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# Global Stabilization of Switched Control Systems with Time Delay

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## Abstract

In this paper, the stabilization problem of switched control systems with time delay is investigated for both linear and nonlinear cases. First, a new global stabilizability concept with respect to state feedback and switching law is given. Then, based on multiple Lyapunov functions and delay inequalities, the state feedback controller and the switching law are devised to make sure that the resulting closed-loop switched control systems with time delay are globally asymptotically stable and exponentially stable.

**Key words:** Switched control systems; stabilizability; time delay.

## 1 Introduction

A switched system is a hybrid system comprised of continuous-time or discrete-time subsystems and a rule that supervises the switching between subsystems. Switched systems can be found in many areas, such as computer science, control systems, electrical engineering and technology, automotive industry, and air traffic management and control [1]-[4]. For switched systems, most important and challenging problems are on stability and stabilization (i.e., is it possible to design (or find) a switching law under which the resulting switched systems are stable?). Hence, the recent focus of switched control

systems is on the design of a switched law and a controller under which the controlled systems are stable. In the last two decades, stability analysis of switched systems and switching control design have attracted considerable attention among control theorists, computer scientists, mathematicians and practicing engineers. Many interesting and important results have been established. See, for example, [3]-[14] and the references therein.

On the other hand, time delay phenomenon exists in many practical switched systems, see, for example, [9]-[10], [14]-[15]. Stability and stabilization problem is also an important and challenging problem for switched systems with time delay. In [15], exponential stability of switched systems with time-delay is established based on average dwell time and Lyapunov functions methods. Using a multiple Lyapunov function method, exponential stability of some special linear switched system is investigated in [16]. Moreover, a linear matrix inequality (LMI) method is applied to study the stability problem of switched systems in [17]-[20]. In [21], asymptotic stability and stabilization of a class of switched control systems is studied, where a delay-dependent stability criterion is formulated in term of LMIs by using quadratic Lyapunov functions and inequality analysis technique. For discrete-time systems, some interesting results can be found in [4], [22] and [23]. Other methods, such as dwell time and average dwell time, are used in the study of switched systems. For example, stability of some slow-switched control systems has been studied in [24], and stabilization problems for switched systems have been discussed in [25]-[28], [38], [39].

For the results mentioned above, the switching signal does not involve time delay. However, in real world, a switched control system may have several controllers and not all the controllers are required to switch at the same time. This class of systems can be described by switched control systems in which the switching signal has time delay. These switched control systems are much more complex than a conventional switched system and hence only few works are available in the literature, such as, [29] and [30], where some sufficient conditions for stability are derived for some switched linear systems with time delay appearing in the switching signal. It appears that no results on the stabilization problem are available in the literature for nonlinear switched control systems with distributed time delays and time delay appearing in the switching signal. This has motivated our research. In addition, our results obtained can be applied to the cases with asynchronous switching in actual operation, such as the stabilization of chemical systems [36] and multi-agent systems [33]-[35]. The asynchronous switching in sys-

tems is always caused by the controller lags of the system.

In this paper, we consider static state feedback control for nonlinear switched system with distributed time delays and with time delay appearing in the switching signal. In this study, the time delay of the state may be different from the time delay appearing in the switching signal of the feedback controller. We first introduce some new concepts on stabilization in relation to controller and switching law. Then, by using the method of Lyapunov functions and delay inequalities, some delay-dependent conditions are derived for the design of state feedback controller and a switching law to guarantee the stability of the resulting closed-loop switched control systems. The advantages of this paper are twofold. First, the system model includes both integral terms, called distributed delay, and asynchronous control time lags. This model can cover most of the existing models for switched linear/nonlinear systems. Second, the stability issue under investigation is with respect to both state feedback control and switching signal, which is a much more general problem than previous results.

The outline of the paper is as follows. Section 2 presents some definitions and some technical lemmas needed for the proof of the main results. The design of a switching law for global asymptotical or exponential stabilizability of linear and nonlinear switched control systems are obtained in Section 3 and Section 4, respectively. Finally, some concluding remarks are made in Section 5.

## 2 Preliminaries

The following switched control system is considered:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u + f_{\sigma(t)}(x(t), x_t), \quad (1)$$

where  $x(t) \in R^n$ ,  $u \in R^n$  is state feedback control,  $A_{\sigma(t)} \in R^{n \times n}$ , and  $B_{\sigma(t)} \in R^{n \times n}$  are state matrices, the right continuous function  $\sigma(t) : R \rightarrow \Theta = \{1, 2, \dots, N\}$  is the switching signal,  $x_t \in C_\tau = \{\varphi | \varphi \in C([- \tau, 0], R^n)\}$ ,  $\tau > 0$  is a constant and  $f_{\sigma(t)} \in C(R^n \times C_\tau, R^n)$  satisfy  $f_{\sigma(t)}(0, 0) = 0$ .

In this paper,  $N_+ = \{1, 2, \dots\}$ ,  $R_+ = [0, +\infty)$ ,  $E$  is an identity matrix of appropriate dimension,  $|A|$  denotes the usual norm of a matrix  $A \in R^{n \times m}$ ,  $\|x_t\| = \sup_{t-\tau \leq \theta \leq t} |x(\theta)|$  denotes the sup norm of the function  $x_t \in C_\tau$ . If  $A \in R^{n \times n}$ ,  $\lambda(A)$  denotes eigenvalue of  $A$ . Let  $\sigma = \{(i_1, t_1), \dots, (i_k, t_k), \dots\}$

be the switching law, meaning that when  $t \in [t_k, t_{k+1})$ , the  $i_{k+1}^{th}$  subsystem is active, i.e.

$$\dot{x}(t) = A_{k+1}x(t) + B_{k+1}u + f_{k+1}(x(t), x_t), \quad t \in [t_k, t_{k+1}).$$

Let  $x(t, t_0, \varphi, \sigma)$  be the solution of system (1) under the switching law  $\sigma$ , starting from  $(t_0, \varphi)$ .

It is well known that, a switching law often plays an important role in the study of the stability of a switched system. Even if the subsystems are stable, different switching laws may results in totally different properties of the overall switched systems. In this paper, our focus is on to design a controller  $u$  and a switching law  $\sigma$  so that the resulting closed-loop switched systems are guaranteed to possess desired properties.

Next, we propose some concepts on the stabilization for the switched control system.

**Definition 2.1.** *Consider the switched control system (1).*

- 1) *It is said to be stabilizable with respect to (w.r.t) the state feedback control  $u$  and the switching law  $\sigma$  (SWUS) if it is stable under the state feedback  $u$  and the switching law  $\sigma$ . That is, for any  $\varepsilon > 0$ , there exists a  $\delta > 0$ , such that for any  $\varphi \in C_\tau$ ,  $\|\varphi\| < \delta$  implies  $\|x(t, t_0, \varphi, \sigma)\| < \varepsilon$ .*
- 2) *It is said to be asymptotical stabilizable under the feedback control  $u$  and the switching law  $\sigma$  (ASWUS) if it is SWUS and there exists a  $\delta > 0$  such that for any  $\varphi \in C_\tau$ ,  $\|\varphi\| < \delta$  implies*

$$\lim_{t \rightarrow +\infty} x(t, t_0, \varphi, \sigma) = 0.$$

- 3) *It is said to be globally asymptotical stabilizable under the feedback control  $u$  and the switching law  $\sigma$  (GASWUS) if it is ASWUS and for any  $\varphi \in C_\tau$ , has*

$$\lim_{t \rightarrow +\infty} x(t, t_0, \varphi, \sigma) = 0.$$

- 4) *It is said to be exponential stabilizable under the feedback control  $u$  and the switching law  $\sigma$  (ESWUS) if there exist constants  $M > 0$ ,  $\delta > 0$ ,  $\lambda > 0$ , such that for any  $\varphi \in C_\tau$ ,  $\|\varphi\| < \delta$  implies*

$$|x(t, t_0, \varphi, \sigma)| \leq Me^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

- 5) It is said to be globally exponential stabilizable under the feedback control  $u$  and the switching law  $\sigma$  (GESWUS) if there exist constants  $M > 0$ ,  $\lambda > 0$ , such that for any  $\varphi \in C_\tau$ , it holds that

$$|x(t, t_0, \varphi, \sigma)| \leq M \|\varphi\| e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

Without loss of generality, we choose, in this paper,  $t_0 = 0$  and denote  $x(t, 0, \varphi, \sigma) := x(t)$ .

To study the stabilizability of system (1), we need the following lemmas.

**Lemma 2.1.** (Gronwall inequality) Suppose that  $g, u \in C([t_0, t_1], R_+)$ , and that  $c$  is a nonnegative real constant. If

$$u(t) \leq c + \int_{t_0}^t g(s)u(s)ds, \quad t \in [t_0, t_1],$$

then

$$u(t) \leq ce^{\int_{t_0}^t g(s)ds}.$$

**Lemma 2.2.**<sup>[31]</sup> Suppose that  $u \in C^1([t_0 - \tau, \infty), R_+)$ ,  $\alpha > \beta > 0$  are constants and that

$$\dot{u}(t) \leq -\alpha u(t) + \beta \|u_t\|, \quad t \geq t_0.$$

Then, there exists a  $\hat{\lambda} > 0$  such that

$$u(t) \leq \|u_{t_0}\| e^{-\hat{\lambda}(t-t_0)},$$

where  $\hat{\lambda}$  is the unique positive solution of the equation  $\hat{\lambda} = \alpha - \beta e^{\hat{\lambda}\tau}$ .

**Lemma 2.3.** Suppose that  $u \in C^1([t_0 - \tau, \infty), R_+)$ ,  $0 < \alpha \leq \beta$  are constants and that

$$\dot{u}(t) \leq -\alpha u(t) + \beta \|u_t\|, \quad t \geq t_0.$$

Then

$$u(t) \leq \|u_{t_0}\| e^{(\beta e^{\alpha\tau} - \alpha)(t-t_0)}.$$

**Proof.** From Lemma 2.2, we have

$$\frac{d}{dt} e^{\alpha(t-t_0)} u(t) \leq e^{\alpha(t-t_0)} \beta \|u_t\|.$$

Integrating both sides gives

$$\begin{aligned} e^{\alpha(t-t_0)}u(t) &\leq u(t_0) + \beta \int_{t_0}^t e^{\alpha(s-t_0)}\|u_s\|ds \\ &\leq u(t_0) + \beta \int_{t_0}^t e^{\alpha\tau} \sup_{s-\tau \leq \theta \leq s} [e^{\alpha(\theta-t_0)}u(\theta)]ds. \end{aligned}$$

Since the right hand side of the inequality above is an increasing function, we have

$$\sup_{t-\tau \leq \theta \leq t} [e^{\alpha(\theta-t_0)}u(\theta)] \leq \|u_{t_0}\| + \beta \int_{t_0}^t e^{\alpha\tau} \sup_{s-\tau \leq \theta \leq s} [e^{\alpha(\theta-t_0)}u(\theta)]ds.$$

By Gronwall inequality, it follows that

$$\sup_{t-\tau \leq \theta \leq t} [e^{\alpha(\theta-t_0)}u(\theta)] \leq \|u_{t_0}\| e^{\beta e^{\alpha\tau}(t-t_0)}.$$

Since

$$\sup_{t-\tau \leq \theta \leq t} [e^{\alpha(\theta-t_0)}u(\theta)] \geq e^{\alpha(t-t_0)}u(t),$$

we have

$$u(t) \leq \|u_{t_0}\| e^{(\beta e^{\alpha\tau} - \alpha)(t-t_0)}.$$

□

**Lemma 2.4.** <sup>[32]</sup> Suppose that  $Z \in R^{n \times n}$  is a positive definite matrix,  $\epsilon > 0$  is a scalar and  $x, y \in R^n$ . Then

$$2x^T y \leq x^T Z^{-1}x + y^T Z y,$$

and, as a special case,

$$2x^T y \leq \epsilon^{-1}x^T x + \epsilon y^T y.$$

### 3 Linear Switched Control Systems

In this section, we consider a linear switched control system, where the linear state feedback control is given by  $u = K_{\gamma(t)}x$  and  $f_{\sigma(t)}(x(t), x_t) =$

$C_{\sigma(t)} \int_{t-\tau}^t x(s)ds$ , with  $\gamma(t) = \sigma(t - \tau)$  and  $C_{\sigma(t)} \in R^{n \times n}$ . Then, the resulting switched system (1) becomes:

$$\begin{cases} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u + C_{\sigma(t)} \int_{t-\tau}^t x(s)ds, \\ u(t) &= K_{\gamma(t)}x(t), \\ \gamma(t) &= \sigma(t - \tau), \end{cases}$$

The closed-loop system may be written more specifically as:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma(t-\tau)}x(t) + C_{\sigma(t)} \int_{t-\tau}^t x(s)ds. \quad (2)$$

System (2) in form of intero-differential equations contains not only integral terms but also delayed switching controllers. These controllers' description is consistent with that for the asynchronous switching controller in [36]. In addition, it can be seen that system (2) can cover the switching linear systems with asynchronous switching described in [36]. Our aim is to choose a control matrix  $K_{\sigma(t-\tau)}$  and a switching law  $\sigma$  such that the resulting switched system (2) is GASWUS or GESWUS.

To continue, we assume that the following conditions are satisfied.

**Assumption  $\mathbb{A}_1$**

- i) There exist  $\alpha > 0$ ,  $\beta > 0$ , and  $M \geq 1$  such that  $\|e^{(A_i+B_iK_i)t}\| \leq Me^{-\alpha t}$ , and  $\|e^{\|A_i+B_iK_j\|t}\| \leq Me^{-\beta t}$  hold for all  $i \neq j, i, j \in \Theta$ ;
- ii)  $\exists C > 0$  such that  $|C_i| \leq C$  for all  $i \in \Theta$ , and  $-\lambda \triangleq CM\tau e^{\alpha\tau} - \alpha < 0$ ;
- iii) For the switching law  $\sigma = \{(i_1, t_1), \dots, (i_m, t_m), \dots\}$ , it holds that  $t_m - t_{m-1} \geq 2\tau, m \in N_+$ .

Now, we have the following result.

**Theorem 3.1.** *Suppose that Assumption  $\mathbb{A}_1$  holds. Let  $M_m := m(2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \lambda))$ ,  $m \in N_+$ . Then, the switched control system (2) is:*

- (a) *SWUS, if there exists a constant  $\tilde{M}$  such that the switching law  $\sigma$  satisfies  $M_m - \lambda t_m \leq \tilde{M}$ ,  $m \in N_+$ ;*
- (b) *GASWUS, if  $\lim_{m \rightarrow \infty} (M_m - \lambda t_m) = -\infty$ ;*
- (c) *GESWUS, if  $\limsup_{m \rightarrow \infty} \frac{M_m - \lambda t_m}{t_m} < 0$ .*



**Proof.** Consider the closed-loop switched control system (2). For a given switching signal  $\sigma = \{(i_1, t_1), \dots, (i_m, t_m), \dots\}$ , when  $t \in [0, t_1)$ , the  $i_1^{th}$  subsystem is active, meaning that

$$\dot{x}(t) = A_{i_1}x(t) + B_{i_1}K_{i_1}x(t) + C_{i_1} \int_{t-\tau}^t x(s)ds.$$

Its solution is

$$x(t) = e^{(A_{i_1}+B_{i_1}K_{i_1})t}x(0) + \int_0^t e^{(A_{i_1}+B_{i_1}K_{i_1})(t-s)} \int_{s-\tau}^s C_{i_1}x(\psi)d\psi ds.$$

Taking the norm on both sides, we

obtain

$$\|x(t)\| \leq Me^{-\alpha t}\|x(0)\| + CM\tau \int_0^t e^{-\alpha(t-s)}\|x_s\|ds.$$

Multiplying both sides by  $e^{\alpha t}$ , we have

$$e^{\alpha t}\|x(t)\| \leq M\|x(0)\| + CM\tau \int_0^t e^{\alpha s}\|x_s\|ds.$$

Since the right hand side of the inequality above is an increasing function, it follows that

$$\begin{aligned} e^{\alpha t}\|x_t\| &\leq e^{\alpha\tau} \sup_{t-\tau \leq \theta \leq t} e^{\alpha\theta}\|x(\theta)\| \\ &\leq Me^{\alpha\tau}\|\varphi\| + e^{\alpha\tau}CM\tau \int_0^t e^{\alpha s}\|x_s\|ds. \end{aligned}$$

Thus, by Lemma 2.1, we obtain

$$\|x_t\| \leq Me^{\alpha\tau}\|\varphi\|e^{-\lambda t}, t \in [0, t_1).$$

From this inequality, it follows from the continuity property of the solution that

$$\|x_{t_1}\| \leq Me^{\alpha\tau}\|\varphi\|e^{-\lambda t_1}.$$

Now, for the case  $t \in [t_{m-1}, t_{m-1} + \tau)$ , where  $m \in N_+$ , and  $m > 1$ . The active subsystem is:

$$\dot{x}(t) = A_{i_m}x(t) + B_{i_m}K_{i_{m-1}}x(t) + C_{i_m} \int_{t-\tau}^t x(s)ds$$

and for  $t \in [t_{m-1} + \tau, t_m)$ , the active subsystem is:

$$\dot{x}(t) = A_{i_m} x(t) + B_{i_m} K_{i_m} x(t) + C_{i_m} \int_{t-\tau}^t x(s) ds.$$

By a similar argument, we can show that

$$\|x_t\| \leq \begin{cases} M e^{\beta\tau} \|x_{t_{m-1}}\| e^{(CM\tau e^{\beta\tau} - \beta)(t - t_{m-1})}, & t \in [t_{m-1}, t_{m-1} + \tau), \\ M e^{\alpha\tau} \|x_{t_{m-1} + \tau}\| e^{-\lambda(t - t_{m-1} - \tau)}, & t \in [t_{m-1} + \tau, t_m) \end{cases}$$

and

$$\|x_{t_{m-1} + \tau}\| \leq M \|x_{t_{m-1}}\| e^{CM\tau e^{\beta\tau}}, \quad (3)$$

$$\|x_{t_m}\| \leq e^{(\alpha + \lambda)\tau} M \|x_{t_{m-1} + \tau}\| e^{-\lambda(t_m - t_{m-1})}. \quad (4)$$

Then, it follows from (3) and (4) that

$$\|x_t\| \leq \begin{cases} e^{M_m} \|\varphi\| e^{-\lambda t_m} e^{CM\tau e^{\beta\tau} - \beta|\tau + \beta\tau}, & t \in [t_m, t_m + \tau), \\ e^{M_m} \|\varphi\| e^{-\lambda t} e^{\alpha\tau}, & t \in [t_m + \tau, t_{m+1}). \end{cases}$$

In conclusion, for  $t \in [t_m, t_{m+1})$ , where  $m = 1, 2, \dots$ , we have

$$\|x_t\| \leq \hat{M} \|\varphi\| e^{M_m - \lambda t_m} e^{-\lambda(t - t_m)}, \quad (5)$$

where  $\hat{M} \in R_+$  is a positive constant.

By (5), it can be shown that

- (a) If the switching law satisfies  $M_m - \lambda t_m \leq \tilde{M}$ ,  $m \in N_+$ , then (5) implies  $\|x_t\| \leq \hat{M} \|\varphi\| e^{\tilde{M}}$ ,  $t \geq 0$ , meaning that (2) is SWUS;
- (b) If the switching law satisfies  $\lim_{m \rightarrow \infty} (M_m - \lambda t_m) = -\infty$ , then (2) is SWUS and (5) implies  $\lim_{t \rightarrow \infty} |x(t, 0, \varphi, \sigma)| = 0$ , for all  $\varphi \in C_\tau$ , meaning that (2) is GSWUS;
- (c) If the switching law satisfies  $\limsup_{m \rightarrow \infty} \frac{M_m - \lambda t_m}{t_m} < 0$ , then there exist  $\varepsilon > 0$  and  $N \in N_+$  such that for all  $m \geq N$ ,  $M_m - \lambda t_m < -\varepsilon t_m$ , (without loss of generality, we may choose  $\varepsilon \leq \lambda$ ). Thus, there exists a positive constant  $\Pi$  such that  $\|x_t\| \leq \Pi \|\varphi\| e^{-\varepsilon t}$ ,  $t \geq 0$ , meaning that (2) is GESWUS.

□

**Corollary 3.1.** *Suppose that the conditions of Theorem 3.1 hold. Then the switched control system (2) is:*

(a) *SWUS, if the switching law  $\sigma$  satisfies*

$$t_m - t_{m-1} \geq \frac{2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau)}{\lambda}, \quad m \in N_+;$$

(b) *GASWUS, if the switching law  $\sigma$  satisfies*

$$t_m - t_{m-1} \geq \frac{2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau)}{\lambda} + \frac{1}{m}, \quad m \in N_+;$$

(c) *GESWUS, if the switching law  $\sigma$  satisfies*

$$\limsup_{m \rightarrow \infty} \frac{2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau)}{t_m - t_{m-1}} = \lambda - \varepsilon,$$

where  $\varepsilon > 0$  is a constant.

**Proof.** For  $t \in [t_m, t_{m+1})$ , from (5), it follows that

$$\begin{aligned} \|x_t\| &\leq \hat{M} \|\varphi\| e^{M_m - \lambda t_m} e^{-\lambda(t-t_m)} \\ &\leq \hat{M} \|\varphi\| e^{\sum_{i=1}^m [2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau) - \lambda(t_i - t_{i-1})]} e^{-\lambda(t-t_m)}. \end{aligned}$$

Thus, it holds that

(a) If the switching law satisfies

$$t_m - t_{m-1} \geq \frac{2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau)}{\lambda}, \quad m \in N_+,$$

then

$$\sum_{i=1}^m [2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau) - \lambda(t_i - t_{i-1})] \leq 0$$

and

$$\|x_i\| \leq \hat{M} \|\varphi\|.$$

This implies that the switched control system (2) is SWUS.

(b) If the switching law satisfies

$$t_m - t_{m-1} \geq \frac{2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau)}{\lambda} + \frac{1}{m}, \quad m \in N_+,$$

then

$$\sum_{i=1}^m [2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau) - \lambda(t_i - t_{i-1})] \leq -\lambda \sum_{i=1}^m \frac{1}{i}$$

and for all  $\varphi \in C_\tau$ ,

$$\lim_{t \rightarrow \infty} |x(t; 0, \varphi, \sigma)| = 0.$$

This means that the switched control system (2) is GASWUS.

(c) If the switching law satisfies

$$\limsup_{m \rightarrow \infty} \frac{2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau)}{t_m - t_{m-1}} = \lambda - \varepsilon,$$

then there exists a  $N \in N_+$ , such that for  $m \geq N$ ,

$$\frac{2 \ln M + \tau(CMe^{\beta\tau} + \alpha + \tau) - \lambda(t_m - t_{m-1})}{t_m - t_{m-1}} \leq -\frac{\varepsilon}{2},$$

and thus

$$\|x_t\| \leq \mathbb{M} \hat{M} \|\varphi\| e^{-\frac{\varepsilon}{2}t},$$

where  $\mathbb{M}$  is a positive number. This implies that the switched control system (2) is GESWUS.

□

## 4 Nonlinear Systems

In this section, we consider the following nonlinear switched control system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + f_{\sigma(t)}(x(t), \int_{t-\tau_1}^t x(s)ds). \quad (6)$$

The aim is to design a switching law and a state feedback controller

$$u(t) = K_{\gamma(t)}x(t), \quad t \geq 0,$$

where  $K_{\gamma(t)} \in R^{n \times n}$  and  $\gamma(t)$  is the detection function of  $\sigma(t)$ , (i.e.,  $\gamma(t) = \sigma(t - \tau_2)$ ), such that the closed-loop switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma(t-\tau_2)}x(t) + f_{\sigma(t)}(x(t), \int_{t-\tau_1}^t x(s)ds) \quad (7)$$

is globally asymptotically stable or globally exponentially stable, where  $\tau_1, \tau_2 \geq 0$ .

To continue, let  $x(t; t_0, \varphi)$  be the solution of the system,

$$\dot{x}(t) = Ax(t) + f(x(t), \int_{t-\tau}^t x(s)ds), \quad (8)$$

with  $x_{t_0} = \varphi$ ,  $\varphi \in C_\tau$ .

**Lemma 4.1.** *Suppose that there exist positive numbers  $\epsilon, \lambda_1, \lambda_2$  and matrices  $E_1 = E_1^T, E_2 \in R^{n \times n}$ , such that*

$$2x^T(t)f(x(t), \int_{t-\tau}^t x(s)ds) \leq x^T(t)E_1x(t) + \int_{t-\tau}^t x^T(t)E_2x(s)ds.$$

$$\lambda(A + A^T + E_1 + \tau\epsilon^{-1}E) \leq -\lambda_1, \lambda(E_2^T E_2) \leq \lambda_2.$$

where  $\lambda(\cdot)$  stands for the maximum eigenvalue of the corresponding matrix.

Then

(i)  $\lambda_1 > \tau\epsilon\lambda_2 \geq 0$  implies

$$|x(t)| \leq \|\varphi\|e^{-\lambda(t-t_0)}, \quad (9)$$

where  $\lambda$  is the unique positive solution of the equation  $2\lambda = \lambda_1 - \tau\epsilon\lambda_2 e^{2\lambda\tau}$ .

(ii)  $0 \leq \lambda_1 \leq \tau\epsilon\lambda_2$  implies

$$|x(t)| \leq \|\varphi\|e^{\frac{1}{2}(\tau\epsilon\lambda_2 e^{\lambda_1\tau} - \lambda_1)(t-t_0)}. \quad (10)$$

**Proof.** Let  $V = x^T x$ . Then, by Lemma 2.4, we have

$$\begin{aligned}
\dot{V}|_{(8)} &= x^T(t)(A + A^T)x(t) + 2x^T(t)f(x(t), \int_{t-\tau}^t x(s)ds) \\
&\leq x^T(t)(A + A^T)x(t) + x^T(t)E_1x(t) + \int_{t-\tau}^t x^T(t)E_2x(s)ds \\
&\leq x^T(t)(A + A^T + E_1)x(t) + \int_{t-\tau}^t [\epsilon^{-1}x^T(t)x(t) + \epsilon x^T(s)E_2^T E_2x(s)]ds \\
&\leq x^T(A + A^T + E_1 + \tau\epsilon^{-1}E)x(t) + \epsilon \int_{t-\tau}^t x^T(s)E_2^T E_2x(s)ds \\
&\leq -\lambda_1 V(t) + \tau\epsilon\lambda_2 \|V_t\|.
\end{aligned}$$

If  $\lambda_1 > \tau\epsilon\lambda_2 \geq 0$ , then it follows from Lemma 2.2 that

$$V(t) \leq \|V_{t_0}\| e^{-2\lambda(t-t_0)},$$

which shows the validity of (9).

If  $\lambda_1 \leq \tau\epsilon\lambda_2$ , then it follows from Lemma 2.3 that

$$|V(t)| \leq \|V_{t_0}\| e^{(\tau\epsilon\lambda_2 e^{\lambda_1\tau} - \lambda_1)(t-t_0)},$$

and hence

$$|x(t)| \leq \|\varphi\| e^{\frac{1}{2}(\tau\epsilon\lambda_2 e^{\lambda_1\tau} - \lambda_1)(t-t_0)}.$$

□

Now, consider system (7) under the following assumptions.

**Assumption  $\mathbb{A}_2$**

- i) There exist positive numbers  $\epsilon_i, \lambda_{1i}, \lambda_{2i}, \lambda_{3i}$  and matrices  $E_{1i} = E_{1i}^T, E_{2i} \in R^{n \times n}, i \in \Theta$ , such that for all  $i \in \Theta$

$$2x^T(t)f_i(x(t), \int_{t-\tau_1}^t x(s)ds) \leq x^T(t)E_{1i}x(t) + \int_{t-\tau_1}^t x^T(t)E_{2i}x(s)ds;$$

- ii)  $\lambda((A_i + B_i K_i)^T + (A_i + B_i K_i) + E_{1i} + \tau_1 \epsilon_i^{-1} E) \leq -\lambda_{1i},$   
 $\lambda((A_i + B_i K_j)^T + (A_i + B_i K_j) + E_{1i} + \tau_1 \epsilon_i^{-1} E) \leq -\lambda_{2i}, \quad i \neq j,$   
 $\lambda(E_{2i}^T E_{2i}) \leq \lambda_{3i}, \quad i, j \in \Theta.$

## 4.1 The case $\tau_1 \geq \tau_2$

**Theorem 4.1.** *Suppose that Assumption  $\mathbb{A}_2$  hold and that*

- (i)  $\lambda_{1i} - \tau_1 \epsilon_i \lambda_{3i} > 0$ ,  $\lambda_{(i)}$  is the unique positive solution of the equation  $2\lambda_{(i)} = \lambda_{1i} - \tau_1 \epsilon_i \lambda_{3i} e^{2\lambda_{(i)}\tau}$  and  $\lambda = \min_{i \in \Theta} \{\lambda_{(i)}\}$ ;
- (ii)  $\lambda_{2i} - \tau_1 \epsilon_i \lambda_{3i} > 0$ ,  $\hat{\lambda}_{(i)}$  is the unique positive solution of the equation  $2\hat{\lambda}_{(i)} = \lambda_{2i} - \tau_1 \epsilon_i \lambda_{3i} e^{2\hat{\lambda}_{(i)}\tau}$  and  $\hat{\lambda} = \min_{i \in \Theta} \{\hat{\lambda}_{(i)}\}$ ; and
- (iii) the switching law  $\sigma = \{(i_1, t_1), \dots, (i_k, t_k), \dots\}$  satisfies  $t_k - t_{k-1} \geq \tau_1 + \tau_2$ .

Then, the switched control system (7) is GESWUS.

**Proof.** Let  $\sigma(t)$  be a given switching signal. For the case when  $t \in [0, t_1)$ , the  $i_1^{th}$  subsystem is active, i.e.,

$$\dot{x}(t) = A_{i_1}x(t) + B_{i_1}K_{i_1}x(t) + f_{i_1}(x(t), \int_{t-\tau_1}^t x(s)ds),$$

Then, by the conditions of the theorem and Lemma 4.1(i), it follows that

$$|x(t)| \leq \|\varphi\|e^{-\lambda t}.$$

We claim that, for any  $m \in N_+$  and  $t \in [t_m, t_{m+1})$ , the solution of system (7) satisfies:

$$(*)_1 : |x(t)| \leq \|\varphi\|e^{-\lambda(t_m - m\tau_1)}e^{-\hat{\lambda}(t - t_m)}, \quad t \in [t_m, t_m + \tau_2),$$

$$(*)_2 : |x(t)| \leq \|\varphi\|e^{-\lambda(t - (m+1)\tau_1)}, \quad t \in [t_m + \tau_2, t_{m+1}).$$

Indeed, when  $t \in [t_1, t_1 + \tau_2)$ , the active subsystem is:

$$\dot{x}(t) = A_{i_2}x(t) + B_{i_2}K_{i_1}x(t) + f_{i_2}(x(t), \int_{t-\tau_1}^t x(s)ds).$$

Thus, by the conditions of theorem and Lemma 4.1(i), we obtain

$$|x(t)| \leq \|x_{t_1}\|e^{-\hat{\lambda}(t-t_1)} \leq \|\varphi\|e^{-\lambda(t_1-\tau_1)}e^{-\hat{\lambda}(t-t_1)}, \quad t \in [t_1, t_1 + \tau_2).$$

For  $t \in [t_1 + \tau_2, t_2)$ , we have

$$\begin{aligned}
|x(t)| &\leq \|x_{t_1+\tau_2}\|_\tau e^{-\lambda(t-t_1-\tau_2)} \\
&= \max\{\sup_{t_1+\tau_2-\tau_1 \leq \theta \leq t_1} |x(\theta)|, \sup_{t_1 \leq \theta \leq t_1+\tau_2} |x(\theta)|\} e^{-\lambda(t-t_1-\tau_2)} \\
&\leq \max\{\|\varphi\| e^{-\lambda(t_1+\tau_2-\tau_1)}, \|x_{t_1}\|\} e^{-\lambda(t-t_1-\tau_2)} \\
&\leq \max\{\|\varphi\| e^{-\lambda(t_1+\tau_2-\tau_1)}, \|\varphi\| e^{-\lambda(t_1-\tau_1)}\} e^{-\lambda(t-t_1-\tau_2)} \\
&= \|\varphi\| e^{-\lambda(t-\tau_2-\tau_1)} \leq \|\varphi\| e^{-\lambda(t-2\tau_1)}.
\end{aligned}$$

It means that  $(*)_1$  and  $(*)_2$  hold for  $m = 1$ .

Suppose that  $(*)_1$  and  $(*)_2$  hold for  $m \in N_+$  and we will show that  $(*)_1$  and  $(*)_2$  hold for  $m + 1$ .

For  $t \in [t_{m+1}, t_{m+1} + \tau_2)$ , we have

$$\begin{aligned}
|x(t)| &\leq \|x_{t_{m+1}}\| e^{-\lambda(t-t_{m+1})} \\
&\leq \|x_0\| e^{-\lambda(t_{m+1}-(m+1)\tau_1)} e^{-\lambda(t-t_{m+1})}.
\end{aligned}$$

On this basis, it follows that for  $t \in [t_{m+1} + \tau_2, t_{m+2})$ ,

$$\begin{aligned}
|x(t)| &\leq \|x_{t_{m+1}+\tau_2}\| e^{-\lambda(t-t_{m+1}-\tau_2)} \\
&\leq \max\{\sup_{t_{m+1}+\tau_2-\tau_1 \leq \theta \leq t_{m+1}} |x(\theta)|, \sup_{t_{m+1} \leq \theta \leq t_{m+1}+\tau_2} |x(\theta)|\} e^{-\lambda(t-t_{m+1}-\tau_2)} \\
&\leq \max\{\|\varphi\| e^{-\lambda(t_{m+1}+\tau_2-(m+2)\tau_1)}, \|\varphi\| e^{-\lambda(t_{m+1}-(m+1)\tau_1)}\} e^{-\lambda(t-t_{m+1}-\tau_2)} \\
&= \|\varphi\| e^{-\lambda(t-(m+2)\tau_1)}.
\end{aligned}$$

This means that  $(*)_1$  and  $(*)_2$  hold for  $m + 1$ . By mathematical induction principle, we conclude that for all  $m \in N_+$ ,  $(*)_1$  and  $(*)_2$  hold.

Now, from  $(*)_1$  and  $(*)_2$ , we have

$$|x(t)| \leq \|\varphi\| e^{\lambda\tau_1} e^{-\lambda(t-m\tau_1)}, \quad t \in [t_m, t_{m+1}). \quad (11)$$

Since the switching law satisfies  $t_k - t_{k-1} \geq \tau_1 + \tau_2$ , thus

$$\liminf_{k \rightarrow \infty} \frac{t_k - k\tau_1}{t_k} > 0,$$



so there exist  $\varepsilon > 0$  and  $N \in N_+$ , such that for  $k \geq N$ ,  $t_k - k\tau_1 > \varepsilon t_k$ . Thus

$$-\lambda(t_k - k\tau_1) < -\lambda\varepsilon t_k, k \geq N.$$

and hence

$$\begin{aligned} |x(t)| &\leq \|\varphi\| e^{\lambda\tau_1} e^{-\lambda(t-m\tau_1)} \\ &\leq \|\varphi\| e^{\lambda\tau_1} e^{-\lambda(t_m-m\tau_1)} e^{-\lambda(t-t_m)} \\ &\leq \|\varphi\| e^{\lambda\tau_1} e^{-\lambda\varepsilon t_m} e^{-\lambda(t-t_m)}, \quad m \geq N \end{aligned}$$

Therefore, the conclusion of the theorem follows readily.  $\square$

**Corollary 4.1.** *Suppose that the conditions of Theorem 4.1 hold and that  $\tau_2 = 0$ . Then, the switched control system (7) is:*

- a) *SWUS, if the switching law  $\sigma$  satisfies  $\lim_{k \rightarrow \infty} (t_k - k\tau_1) < \infty$ ;*
- b) *GASWUS, if the switching law  $\sigma$  satisfies  $\lim_{k \rightarrow \infty} (t_k - k\tau_1) = \infty$ ;*
- c) *GESWUS, if the switching law  $\sigma$  satisfies  $\liminf_{k \rightarrow \infty} \frac{t_k - k\tau_1}{t_k} > 0$ .*

**Corollary 4.2.** *Suppose that the conditions of Theorem 4.1 hold. Then, the switched control system (7) is GESWUS, if the switching law  $\sigma$  satisfies*

$$\liminf_{m \rightarrow \infty} \frac{t_m - t_{m-1} - \tau_1}{t_m - t_{m-1}} = \varepsilon > 0.$$

**Proof.** From (11), it follows that

$$\begin{aligned} |x(t)| &\leq \|\varphi\| e^{\lambda\tau_1} e^{-\lambda(t-m\tau_1)} \\ &\leq \|\varphi\| e^{-\sum_{i=1}^m \lambda(t_i - t_{i-1} - \tau_1)} e^{-\lambda(t-t_m - \tau_1)}. \end{aligned}$$

Since the switching law satisfies

$$\liminf_{m \rightarrow \infty} \frac{t_m - t_{m-1} - \tau_1}{t_m - t_{m-1}} = \varepsilon,$$

there exists an  $N \in N_+$ , such that for  $m \geq N$  and  $\lambda > 0$ ,

$$-\lambda \frac{t_m - t_{m-1} - \tau_1}{t_m - t_{m-1}} \leq -\frac{\varepsilon}{2} \lambda,$$

and hence

$$\begin{aligned}
\|x_t\| &\leq \|\varphi\| e^{-\lambda \sum_{i=1}^m (t_i - t_{i-1} - \tau_1)} e^{-\lambda(t - t_m - \tau_1)} \\
&\leq \|\varphi\| e^{-\lambda \sum_{i=1}^N (t_i - t_{i-1} - \tau_1)} e^{-\lambda \sum_{i=N}^m (t_i - t_{i-1} - \tau_1)} e^{-\lambda(t - t_m - \tau_1)} \\
&\leq \|\varphi\| e^{-\lambda \sum_{i=1}^N (t_i - t_{i-1} - \tau_1)} e^{-\frac{\varepsilon}{2} \lambda (t_m - t_N)} e^{-\lambda(t - t_m - \tau_1)} \\
&\leq \mathbb{M} \|\varphi\| e^{-\frac{\varepsilon}{2} \lambda t},
\end{aligned}$$

where  $\mathbb{M}$  is a positive number. Thus the switched system is GESWUS.  $\square$

Using the similar proof method of Theorem 4.1 and Lemma 4.1(ii), we can prove the following results.

**Theorem 4.2.** *Suppose that Assumption  $\mathbb{A}_2$  holds and that*

- (i)  $\lambda_{1i} - \tau_1 \epsilon_i \lambda_{3i} > 0$  and  $\lambda_{(i)}$  is the unique positive solution of the equation  $2\lambda_{(i)} = \lambda_{1i} - \tau_1 \epsilon_i \lambda_{3i} e^{2\lambda_{(i)} \tau}$  and  $\lambda = \min_{i \in \Theta} \{\lambda_{(i)}\}$ ;
- (ii)  $\lambda_{2i} - \tau_1 \epsilon_i \lambda_{3i} \leq 0$ ,  $i \in \Theta$ ; and
- (iii) the switching law  $\sigma = \{(i_1, t_1), \dots, (i_k, t_k), \dots\}$  satisfies  $t_k - t_{k-1} \geq \tau_1 + \tau_2$ .

Then, the switched control system (7) is:

- a) SWUS, if there exists  $M \in \mathbb{R}$ , such that for all  $m \in \mathbb{N}_+$ ,  $m(\Delta + \lambda \tau_2) - \lambda(t - m\tau_1) \leq M$ ;
- b) GASWUS, if  $\lim_{m \rightarrow \infty} [m(\Delta + \lambda \tau_2) - \lambda(t - m\tau_1)] = -\infty$ ; and
- c) GESWUS, if  $\liminf_{k \rightarrow \infty} \frac{m(\Delta + \lambda \tau_2) - \lambda(t - m\tau_1)}{t} < 0$ ,

where  $2\Delta = \max_{i \in \Theta} \{(\lambda_{3i} \epsilon_i \tau_1 e^{\lambda_{2i} \tau_1} - \lambda_{2i}) \tau_2\}$ .

**Corollary 4.3.** *Suppose that the conditions of Theorem 4.2 hold. Then, the switched control system (7) is:*

- a) SWUS, if there exists  $M > 0$  such that  $\sum_{k=1}^m (t_k - t_{k-1} - \tau_1 - \tau_2 - \frac{\Delta}{\lambda}) \geq -M$ ;
- b) GASWUS, if  $\lim_{k \rightarrow \infty} \sum_{k=1}^m (t_k - t_{k-1} - \tau_1 - \tau_2 - \frac{\Delta}{\lambda}) = +\infty$ ; and
- c) GESWUS, if  $\liminf_{k \rightarrow \infty} \frac{\sum_{k=1}^m (t_k - t_{k-1} - \tau_1 - \tau_2 - \frac{\Delta}{\lambda})}{t} > 0$ ,

where  $2\Delta = \max_{i \in \Theta} \{(\lambda_{3i} \epsilon_i \tau_1 e^{\lambda_{2i} \tau_1} - \lambda_{2i}) \tau_2\}$ .

## 4.2 The case of $\tau_2 \geq \tau_1$

To begin, we notice that since  $\tau_2 \geq \tau_1$ ,  $\|x_t\| = \sup_{t-\tau_2 \leq \theta \leq t} |x(\theta)|$ .  
By similar argument, we can prove following results.

**Theorem 4.3.** *Suppose that Assumption  $\mathbb{A}_2$  holds and that*

- (i)  $\lambda_{1i} - \tau_1 \epsilon_i \lambda_{3i} > 0$  and  $\lambda_{(i)}$  is the unique positive solution of the equation  $2\lambda_{(i)} = \lambda_{1i} - \tau_1 \epsilon_i \lambda_{3i} e^{2\lambda_{(i)}\tau}$  and  $\lambda = \min_{i \in \Theta} \{\lambda_{(i)}\}$ ; and
- (ii)  $\lambda_{2i} - \tau_1 \epsilon_i \lambda_{3i} > 0$  and  $\hat{\lambda}_{(i)}$  is the unique positive solution of the equation  $2\hat{\lambda}_{(i)} = \lambda_{2i} - \tau_1 \epsilon_i \lambda_{3i} e^{2\hat{\lambda}_{(i)}\tau}$  and  $\hat{\lambda} = \min_{i \in \Theta} \{\hat{\lambda}_{(i)}\}$ .

Then, the switched control system (7) is:

- a) *SWUS*, if the switching law  $\sigma = \{(i_1, t_1), \dots, (i_m, t_m), \dots\}$  satisfies  $\tau_2 \leq t_m - t_{m-1} \leq 2\tau_2$ , for  $m \in N_+$ ;
- b) *GASWUS*, if the switching law  $\sigma = \{(i_1, t_1), \dots, (i_m, t_m), \dots\}$  satisfies  $t_m - t_{m-1} \geq 2\tau_2$  for  $m \in N_+$  and  $\lim_{m \rightarrow \infty} (t_m - 2m\tau_2) = \infty$ ; and
- c) *GESWUS*, if the switching law  $\sigma = \{(i_1, t_1), \dots, (i_m, t_m), \dots\}$  satisfies  $t_m - t_{m-1} \geq 2\tau_2$ ,  $m \in N_+$  and  $\liminf_{m \rightarrow \infty} \frac{t_m - 2m\tau_2}{t_m} > 0$ .

**Proof.** Let  $\sigma = \{(i_1, t_1), \dots, (i_m, t_m), \dots\}$  be a given a switching signal. For  $t \in [0, t_1)$ , the  $i_1^{th}$  subsystem is active, i.e.,

$$\dot{x}(t) = A_{i_1}x(t) + B_{i_1}K_{i_1}x(t) + f_{i_1}(x(t), \int_{t-\tau_1}^t x(s)ds),$$

Thus, by the conditions of the theorem and Lemma 4.1(i), it is clear that

$$|x(t)| \leq \|\varphi\|e^{-\lambda t}.$$

- a) We claim that, for any  $m \in N_+$ , the solution of system (7) satisfies:

$$(*)_3: |x(t)| \leq \|\varphi\|e^{-\lambda(t_1-\tau_2)}e^{-\hat{\lambda}(t-t_m)}, \quad t \in [t_m, t_m + \tau_2),$$

$$(*)_4: |x(t)| \leq \|\varphi\|e^{-\lambda(t_1-2\tau_2)}e^{-\lambda(t-t_m)}, \quad t \in [t_m + \tau_2, t_{m+1}).$$

Indeed, for  $t \in [t_1, t_1 + \tau_2)$ , the active subsystem is:

$$\dot{x}(t) = A_{i_2}x(t) + B_{i_2}K_{i_1}x(t) + f_{i_2}(x(t), \int_{t-\tau_1}^t x(s)ds),$$

Thus, by the conditions of the theorem and Lemma 4.1(i), we obtain

$$|x(t)| \leq \|x_{t_1}\| e^{-\hat{\lambda}(t-t_1)} = \|\varphi\| e^{-\lambda(t_1-\tau_2)} e^{-\hat{\lambda}(t-t_1)}.$$

Now, for  $t \in [t_1 + \tau_2, t_2)$ , we have

$$\begin{aligned} |x(t)| &\leq \|x_{t_1+\tau_2}\| e^{-\lambda(t-t_1-\tau_2)} \\ &= \sup_{t_1 \leq \theta \leq t_1+\tau_2} \|\varphi\| e^{-\lambda(t_1-\tau_2)} e^{-\hat{\lambda}(\theta-t_1)} e^{-\lambda(t-t_1-\tau_2)} \\ &= \|\varphi\| e^{-\lambda(t_1-2\tau_2)} e^{-\lambda(t-t_1)}. \end{aligned}$$

It means that  $(*)_3$  and  $(*)_4$  hold for  $m = 1$ .

Suppose that  $(*)_3$  and  $(*)_4$  hold for  $m \in N_+$  and we shall show prove that  $(*)_3$  and  $(*)_4$  hold for  $m + 1$ .

For  $t \in [t_{m+1}, t_{m+1} + \tau_2)$ , we have

$$\begin{aligned} |x(t)| &\leq \|x_{t_{m+1}}\| e^{-\hat{\lambda}(t-t_{m+1})} \\ &= \sup_{t_{m+1}-\tau_2 \leq \theta \leq t_{m+1}} \|x(\theta)\| e^{-\hat{\lambda}(t-t_{m+1})} \\ &= \max\{\sup_{t_{m+1}-\tau_2 \leq \theta \leq t_m+\tau_2} \|x(\theta)\|, \sup_{t_m+\tau_2 \leq \theta \leq t_{m+1}} \|x(\theta)\|\} e^{-\hat{\lambda}(t-t_{m+1})} \\ &= \max\{\sup_{t_{m+1}-\tau_2 \leq \theta \leq t_m+\tau_2} \|\varphi\| e^{-\lambda(t_1-\tau_2)} e^{-\hat{\lambda}(\theta-t_m)}, \\ &\quad \sup_{t_m+\tau_2 \leq \theta \leq t_{m+1}} \|\varphi\| e^{-\lambda(t_1-2\tau_2)} e^{-\lambda(\theta-t_m)}\} e^{-\hat{\lambda}(t-t_{m+1})} \\ &\leq \max\{\|\varphi\| e^{-\lambda(t_1-\tau_2)} e^{-\hat{\lambda}(t_{m+1}-\tau_2-t_m)}, \|\varphi\| e^{-\lambda(t_1-\tau_2)}\} e^{-\hat{\lambda}(t-t_{m+1})} \\ &\leq \|\varphi\| e^{-\lambda(t_1-\tau_2)} e^{-\hat{\lambda}(t-t_{m+1})}. \end{aligned}$$

Now, for  $t \in [t_{m+1} + \tau_2, t_{m+2})$ , we obtain

$$\begin{aligned} |x(t)| &\leq \|x_{t_{m+1}+\tau_2}\| e^{-\lambda(t-t_{m+1}-\tau_2)} \\ &= \sup_{t_{m+1} \leq \theta \leq t_{m+1}+\tau_2} \|\varphi\| e^{-\lambda(t_1-\tau_2)} e^{-\hat{\lambda}(\theta-t_{m+1})} e^{-\lambda(t-t_{m+1}-\tau_2)} \\ &\leq \|\varphi\| e^{-\lambda(t_1-\tau_2)} e^{-\lambda(t-t_{m+1}-\tau_2)} \\ &\leq \|\varphi\| e^{-\lambda(t_1-2\tau_2)} e^{-\lambda(t-t_{m+1})}. \end{aligned}$$

This means that for  $m + 1$ ,  $(*)_3$  and  $(*)_4$  hold. Thus, by mathematical induction principle, it is clear that for all  $m \in N_+$ ,  $(*)_3$  and  $(*)_4$  hold.

From  $(*)_3$  and  $(*)_4$ , it follows that

$$|x(t)| \leq \|\varphi\| e^{-\lambda(t_1 - \tau_2)}, \quad t \in [t_m, t_{m+1}),$$

which implies that the the switched control system (7) is SWUS.

b) In this case, the switching law  $\sigma$  satisfies the condition  $t_m - t_{m-1} \geq 2\tau_2$ .

We claim that, for any  $m \in N_+$ , the solution of system (7) satisfies:

$$(*)_{3_1} : |x(t)| \leq \|\varphi\| e^{-\lambda(t_m - 2m\tau_2 + \tau_2)} e^{-\hat{\lambda}(t - t_m)}, \quad t \in [t_m, t_m + \tau_2),$$

$$(*)_{4_1} : |x(t)| \leq \|\varphi\| e^{-\lambda(t - 2m\tau_2)}, \quad t \in [t_m + \tau_2, t_{m+1}).$$

Indeed, for  $t \in [t_1, t_1 + \tau_2)$ , the active subsystem is:

$$\dot{x}(t) = A_{i_2}x(t) + B_{i_2}K_{i_1}x(t) + f_{i_2}(x(t), \int_{t-\tau_1}^t x(s)ds),$$

Thus, by the conditions of the theorem and Lemma 4.1(i), it follows that

$$|x(t)| \leq \|x_{t_1}\| e^{-\hat{\lambda}(t - t_1)} \leq \|\varphi\| e^{-\lambda(t_1 - \tau_2)} e^{-\hat{\lambda}(t - t_1)}.$$

Now, for  $t \in [t_1 + \tau_2, t_2)$ , we have

$$\begin{aligned} |x(t)| &\leq \|x_{t_1 + \tau_2}\| e^{-\lambda(t - t_1 - \tau_2)} \\ &= \sup_{t_1 \leq \theta \leq t_1 + \tau_2} \|\varphi\| e^{-\lambda(t_1 - \tau_2)} e^{-\hat{\lambda}(\theta - t_1)} e^{-\lambda(t - t_1 - \tau_2)} \\ &\leq \|\varphi\| e^{-\lambda(t_1 - \tau_2)} e^{-\lambda(t - t_1 - \tau_2)} = \|\varphi\| e^{-\lambda(t - 2\tau_2)}. \end{aligned}$$

It means that  $(*)_{3_1}$  and  $(*)_{4_1}$  hold for  $m = 1$ .

Suppose that  $(*)_{3_1}$  and  $(*)_{4_1}$  hold for  $m \in N_+$  and we shall show that  $(*)_{3_1}$  and  $(*)_{4_1}$  hold for  $m + 1$ .

For  $t \in [t_{m+1}, t_{m+1} + \tau_2)$ , we have

$$\begin{aligned}
|x(t)| &\leq \|x_{t_{m+1}}\| e^{-\hat{\lambda}(t-t_{m+1})} \\
&= \sup_{t_{m+1}-\tau_2 \leq \theta \leq t_{m+1}} \|x(\theta)\| e^{-\hat{\lambda}(t-t_{m+1})} \\
&\leq \sup_{t_{m+1}-\tau_2 \leq \theta \leq t_{m+1}} \|\varphi\| e^{-\lambda(\theta-2m\tau_2)} e^{-\hat{\lambda}(t-t_{m+1})} \\
&\leq \|\varphi\| e^{-\lambda(t_{m+1}-2(m+1)\tau_2+\tau_2)} e^{-\hat{\lambda}(t-t_{m+1})}.
\end{aligned}$$

Now, for  $t \in [t_{m+1} + \tau_2, t_{m+2})$ , it follows that

$$\begin{aligned}
|x(t)| &\leq \|x_{t_{m+1}+\tau_2}\| e^{-\lambda(t-t_{m+1}-\tau_2)} \\
&= \sup_{t_{m+1} \leq \theta \leq t_{m+1}+\tau_2} \|\varphi\| e^{-\lambda(t_{m+1}-2(m+1)\tau_2+\tau_2)} e^{-\hat{\lambda}(\theta-t_{m+1})} e^{-\lambda(t-t_{m+1}-\tau_2)} \\
&\leq \|\varphi\| e^{-\lambda(t_{m+1}-2(m+1)\tau_2+\tau_2)} e^{-\lambda(t-t_{m+1}-\tau_2)} \\
&\leq \|\varphi\| e^{-\lambda(t-2(m+1)\tau_2)}.
\end{aligned}$$

This means that for  $m+1$ ,  $(*)_{3_1}$  and  $(*)_{4_1}$  hold. Therefore, by mathematical induction principle, it is clear that for all  $m \in N_+$ ,  $(*)_{3_1}$  and  $(*)_{4_1}$  hold.

From  $(*)_{3_1}$  and  $(*)_{4_1}$ , we can show that

$$|x(t)| \leq \|\varphi\| e^{-\lambda(t-2m\tau_2)}, \quad t \in [t_m, t_{m+1}), \quad (12)$$

which implies that the switched control system (7) is GASWUS.

c) In this case, the switching law  $\sigma$  satisfies the condition  $t_m - t_{m-1} \geq 2\tau_2$ .

It follows from (12) that

$$|x(t)| \leq \|\varphi\| e^{-\lambda(t-2m\tau_2)}, \quad t \in [t_m, t_{m+1}).$$

If

$$\liminf_{k \rightarrow \infty} \frac{t_m - 2m\tau_2}{t_m} > 0,$$

then there exist  $\varepsilon > 0$  and  $N \in N_+$ , such that for all  $m > N$ , have

$$\frac{t_m - 2m\tau_2}{t_m} \geq \frac{\varepsilon}{\lambda} \Rightarrow -\lambda(t_m - 2m\tau_2) \leq -\varepsilon t_m$$

and hence (without loss of generality, we may choose  $\varepsilon \leq \lambda$ )

$$\begin{aligned} |x(t)| &\leq \|\varphi\| e^{-\lambda(t-2m\tau_2)} = \|\varphi\| e^{-\lambda(t_m-2m\tau_2)} e^{-\lambda(t-t_m)} \\ &\leq \|\varphi\| e^{-\varepsilon t_m} e^{-\varepsilon(t-t_m)} = \|\varphi\| e^{-\varepsilon t}. \end{aligned}$$

Thus, the switched control system (7) is GESWUS. □

**Corollary 4.4.** *Suppose that the conditions of Theorem 4.3 hold. Then, the switched control system (7) is:*

- (a) *GASWUS, if the switching law  $\sigma$  satisfies  $t_m - t_{m-1} \geq 2\tau_2 + \frac{1}{m}$ ,  $m \in N_+$ ; and*
- (b) *GESWUS, if the switching law  $\sigma$  satisfies*

$$\frac{t_m - t_{m-1} - 2\tau_2}{t_m - t_{m-1}} \geq \varepsilon > 0.$$

Using the similar proof method of Theorem 4.3 and Lemma 4.1(ii), we can prove the following result:

**Theorem 4.4.** *Suppose that Assumption  $\mathbb{A}_2$  holds and that*

- (i)  $\lambda_{1i} - \tau_1 \epsilon_i \lambda_{3i} > 0$  and  $\lambda_{(i)}$  is the unique positive solution of the equation  $2\lambda_{(i)} = \lambda_{1i} - \tau_1 \epsilon_i \lambda_{3i} e^{-2\lambda_{(i)}\tau}$  and  $\lambda = \min_{i \in \Theta} \{\lambda_{(i)}\}$ ;
- (ii)  $\lambda_{2i} - \tau_1 \epsilon_i \lambda_{3i} \leq 0$ ,  $i \in \Theta$ ; and
- (iii) Switching law  $\sigma = \{(i_1, t_1), \dots, (i_k, t_k), \dots\}$  satisfies  $t_k - t_{k-1} \geq 2\tau_2$ .

Then, the switched control system (7) is:

- a) *SWUS, if there exists a  $M > 0$  such that for all  $m \in N_+$ ,  $-\lambda t_m + m(2\lambda + \Delta)\tau_2 < M$ ;*
- b) *GASWUS, if  $\lim_{m \rightarrow \infty} -\lambda t_m + m(2\lambda + \Delta)\tau_2 = -\infty$ ;*
- c) *GESWUS, if  $\liminf_{t \rightarrow \infty} \frac{-\lambda t_m + m(2\lambda + \Delta)\tau_2}{t_m} < 0$ ,  $t \in [t_m, t_{m+1})$ , where  $2\Delta = \max_{i \in \Theta} \{(\lambda_{3i} \epsilon_i \tau_1 e^{\lambda_{2i} \tau_1} - \lambda_{2i})\tau_2\}$ .*

**Corollary 4.5.** *Suppose that the conditions of Theorem 4.4 hold. Then, the switched control system (7) is:*

(a) *SWUS, if the switching law  $\sigma$  satisfies*

$$\lambda(t_m - t_{m-1}) - (2\lambda + \Delta)\tau_2 \geq 0, \quad m \in N_+;$$

(b) *GASWUS, if the switching law  $\sigma$  satisfies*

$$\lambda(t_m - t_{m-1}) \geq (2\lambda + \Delta)\tau_2 + \frac{1}{m}, \quad m \in N_+;$$

(c) *GESWUS, if there exist  $\varepsilon > 0, N \in N_+$  such that for all  $m \geq N$ , the switching law  $\sigma$  satisfies*

$$\frac{\lambda(t_m - t_{m-1}) - (2\lambda + \Delta)\tau_2}{t_m - t_{m-1}} \geq \varepsilon.$$

## 5 Conclusion

In this paper, we study the stabilization problem of switched control systems with switching signal time delay. New concepts of globally asymptotical or exponential stabilizability under state feedback controllers and switching laws are presented. Then, by using the method of Lyapunov functions and delay inequalities, appropriate state feedback controllers and switching laws are devised under which the resulting closed-loop switched systems are globally asymptotically stable and exponentially stable.

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