $\vartheta = \gamma = 1$ and the initial status of u_1.

To show the physical insight of these solutions, we take the solution (u_1, v_1) as an example. Figures 4.5 and 4.6 display the graphs of u_1 and v_1 with $m = 0.99$, $\nu = -2$ and $\vartheta = \gamma = 1$. Clearly, the solution describes the propagation of waves with horizontal velocity u_1 along the negative x-direction.

4.7 Concluding remarks

In this chapter, we have presented a generalized expansion method for generating traveling wave solutions of non-linear partial differential equations. This method has been successfully applied to the Boussinesq equation, the modified KdV equation and the shallow water long wave approximate equations, and many new results have been obtained. For each equation investigated, we are able to replicate solutions previously derived in the literature, and discover many new ones. Extensions to two and three dimensional partial differential equations are possible. Other non-linear partial differential equations can be tackled if an appropriate transformation can be found. For example, in [45], the transformation $u = \ln v$ was applied to the Dodd-Bullough-Mikhailov equation to yield a non-linear partial differential equation involving powers of v and its derivatives.

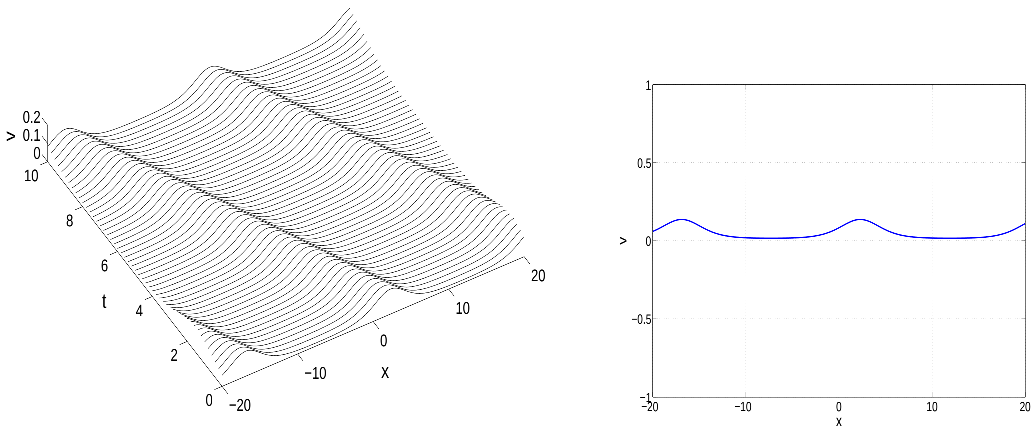

Figure 4.6: The plot of the solution v_1 to the shallow water long wave approximate equations (4.37) with $m = 0.99$, $\nu = -2$ and $\vartheta = \gamma = 1$ and the initial status of v_1.

Chapter 5

Linear B-spline finite element method for the improved Boussinesq equation

5.1 Introductory remarks

The improved Boussinesq equation (1.3) has been studied extensively over the past two decades (see [1, 18, 20, 33, 47, 50]). Iskandar and Jain [50] were the first to investigate the improved Boussinesq equation (1.3) numerically. Applying a linearization technique and finite difference approximations, Iskandar and Jain derived a three-level iterative scheme with second order local truncation error. The scheme was used to investigate head-on collisions between solitary waves. Later, Zoheiry [33] developed an improved scheme with a Crank-Nicolson modification. For this scheme, each time step is accompanied by an iterative process that ensures the accuracy requirements are satisfied. Hence, whilst accuracy is maintained, efficiency is compromised.

In [47], Adomian's decomposition method was applied to the Cauchy problem for the improved Boussinesq equation (1.3). Using this method, the solution is expressed as a convergent series, and an approximation is obtained by truncating the series after a sufficient number of terms. However, the computation of each term in this series is cumbersome, requiring the integration and differentiation of several complex expressions. The symbolic manipulation package Maple was used and numerical results were calculated and compared with the analytical solution, but only for a very small value of t. It remains to be seen how this method performs for large values of t. In fact, in order to maintain accuracy as t increases, it is expected that a large number of more complicated terms will need to be calculated. In [1], the Adomian decomposition-Padé technique has been presented and it has been shown that this technique gives the approximate solution with faster convergence rate and higher accuracy than using Adomian's decomposition method alone. However, the disadvantage of Adomian's decomposition method still remains, that is, the error increases rapidly as t increases.

In [18], Bratsos considered the improved Boussinesq equation (1.3) with boundary conditions imposed on the first spatial derivative. Finite difference approximations were used to reduce the improved Boussinesq equation (1.3) to a system of ordinary differential equations. Using a Padé approximation, a three level implicit time-step scheme was developed. Relevant stability bounds were also derived. In addition, Bratsos has employed an implicit finite-difference method associated with a predictor-corrector scheme to solve the initial boundary value problem governed by the improved Boussinesq equation (1.3) (see [20]).

In this chapter, we develop a Galerkin-based finite element method for a class of initial boundary value problems governed by the improved Boussinesq equation (1.3). The spatial axis is partitioned into a set of finite elements and the solution is expressed in terms of the linear B-spline basis functions. On this basis, a system involving only ordinary derivatives is obtained. Then, the structure of the system coefficient matrices is exploited to transform the problem into an explicit initial value problem. Accordingly, many standard numerical integration algorithms are applicable. In this manner, an approximate solution to the problem can be generated. In contrast to existing methods, this method is simple to implement and capable of handling the non-linearity in the governing equation. We present the results of four numerical experiments to validate the method and demonstrate its capability in simulating complex wave phenomena.

5.2 Problem statement

Consider the initial boundary value problem consisting of the improved Boussinesq equation

$$u_{tt} = u_{xx} + u_{xxtt} + \left(u^2\right)_{xx}, \quad x \in (a, b), \quad t > 0, \tag{5.1}$$

the initial conditions

$$u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad \text{for all } x \in (a, b), \tag{5.2}$$

and the boundary conditions

$$u(a, t) = 0, \quad u(b, t) = 0, \quad \text{for all } t \in (0, \infty), \tag{5.3}$$

where u^0, $u^1 : (a, b) \to \mathbb{R}$ are given functions.

For any fixed t, we multiply (5.1) by a test function $v \in H_0^1(a, b) = \{w \in L^2(a, b) : w_x \in L^2(a, b), w(a) = w(b) = 0\}$, integrate the product over $[a, b]$ using integration by parts, and then apply the boundary conditions (5.3) to yield

$$\int_a^b \left(u_{tt}v + u_x v_x + u_{xtt}v_x + (u^2)_x v_x\right) dx = 0, \tag{5.4}$$

where the function arguments are suppressed for clarity. Equation (5.4) is required to hold for all admissible test functions. On this basis, we define the following variational problem.

Problem 5.1. Find a $u \in H_0^1(a, b)$ such that (5.2) is satisfied and, for each $t > 0$,

$$(u_{tt}, v) + (u_x, v_x) + (u_{xtt}, v_x) + \left((u^2)_x, v_x\right) = 0, \quad \text{for all } v \in H_0^1(a, b), \tag{5.5}$$

where

$$(u, v) = \int_a^b u(x)v(x)dx.$$

5.3 Numerical method

We partition the x-axis into n finite elements by choosing a set of evenly-spaced knots $\{x_i\}_{i=0}^n$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and $x_{i+1} - x_i = h$,

$i = 0, \ldots, n-1$. Consider an approximate solution of Problem 5.1 of the form:

$$U^n(x,t) = \sum_{i=0}^{n} u_i(t)\phi_i(x), \tag{5.6}$$

where

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i, x_{i+1}], \\ 0, & \text{elsewhere.} \end{cases}$$

According to (5.6), $u_i(t) = U^n(x_i, t)$, $i = 0, \ldots, n$.

Applying the boundary conditions (5.3) gives $u_0(t) = 0$ and $u_n(t) = 0$ for all $t \in (0, \infty)$. Hence, (5.6) can be simplified to

$$U^n(x,t) = \sum_{i=1}^{n-1} u_i(t)\phi_i(x). \tag{5.7}$$

We then follow the standard Galerkin approach and choose test functions $v = \phi_i$, $i = 1, \ldots, n-1$. On this basis, (5.5) must hold with $v = \phi_i$, $i = 1, \ldots, n-1$. Substituting (5.7) into (5.5) gives

$$\sum_{j=1}^{n-1} \left((\phi_i, \phi_j)\, \ddot{u}_j + \left(\phi_i', \phi_j'\right) u_j + \left(\phi_i', \phi_j'\right) \ddot{u}_j + 2 \sum_{k=1}^{n-1} \left(\phi_i' \phi_k', \phi_j\right) u_k u_j \right) = 0 \tag{5.8}$$

for each $i = 1, \ldots, n-1$, where $'$ and $\dot{}$ denote differentiation with respect to x and t, respectively.

In matrix notation, the system of equations (5.8) can be written as

$$(A + B)\ddot{\mathbf{U}}(t) + B\mathbf{U}(t) + C\big(\mathbf{U}(t)\big)\mathbf{U}(t) = \mathbf{0}, \tag{5.9}$$

where $\mathbf{0} \in \mathbb{R}^{n-1}$ is a zero vector and $\mathbf{U}(t) = \big[u_1(t), u_2(t), \ldots, u_{n-1}(t)\big]^T$. The $(n-1) \times (n-1)$ matrices A, B and $C\big(\mathbf{U}(t)\big)$ are given as follows:

$$A = \big[(\phi_i, \phi_j)\big] = \frac{h}{6} \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ 0 & 1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4 \end{bmatrix},$$

$$B = \big[(\phi_i', \phi_j')\big] = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix},$$

and

$$
\begin{aligned}
C\big(\mathbf{U}(t)\big) &= \left[2\sum_{k=1}^{n-1}(\phi_i'\phi_k', \phi_j)u_k\right] \\
&= \frac{1}{h}\begin{bmatrix}
2u_1 - u_2 & u_1 - u_2 & 0 & \cdots & 0 \\
u_2 - u_1 & -u_1 + 2u_2 - u_3 & u_2 - u_3 & \cdots & 0 \\
0 & u_3 - u_2 & -u_2 + 2u_3 - u_4 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -u_{n-2} + 2u_{n-1}
\end{bmatrix}.
\end{aligned}
$$

Note that C is a time dependent matrix, whilst A and B are constant. By virtue of the structure of A and B, we have the following theorem.

Theorem 5.1. The matrix $A + B$ is invertible.

Proof. Let $\mathbf{y} \in \mathbb{R}^{n-1}$ be a non-zero vector and define $w(x) = \sum_{i=1}^{n-1} y_i\phi_i(x)$. Then we have

$$
\begin{aligned}
\mathbf{y}^T B\mathbf{y} &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} y_i b_{ij} y_j \\
&= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} y_i \left(\int_a^b \phi_i'(x)\phi_j'(x)dx\right) y_j \\
&= \int_a^b \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} y_i\phi_i'(x)\phi_j'(x)y_j dx \\
&= \int_a^b \left(\sum_{i=1}^{n-1} y_i\phi_i'(x)\right)^2 dx \\
&= \int_a^b (w'(x))^2\, dx \\
&\geq 0.
\end{aligned}
$$

Since w' is piecewise continuous, equality holds if, and only if, $w'(x) = 0$ for all $x \in [a, b]$. Now, since $w(a) = 0$, $w'(x) = 0$ for all $x \in [a, b]$ if, and only if, $w \equiv 0$. This, in turn, requires $\mathbf{y} = \mathbf{0}$, which contradicts the assumption that $\mathbf{y}$ is non-zero. Hence, $\mathbf{y}^T B\mathbf{y} > 0$ for all non-zero $\mathbf{y}$ and so B is positive definite. In a similar manner, one can ascertain the positive definiteness of A. Since both A and B are positive definite, it readily follows that $A + B$ is positive definite and therefore invertible. ■

From Theorem 5.1, it follows that we can invert the matrix $A+B$ in (5.9) to isolate the second derivative term. Since $A + B$ is tridiagonal, this inversion can be performed efficiently using a special algorithm (see Section 6.6 of [21]). Introducing the new variable $\mathbf{V}(t) = \dot{\mathbf{U}}(t)$, it is clear that the system (5.9) is equivalent to the following first order system of ordinary differential equations:

$$\dot{\mathbf{U}}(t) = \mathbf{V}(t), \tag{5.10}$$

$$\dot{\mathbf{V}}(t) = -(A+B)^{-1}\Big[B\mathbf{U}(t) + C\big(\mathbf{U}(t)\big)\mathbf{U}(t)\Big]. \tag{5.11}$$

Initial conditions for (5.10) and (5.11) are obtained by considering (5.2). As such, we have

$$\mathbf{U}(0) = \big[u^0(x_1), \ldots, u^0(x_{n-1})\big]^T \tag{5.12}$$

and

$$\mathbf{V}(0) = \big[u^1(x_1), \ldots, u^1(x_{n-1})\big]^T. \tag{5.13}$$

The system of ordinary differential equations (5.10) and (5.11) with initial conditions (5.12) and (5.13) defines a standard initial value problem. This problem can be solved using a standard numerical integration algorithm (for example, a Runge-Kutta method).

5.4 Numerical examples

In this section, we implement the procedure developed in Section 5.3 and solve some concrete problems. Firstly, in Example 5.1, we validate the procedure by comparing our numerical results with the exact solution. Then, in Examples 5.2, 5.3 and 5.4, we demonstrate the capacity of this technique to simulate wave splitting, wave interaction and blow-up behavior.

The differential equations (5.10) and (5.11) are solved using the Runge-Kutta-Verner variable step-size method (see Section 5.5 of [21]). Thus, the time-step is actually dynamic and is modified within the preset maximum and minimum bounds to ensure that the given error tolerances are satisfied. In Examples 5.1, 5.2 and 5.3, the error tolerance is 1.0×10^{-7}; in Example 5.4, it is 1.0×10^{-4}. All program codes for the examples below were written in Fortran 95.

Example 5.1. (Numerical validation)

Note that, on an unbounded region with boundary conditions $u(x,t)\to 0$ as $x\to\pm\infty$, the improved Boussinesq equation (5.1) admits analytical solutions of the form

$$u(x,t)=\eta\text{sech}^2\left(\frac{1}{\nu}\sqrt{\frac{\eta}{6}}\,(x-\nu t-x_0)\right), \tag{5.14}$$

where x_0 is the initial position of the solitary wave, $\eta>0$ is the wave amplitude and $\nu=\pm\sqrt{1+\frac{2}{3}\eta}$ is the wave speed. The validity of (5.14) is expected to hold for bounded regions which are sufficiently large.

Set $\eta=0.5$, $x_0=0$ and $\nu=\sqrt{1+\frac{2}{3}\eta}$ with

$$u^0(x)=\eta\text{sech}^2\left(\frac{1}{\nu}\sqrt{\frac{\eta}{6}}\,(x-x_0)\right)$$

and

$$u^1(x)=2\eta\sqrt{\frac{\eta}{6}}\text{sech}^2\left(\frac{1}{\nu}\sqrt{\frac{\eta}{6}}\,(x-x_0)\right)\tanh\left(\frac{1}{\nu}\sqrt{\frac{\eta}{6}}\,(x-x_0)\right).$$

Under these conditions, the exact solution to Problem 5.1 is given by (5.14). In applying the procedure of Section 5.3, we discretize the problem on $x\in[-30,150]$ using evenly-spaced knots with a distance of h between consecutive nodes. In general, the numerical error will depend on h and the time-step size Δt. Here, Δt is chosen automatically by the integration routine to satisfy bounds on the local truncation error, while h is determined through a convergence analysis. The numerical solution is compared with the exact solution at $t=10$ for different values of h in

Table 5.1: Comparison of the numerical results and exact solution for Example 5.1.

x	Numerical solution					Exact solution
	$h = 1.000$	$h = 0.500$	$h = 0.250$	$h = 0.100$	$h = 0.05$	
5.0	0.073052	0.071010	0.070492	0.070347	0.070327	0.070320
6.0	0.111137	0.110728	0.110658	0.110641	0.110638	0.110637
7.0	0.165915	0.168348	0.169026	0.169220	0.169248	0.169258
8.0	0.240392	0.246098	0.247597	0.248021	0.248082	0.248102
9.0	0.331384	0.339093	0.341042	0.341589	0.341667	0.341694
10.0	0.423374	0.429964	0.431557	0.431999	0.432062	0.432083
11.0	0.487991	0.490172	0.490623	0.490742	0.490759	0.490765
12.0	0.497708	0.494722	0.493915	0.493686	0.493653	0.493642
13.0	0.447066	0.441326	0.439898	0.439499	0.439442	0.439423
14.0	0.357728	0.352426	0.351142	0.350786	0.350735	0.350718
15.0	0.260700	0.257474	0.256703	0.256489	0.256459	0.256448
$\overline{E}$	0.010309	0.002601	0.000651	0.000105	0.000026	

$\overline{E}= \max_{0\leq m\leq n} \{|U^n(x_m, t) - u(x_m, t)|\}$, $u(x, t)$ is the analytical solution.

Table 5.1. To examine the influence of h on the numerical solutions, Figure 5.1 shows the convergence process. It is clear that convergence is achieved at $h = 0.1$ $(-\ln(h) = 2.3)$ and thus this value for h is used here and in Examples 5.2 and 5.3. To investigate the variation of numerical error with time, we plot the error at two points against time in Figure 5.2. The time-step determined by the local truncation error is between 0.25 and 0.7.

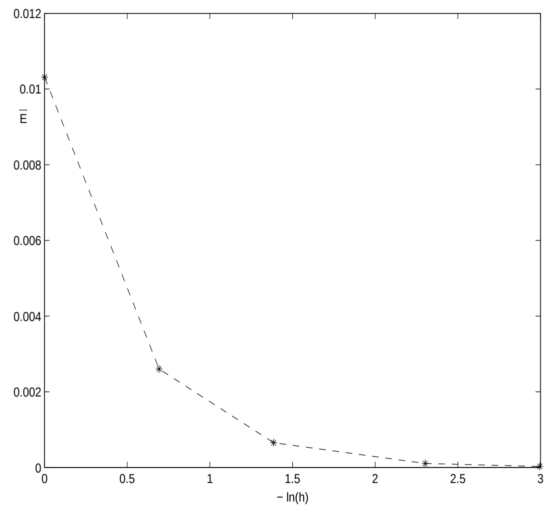

Figure 5.1: Relationship between h and $\overline{E}$ for Example 5.1.

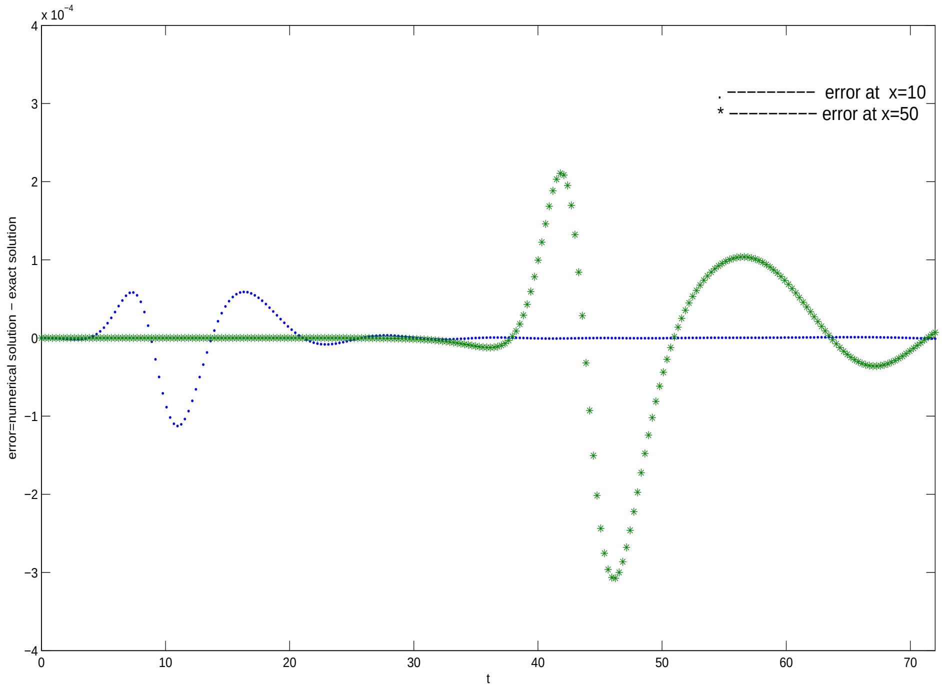

Figure 5.2: Numerical errors versus time at $x = 10$ and $x = 50$ for Example 5.1.

The wave profile of the numerical solution for $t \in [0, 72]$ is shown in Figure 5.3. The results are in good agreement with those presented in [18, 15, 33, 50]. The average speed of this solitary wave is 1.1542, which is quite close to the theoretical value of $\sqrt{1 + \frac{2}{3}\eta} = 1.154701$. We note that our numerical method is much more efficient than those presented in the references. For example, using our method, an accuracy of $\overline{E} = 3.96 \times 10^{-4}$ at $t = 72$ is achieved with $0.25 \leq \Delta t \leq 0.7$. In [18], Δt needs to be in the order of 0.001 to generate results of comparable accuracy.

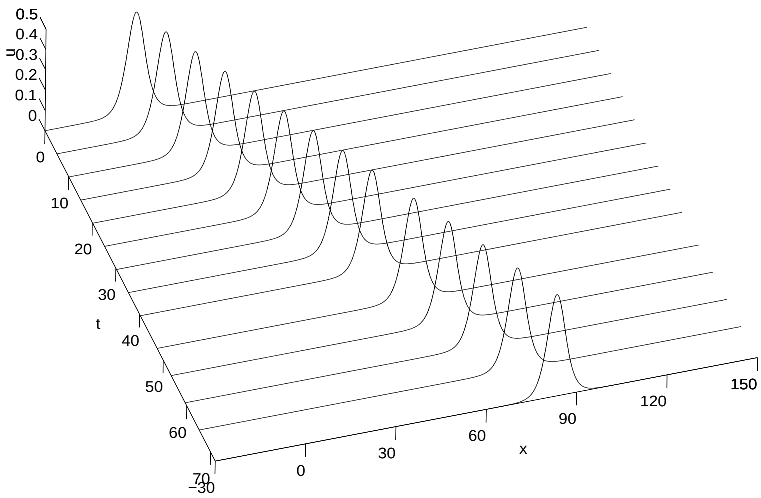

Figure 5.3: Single soliton solution for Example 5.1.

The results in Example 5.1 demonstrate that the method developed in Section 5.3 is highly accurate for quite moderate time-steps and values of h. Having validated the procedure, we will now present some simulations in the remaining examples.

Example 5.2. (Wave break-up)

We consider Problem 5.1 with $u^1(x) = 0$ and $u^0(x)$ defined as in Example 5.1, where now $x_0 = 30$. This problem is solved on $-30 \le x \le 90$ for $0 \le t \le 40$ using the method of Section 5.3 with $\Delta t \in [0.25, 0.7]$. The initial stationary wave and the numerical solution are displayed together in Figure 5.4. The diagram shows the initial stationary wave of amplitude 0.5 breaking into two smaller diverging solitary waves. The break-up is completed at approximately $t = 10$, and the amplitudes of these two solitary waves are approximately equal to 0.26. It is also noted that the solution is symmetric about the plane $x = 30$.

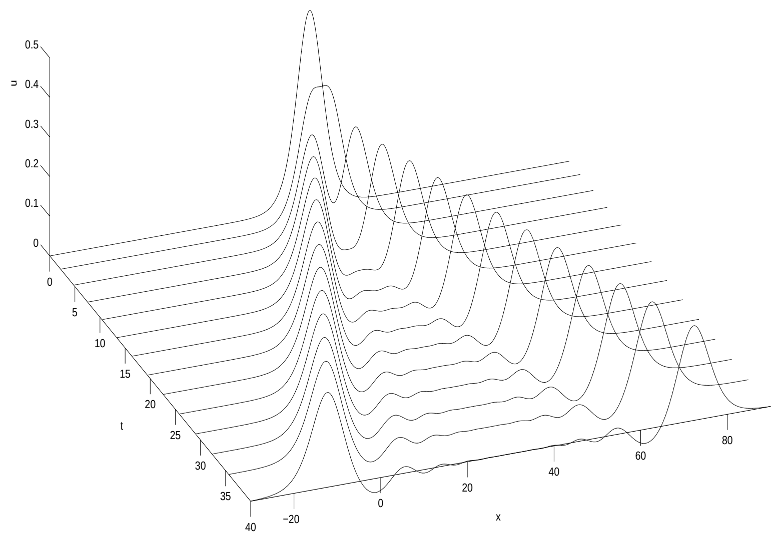

Figure 5.4: Wave break-up solution for Example 5.2.

Example 5.3. (Wave collision)

As in [50], we investigate the interaction of two soliton waves moving on a collision course. Here, $x \in [-60, 90]$ and $t \in [0, 40]$ with

$$u^0(x) = \eta_1 \text{sech}^2\left(\frac{1}{\nu_1}\sqrt{\frac{\eta_1}{6}}(x+x_0)\right) + \eta_2 \text{sech}^2\left(\frac{1}{\nu_2}\sqrt{\frac{\eta_2}{6}}(x-x_0)\right)$$

and

$$\begin{aligned} u^1(x) = \quad & 2\eta_1\sqrt{\frac{\eta_1}{6}}\text{sech}^2\left(\frac{1}{\nu_1}\sqrt{\frac{\eta_1}{6}}(x+x_0)\right)\tanh\left(\frac{1}{\nu_1}\sqrt{\frac{\eta_1}{6}}(x+x_0)\right) \\ & -2\eta_2\sqrt{\frac{\eta_2}{6}}\text{sech}^2\left(\frac{1}{\nu_2}\sqrt{\frac{\eta_2}{6}}(x-x_0)\right)\tanh\left(\frac{1}{\nu_2}\sqrt{\frac{\eta_2}{6}}(x-x_0)\right), \end{aligned}$$

where $\nu_1 = \sqrt{1+\frac{2}{3}\eta_1}$, $\nu_2 = \sqrt{1+\frac{2}{3}\eta_2}$, $x_0 = 20.0$, $\eta_1 = 1.0$, $\eta_2 = 0.5$ and $\Delta t \in [0.15, 0.7]$. Figure 5.5 displays the head-on collision. The collision starts at approximately $t = 5.29484$. Before the collision of the two waves, the speed and amplitude of one of the waves are 1.28431 and 0.99998, respectively; while the speed and amplitude of the other wave are -1.1521 and 0.49999, respectively. A negative speed indicates that the wave travels in the negative $x-$direction. When the two waves interact, they become a single wave. At approximately $t = 15.95779$, the amplitude of the solitary wave achieves its maximal value of 1.32705. When $t = 22.32919$, the collision is finished, and the amplitude of the larger wave is 0.97714; while the amplitude of the smaller wave becomes 0.49071. According to the contour map in Figure 5.5, the secondary solitons are visable. Hence, the collision is inelastic. Figure 5.6 shows another example of inelastic collision in which $x_0 = 20.0$, $\eta_1 = 0.5$, $\eta_2 = 2$ and $\Delta t \in [0.09, 0.7]$.

Now, we give some examples of waves of equal magnitude colliding. When $\eta_1 = \eta_2 = 0.4$, the collision, shown in Figure 5.7, is elastic, while, for $\eta_1 = \eta_2 = 1$, the interaction, illustrated in Figure 5.8, is inelastic. The results are in good agreement with those reported in [20]. However, according to the contour map in Figure 5.9, the collision with $\eta_1 = \eta_2 = 0.5$ is still elastic. Hence, we can conclude that, for the case of equal magnitude colliding, if the amplitude is less than or equal to 0.5, the collision is elastic.

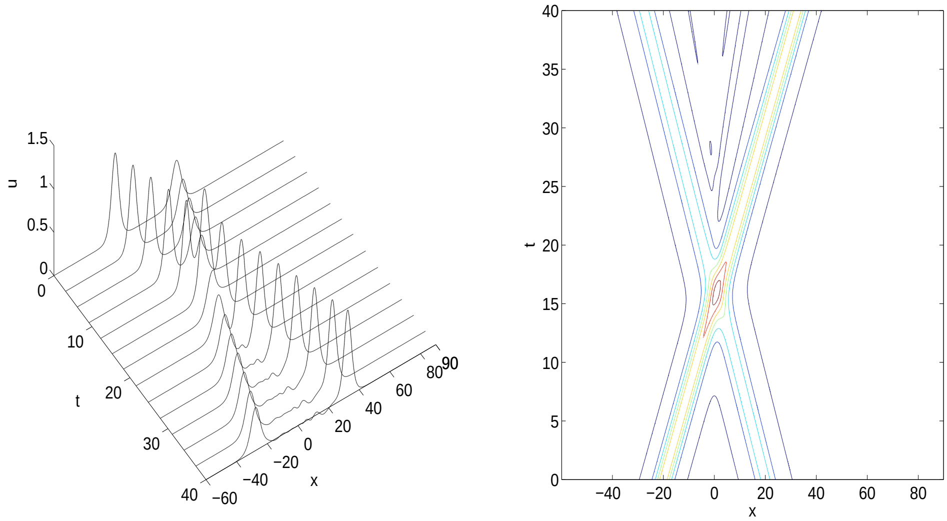

Figure 5.5: Inelastic collision with $\eta_1 = 1.0$ and $\eta_2 = 0.5$ in Example 5.3. The contour line on the right illustration starts from 0.01 and the level step is 0.2.

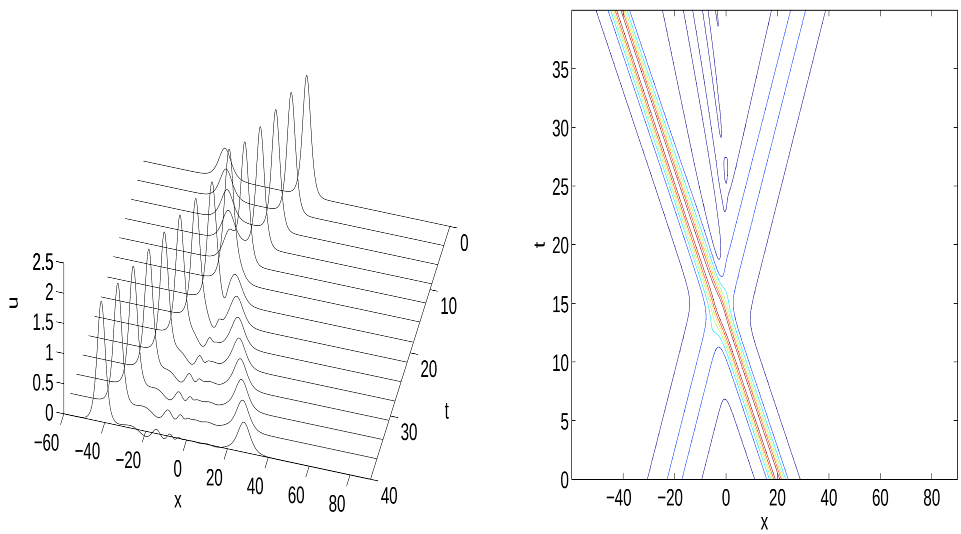

Figure 5.6: Inelastic collision with $\eta_1 = 0.5$ and $\eta_2 = 2.0$ in Example 5.3. The contour line on the right illustration starts from 0.01 and the level step is 0.3.

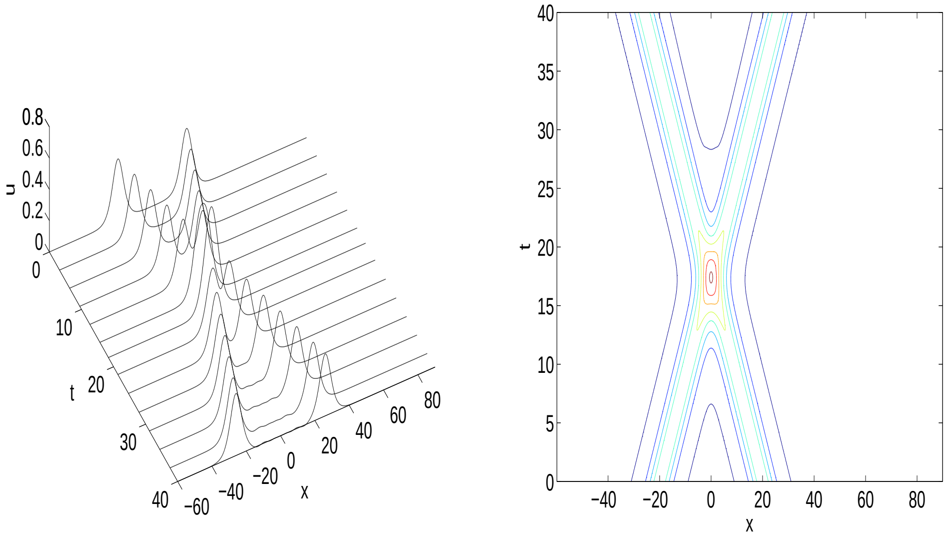

Figure 5.7: Elastic collision with $\eta_1 = 0.4$ and $\eta_2 = 0.4$ in Example 5.3. The contour line starts from 0.01 and the level step is 0.1.

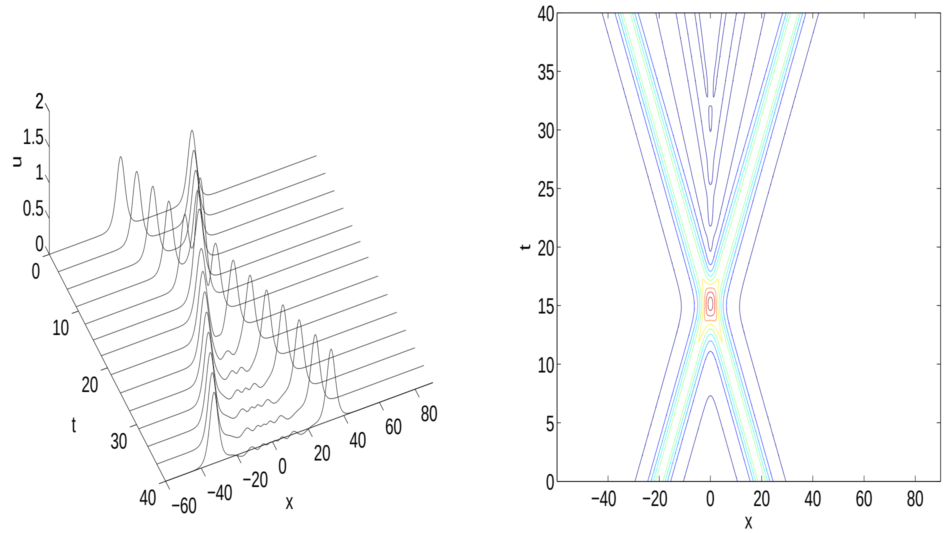

Figure 5.8: Inelastic collision with $\eta_1 = 1.0$ and $\eta_2 = 1.0$ in Example 5.3. The contour line starts from 0.01 and the level step is 0.2.

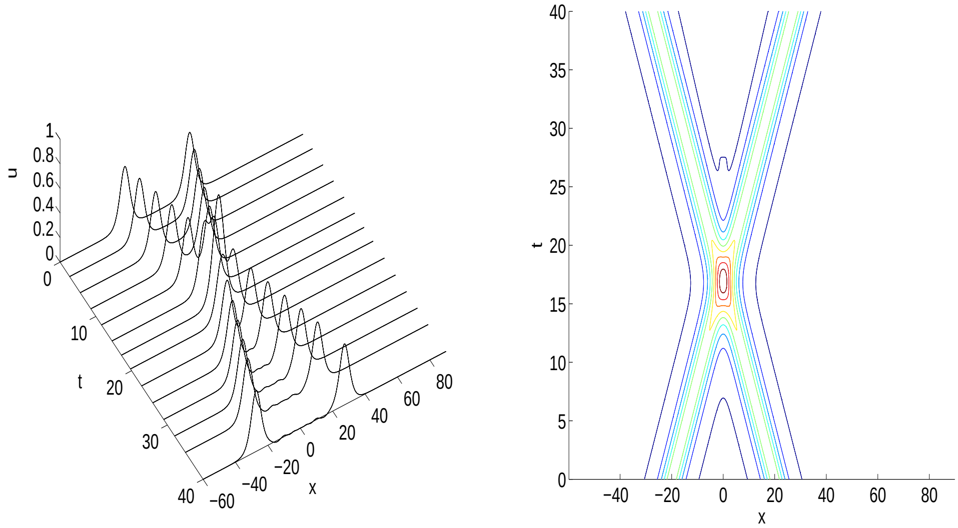

Figure 5.9: Elastic collision with $\eta_1 = 0.5$ and $\eta_2 = 0.5$ in Example 5.3. The contour line starts from 0.01 and the level step is 0.1.

Example 5.4. (Solution blow-up)

In this example, we simulate the solution blow-up discussed in [108, 111]. The improved Boussinesq equation (5.1) is considered on $x \in [0, 1]$ with the initial boundary conditions (5.2) and (5.3) defined by $u^0(x) = -3\sin(\pi x)$ and $u^1(x) = -\sin(\pi x)$. Under these assumptions, it is known from [108] that there exists a $T^0 > 0$ such that a unique local solution $u \in C^2([0, T^0); H^2(0, 1) \cap H_0^1(0, 1))$ exists, with

$$\|u(\cdot, t)\|_{L^2(0,1)} \to +\infty, \qquad \text{as } t \to T^0,$$

and

$$I(t) = \int_0^1 u(x, t)\sin(\pi x)dx \to -\infty, \qquad \text{as } t \to T^0.$$

To solve this problem numerically using the procedure developed in Section 5.3, we discretize the space domain into evenly-spaced knots with $h = 0.005$. Note that in this example, we had to set the minimum time-step very small (0.00001) to generate reasonable results until $t = 1.8$. The numerical solution at various values of t is shown in Figure 5.10. $I(t)$ is tabulated for these values in Table 5.2.

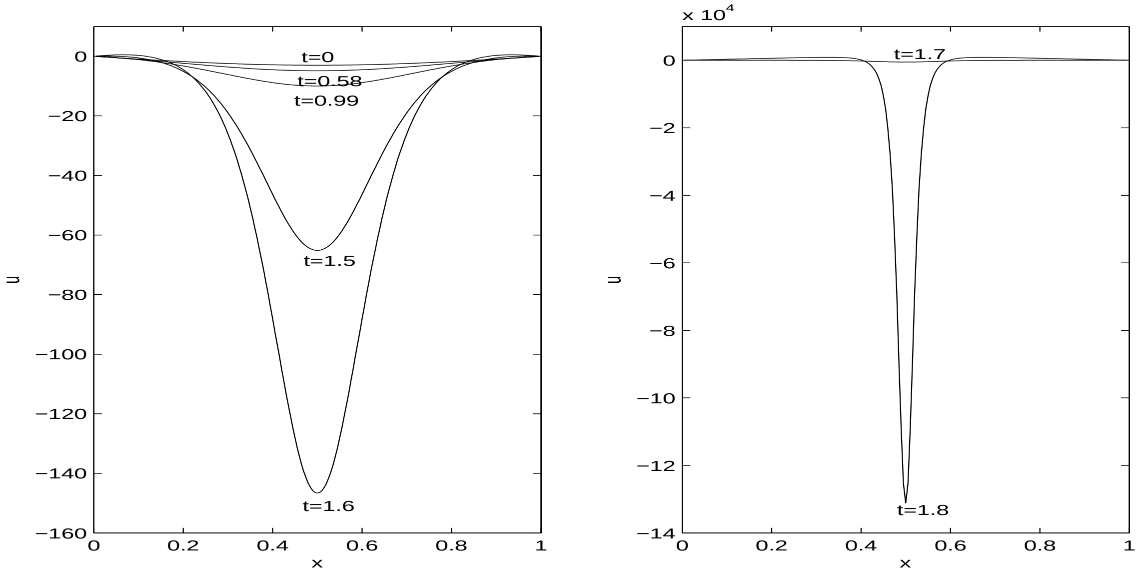

Figure 5.10: Solution blow-up in Example 5.4.

Table 5.2: Numerical results for Example 5.4.

t	0.0	0.58	0.99	1.50	1.60	1.70	1.80
$u(0.5,t)$	-3.00	-4.86	-9.97	-65.16	-146.64	-535.13	-131146.69
$I(t)$	-0.0075	-0.0113	-0.0206	-0.0951	-0.18	-0.49	-30.81

5.5 Concluding remarks

In this chapter, we have developed an efficient and practical finite element scheme for solving initial boundary value problems for the improved Boussinesq equation. Our numerical results were generated using an adaptive Runge-Kutta-Verner method. This method proved highly accurate. Excellent agreement between the analytical and numerical solutions was obtained in Example 5.1 for relatively large time-steps, and wave interaction and wave break-up were successfully simulated in Examples 5.2 and 5.3. Additionally, we verified numerically a type of solution blow-up that has been shown to exist theoretically. The advantage of our scheme is that it can be implemented easily using existing ordinary differential equation solvers. Many such solvers of excellent quality are available. A special time-stepping scheme does not need to be developed to handle the non-linearity inherent in the improved Boussinesq equation.

Chapter 6

Summary and further research

6.1 Summary

In this thesis, we have studied Boussinesq-type equations, including the existence and nonexistence of a global solution for a generalized Boussinesq equation, exact solutions for the Boussinesq equation and numerical solutions to a class of initial boundary value problems governed by the improved Boussinesq equation. Based on previous work in the field and methods of research, we have derived some important results. The main results achieved are summarized as follows.

(1) We have studied a generalized Boussinesq equation

$$u_{tt} - \alpha u_{xx} + u_{xxxx} + \big[f(u)\big]_{xx} = 0$$

and its corresponding Euclidean scalar field equation

$$-\phi_{xx} + \alpha\phi - f(\phi) = 0,$$

where α is a positive constant and f satisfies the conditions listed in either Case 1 or Case 2 (Cases 1 and 2 are as defined in Section 3.2 of Chapter 3). We have shown that there exists a ground state of the Euclidean scalar field equation. Based on the ground state, a constant d is determined by (3.14). Then, according to the constant d, two sets have been constructed. It has been shown that these two sets are invariant under the flow generated by the generalized Boussinesq equation if the initial data satisfy some conditions. By virtue of the local existence theorem derived by Liu [65], we have established sufficient conditions for Cauchy problems involving the generalized Boussinesq equation such that the solution exists globally or blows up in finite time. More precisely, if the initial wave belongs to the first invariant set, then the solution exists globally, while the solution blows up in finite time if the initial wave belongs to the second invariant set and some additional conditions have been satisfied.

(2) A generalized expansion method for constructing exact solutions of non-linear partial differential equations has been proposed, in which the solutions of partial differential equations can be derived from solutions of an auxiliary ordinary differential equation. We have obtained some new Jacobi elliptic and exponential solution classes for an auxiliary ordinary differential equation. Our new results ensure that the proposed expansion method is a significant generation of the expansion methods in the literature. Moreover, the proposed expansion method has been successfully applied to the Boussinesq equation, the modified Boussinesq equation, the modified KdV equation and the shallow water long wave approximate equations. For each equation considered, we are capable of replicating solutions previously derived in the literature and discovering many new ones.

(3) Applying the finite element method with linear B-spline basis functions, an efficient numerical scheme has been established for solving initial boundary value problems for the improved Boussinesq equation. Using the finite element method, the original problem is converted into a Cauchy problem for an ordinary differential system. Then, numerical results can be generated by using an adaptive Runge-Kutta-Verner method. Four numerical experiments have been presented to validate the method and demonstrate its capability in simulating complex wave phenomena. Excellent agreement between the analytical and numerical solutions has been obtained for relatively large time-steps, and wave interaction and wave break-up have been successfully simulated. Furthermore, we have successfully simulated a type of solutions which has been shown theoretically to blow up in finite time in [108, 111].

6.2 Further research

In this project, we use some techniques to investigate Boussinesq-type equations and achieve some important results. Based on the obtained results, there are some problems for further research.

As mentioned in Chapter 2, the instability of solitary wave solutions for the generalized Boussinesq equation has been investigated only for the case $f(s) = |s|^{p-1}s$ ($p > 1$). Note that the instability for solitary waves is generally derived from the blow-up theorem. In this thesis, we consider the generalized Boussinesq equation when f is in a general form and establish sufficient conditions under which the solution blows up in finite time. Using the new blow-up theorem, we can investigate the instability for solitary wave solutions of the generalized Boussinesq equation when f is in a general form.

For the expansion method proposed in Chapter 4, we can apply it to the invariant Boussinesq equations. Furthermore, it is possible to apply the method to other sets of Boussinesq-type equations. On the other hand, motivated by the interesting transformation (2.11), we can investigate some similar transformation which can be used to generate new exact solutions for non-linear partial differential equations. In addition, we also can use the results given in Section 4.2 of Chapter 4 to construct exact solutions for non-linear partial differential equations with boundary conditions.

For the numerical methods, we can generalize the technique used to derive the proposed numerical scheme in Chapter 5 for the improved Boussinesq equation in $1 + 1$ dimensions to $2 + 1$ or $3 + 1$ dimensional space. Furthermore, we can apply the technique to Boussineq-type systems to stimulate some complex wave propagations. In addition, other kind of numerical techniques can be applied to solve Boussinesq-type equations.

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