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A canonical theory for short GPS baselines. Part IV: precision versus reliability

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Abstract. This contribution is the last of four parts and deals with the link between baseline precision and ambiguity reliability. It is shown analytically how and to what extent the baseline-ambiguity correlation is related to the gain in baseline precision, to the volume of the ambiguity search space, and to the impact of potential integer ambiguity biases. Also, an ambiguity DOP measure is introduced together with its closed-form formulae for the three different single-baseline models.

1 Introduction

The present contribution is a continuation of Teunissen (1996 a, b, c), which will henceforth be referred to as, respectively, *Parts I*, *II*, and *III*. As in the previous three parts, the present contribution considers three different versions of the single-baseline model. They are the geometry-based model, the time-averaged model, and the geometry-free model. For the *geometry-based* model, the linearized set of double-differenced (DD) observation equations reads as

$$\begin{aligned} D^T\phi_j(i) &= D^T A_i b + \lambda_j a_j \ , \\ D^T p_j(i) &= D^T A_i b \ , \end{aligned} \tag{1}$$

with $j = 1, 2$ and where $i = 1, \ldots, k$ denotes the epoch number and k equals the total number of epochs; ϕ_1, ϕ_2, p_1 and p_2 are the m-vectors containing the (observed minus computed) metric *single-differenced* (SD) phase and code observables on L_1 and L_2; D^T is the $(m-1) \times m$ DD matrix operator; A_i is the $m \times 3$ SD design matrix that captures the relative receiver-satellite geometry at epoch i; b is the 3-vector that contains the unknown increments of the three-dimensional baseline; λ_1 and λ_2 are the wavelengths of L_1 and L_2; and a_1 and a_2 are the two $(m-1)$-vectors that contain the unknown *integer* DD ambiguities. Time correlation is assumed to be absent and the time-invariant weight matrix (inverse variance matrix) at epoch i, is assumed to be given as the block diagonal matrix

$$Q^{-1} = \mathrm{diag}(\alpha_1, \alpha_2, \beta_1, \beta_2) \otimes (D^T D)^{-1} \ , \tag{2}$$

where '$\otimes$' denotes the Kronecker product. The scalars α_1, α_2, β_1, and β_2 are the weights of the L_1 and L_2 phase and code observables.

The *time-averaged* model follows from taking the time-average of the vectorial observation equations of Eq.(1). The *geometry-free* model follows from the geometry-based model if we disregard the presence of the receiver-satellite geometry. Hence it follows if we replace $A_i b$ in Eq.(1) by the SD range vector r_i.

In *Part I* we studied the gain in baseline precision due to ambiguity fixing and in *Part II* we studied the precision and correlation of the ambiguities. The three stationary values γ_i, $i = 1, 2, 3$, of the variance ratio of the baseline precision before and after ambiguity fixing, based on phase data only, were referred to as the gain numbers. With the gain number concept, we were able to describe the baseline precision in canonical form. The gain numbers are infinite when only a single observation epoch is used, they are large for short observation time-spans and get smaller, following an inverse-square law in the observation time-span, as time progresses. The gain numbers were also used to express the ambiguity precision and correlation. One of the results obtained was that the DD ambiguities are generally of a poor precision, while at the same time highly correlated. This holds true for each of the three single-baseline models.

In *Part III* we studied the geometry of the ambiguity search space. Again the gain-number concept allowed us to formulate canonical forms of the different search spaces. It was shown for each of the three single-baseline models how the size, shape and orientation of the search space change when the observation weights, the number of satellites tracked, the number of observation epochs used, or the relative receiver-satellite geometry changes. We also explained the phenomenon of search halting and showed how decorrelating ambiguity transformations allow one appropriately to mold the shape of the

search space such that the problem of search halting is largely eliminated.

In the present contribution, the last of the four parts, we, so to say, close the loop between ambiguities and the baseline. This will be done by establishing a link between baseline *precision* and ambiguity *reliability*. In Sect. 2 we first discuss the sensitivity of the baseline to changes in the ambiguities. This sensitivity is expressed in the baseline-ambiguity correlation, which in turn is shown to depend on the gain numbers. We also show, in analogy with the theory of minimal detectable biases, how to compute integer ambiguity biases which have the least chance of being detected in the validation stage. In Sect. 3, we introduce a DOP measure for the ambiguities and present closed-form formulae for it. This is done for each of the three different single-baseline models. Finally in Sect. 4, we consider the volume of the ambiguity search space and show that it plays an important part in both the estimation and validation of the integer ambiguities. It can be used to downsize the search space in order to avoid its containing an abundance of unnecessary grid points. This allows one to ease the burden of the search. For search spaces not too elongated, it also provides a measure for the peakedness of the discrete distribution of the integer least-squares ambiguities and thus for their reliability. Through the volume it is shown how the relation between baseline precision and ambiguity reliability can be seen to act as a pair of scales.

2 Baseline sensitivity

In this section we will study the sensitivity of the baseline for changes in the ambiguities. First we will consider the dependence of the *floated* baseline on the least-squares ambiguities. Then we will consider the dependence of the *fixed* baseline on the fixed ambiguities. Here we also establish a link between the baseline sensitivity and the ambiguity search space. Finally, we consider ambiguity validation and present an approximate way to infer *a priori* the likelihood of successful validation.

Floated baseline

It will be intuitively clear that the correlation between the floated baseline and the least-squares ambiguities, $\hat{b}$ and $\hat{a}$, must be related to the gain in baseline precision. If the baseline and the ambiguities were not correlated, a fixing of the ambiguities would have no impact on the baseline and the baseline would remain unchanged. Hence, in that case there would not be any gain in baseline precision. On the other hand, if the baseline and ambiguities were fully correlated and thus functionally related, then knowing the ambiguities would imply a full knowledge of the baseline. So, in this case the baseline would become deterministic and the gain in baseline precision would become infinitely large. From this reasoning it follows that one can expect the correlation to be the driving force of the gain in baseline precision: zero correlation corresponding with minimum gain and maximum correlation corresponding with infinite gain. The following theorem makes this clear.

Theorem 1 (*Ambiguity-baseline correlation*)

Let the ambiguity-baseline correlation coefficient be defined as

$$\rho(a,b) = \frac{a^T Q_{\hat{a}\hat{b}} b}{\sqrt{a^T Q_{\hat{a}} a}\sqrt{b^T Q_{\hat{b}} b}} \ .$$

Then the solution to

$$\rho_i = \mid \rho(a_i,b_i) \mid = \max_a \max_b \mid \rho(a,b) \mid$$

subject to $a^T Q_{\hat{a}} a_j = 0$, $b^T Q_{\hat{b}} b_j = 0$ for $j = 1,\ldots,(i-1)$, is given as

$$\rho_i = \begin{cases} \frac{1}{\sqrt{\epsilon+1}}\sqrt{\frac{\gamma_{4-i}-1}{\gamma_{4-i}}} & i = 1,2,3 \ , \\ 0 & i = 4,\ldots,2(m-1) \ , \end{cases} \tag{3}$$

with the phase-code variance ratio $\epsilon = (\beta_1+\beta_2)/(\alpha_1+\alpha_2)$ and, in ascending order, the three gain numbers $\gamma_1,\gamma_2,\gamma_3$.
Proof: see Appendix. □

This theorem shows that only three correlation coefficients are possibly nonzero, while $(2m-5)$ are identically zero. These zero correlation coefficients can be explained if we recall, from Theorem 1 of *Part II*, the least-squares solution for the ambiguities. It reads

$$\hat{a}_1 = \frac{1}{\lambda_1} D^T[\bar{\phi}_1 - \bar{A}\hat{b}] \ ,$$

$$\hat{a}_2 = \frac{1}{\lambda_2} D^T[\bar{\phi}_2 - \bar{A}\hat{b}] \ .$$

Also recall that the time-averaged phase vectors, $\bar{\phi}_1$ and $\bar{\phi}_2$, are not correlated with the baseline vector $\hat{b}$. Let $f^T\hat{a}_1, f \in R^{m-1}$, be a function of the L_1 ambiguities. This function is then invariant for changes in the baseline if f lies in the null space of $\bar{A}^T D$: $f \in N(\bar{A}^T D)$. Since the dimension of this null space equals $(m-4)$, there are $(m-4)$ linear independent functions of the L_1 ambiguities that are insensitive to changes in the baseline and thus also uncorrelated to the baseline. Now consider the $(m-1)$ functions $\lambda_1\hat{a}_1 - \lambda_2\hat{a}_2$. Also they are invariant to changes in the baseline and uncorrelated with it. Hence we have now identified the set of $(2m-5)$ linear independent functions of the ambiguities not correlated with the baseline. Of course, also the $(m-4)$ functions $f^T\hat{a}_2$, with $f \in N(\bar{A}^T D)$, are uncorrelated with the baseline. These functions, however, are linearly dependent on the stated set and thus already taken care of. Note that the set of $(2m-5)$ functions consists of those ambiguity functions that can be estimated with the highest possible precision. These are hence the ambiguity functions that correspond with the subspaces in

which the shortest principal axes of the ambiguity search space lie.

The remaining three correlation coefficients are generally nonzero. The theorem shows, through the gain numbers how they depend on the relative receiver-satellite geometry, and through the variance ratio ϵ how they depend on the precision of the GPS observables. These correlation coefficients get larger the larger the gain numbers become. For an infinite gain, the correlation coefficients can only be pulled away from 1 by improving the precision of the code observables relative to the precision of the phase observables. But the precision of code must then really improve significantly for it to have a significant effect on the correlation coefficient. In practice, ϵ is so small and the gain numbers large when short observation time-spans are used, that the correlation coefficients will be very close to 1. Hence the ambiguities and baseline will then be highly correlated.

The whole purpose of ambiguity fixing is of course to benefit from this high correlation. Due to the high correlation, the baseline is very sensitive to changes in the ambiguities. As a consequence, a fixing of the ambiguities results in a greatly improved baseline, *precision*-wise. But what happens when we fix to the wrong ambiguities? How sensitive is the baseline to biases in the fixed ambiguities? This type of sensitivity is as important as the other, since clearly, it does not make much sense to have a highly precise baseline, but one which otherwise is completely off target.

Fixed baseline

In order to study this second type of sensitivity, we need to know the impact of ambiguity biases on the *fixed* baseline. The fixed baseline and ambiguities are related as

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \ .$$

Let $\check{a} = E\{\hat{a}\} + \nabla\check{a}$ and $E\{\check{b}\} = E\{\hat{b}\} + \nabla\check{b}$, where $E\{.\}$ is the expectation operator, $\nabla\check{a}$ is the ambiguity bias, and $\nabla\check{b}$ is the corresponding bias in the fixed baseline. Then

$$\nabla\check{b} = Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} \nabla\check{a} \ . \tag{4}$$

In order to measure the significance of this baseline bias, we use the squared *Bias-to-noise ratio* (BNR):

$$\| \nabla\check{b} \|^2 = \nabla\check{b}^T Q_{\check{b}}^{-1} \nabla\check{b} \ . \tag{5}$$

Since $Q_{\hat{a}|\hat{b}}^{-1} = Q_{\hat{a}}^{-1} + Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} Q_{\check{b}}^{-1} Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}$, where $Q_{\hat{a}|\hat{b}}$ is the conditional ambiguity variance matrix, conditioned on knowing the baseline, it follows that

$$\nabla\check{a}^T Q_{\hat{a}|\hat{b}}^{-1} \nabla\check{a} = \nabla\check{a}^T Q_{\hat{a}}^{-1} \nabla\check{a} + \nabla\check{b}^T Q_{\check{b}}^{-1} \nabla\check{b} \ . \tag{6}$$

In this Pythagorean decomposition, we recognize, besides the baseline BNR, also

$$\| \nabla\check{a} \|^2 = \nabla\check{a}^T Q_{\hat{a}}^{-1} \nabla\check{a} \ , \tag{7}$$

the noncentrality parameter of the distribution of the quadratic form

$$T = (\hat{a} - \check{a})^T Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \ ,$$

assuming that the least-squares ambiguities are normally distributed and $\check{a}$ nonstochastic. That is, it is the shift the mean of T undergoes when $E\{\hat{a}\} \neq \check{a}$.

Equation (6) shows that the ambiguity BNR $\| \nabla\check{a} \|^2$ gets distributed over $\nabla\check{a}^T Q_{\hat{a}|\hat{b}}^{-1} \nabla\check{a}$ and the baseline BNR. In the ideal case of course, we would like all of the bias to be contributed to $\nabla\check{a}^T Q_{\hat{a}|\hat{b}}^{-1} \nabla\check{a}$ and nothing to $\| \nabla\check{b} \|^2$. But for an arbitrary ambiguity bias, this can clearly only happen when $Q_{\hat{a}|\hat{b}} = Q_{\hat{a}}$, that is, when the floated baseline and the ambiguities would *not* correlate. This however is not the case, as we have already seen. Thus in general one can expect the baseline BNR to be nonzero and the ambiguity-baseline correlation a factor that influences it. The following corollary shows how this correlation puts a bound to the baseline BNR relative to the ambiguity BNR.

Corollary 1 (*Baseline to ambiguity bias*)

Let the baseline BNR and the ambiguity BNR be defined as in Eqs.(5) and (7), and let $\rho_{\max}$ be the largest ambiguity-baseline correlation coefficient in absolute value. Then

$$\max_{\nabla\check{a}} \frac{\| \nabla\check{b} \|^2}{\| \nabla\check{a} \|^2} = \frac{\rho_{\max}^2}{1 - \rho_{\max}^2} \ . \tag{8}$$

Proof: see Appendix. □

It follows from this corollary that the baseline BNR is bounded from above as

$$\nabla\check{b}^T Q_{\check{b}}^{-1} \nabla\check{b} \leq \frac{\rho_{\max}^2}{1 - \rho_{\max}^2} \nabla\check{a}^T Q_{\hat{a}}^{-1} \nabla\check{a} \ .$$

This shows, since the correlation coefficient is very close to 1 for large gains, that the baseline BNR can be a multiple of the ambiguity BNR. The conclusion reached so far reads therefore, that although the high correlation between the baseline and ambiguities allows for a drastic improvement in baseline precision, it at the same time allows the possibility of a very large baseline BNR. The first property is what we strive for, the second property, however, is what we need to avoid.

So far, no specific assumptions were made about the bias in the ambiguities. We will now consider for which ambiguity biases the baseline BNR is affected and for which it is not. In order to study this, we will make use of the following canonical decomposition of the baseline BNR.

Theorem 2 (*The Baseline BNR*)

The baseline BNR can be decomposed as

$$\| \nabla\check{b} \|^2 = \frac{(\alpha_1 + \alpha_2)k}{\epsilon + 1} \check{\nabla}^T D^+ U (I_3 - \Gamma^{-1}) U^T D^{+T} \check{\nabla} \ , \tag{9}$$

where $\check{\nabla}$ is the weighted average of the L_1 and L_2 ambiguity biases, expressed in units of range rather than cycles,

$$\check{\nabla} = \lambda_1 \frac{\alpha_1}{\alpha_1 + \alpha_2} \nabla \check{a}_1 + \lambda_2 \frac{\alpha_2}{\alpha_1 + \alpha_2} \nabla \check{a}_2 \ ,$$

and where D^+ is the pseudo-inverse of D, U the matrix of orthonormal columns as defined in Theorem 1 of *Part III*, and Γ the diagonal matrix of gain numbers.
Proof: see the Appendix. □

Let us first consider the cases for which the baseline BNR equals zero. It equals zero when all gain numbers are equal to 1. This is not likely to happen in practice however. In fact, for short observation time-spans, the gain numbers will be large. Hence, no significant reduction of the impact of the ambiguity biases on the baseline can be expected from $(I_3 - \Gamma^{-1})$.

The baseline BNR is also equal to zero when $D^{+T}\check{\nabla} \in R(U)^{\perp}$. Recall from Theorem 1 of *Part III* that $R(U)^{\perp} = R(V, w)$, where the orthonormal columns of (V, w) correspond with the principal axes of the ambiguity search space that have the *smallest* length. This shows that the ambiguity biases for which the baseline BNR is insensitive are precisely those for which the quadratic form that defines the ambiguity search space is most sensitive. The converse of this statement also holds true; that is, the ambiguity biases for which the baseline BNR is most sensitive, are precisely those for which the quadratic form defining the ambiguity search space is less sensitive. It will be clear that this situation is potentially a disastrous one. It implies, namely, that the ambiguity biases which have the poorest chance of being detected by means of statistical tests are at the same time those which have the largest impact on the baseline. Fortunately, the situation is somewhat less dramatic as it may seem at first sight. This is due to the fact that the ambiguity biases $\nabla \check{a}_1$ and $\nabla \check{a}_2$ cannot take on arbitrary positions in R^{m-1}. Instead, they are confined to the grid points of Z^{m-1}. This therefore reduces the chance of having $D^{+T}\check{\nabla} \in R(U)$.

It follows from Eq.(9) and $R(U) = R(P\bar{A})$, that it is the *average* receiver-satellite geometry which is instrumental in the propagation of the ambiguity biases. When the gains are sufficiently large, we may set $\Gamma^{-1} = 0$, and recognizing that UU^T is the orthogonal projector that projects onto the range space $R(P\bar{A})$, we get to a good approximation

$$\| \nabla \check{b} \|^2 \simeq \frac{(\alpha_1 + \alpha_2)k}{\epsilon + 1} \check{\nabla}^T D^+ \bar{A} (\bar{A}^T P \bar{A})^{-1} \bar{A}^T D^{+T} \check{\nabla} \ . \quad (10)$$

Using the cosine rule and the projector property of UU^T, this may also be written as

$$\| \nabla \check{b} \|^2 \simeq \frac{(\alpha_1 + \alpha_2)k}{\epsilon + 1} \check{\nabla}^T (D^T D)^{-1} \check{\nabla} \cos^2 \omega \ , \quad (11)$$

where ω is the angle between $D^{+T}\check{\nabla}$ and $R(P\bar{A})$. Thus $\omega = 0$ when $\check{\nabla} \in R(D^T \bar{A})$.

Now let us assume that the ambiguity bias is such that a bias of a single cycle occurred in the ith entry of the L_1 integer least-squares solution. Then $\nabla \check{a}_2 = 0$ and $\nabla \check{a}_1 = c_i$, where c_i is the unit vector having 1 as its ith entry. We also assume that the precision of the L_1 phase data equals that of the L_2 phase data: $\alpha_1 = \alpha_2$. Then, since $(D^T D)^{-1} = I_{m-1} - \frac{1}{m} e_{m-1} e_{m-1}^T$ and thus $c_i^T (D^T D)^{-1} c_i = 1 - \frac{1}{m}$, the baseline BNR follows as

$$\| \nabla \check{b} \|^2 \simeq \frac{1}{2} \lambda_1^2 \alpha_1 \frac{k}{\epsilon + 1} (1 - \frac{1}{m}) \cos^2 \omega_i \ .$$

This shows, since $\lambda_1 \sqrt{\alpha_1}$ is already so large ($\lambda_1 \sqrt{\alpha_1} \simeq 46$ for a standard deviation of 3 mm of an undifferenced phase), that $\cos^2 \omega_i$ has to be very small indeed in order for the baseline BNR not to become unacceptably large.

On the ambiguity bias

The preceding shows that the utmost care has to be exercized in making sure that the biases in the ambiguities stay sufficiently bounded. Only then is one in a position to profit from the large gain in baseline precision. Validation of the integer least-squares ambiguities is therefore an important aspect of the whole process of computing and fixing the ambiguities. In the literature, different approaches are in use for validating the ambiguities. Examples can be found in the textbooks Leick (1995), Hofmann-Wellenhof et al. (1996), Kleusberg and Teunissen (1996).

The *actual* validation depends on $\hat{a}$ and therefore on the data. It is however also of importance, in particular from the design perspective, to be able to infer a priori, thus without the need for actual data, whether one can expect to have a successful validation or not; this is in analogy with the theory of reliability (Baarda 1968). It will be clear that the precision of the ambiguities plays a decisive role in this respect. In the one-dimensional case, one could proceed as follows. Take the standard deviation of the ambiguity and compare it with the distance between two grid points: 1 cycle. Then, if the standard deviation is sufficiently small with respect to 1, one can decide that the distribution of $\hat{a}$ is sufficiently peaked. If one assumes in addition that the least-squares ambiguities are normally distributed and unbiased, the expectation $E\{\hat{a}\}$ and thus the distribution of $\hat{a}$ will be centered at a grid point. From this and the peakedness of the distribution follows then that the probability of choosing the wrong grid point will be sufficiently small. The distance of the grid points to $E\{\hat{a}\}$ will then be sufficiently large in relation to the peakedness of the distribution. Or in other words, the *discrete* distribution of the integer least-squares solution will have sufficient probability mass located at one single grid point.

In two dimensions and higher, the situation becomes much more complicated. One could think of simply copying the one-dimensional approach and apply it to each of the individual ambiguities. This works however only when the ambiguities are uncorrelated or when the standard deviations of the ambiguities are already small enough with respect to the unit distance of the grid. But as we know from *Part II*, the DD ambiguities are highly correlated and their individual standard deviations are usually very large. In principle, one is therefore forced to

follow a different route for the higher-dimensional case. One possible approach is the following.

Again we assume that the least-squares ambiguities are normally distributed and unbiased. This implies that the multivariate normal distribution is centered at a grid point. The idea is now again to check whether this distribution is sufficiently peaked with respect to the separation of the grid points. But instead of using the precision measures of the ambiguities directly, as was done in the one-dimensional case, we use them indirectly by computing the distance between two grid points. Here, distance is measured in the metric of the ambiguity variance matrix. An additional complication when compared to the one-dimensional case, is that this distance differs for different directions. We therefore choose to consider the smallest distance between two grid points, since it provides the direction in which one will have the greatest difficulty in discriminating between two grid points. Thus we solve the minimization problem

$$\min_{\nabla a} \nabla a^T Q_{\hat{a}}^{-1} \nabla a \neq 0 \quad \text{with} \quad \nabla a \in Z^{2(m-1)} \quad . \tag{12}$$

This is again an integer least-squares problem. But it differs in two ways from the one we have met until now. First, the trivial solution $\nabla a = 0$ needs to be excluded. Secondly, Eq. (12) has at least two solutions, since it can not discriminate between reflections about the origin. This however is no point of concern, since it is the distance that counts.

Once a solution of Eq. (12) has been computed, say $\check{\nabla} a$, two steps can be taken. First, by interpreting $\check{\nabla} a$ as an integer ambiguity bias, one could use Eqs. (4) and (5) to compute its impact on the baseline. This will give us the baseline bias, should an ambiguity bias as $\check{\nabla} a$ occur. But as our earlier analysis showed, the conclusion will probably be that the impact is unacceptably large. For the second step, we use $\nabla \check{a}$ to compute the minimum of Eq. (12) and interpret this minimum as a noncentrality parameter. The decision as to whether the distribution is sufficiently peaked or not can then be based on the value this noncentrality parameter takes. If its value turns out to be insignificant, then the smallest distance between two grid points is too small for one to expect a successful validation. The integer least-squares ambiguities are then simply not reliable enough. If, on the other hand, the noncentrality parameter turns out to be sufficiently significant, then the smallest distance between two grid points is still large enough for one to be confident that the integer least-squares ambiguities are reliable. In that case, one may *expect* that the actual validation will turn out to be successful.

3 Ambiguity DOP

Although the stochastic nature of the least-squares ambiguities is described in all its detail by the complete variance-covariance matrix of the ambiguities, it would be helpful if we could also make use of a somewhat simpler measure and one that still captures the main characteristics of the ambiguities. In many applications of GPS it is already common practice to use *dilution of precision* (DOP) measures to describe the impact of the receiver-satellite geometry on the precision of the parameters under study. Various DOP measures are in use, such as the *position* (PDOP), the *vertical* (VDOP), or the *horizontal* dilution of precision (HDOP). The DOP measures in use are very simple functions of the appropriate variance-covariance matrices. PDOP for instance is defined as the square-root of the trace of the position variance-covariance matrix. Although various DOP measures exist, no DOP measure yet exists for the ambiguities. In this section we will propose a DOP measure for the ambiguities. It is the geometric mean of the ambiguity conditional standard deviations. Its properties will be discussed, and a closed-form formula derived for it which applies to our three different single GPS baseline models.

3.1 A proposal

First we will formulate criteria which an ambiguity DOP (ADOP) measure should preferably satisfy. Then we show that these criteria are met by the geometric mean of the ambiguity conditional standard deviations. Finally we present some different ways of computing the ADOP and show how it bounds the ambiguity bias.

Criteria As with the existing DOP measures, the DOP measure for the ambiguities should preferably be a scalar measure, which is simple to compute and which captures the main intrinsic characteristics of the ambiguity variance matrix. The existing DOP measures are all based on the trace of an appropriate variance-covariance matrix. For the ambiguities however, the trace cannot be considered adequate. There are two main reasons for this. First, the trace lacks some important properties of invariance, and second, it is only based on the diagonal entries of the variance matrix.

Let us first consider the property of invariance. Let Q be a variance matrix and let $\mathrm{tr}(Q)$ be its trace. Furthermore, let Z^T be a transformation matrix that transforms the parameters under study. The variance matrix of the transformed parameters is then given as $Z^T Q Z$ and its trace as $\mathrm{tr}(Z^T Q Z)$. Since $\mathrm{tr}(Z^T Q Z) = \mathrm{tr}(Q Z^T Z)$, it follows that the trace remains invariant under the transformation when the transformation matrix is orthogonal. For the position DOP this may seem to be a useful property, since it shows its invariance for a rotation of the coordinate frame of reference. For the ambiguities however, it is not a useful property; here the simplest change in the frame of reference occurs when one changes the choice of reference satellite. But already this simple transformation fails to keep the trace unchanged. It is precisely due to this lack of invariance that we had to introduce the notion of 'double averaging' when studying the average precision of the DD ranges in *Part II*. But even when we can correct the trace for this lack of invariance, we are still left with the most general changes the frame of reference of the ambiguities may undergo. As we know, these are the transformations which are volume preserving and have integers as their

entries. And although these transformations have the volume preserving property in common with the orthogonal transformations, they do not leave the trace unchanged.

A second reason why the trace should be rejected as a measure on which to base the ambiguity DOP is due to the fact that it fails to take the existing correlation between the ambiguities into account. This would not be too harmful if the correlations were small to moderate and if they only played a secondary role in the process of ambiguity estimation and validation. In *Part II* we showed however, that the ambiguities are extremely correlated, and in *Part III* that the correlations are the driving force in giving the ambiguity search space its very elongated shape.

Based on these considerations, it will be clear that we cannot accept the trace of the ambiguity variance matrix $Q_{\hat{a}}$ as an appropriate measure. Instead of the trace, we therefore propose to base the ADOP on the determinant and introduce it as

$$\text{ADOP} = \sqrt{|\, Q_{\hat{a}} \,|}^{\frac{1}{n}} \quad \text{(cycle)} \;, \tag{13}$$

where n is the order of the ambiguity variance matrix. This measure has none of the drawbacks mentioned. Since all ambiguity transformations are volume preserving, and so also those that merely change the choice of reference satellite, we have $|\, Z^T Q_{\hat{a}} Z \,| = |\, Q_{\hat{a}} \,|$. Furthermore, since the determinant makes use of all entries of the variance matrix, it includes the correlations as well. In fact, we have the decomposition

$$|\, Q_{\hat{a}} \,| = |\, R_{\hat{a}} \,| \, \Pi_{i=1}^{n} \sigma_{a_i}^2 \;, \tag{14}$$

where $R_{\hat{a}}$ is the ambiguity *correlation* matrix and $\sigma_{a_i}^2$ are the ambiguity variances. It shows once more why a measure based on the ambiguity variances alone would be inappropriate. The ambiguity variances are quite large, but their correlations are quite large too; hence, the determinant of the correlation matrix will be very small. Both these effects are incorporated in the determinant of the ambiguity variance matrix. Note that in the two-dimensional case, the determinant of the correlation matrix reduces to $(1 - \rho^2)$, with ρ being the correlation coefficient.

Its computation At first sight it may seem that the computation of the determinant of the ambiguity variance matrix involves quite some more work than needed when simply summing its diagonal entries. But as it will be shown, one can economize significantly on the required computations.

For the computation of the determinant of $Q_{\hat{a}}$ one can either use its eigenvalues or its conditional variances. We have

$$|\, Q_{\hat{a}} \,| = \begin{cases} \Pi_{i=1}^{n} \lambda_{a,i} \;, \\ \Pi_{i=1}^{n} \sigma^2_{a_{i|1,\ldots,(i-1)}} \;, \\ \Pi_{i=1}^{n} \sigma^2_{a_{i|i+1,\ldots,n}} \;. \end{cases} \tag{15}$$

The first product consists of the product of all eigenvalues. The last two are products of conditional variances. In the first product of conditional variances, the conditioning is done on the previous ambiguities, whereas in the second product, the conditioning is done on the ambiguities following. The conditional variances can be obtained from a *triangular decomposition* of either $Q_{\hat{a}}$ or its inverse. Let the LDL^T decomposition of $Q_{\hat{a}}$ be given as $Q_{\hat{a}} = LDL^T$. The conditional variances $\sigma^2_{a_{i|1,\ldots,(i-1)}}$ are then given as the entries of the diagonal matrix D (not to be confused with the DD matrix operator). The conditional variances $\sigma^2_{a_{i|i+1,\ldots,n}}$ follow if instead of a *lower* triangulation, an *upper* triangulation of $Q_{\hat{a}}$ is used. Thus if $Q_{\hat{a}} = U\tilde{D}U^T$, where U is upper unit triangular and $\tilde{D}$ is diagonal, then the entries of this diagonal matrix equal $\sigma^2_{a_{i|i+1,\ldots,n}}$. An upper triangular decomposition of $Q_{\hat{a}}$ is equivalent to using a lower triangular decomposition for its inverse $Q_{\hat{a}}^{-1}$. Thus $Q_{\hat{a}}^{-1} = U^{-T}\tilde{D}^{-1}U^{-1}$, with U^{-T} lower unit triangular. The lower triangulation can easily be obtained from a Cholesky decomposition. For instance, the Cholesky factor of $Q_{\hat{a}}^{-1}$ is given as $G = U^{-T}\sqrt{\tilde{D}^{-1}}$.

In Eq.(15) the variance matrix of the DD ambiguities is used. But, since the determinant is invariant for ambiguity transformations, one can also use the variance matrix of the decorrelated ambiguities, as it is produced by the LAMBDA method. Thus

$$|\, Q_{\hat{a}} \,| = |\, Q_{\hat{z}} \,| = \begin{cases} \Pi_{i=1}^{n} \lambda_{z,i} \;, \\ \Pi_{i=1}^{n} \sigma^2_{z_{i|1,\ldots,(i-1)}} \;, \\ \Pi_{i=1}^{n} \sigma^2_{z_{i|i+1,\ldots,n}} \;, \end{cases} \tag{16}$$

where $\hat{z}$ denotes the vector of transformed ambiguities. Note that although the product of eigenvalues is invariant for ambiguity transformations, the individual eigenvalues are not. The same holds true for the conditional variances.

In general, the computation of the determinant based on the conditional variances is preferable over the computation based on the eigenvalues. Not only is the computation of the n eigenvalues in general more involved, but the triangular decompositions or the Cholesky decompositions are usually already available, in particular when solving least-squares problems. The computation of the determinant becomes particularly cheap when the search for the integer least-squares ambiguities is based on the bounds that follow from a sequential conditional least-squares adjustment. In that case the conditional variances of the transformed ambiguities are readily available. Yet another alternative will be given in Sect. 3.2.

Bounding the ambiguity bias The geometric mean of the ambiguity conditional variances can be used to put a bound on the ambiguity bias. We will formulate it for the decorrelated ambiguities. Let us assume that an integer ambiguity bias $\nabla z \in Z^n$ is acceptable if and only if it is bounded as $\| \nabla z \|^2 \leq \chi^2$. It follows then that the

geometric mean of its sequential conditional least-squares components, $\nabla z_{i|1,\ldots,(i-1)}$, is bounded as

$$(\Pi_{i=1}^{n} \mid \nabla z_{i|1,\ldots,(i-1)} \mid)^{\frac{1}{n}} \leq \chi \text{ ADOP}. \tag{17}$$

This shows that the set $\| \nabla z \|^2 \leq \chi^2$, $\nabla z \in Z^n$, will only have the trivial solution $\nabla z = 0$, when χ ADOP < 1 and all conditional variances $\sigma^2_{z_{i|1,\ldots,(i-1)}}$ are equal. We know from *Part III* that the conditional variances of the *DD* ambiguities are far from equal. For the geometry-based model for instance, the first three conditional variances will be large, while the remaining conditional variances will be very small. But this discontinuity disappears when the decorrelated ambiguities are used, in which case one obtains an almost flat spectrum of conditional variances.

3.2 A closed-form formula

In this subsection we present a closed-form formula for the determinant of the ambiguity variance matrix and show how it is affected when different types of data redundancy are used. The following theorem presents the formula for the geometry-based single GPS baseline model.

Theorem 3 (*Determinant of ambiguity variance matrix*)

Let $Q_{\hat{a}}$ be the $2(m-1) \times 2(m-1)$ variance matrix of the dual-frequency least-squares ambiguities which follows when using the geometry-based model. Its determinant is then given as

$$\mid Q_{\hat{a}} \mid = m^2 \left(\frac{1}{\alpha_1 k \lambda_1^2}\right)^{m-1} \left(\frac{1}{\alpha_2 k \lambda_2^2}\right)^{m-1} \Pi_{i=1}^{3} \left(1 + \frac{\gamma_i - 1}{\epsilon \gamma_i + 1}\right), \tag{18}$$

with $\epsilon = (\beta_1 + \beta_2)/(\alpha_1 + \alpha_2)$.
Proof: see Appendix. □

Note that the determinant is indeed, as it should be, independent of the arbitrary choice of reference satellite. But the formula also holds true for all ambiguity variance matrices obtained from $Q_{\hat{a}}$ by means of an admissible ambiguity transformation. Thus instead of $\mid Q_{\hat{a}} \mid$, one may also read the determinant of the variance matrix of the decorrelated ambiguities.

The theorem has been formulated for the geometry-based model, assuming that dual-frequency phase and code data are used. For the time-averaged and geometry-free model however, the determinants can be obtained directly from the theorem as well. For the time-averaged model, one simply has to take the limits $\gamma_i \to \infty$; and to obtain the corresponding result for the geometry-free model, one in addition to these limits also has to raise $(1 + 1/\epsilon)$ to the power of $(m-1)$ instead of 3.

The theorem is also easily adapted to hold for those cases where less observation data are used. For dual-frequency phase data without code data or with code data on only L_1 or L_2, one simply has to set $\beta_1 = \beta_2 = 0$, or $\beta_2 = 0$ or $\beta_1 = 0$. When phase data are used on only L_1 (or L_2), one has to replace m^2 by m, set $1/(\alpha_2 k \lambda_2^2)^{m-1}$ (or $1/(\alpha_1 k \lambda_1^2)^{m-1}$) equal to 1 and set α_2 (or α_1) equal to zero.

The theorem clearly shows the contribution of the various types of data redundancy:

1. satellite redundancy (m),
2. observation epochs redundancy (k),
3. redundancy in frequency (L_i),
4. observation type redundancy (ϕ_i, p_i),
5. change in receiver-satellite geometry (γ_i).

In order to have an easy reference and to be able to compare and discuss the results for the three types of single baseline model, the determinants of the three models are summarized in Table 1, whose results we discuss in the following order: first the geometry-free model, then the time-averaged model, and finally the geometry-based model.

Geometry-free model The determinant of the geometry-free model exists already for $m = 2$. This shows, when one opts to dispense with the receiver-satellite geometry, that individual ambiguity estimation is possible in principle from data of two satellites only. The determinant fails to exist however, when $\epsilon = 0$. This shows that code data are needed per se.

For the single-frequency case, the determinant equals a product of a very small term and a very large term. The small term is dominated by the phase variance, whereas the large term is dominated by the ratio of the code variance with the phase variance. As a result, the product will be dominated by the variance of code. This shows that the determinant will be very large indeed, unless a sufficient number of epochs is taken into account. Hence, a successful validation of the integer

Table 1. The determinants of the single- and dual-frequency ambiguity variance matrices, for the geometry-free model, the time-averaged model and the geometry-based model

$\mid Q_{\hat{a}} \mid$	geometry-free ($m \geq 2$)	time-averaged ($m \geq 4$)	geometry-based ($m \geq 4$)
L_1	$m\left(\frac{1/\alpha_1}{k\lambda_1^2}\right)^{m-1} \left(1 + \frac{1}{\epsilon}\right)^{m-1}$	$m\left(\frac{1/\alpha_1}{k\lambda_1^2}\right)^{m-1} \left(1 + \frac{1}{\epsilon}\right)^{3}$	$m\left(\frac{1/\alpha_1}{k\lambda_1^2}\right)^{m-1} \left(1 + \frac{1}{\epsilon}\right)^{3} \Pi_{i=1}^{3} \frac{\epsilon\gamma_i}{\epsilon\gamma_i + 1}$
L_1/L_2	$\left[m\left(\frac{1\sqrt{\alpha_1\alpha_2}}{k\lambda_1\lambda_2}\right)^{m-1}\right]^2 \left(1 + \frac{1}{\epsilon}\right)^{m-1}$	$\left[m\left(\frac{1\sqrt{\alpha_1\alpha_2}}{k\lambda_1\lambda_2}\right)^{m-1}\right]^2 \left(1 + \frac{1}{\epsilon}\right)^{3}$	$\left[m\left(\frac{1\sqrt{\alpha_1\alpha_2}}{k\lambda_1\lambda_2}\right)^{m-1}\right]^2 \left(1 + \frac{1}{\epsilon}\right)^{3} \Pi_{i=1}^{3} \frac{\epsilon\gamma_i}{\epsilon\gamma_i + 1}$

ambiguities can be expected only when enough samples of data are used.

In the dual-frequency case, we clearly see the beneficial role of including the second frequency. The very large term of the single-frequency case is *not* squared, but instead simply copied. However, apart from the change in wavelength, the very small term *is* squared. Hence, when taken to the appropriate power, one can expect the dual-frequency determinant to be much smaller than the single-frequency determinant.

Time-averaged model The determinant of the time-averaged model exists only when $m \geq 4$. This is of course due to the fact that the parameters in the model include, apart from the ambiguities, also the three-dimensional baseline. Again, the determinant fails to exist when $\epsilon = 0$. This can be explained by the fact that the floated baseline solution of the time-averaged model is a code-only solution. Hence, the phase data do not contribute to the floated baseline and consequently code data are needed per se.

The determinant of the time-averaged model also equals the product of a very small term and very large term. Note however, that when compared to the result of the geometry-free model the large term is only taken to the power of 3 and not to the power of $(m-1)$. Hence, the determinant of the time-averaged model will be smaller than that of the geometry-free model when $m > 4$, and the two will be equal when $m = 4$. The fact that the power of the very large term stays restricted to 3 is due to the time invariance of the baseline in the time-averaged model.

As with the geometry-free model, the beneficial impact of the second frequency is clearly visible. The large term is copied, while the very small term is approximately squared.

Finally we note that the determinant for the time-averaged model is completely independent of the receiver-satellite geometry. This is quite remarkable, since the design matrix of the time-averaged model and also the ambiguity variance matrix itself are strongly dependent on it. In fact, they are constructed from the average over time of the receiver-satellite geometries. The receiver-satellite geometry independence of the determinant can be explained if we consider the results of the geometry-based model.

Geometry-based model The results of the geometry-based model reduce to that of the time-averaged model when the gain numbers tend to infinity. The gain numbers become infinite when the model is based on only one single observation epoch, that is, when their is *no change* in the receiver-satellite geometry. This happens for the *single-epoch* geometry-based model. But it also happens for the time-averaged model, since it also depends on the receiver-satellite geometry of only one single epoch, namely the one that corresponds with the time average of the observation time-span.

Note that in contrast to the determinants of the geometry-free model and the time-averaged model, no code data are needed per se with the geometry-based model, provided of course that the gain numbers are finite. Also note that the inclusion of the change in receiver-satellite geometry has resulted in determinants that are smaller than their counterparts of the geometry-free model and the time-averaged model. The two determinants reach their minimum when all three gain-numbers equal 1. This will not happen in practice, however, when short observation time-spans are used.

The amount in which the two determinants differ from their counterparts of the time-averaged model is governed by $1/(\epsilon\gamma_i + 1)$. This shows that the two pairs of determinants will not differ too much when $\epsilon\gamma_i \gg 0$.

4 Volume of search space

In the previous section we considered the determinant of the ambiguity variance matrix. In this section we will give it a geometric interpretation by connecting it to the volume of the ambiguity search space. The purpose of this section is twofold. First we will show in Sect. 4.1 that the volume (and thereby also the determinant) is a very useful tool for appropriately downsizing the ambiguity search space. In this way one can avoid having an abundance of unnecessary grid points in the search space when performing the search for the integer least-squares ambiguities. Second, we will establish links between the volume of the search space on the one hand, and the ambiguity-baseline correlation and the volumes of the confidence ellipsoids of the baseline on the other. This will be done in Sect. 4.2. It will allow us to understand the factors that contribute to the size of the search space and it in particular shows how in the process of GPS ambiguity resolution, the gain in baseline *precision* competes with the *reliability* of the integer ambiguity solution.

4.1 Downsizing the search space

The volume of the ambiguity search space

$$(\hat{a} - a)^T Q_{\hat{a}}^{-1} (\hat{a} - a) \leq \chi^2 \tag{19}$$

is given as

$$V_n = \chi^n U_n \sqrt{|\, Q_{\hat{a}} \,|}\ , \tag{20}$$

where n is the order of the ambiguity variance matrix and U_n is the volume of the unit sphere in R^n. Thus the volume depends on χ^2, a scale factor with which one can tune the size of the search space, on U_n, which is known once the dimension n is known, and on the determinant of the ambiguity variance matrix.

The computation of the determinant has been discussed in the previous section. The volume of the unit sphere in R^n can be computed as

$$U_n = \pi^{n/2} / \Gamma(n/2 + 1)\ , \tag{21}$$

with the gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, x > 0$. For our purposes it is not needed to evaluate the integral explicitly. One can make use of the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, for $x > 0$, with the initial values $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$.

It has been shown in Teunissen et al. (1995) and Jonge and Tiberius (1996) that the volume provides, on average, a very good approximation of the number of grid points contained in the search space. This implies that the volume can be used appropriately to downsize the ambiguity search space. We present two approaches, one that can be used prior to the actual measurement stage and one that makes use of the measurements. The first approach is the simpler, but also the cruder; it goes as follows. Depending on the approximate number of grid points required, the volume is set. Then the volume of the unit sphere and the determinant of the variance matrix are computed. From V_n, U_n, and $|\, Q_{\hat{a}} \,| = |\, Q_{\hat{z}} \,|$, the appropriate value of χ^2 for scaling the search space then follows from invoking Eq. (20). Of course, since the volume is only an indicator for the number of grid points, it is not so much the precise value that counts, but more its order of magnitude.

The second approach makes use of the least-squares ambiguities and guarantees that the search space will contain at least one, or if needed, at least two grid points, and most likely not many more. This approach takes the full advantage of the high precision and low correlation of the transformed ambiguities as they are provided by the LAMBDA method. We discuss two versions of it. We first discuss the case which guarantees that at least one single grid point is contained in the search space. The idea is the following. Starting from the (real-valued) least-squares estimate of the transformed ambiguities, $\hat{z}$, we round each of its n entries to the nearest integer. This will give an integer vector, which is then substituted for z into the quadratic form that defines the transformed search space. The value of χ^2 is then taken to be equal to the value of the quadratic form. This approach guarantees that the search space will at least contain one grid point. Also, the number of grid points contained in it will not be too large. This is due to the high precision and low correlation of the transformed ambiguities. In fact, it often happens that the search space so obtained only contains one grid point, since in many cases the rounded integer vector already equals the integer least-squares estimate $\check{z}$. But note that *this* is not guaranteed, since the variance matrix of $\hat{z}$ is not completely diagonal.

For *validation* purposes, often not only the most likely integer vector is needed, but also the second most likely integer vector. Hence in this case we would need to choose χ^2 such that the search space contains at least two grid points and preferably not many more than two. Again the idea of integer-rounding the entries of $\hat{z}$ can be used. But now we proceed as follows. As before, we first round all the entries of $\hat{z}$ to their nearest integer. This gives one integer vector. Then another n integer vectors are constructed by rounding all entries of $\hat{z}$ but one to their nearest integer, and one entry to its next-nearest integer. Thus we have now obtained $(n+1)$ integer vectors, all of which have their own corresponding χ^2-value. By setting χ^2 equal to the smallest-but-one of these values, it is guaranteed that the corresponding search space at least contains two grid points, and most likely not many more than two.

The second approach performs very well when applied to the decorrelated ambiguities. After the downsizing has been applied, the actual number of grid points found in the search space is small and close to the volume of the search space. The approach works very poorly though, when it is applied to the original DD ambiguities. But this is understandable when one considers the poor precision and high correlation of the DD ambiguities.

4.2 Volume, correlation, and gain

As noted earlier, validation can be expected to be successful when the *discrete* distribution of the integer least-squares ambiguities is sufficiently peaked. That is, when sufficient probability mass is centered at one single grid point and the residual probability masses, which are distributed over the remaining grid points, are sufficiently small so that they may safely be neglected. In fact, this is what one tries to verify at the actual validation stage, by computing and comparing the most likely and second most likely integer ambiguities. But as also observed earlier, if the assumptions underlying the GPS model are correct, one should already at the designing stage be able to infer whether the 'strength' of the model is such as to produce a sufficiently peaked distribution for the integer least-squares ambiguities. For nonelongated or moderately elongated search spaces, such as those which may be obtained for our three single-baseline models through the use of the decorrelated ambiguities, the peakedness of the distribution is mirrored in the volume of the search space. Hence, the volume can then also play the role of a *reliability* measure. The smaller the volume, the more peaked the distribution, and thus the more reliable the integer least-squares solution becomes.

In Sect. 1 we studied the sensitivity of the baseline to changes in the ambiguities and showed how it was related to the ambiguity-baseline correlation. One of the conclusions was that the *fixed* baseline is very sensistive to ambiguity biases when the ambiguity-baseline correlation is large. Theorem 4 shows that a large ambiguity-baseline correlation is also likely to result in a large volume of the search space and thus in a poor reliability of the integer ambiguities. Thus not only is the fixed baseline sensitive to ambiguity biases when the ambiguity-baseline correlation is large, the ambiguity biases are then also most likely to occur.

In Sect. 1, Theorem 1, we also established the relation between the ambiguity-baseline correlation and the gain numbers. Hence, it is also possible to link the volume of the search space to the precision of the baseline before and after ambiguity fixing. This is shown in Theorem 4, by relating the volume of the search space to the volumes of the two confidence ellipsoids of the baseline.

Theorem 4 (*Volume*, *correlation*, *and gain*)

Let $V_{2(m-1)}$ be the volume of the geometry-based dual-frequency ambiguity search space based on m satellites,

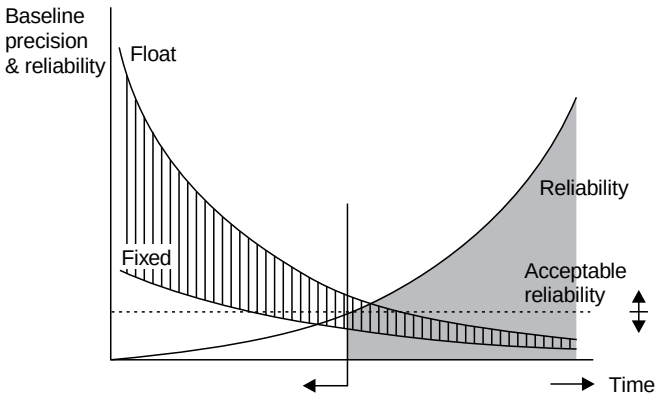

Fig. 1. Baseline precision and ambiguity reliability versus time

let $\rho_{ab,i}$ be the ambiguity-baseline correlation coefficients and let $Q_{\hat{b}}(\phi,p)$ and $Q_{\check{b}}(\phi,p)$ be the variance matrices of the floated and fixed baselines. Then

$$(i)\ V_{2(m-1)} = \frac{m}{(m-1)!}\left(\frac{\pi\chi^2}{\sqrt{\alpha_1\alpha_2}\lambda_1\lambda_2 k}\right)^{m-1}\Pi_{i=1}^{3}\frac{1}{\sqrt{1-\rho_{ab,i}^2}}\ ,$$

$$(ii)\ V_{2(m-1)} = \frac{m}{(m-1)!}\left(\frac{\pi\chi^2}{\sqrt{\alpha_1\alpha_2}\lambda_1\lambda_2 k}\right)^{m-1}\sqrt{\frac{|\,Q_{\hat{b}}(\phi,p)\,|}{|\,Q_{\check{b}}(\phi,p)\,|}}\ .$$

Proof: See Appendix. □

Both expressions hold for the dual-frequency case, but similar expressions can be given for the single-frequency case as well. Similar expressions can also be given for the time-averaged and the geometry-free model.

The first expression relates the volume of the search space to the ambiguity-baseline correlation and the second relates the same volume to the volumes of the two confidence ellipsoids of the baseline. The square root of the determinantal ratio is namely also the ratio of volumes of the confidence ellipsoid of the baseline before and after fixing.

Both expressions show two distinct components, one component that depends on the receiver-satellite geometry and one that does not. The first expression clearly shows that the volume blows up when the correlation coefficients approach 1. The only possible way that this effect can be counteracted is by having the triple product multiplied by a very small number. And as the theorem shows, it is here where we see the very beneficial role of having dual-frequency data, with very precise carrier-phases, tracking as many satellites as possible.

The second expression, apart from showing a similar structure to the first, is of importance in its own right. It features the relation between ambiguity *reliability* and baseline *precision*. Hence it captures in one simple formula two of the most essential aspects of GPS ambiguity resolution. For the purpose of having a successful validation, we need the volume to be *as small as possible*. At the same time however, we also would like the gain in baseline precision and thus the volume ratio of the baseline confidence ellipsoids to be *as large as possible*. According to the preceding formula, these are clearly two conflicting demands which cannot be met separately. Since validation takes priority over the gain, we have to be satisfied with a smaller gain in baseline precision. After all, it does not make sense to have a baseline, which is very precise but which otherwise is completely off target.

These two aspects, ambiguity reliability and gain in baseline precision, are shown in the sketch of Fig. 1. The figure shows, as the observation time progresses, that both the floated baseline and the fixed baseline improve in precision, but that the difference between the two, and thus the gain, gets smaller. The figure also shows that the ambiguity reliability improves with time. Depending on the model used and the application at hand, it may happen that the required ambiguity reliability occurs at a point in time for which the gain is so small that the precision of the floated baseline is practically as good as the precision of the fixed baseline. In that case, ambiguity resolution, although successful in itself, will not help much in improving the baseline precision. The whole art of ambiguity resolution is therefore to devise models and measurement techniques that simultaneously allow for (*a*) a sufficiently large ambiguity reliability; (*b*) a sufficiently large precision gain, and (*c*) a sufficiently large time-span for the float solution to overcome the gain. It is the strength of the GPS system, that this is indeed feasible for many important applications and that, for the models considered in this contribution, it can be established within a time-frame that also allows very economical surveying. And as the volume formula shows, the typical GPS characteristics of having very precise phase data and the possibility of tracking quite a number of satellites, are main contributors in this respect.

5 Summary

In this contribution we considered the baseline in relation to the ambiguities. It was shown how the baseline-ambiguity correlation depends on the phase-code variance ratio and on the change over time in the receiver-satellite geometry. The conclusion was reached that the baseline and the ambiguities will generally be highly correlated. As a consequence, undetected biases in the integer ambiguities will propagate almost unfiltered into the solution of the fixed baseline, thus emphasizing the importance of ambiguity validation. It was shown how the integer ambiguity bias which has the least chance of being detected, could be found from solving an integer least-squares problem. This allows one to infer a priori whether one is likely to have a successful ambiguity validation or not.

We introduced an ambiguity transformation invariant DOP measure for the ambiguities and showed that it captures both the precision and correlation characteristics of the ambiguities. Closed-form formulae for it were given for the geometry-free model, the time-averaged model, and the geometry-based model. A mutual comparison of these formulae clearly showed for

each of the three single-baseline models the impact of different types of data redundancy.

We also linked the ambiguity DOP measure to the volume of the search space and showed how it could be used to downsize the search space, such that an abundance of unnecessary grid points in the search space is avoided. In order to have a search space with as few unnecessary grid points as possible, while still being able to guarantee that at least one, or if needed, two grid points are contained in it, one needs to use the decorrelated ambiguities instead of the original DD ambiguities. This is due to the high precision and low correlation of the transformed ambiguities.

Finally we showed how the volume is related to the ambiguity-baseline correlation and to the gain in baseline precision. It is revealed by one simple formula, that baseline *precision* and ambiguity *reliability* act as a pair of scales. The gain in baseline precision gets larger the more the baseline and the ambiguities become correlated. Thus ambiguity fixing is most beneficial, precision-wise, when the baseline-ambiguity correlation is large. However, also the volume gets larger when the baseline and the ambiguities become more correlated. Thus a larger baseline-ambiguity correlation also implies less reliable integer ambiguities. In order for ambiguity resolution to be successful as well as beneficial, one thus needs to reach a balance between these two effects. That this is feasible for the three different single-baseline models considered is in a large part due to the high precision of the phase data and to the number of satellites that can be tracked.

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7 Appendix

Proof of Theorem 1 (*Ambiguity-baseline correlation*)
The proof of this theorem consists of two steps. We will first bring $\rho(a,b)$ into a simpler form. For that purpose we introduce the $2(m-1)\times 3$ matrix

$$R = Q_{\hat{a}}^{-1/2} Q_{\hat{a}\hat{b}} Q_{\hat{b}}^{-1/2} \quad . \tag{22}$$

Let its singular value decomposition be given as

$$R = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T \quad . \tag{23}$$

Introducing the reparametrizations

$$a = Q_{\hat{a}}^{-1/2} U u \ , \ b = Q_{\hat{b}}^{-1/2} V v \ ,$$

it follows that

$$a^T Q_{\hat{a}\hat{b}} b = u^T \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} v \ , \ a^T Q_{\hat{a}} a = u^T u \ , \ b^T Q_{\hat{b}} b = v^T v \ .$$

This shows that the correlation coefficients ρ_i follow from solving

$$\rho_i = \max_u \max_v \left| \frac{u^T \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} v}{\sqrt{u^T u v^T v}} \right| = \left| \frac{u_i^T \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} v_i}{\sqrt{u_i^T u_i v_i^T v_i}} \right| \ , \tag{24}$$

subject to

$$u^T u_j = 0 \quad j = 1, \ldots, (i-1) \ ,$$
$$v^T v_j = 0 \quad j = 1, \ldots, (i-1) \ .$$

In order to solve Eq. (24), we still need to derive the entries of the diagonal matrix Σ. It follows from Eqs. (22) and (23) that

$$Q_{\hat{b}}^{-1/2} Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} Q_{\hat{b}}^{-1/2} = V \Sigma^2 V^T \quad . \tag{25}$$

Since we know that the variance matrix of the fixed baseline is related to the variance matrix of the floated baseline as

$$Q_{\check{b}} = Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} \ ,$$

it follows that

$$Q_{\hat{b}}^{-1/2} Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} Q_{\hat{b}}^{-1/2} = I_3 - Q_{\hat{b}}^{-1/2} Q_{\check{b}} Q_{\hat{b}}^{-1/2} \quad .$$

With Eq.(25), this gives

$$V \Sigma^2 V^T = Q_{\hat{b}}^{-1/2} [I_3 - Q_{\check{b}} Q_{\hat{b}}^{-1}] Q_{\hat{b}}^{1/2} \quad . \tag{26}$$

We know from Theorem 7 of *Part I* that

$$Q_{\hat{b}} = [F(\epsilon I_3 + \Gamma^{-1}) F^T]^{-1} \ , \ Q_{\check{b}} = [F(\epsilon + 1) F^T]^{-1} \quad .$$

Substitution into Eq.(26) gives

$$V \Sigma^2 V^T = Q_{\hat{b}}^{-1/2} F^{-T} \left[\frac{1}{\epsilon + 1} (I_3 - \Gamma^{-1}) \right] F^T Q_{\hat{b}}^{1/2} \quad .$$

Hence, it follows from solving the characteristic equation

$$| \ V \Sigma^2 V^T - \lambda I_3 \ | = 0$$

that the diagonal entries of Σ^2 correspond with the diagonal entries of $(I_3 - \Gamma^{-1})/(\epsilon + 1)$. This result, together with Eq.(24) proves the theorem.
End of proof. □

Proof of Corollary 1 (*Baseline to ambiguity bias*)
The maximum of the ratio $\| \nabla \check{b} \|^2 / \| \nabla \check{a} \|^2$ equals the maximum eigenvalue of the generalized eigenvalue problem

$$| \ Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} Q_{\check{b}}^{-1} Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} - \mu Q_{\hat{a}}^{-1} \ | = 0 \ . \tag{27}$$

Since $Q_{\hat{a}|\hat{b}}^{-1} = Q_{\hat{a}}^{-1} + Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} Q_{\check{b}}^{-1} Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}$, Eq.(27) is equivalent to

$$| \ I_{2(m-1)} - (\mu + 1) Q_{\hat{a}}^{-1/2} Q_{\hat{a}|\hat{b}} Q_{\hat{a}}^{-1/2} \ | = 0 \ . \tag{28}$$

But since the nonzero eigenvalues of $Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} Q_{\hat{b}}^{-1} Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} = I_{2(m-1)} - Q_{\hat{a}}^{-1/2} Q_{\hat{a}|\hat{b}} Q_{\hat{a}}^{-1/2}$ are given, according to Theorem 1, as ρ_i^2, it follows with Eq. (28) that

$$\mu_{\max} = \frac{\rho_{\max}^2}{1-\rho_{\max}^2} \quad .$$

End of proof. □

Proof of Theorem 2 (*The baseline BNR*)
The fixed baseline is computed as

$$\check{b} = Q_{\check{b}} \sum_{i=1}^{k} A_i^T D^{+T} d(i) \quad ,$$

with

$$d(i) = \sum_{j=1}^{2} [\alpha_j (D^T \phi_j(i) - \lambda_j \check{a}_j) + \beta_j D^T p_j(i)] \quad .$$

Hence, the baseline bias due to biases in the fixed ambiguities reads

$$\nabla \check{b} = -Q_{\check{b}} k \bar{A}^T D^{+T} \check{\nabla} \quad ,$$

where $\check{\nabla}$ is the weighted average of the L_1 and L_2 ambiguity biases, expressed in units of range rather than cycles. The baseline BNR reads therefore

$$\nabla \check{b}^T Q_{\check{b}}^{-1} \nabla \check{b} = (\alpha_1 + \alpha_2)^2 k^2 \check{\nabla}^T D^+ \bar{A} Q_{\check{b}}^{-1} \bar{A}^T D^{+T} \check{\nabla} \quad . \qquad (29)$$

According to Theorem 7 of *Part I* and Theorem 1 of *Part III*, we have

$$(i)\ Q_{\check{b}} = [F(\epsilon+1)F^T]^{-1} \quad ,$$
$$(ii)\ [(\beta_1+\beta_2)k\bar{A}^T P \bar{A}]^{-1} = [F\epsilon(I_3 - \Gamma^{-1})F^T]^{-1} \quad ,$$
$$(iii)\ U = P\bar{A}F^{-T}[F^{-1}\bar{A}^T P \bar{A} F^{-T}]^{-1/2} \quad .$$

From the last two equations, it follows that

$$D^+ \bar{A} = D^+ U [\, \tfrac{1}{(\alpha_1+\alpha_2)k}(I_3 - \Gamma^{-1})]^{1/2} F^T \quad .$$

This combined with the first equation, shows that

$$D^+ \bar{A} Q_{\check{b}}^{-1} \bar{A}^T D^{+T} = \frac{1}{(\alpha_1+\alpha_2)k} \frac{1}{\epsilon+1} D^+ U (I_3 - \Gamma^{-1}) U^T D^{+T} \quad .$$

Substitution into Eq.(29) proves the theorem.
End of proof. □

Proof of Theorem 3 (*Determinant of ambiguity variance matrix*)
To solve for the determinant of the variance matrix, we start from Theorem 1 of *Part I*. Using this theorem, we may write

$$| Q_{\hat{a}} | = | M_1 \otimes D^T D + M_2 \otimes D^T \bar{A} Q_{\hat{b}}(\phi,p) \bar{A}^T D | \quad , \qquad (30)$$

with

$$M_1 = \mathrm{diag}(1/(\alpha_1 k \lambda_1^2), 1/(\alpha_2 k \lambda_2^2)) \quad ,$$

$$M_2 = \begin{bmatrix} \frac{1}{\lambda_1^2} & \frac{1}{\lambda_1 \lambda_2} \\ \frac{1}{\lambda_1 \lambda_2} & \frac{1}{\lambda_2^2} \end{bmatrix} \quad ,$$

and where '⊗' denotes the Kronecker product. Using the determinant factorization

$$| E - DB^{-1}C \,||\, B | = | E \,||\, B - CE^{-1}D | \quad ,$$

with

$$B := I_2 \otimes I_3 \quad , \qquad C := I_2 \otimes Q_{\hat{b}}(\phi,p) \bar{A}^T D \quad ,$$
$$D := -M_2 \otimes D^T \bar{A} \quad , \qquad E := M_1 \otimes D^T D \quad ,$$

we may write Eq.(30) also as

$$| Q_{\hat{a}} | = | M_1 \otimes D^T D \,||\, I_2 \otimes I_3 + M_1 M_2 \otimes Q_{\hat{b}}(\phi,p) \bar{A}^T P \bar{A} | \quad . \qquad (31)$$

Since

$$| M_1 \otimes D^T D | = | M_1 |^{m-1} | D^T D |^2 \quad ,$$
$$| M_1 | = 1/(\alpha_1 \alpha_2 \lambda_1^2 \lambda_2^2 k^2) \quad ,$$
$$| D^T D | = m \quad ,$$

it follows that

$$| M_1 \otimes D^T D | = m^2 \left(\frac{1}{\alpha_1 k \lambda_1^2} \right)^{m-1} \left(\frac{1}{\alpha_2 k \lambda_2^2} \right)^{m-1} \quad . \qquad (32)$$

In order to solve for the second determinant on the right-hand side of Eq.(31), we make use of the canonical decomposition of Theorem 7 of *Part I*. According to this theorem, we have

$$Q_{\hat{b}}(\phi,p) \bar{A}^T P \bar{A} = \frac{1}{(\beta_1+\beta_2)k} Q_{\hat{b}}(\phi,p) Q_{\hat{b}}(\bar{p})^{-1}$$
$$= F^{-T} \Lambda_3 F^T \quad , \qquad (33)$$

with

$$\Lambda_3 = \mathrm{diag}(\lambda_{3,1}, \lambda_{3,2}, \lambda_{3,3}) \quad ,$$
$$\lambda_{3,i} = \frac{1}{(\alpha_1+\alpha_2)k} \frac{\gamma_i - 1}{\epsilon \gamma_i + 1}, \ i = 1,2,3 \quad .$$

We therefore may write

$$| I_2 \otimes I_3 + M_1 M_2 \otimes Q_{\hat{b}}(\phi,p) \bar{A}^T P \bar{A} |$$
$$= \left| \begin{bmatrix} (I_3 + \alpha_1 k \Lambda_3) & \alpha_1 k \frac{\lambda_1}{\lambda_2} \Lambda_3 \\ \alpha_2 k \frac{\lambda_2}{\lambda_1} \Lambda_3 & (I_3 + \alpha_2 k \Lambda_3) \end{bmatrix} \right| \quad .$$

Since the matrix in this last determinant consists of diagonal matrices only, its determinant is not difficult to compute. As a result, we get

$$| I_2 \otimes I_3 + M_1 M_2 \otimes Q_{\hat{b}}(\phi,p) \bar{A}^T P \bar{A} |$$
$$= \Pi_{i=1}^3 (1 + (\alpha_1+\alpha_2) k \lambda_{3,i})$$
$$= \Pi_{i=1}^3 \left(1 + \frac{\gamma_i - 1}{\epsilon \gamma_i + 1} \right) \quad . \qquad (34)$$

Substitution of Eqs.(32) and (34) into Eq.(31), proves the theorem.
End of proof. □

Proof of Theorem 4 (*Volume, correlation and gain*)
The volume of the dual-frequency ambiguity search space reads

$$V_{2(m-1)} = \frac{(\pi\chi^2)^{m-1}}{(m-1)!}\sqrt{|\ Q_{\hat{a}}\ |}\ , \tag{35}$$

with, according to Theorem 3,

$$|\ Q_{\hat{a}}\ | = m^2\left(\frac{1}{\alpha_1 k\lambda_1^2}\right)^{m-1}\left(\frac{1}{\alpha_2 k\lambda_2^2}\right)^{m-1}\Pi_{i=1}^{3}\frac{(1+\epsilon)\gamma_i}{1+\epsilon\gamma_i}\ . \tag{36}$$

Case (i): From Theorem 1, it follows that

$$\gamma_{4-i} = 1/(1-(1+\epsilon)\rho_{ab,i}^2)\ .$$

Substitution into Eq.(35) proves the first case.
Case (ii): According to Theorem 7 of *Part I*, we have

$$Q_{\hat{b}}(\phi,p) = [F(\epsilon I_3 + \Gamma^{-1})F^T]^{-1}\ ,$$
$$Q_{\check{b}}(\phi,p) = [(1+\epsilon)F\,F^T]^{-1}\ ,$$

from which it follows that

$$\frac{|\ Q_{\hat{b}}(\phi,p)\ |}{|\ Q_{\check{b}}(\phi,p)\ |} = \Pi_{i=1}^{3}\frac{(1+\epsilon)\gamma_i}{1+\epsilon\gamma_i}\ .$$

Substitution into Eq.(36) proves the second case.
End of proof. □

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