# Global Convergence Analysis for the NIC Flow

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***Abstract*—Recently, a family of fast subspace tracking algorithms based on a novel information criterion (NIC) has been proposed and investigated. It is known that these new algorithms are associated with a new kind of flow, which will be called the NIC flow in this paper, as in the case of the conventional Oja subspace algorithms with the Oja flow. In this paper, some fundamental questions about this new NIC flow, such as its solution existence and convergence, will be investigated. In addition, the convergence domain will be characterized. Some important results on these issues are obtained via manifold theory.**

***Index Terms*—Global convergence, matrix differential equation, novel information criterion, Oja flow.**

## I. INTRODUCTION

IT is well-known that the convergence analysis of a class of two-layer linear neural networks is closely associated with principal subspace analysis (PSA) [3]. Recently, the subspace algorithm [4], [5], the symmetric error correction algorithm [6], and the back propagation algorithm [7] have been individually investigated. It has been shown in [13] that all these algorithms, which are collectively referred to as the Oja algorithm, are closely related. Much attention has been paid to the initial convergence analysis of the Oja algorithm and its flow [19], [20]. The thorough analysis for this flow with its variants are conducted in [10], [11], and [21] via differential Riccati equation.

In fact, a properly chosen criterion is a very important part in developing new learning algorithms. The two well-known performance functions associated with the subspace algorithm are variance (VAR) function and mean-square-error (MSE) function, which are defined in the sequel. Both of these functions are quadratic functions. A different version of Oja's algorithm (the least mean error algorithm) was presented in [13]. Recently, a new algorithm based on the novel information criterion (NIC) is proposed in [1]. In contrast with the previously mentioned algorithms, the NIC is nonquadratic and has a steep peak around the global maximum. Different algorithms based on the NIC also have been developed, and these algorithms have some advantages over the Oja algorithm, as demonstrated in the simulations of [1].

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All the NIC-based algorithms share the same dynamic flow, which will be called NIC flow in this paper. Although a few properties for the NIC flow have been obtained in [1], some important convergence issues remain open. For example, the convergence domain has not been clearly characterized in [1]. Initial convergence analysis has been done there via Lyapunov approach, and it is still not clear whether the NIC flow converges to a point. In this paper, all of these issues will be investigated. Moreover, the asymptotic convergence will also be addressed when the flow approaches the center manifold. The machinery used here is manifold theory, which is a powerful tool for analyzing nonlinear dynamic systems.

The rest of the paper is organized as follows: Section II introduces some notations and problem formulation. In Section III, we will present some basic knowledge on center manifold theory used in the sequel. The main results will be given in Section IV. Some discussions and conclusions are given in Section V. An Appendix is attached to present some basic concepts on topological space and smooth manifolds.

## II. PRELIMINARIES

### A. Notations and Acronyms

Some notational symbols and acronyms used in this paper are listed below.

| | |
|---|---|
| $\mathcal{E}$ | Expectation of a stochastic variable. |
| $\mathcal{R}^{r\times m}(\mathcal{R}^r)$ | Set of all $r\times m$ real matrices ($r$-dimensional real vectors). |
| $A^{\mathrm{T}}$ | Tranpose of a matrix $A\in\mathcal{R}^{r\times m}$. |
| $tr(A)$ | Trace of $A\in\mathcal{R}^{r\times r}$. |
| $\mathrm{rank}(A)$ | Rank of $A\in\mathcal{R}^{r\times m}$. |
| $\dim(\mathcal{N})$ | Dimension of space $\mathcal{N}$. |
| $\mathrm{Re}(A)$ | Real part pf eigenvalues of matrix $A\in\mathcal{R}^{r\times r}$. |
| $\log(A)$ | Natural logarithm of a symmetric positive definite matrix $A$. |
| $\det(A)$ | Determinant of a matrix $A\in\mathcal{R}^{r\times r}$. |
| $A>(\geq)B$ | Difference matrix $A-B$ is positive (nonnegative) definite. |
| $I_r$ | $r\times r$ identity matrix. |
| $\mathbf{0}$ | Null matrix or vector. |
| $\mathrm{diag}(d_1,d_2,\ldots,d_r)$ | Diagonal matrix with diagonal elements $d_1,d_2,\ldots,d_r$. |
| $T_p\mathcal{N}$ | Tangent space to smooth manifold $\mathcal{N}$ at point $p$. |

In addition, the following acronyms are used in this paper.

| | |
|---|---|
| ODE | Ordinary differential equation. |
| VAR | Variance. |

| | |
|---|---|
| MIC | Mutual information criterion. |
| NIC | Novel information criterion. |
| PSA | Principal subspace analysis. |
| PCA | Principal component analysis. |
| MSE | Mean square error. |
| LMSE | Least mean square error. |

### *B. Problem Formulation*

Suppose $\{\mathbf{x}_k, k = 1,2,\ldots\}$ is a stationary random vector sequences with zero mean and the covariance matrix $\mathbf{R} = \mathcal{E}\{\mathbf{x}_k\mathbf{x}_k^{\mathrm{T}}\} \in \mathcal{R}^{n\times n}$, where $\mathcal{E}$ denotes expectation. $\mathbf{R}$ is assumed to be positive definite. Assume that the orthogonal eigenvectors of $\mathbf{R}$ are $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ in such a way that the corresponding eigenvalues $\lambda_i, i = 1,2,\ldots n$ are in descending order:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} \geq \cdots \geq \lambda_n > 0.$$

Usually, $\mathbf{x}_k$ may be thought of consisting of $r$ independent signal components embedded in an $n$-dimensional noise signal with $r < n$, and the last $n-r$ eigenvalues are caused by noise. The purpose of signal processing is to remove the signal from noise via principle component analysis (PCA) or principal subspace analysis (PSA). The PSA will give an optimal solution to this kind of problem in the sense that it minimizes the mean square error (MSE) between $\mathbf{x}_k$ and its reconstruction or equivalently maximizes the variance of $\mathbf{y}_k$ defined by

$$\mathbf{y}_k = \mathbf{W}^{\mathrm{T}}\mathbf{x}_k \tag{1}$$

in the hope that the columns of the optimal weight matrix $\mathbf{W} \in \mathcal{R}^{n\times r}$ span the same space as $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_r\}$. Conventionally, the PSA is formulated into either of the following two optimization problems:

1) Solve the following maximization problem:

$$\begin{aligned}\max_{\mathbf{W}}\{J_{\mathrm{VAR}}(\mathbf{W})\} &= \max_{\mathbf{W}} \frac{1}{2}\mathcal{E}\left\{\mathbf{y}_k\mathbf{y}_k^{\mathrm{T}}\right\} \\ &= \max_{\mathbf{W}} \frac{1}{2}\mathrm{tr}(\mathbf{W}^{\mathrm{T}}\mathbf{R}\mathbf{W})\end{aligned} \tag{2}$$

with an orthogonality constraint on $\mathbf{W}$

$$\mathbf{W}^{\mathrm{T}}\mathbf{W} = \mathbf{I}_r.$$

2) Solve the following minimization problem:

$$\begin{aligned}&\min_{\mathbf{W}}\{J_{\mathrm{MSE}}(\mathbf{W})\} = \min_{\mathbf{W}} \frac{1}{2}\mathcal{E}\|\mathbf{x}_k - \mathbf{W}\mathbf{y}_k\|^2 \\ &= \min_{\mathbf{W}} \frac{1}{2}\left[\mathrm{tr}(\mathbf{R}) - \mathrm{tr}(2\mathbf{W}^{\mathrm{T}}\mathbf{R}\mathbf{W} - \mathbf{W}^{\mathrm{T}}\mathbf{R}W\mathbf{W}^{\mathrm{T}}\mathbf{W})\right].\end{aligned} \tag{3}$$

It is shown in [13] that these two approaches are actually equivalent. The main advantage of PSA algorithms based on these two approaches is that $J_{\mathrm{MSE}}$ or $J_{\mathrm{VAR}}$ has a global optimum at the principle subspace with all the other stationary points being saddle ones. A disadvantage is its slow convergence due to the slow stochastic gradient search. In order to overcome this slow convergence, different nonsquare extensions for the performance index function have been proposed, which generally produce robust PCA solutions totally different from the standard ones [14]. One recent nonsquare NIC proposed in [1] is as follows: Given $\mathbf{W}$ in the domain $\{\mathbf{W}|\mathbf{W}^{\mathrm{T}}\mathbf{R}\mathbf{W} > 0\}$, define

$$\begin{aligned}\min_{\mathbf{W}}\{J_{\mathrm{NIC}}(\mathbf{W})\} &= \min_{\mathbf{W}} \frac{1}{2} \\ &\times \left\{\mathrm{tr}(\mathbf{W}^{\mathrm{T}}\mathbf{W}) - \mathrm{tr}\left[\log(\mathbf{W}^{\mathrm{T}}\mathbf{R}\mathbf{W})\right]\right\}\end{aligned} \tag{4}$$

where the matrix logarithm is defined in [12], and $\mathbf{R} \in \mathcal{R}^{n\times n}$ is a semi-positive symmetric matrix with eigenvalues as follows:

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > \lambda_{r+1} \geq \cdots \geq \lambda_n \geq 0.$$

It should be noted that $J_{\mathrm{NIC}}(\mathbf{W})$ defined in (4) is a negation of the NIC defined in [1] since the minimization is used here instead of maximization. This will not affect the correctness of results in this paper. This criterion is different from all existing criteria and is closely related to the mutual information criterion (MIC), which is given by

$$J_{\mathrm{MIC}}(\mathbf{W}) = \frac{1}{2}\left\{\mathrm{tr}\left[\log(\sigma^2\mathbf{W}^{\mathrm{T}}\mathbf{W})\right] - \mathrm{tr}\left[\log(\mathbf{W}^{\mathrm{T}}\mathbf{R}\mathbf{W})\right]\right\}$$

where $\sigma^2$ is the variance of a Gaussian noise signal uncorrelated with $\mathbf{x}_k$ [15], [16]. Although NIC and MIC are similar in appearance, their significant difference has been shown in [1]. In addition, different versions of learning algorithms based on the performance index (4) for minimizing the NIC are proposed in [1]. The convergence of the proposed algorithms are closely related to some convergence properties of their dynamic flow, which is given by

$$\dot{\mathbf{W}}(t) = \mathbf{R}\mathbf{W}(t)\left[\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t)\right]^{-1} - \mathbf{W}(t). \tag{5}$$

The analysis for this flow given in [1] has not addressed several important questions on the global property such as the solution existence of the flow, convergence set characterization, and convergence domain characterization, etc.

To see the difference of the NIC flow and the Oja flow, let us rewrite the NIC flow as

$$\begin{aligned}\dot{\mathbf{W}}(t) =& \mathbf{R}\mathbf{W}(t)\left[\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t)\right]^{-1} - \mathbf{W}(t) \\ =& \left[\mathbf{R}\mathbf{W}(t) - \mathbf{W}(t)\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t)\right] \\ &\times \left[\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t)\right]^{-1}.\end{aligned} \tag{6}$$

Comparing the above equation with the Oja flow, which is given by

$$\dot{\mathbf{W}}(t) = \mathbf{R}\mathbf{W}(t) - \mathbf{W}(t)\left[\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t)\right] \tag{7}$$

one can see that an extra factor

$$\left[\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t)\right]^{-1}$$

appeared in the right side of proposed differential (6). Later, we will show that this term is bounded along the trajectory $\mathbf{W}(t)$ of the NIC flow.

## III. Stability Analysis of Dynamic Systems

The center manifold theory is used in this paper due to the fact that the convergence analysis in [1] could not tell us under what conditions the NIC flow will converge to a point. Further, the convergence domain characterization can be partly given via center manifold theory here. It will be shown that the results here are more profound and accurate theoretically compared with those reported in [1]. Next, we will present some basic knowledge on center manifold theory, which is a powerful tool for the stability analysis of nonlinear dynamic systems. In order to explain the center manifold theory clearly, many concepts on topological space and smooth manifolds are needed. We put these related knowledge in an Appendix for convenience. See [17], [18], and references therein for more details.

Consider a nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{8}$$

where $\mathbf{f}$ is defined on an open set $U$ of $\mathcal{R}^n$. Let $\mathbf{x}^0 = \mathbf{0}$ be a point of equilibrium for (8), i.e., $\mathbf{f}(\mathbf{x}^0) = \mathbf{0}$. Then, the local asymptotic stability of this point can often be determined by its Jacob matrix given by

$$\mathbf{F} = \left[\frac{\partial f}{\partial x}\right]_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=x^0}.$$

The stability condition can be stated as follows.

i) If all the eigenvalues of $\mathbf{F}$ are in the open left complex plane, then $\mathbf{x} = \mathbf{0}$ is an asymptotically stable equilibrium of (8).
ii) If one or more eigenvalues of $\mathbf{F}$ are in the right half complex plane, then $\mathbf{x} = \mathbf{0}$ is an unstable equilibrium of (8).

When matrix $\mathbf{F}$ has some eigenvalues with zero real parts, the system (8) is usually referred to as a *critical case* for the asymptotic stability analysis. In this case, the center manifold defined below is a very useful tool to the stability analysis.

Suppose that matrix $\mathbf{F}$ has $n^0$ eigenvalues with zero real parts and the other $n^-$ eigenvalues with negative real parts. Then, it is easy to see that the domain of the linear mapping $\mathbf{F}$ can be decomposed into the direct sum of two invariant spaces, which are noted as $\mathbf{E}^0$ and $\mathbf{E}^-$ with dimension $n^0$ and $n^-$, respectively. Regarding the linear mapping $\mathbf{F}$ as a representation of the differential of the nonlinear mapping $\mathbf{f}: U \rightarrow \mathcal{R}^n$, then from Definition 6.7 in the Appendix, its domain is the tangent space $T_0U$ of $U$ at $x = 0$. Thus

$$T_0U = E^0 \bigoplus E^-. \tag{9}$$

*Definition 3.1:* Let $\mathcal{S} \subset U$ be a smooth manifold with $\mathbf{0} \in \mathcal{S}$. $\mathcal{S}$ is said to be a center manifold for (8) at $\mathbf{x} = \mathbf{0}$ if it satisfies the following two conditions.

i) The tangent space to $\mathcal{S}$ at $\mathbf{0}$ is exactly $\mathbf{E}^0$.
ii) For each $\mathbf{x}_0 \in \mathbf{S}$, there exist $t_1 < 0 < t_2$ such that the integral curve $\mathbf{x}(t)$ of (8) satisfying $\mathbf{x}(0) = \mathbf{x}_0$ will satisfy $\mathbf{x}(t) \in \mathcal{S}$ for all $t \in (t_1, t_2)$.

In what follows, we will assume that the matrix $\mathbf{F}$ has all eigenvalues in the closed left half plane because otherwise, $\mathbf{x} = \mathbf{0}$ must be unstable. In this case, one can choose coordinates in $U$ such that system (8) can be decomposed into

$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} + \mathbf{g}(\mathbf{y}, \mathbf{z}) \\ \dot{\mathbf{z}} &= \mathbf{B}\mathbf{z} + \mathbf{h}(\mathbf{y}, \mathbf{z}) \end{aligned} \tag{10}$$

where

| | |
|---|---|
| $\mathbf{A}$ | stable matrix with dimension $n^-$; |
| $\mathbf{B}$ | matrix of dimension $n^0$ having all its eigenvalues with zero real parts; |
| $\mathbf{G}$ and $\mathbf{h}$ | functions vanishing at $(\mathbf{y}, \mathbf{z}) = (\mathbf{0}, \mathbf{0})$ together with all their first derivatives. |

Actually, this can be done by linearizing (8) at $\mathbf{x} = \mathbf{0}$ and then transforming the first order part via matrix transform.

Now, it is time to state the stability result based on center manifold.

*Lemma 3.1 [17]:*

i) There exists a neighborhood $\mathbf{V} \subset \mathcal{R}^{n^0}$ of $\mathbf{z} = \mathbf{0}$ and a mapping $\pi : V \rightarrow R^{n^-}$ such that
$$\mathcal{S} = \left\{(\mathbf{y}, \mathbf{z}) \in \mathcal{R}^{n^-} \times \mathbf{V} : \mathbf{y} = \pi(\mathbf{z})\right\}$$
is a center manifold for (10).
ii) Suppose $\mathbf{y} = \pi(\mathbf{z})$ is a center manifold for (10) at $(\mathbf{0}, \mathbf{0})$. Let $(\mathbf{y}(t), \mathbf{z}(t))$ be a solution of (10). There exists a neighborhood $U^0$ of $(0, 0)$ and real numbers $M > 0$, $K > 0$ such that if $(\mathbf{y}(0), \mathbf{z}(0)) \in U^0$, then
$$\|\mathbf{y}(t) - \pi(\mathbf{z}(t))\| \le Me^{-Kt}\|\mathbf{y}(0) - \pi(\mathbf{z}(0))\|, \quad t \ge 0$$
as long as $(y(t), z(t)) \in U^0$.

*Remark 3.1:* This lemma shows that any trajectory of the system (10) starting at a point sufficiently close to $(\mathbf{0}, \mathbf{0})$ contained in the center manifold will converge to the center manifold as $t \rightarrow \infty$ with exponential decay.

Actually, a center manifold captures the behavior of a flow near an equilibrium point, as reflected in the following reduction principle.

With the center manifold defined above, let $\mathbf{y}(t) = \pi(\eta(t))$ and $\mathbf{z}(t) = \eta(t)$, where $\eta(t)$ satisfies the ODE

$$\dot{\eta}(t) = \mathbf{B}\eta + \mathbf{h}(\pi(\eta), \eta). \tag{11}$$

Now, one can state the following reduction principle, which is frequently used to decide the stability of an equilibrium for a nonlinear dynamic systems.

*Lemma 3.2 [17]:* Suppose $\eta = 0$ is a stable (resp. asymptotic stable, unstable) equilibrium of (11). Then, $(\mathbf{y}, \mathbf{z}) = (\mathbf{0}, \mathbf{0})$ is a stable (resp. asymptotic stable, unstable) equilibrium of (10).

This lemma indicates that the behaviors of the flow on the center manifold will decide the behavior of the flow on the whole space. Combining Lemmas 3.1 and 3.2, one can obtain the following remark, which will be used in the sequel.

*Remark 3.2 [18]:* Let $p \in \mathcal{N}$ be an equilibrium of point of dynamic flow (8), where $\mathcal{N}$ is a smooth manifold on which (8) is defined, and let $n^0$ and $n^-$ denote the numbers of eigenvalues of $\mathbf{F}$ with $\text{Re}(\mathbf{F}) = 0$ and $\text{Re}(\mathbf{F}) < 0$. Then , there exists a homeomorphism $\phi : U \rightarrow \mathcal{R}^n$ from a neighborhood $U \subset \mathcal{N}$ of $p$ onto a neighborhood of $0 \in \mathcal{R}^n$ such that $\phi$ maps integral curves of flow (8) to integral curves of

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{k}(\mathbf{x_1}), \quad \mathbf{x}_1 \in \mathcal{R}^{n^0} \\ \dot{\mathbf{y}} &= -\mathbf{y}, \quad \mathbf{y} \in \mathcal{R}^{n^-}. \end{aligned}$$

Here, the flow of $\dot{\mathbf{x}}_1 = \mathbf{k}(\mathbf{x_1})$ is equivalent to the flow of (8) on a center manifold passing through $p$.

In the next section, we will use these center manifold results to investigate the global convergence of the NIC flow.

## IV. NIC Flow Convergence Analysis

Before investigating some convergence properties for the NIC flow (5), we present a lemma about the convergence property of $W^{\mathrm{T}}(t)W(t)$ along the trajectory (5).

Let us define a region

$$\mathcal{D} \triangleq \left\{ \mathbf{W} \mid \mathbf{W}^{\mathrm{T}}\mathbf{R}\mathbf{W} > 0 \right\}.$$

The following lemma is directly from (42) in [1].

*Lemma 4.1:* Let $\mathbf{W}(t)$ be the solution of the NIC flow (5) with initial condition $\mathbf{W}(0) \in \mathcal{D}$. Then, for all $t \in [0, \infty)$, the following holds:

$$\mathbf{W}^{\mathrm{T}}(t)\mathbf{W}(t) - \mathbf{I}_r = e^{-2t}\left[\mathbf{W}^{\mathrm{T}}(0)\mathbf{W}(0) - \mathbf{I}_r\right] \tag{12}$$

as long as $\mathbf{W}(t)$ exists.

This lemma indicates that if the solution of the NIC flow exists, then the trajectory $\mathbf{W}^{\mathrm{T}}(t)\mathbf{W}(t)$ will converge to identity matrix with exponential rate 2. However, the existence of the NIC flow has not been confirmed in this lemma. The existence of $\mathbf{W}(t)$ itself is theoretically important as in the case of the Oja flow [11], [21]. On the other hand, this lemma implies that the trajectory $\mathbf{W}^{\mathrm{T}}(t)\mathbf{W}(t)$ will converge to identity matrix with exponential rate 2, which is independent of the weight matrix $\mathbf{R}$. This is different from the Oja flow in which the convergence rate of the trajectory $\mathbf{W}^{\mathrm{T}}(t)\mathbf{W}(t)$ depends on $\mathbf{R}$ [10].

In order to study the convergence property of the NIC flow, the following facts concerning the NIC objective function $\mathbf{J}_{\mathrm{NIC}}(\mathbf{W})$ will be proved.

*Lemma 4.2:*

i) The gradient of $\mathbf{J}_{\mathrm{NIC}}(\mathbf{W})$ is given by

$$\nabla J_{\mathrm{NIC}}(\mathbf{W}) = \mathbf{W} - \mathbf{R}\mathbf{W}(\mathbf{W}^{\mathrm{T}}\mathbf{R}\mathbf{W})^{-1}.$$

ii) The set of stationary points of $J_{\mathrm{NIC}}(\mathbf{W})$ is a union of disjoint compact sets, each of which has the form

$$\mathcal{W} \triangleq \left\{ \mathbf{W} \mid \mathbf{W} = [\mathbf{u}_{i_1}, \dots \mathbf{u}_{i_r}]\,\mathbf{Q},\ \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I} \right\} \tag{13}$$

where $1 \le i_1 < \cdots < i_r \le n$.

*Proof:* i) is direct from [1, Th. 3.1]. Now, we prove part ii). It is known from [1, Th. 3.1] that the union of $\mathcal{W}$ in (13) is the set of stationary points of $J_{\mathrm{NIC}}(\mathbf{W})$. Further, $\mathcal{W}$ is compact due to the fact that the set $\{\mathbf{Q} \mid \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}\}$ is compact. Next, we prove the disjoint property.

Let

$$\mathcal{W}_1 \triangleq \left\{ \mathbf{W} \mid \mathbf{W} = [\mathbf{u}_{i_1}, \dots \mathbf{u}_{i_r}]\,\mathbf{Q},\ \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I} \right\}$$

and

$$\mathcal{W}_2 \triangleq \left\{ \mathbf{W} \mid \mathbf{W} = [\mathbf{u}_{j_1}, \dots \mathbf{u}_{j_r}]\,\mathbf{Q},\ \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I} \right\}$$

with $\{j_1, j_2, \dots, j_r\} \neq \{i_1, i_2, \dots, i_r\}$. Denote

$$\mathbf{I}_w \triangleq \{i_1, i_2, \dots, i_r\}, \quad \mathbf{J}_w \triangleq \{j_1, j_2, \dots, j_r\}$$

and

$$\mathbf{V}_w \triangleq \mathbf{I}_w \cap \mathbf{J}_w,\ \ \hat{\mathbf{I}}_w \triangleq \mathbf{I}_w / \mathbf{V}_w,\ \ \hat{\mathbf{J}}_w \triangleq \mathbf{J}_w / \mathbf{V}_w.$$

Suppose that there exist two square orthogonal matrix $\mathbf{Q}_1$ and $\mathbf{Q}_2$ such that

$$\mathbf{U}_{\mathbf{I}_w}\mathbf{Q}_1 = \mathbf{U}_{\mathbf{J}_w}\mathbf{Q}_2$$

where

$$\mathbf{U}_{\mathbf{I}_w} = [\mathbf{u}_{i_1}, \dots \mathbf{u}_{i_r}], \quad \mathbf{U}_{\mathbf{J}_w} = [\mathbf{u}_{j_1}, \dots \mathbf{u}_{j_r}].$$

Then

$$\mathbf{U}_{\mathbf{I}_w} = \mathbf{U}_{\mathbf{J}_w}\mathbf{Q}_2\mathbf{Q}_1^{\mathrm{T}}. \tag{14}$$

It can be seen from (14) that $[\mathbf{U}_{\mathbf{I}_w}, \mathbf{U}_{\hat{\mathbf{J}}_w}, \mathbf{U}_{\mathbf{V}_w}]$ is not full rank. This is contradictory to the fact that $[\mathbf{u}_{i_1}, \dots \mathbf{u}_n]$ is full rank. □

*Lemma 4.3:* Any sublevel set of $\mathbf{J}_{\mathrm{NIC}}(\mathbf{W})$ is compact, namely, the set

$$\mathcal{S}_a \triangleq \{\mathbf{W} \mid J_{\mathrm{NIC}}(\mathbf{W}) \le a\}$$

is compact for any real number $a \ge 0$.

*Proof:* Obviously, it suffices to show that the set $\mathcal{S}_a$ is bounded from below and above. From the definition of $J_{\mathrm{NIC}}(\mathbf{W})$ in (4), one can see that if $J_{\mathrm{NIC}}(\mathbf{W})$ is finite, then $\mathbf{W} \in \mathcal{D}$. To prove the boundness of $\mathcal{S}_a$, it can be seen that

$$J_{\mathrm{NIC}}(\mathbf{W}) \ge \mathrm{tr}\left\{ (\mathbf{W}^{\mathrm{T}}\mathbf{W}) - \ln\left[\lambda_1^r \det(\mathbf{W}^{\mathrm{T}}\mathbf{W})\right] \right\} \tag{15}$$

due to the fact that $tr(\log(\mathbf{A})) = \log(\det(\mathbf{A}))$ for $\mathbf{A} > 0$. Fix $\mathbf{W} \in \mathbf{S}_a$, and let $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_r > 0$ be eigenvalues of $\mathbf{W}^{\mathrm{T}}\mathbf{W}$. Then, (15) implies that

$$a \ge J_{\mathrm{NIC}}(\mathbf{W}) \ge \sum_{i=1}^{r}(\mu_i - \ln\mu_i) - r\ln\lambda_1.$$

Since $x \ge \ln x$ for any $x > 0$, it follows that

$$\mu_i - \ln\mu_i \le a + r\ln\lambda_1, \quad i = 1, 2, \dots, r.$$

As the function $x - \ln x$ is unbounded in $(0, 1]$ and $[1, +\infty)$ and reaches its minimum at $x = 1$, there exist $\alpha$ and $\beta$ with $0 < \alpha < 1 < \beta$ such that

$$\alpha \le \mu_i \le \beta, \quad i = 1, 2, \dots, r.$$

Since $\mathbf{W}$ is arbitrary in $\mathcal{S}_a$, therefore, $\mathcal{S}_a$ is compact. □

Now, the existence for the NIC flow (5) can be stated below.

*Theorem 4.1:* Consider the NIC flow (5) with an initial condition $\mathbf{W}(0) = \mathbf{W}_0 \in \mathcal{D}$. Then, the NIC flow (5) has a unique solution $\mathbf{W}(t)$ defined for all $t \ge 0$.

*Proof:* Let us assume that the maximal escape time (which is the time after which the NIC flow will not exist) of the ODE (5) is $t_{\max}$, and we intend to prove that $t_{\max} = \infty$. From Lemma 4.1, one can see that

$$\lim_{t \to t_{\max}} \mathbf{W}^{\mathrm{T}}(t)\mathbf{W}(t)$$

exists. Then, there exists a positive parameter $\alpha$ such that $\mathbf{J}_{\mathrm{NIC}}(\mathbf{W}(t)) \le \alpha$ for all $\mathbf{W}(t)$, $t \in [0, t_{\max})$. From Lemma 4.3, $\mathbf{W}(t)$ will be contained in a compact set for all $t \in [0, t_{\max})$. This contradicts the maximality of the interval $[0, t_{\max})$. Therefore, $t_{\max} = \infty$. □

Now, it is clear that the NIC flow (5) always has solution with an initial condition $\mathbf{W}_0 \in \mathcal{D}$. This conclusion is quite optimistic for the application of the NIC flow. In [1], the existence of the NIC flow has not been discussed. The authors there only discussed the convergence property via Lyapunov approach with assumption of its existence. Next, we can investigate the property of $\mathbf{W}(t)$ along the NIC flow (5).

*Lemma 4.4:*

i) Given any initial condition $\mathbf{W}_0 \in \mathcal{D}$

$$\text{rank}W(t) = \text{rank}W_0, \quad t \geq 0.$$

ii) If $\mathbf{W}_0^{\mathrm{T}}\mathbf{R}\mathbf{W}_0 > 0$, then

$$\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t) > 0.$$

*Proof:*

i) It is true that

$$\text{rank}\,[\mathbf{W}(t)] = \text{rank}\,\left[\mathbf{W}(t)^{\mathrm{T}}\mathbf{W}(t)\right].$$

According to Lemma 4.1, we have

$$\text{rank}\left[\mathbf{W}(t)^{\mathrm{T}}\mathbf{W}(t)\right] = \text{rank}\left[\left(\mathbf{I}_r - e^{-2t}\mathbf{I}_r\right) + \mathbf{W}_0^{\mathrm{T}}\mathbf{W}_0\right].$$

This completes the proof of part one.

ii) If $\mathbf{R} > 0$, part two is the direct consequence of part one. With $\mathbf{R} \geq 0$, one can decompose $\mathbf{R}$ into

$$\mathbf{R} = \mathbf{U}^{\mathrm{T}}\begin{bmatrix}\mathbf{R}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}$$

where $\mathbf{R}_1 > 0$, and $\mathbf{U}$ is an orthogonal matrix. Denote $\mathbf{W}_u \triangleq \mathbf{U}^{\mathrm{T}}\mathbf{W}$, and decompose $\mathbf{W}_u$ into

$$\mathbf{W}_u = \begin{bmatrix}\mathbf{W}_{u,1}\\ \mathbf{W}_{u,2}\end{bmatrix}.$$

Then, $\mathbf{W}_{u,1}(0)\mathbf{R}_1\mathbf{W}_{u,1}(0) > 0$ and $\mathbf{W}_{u,1}(t)$ will satisfy a reduced-order NIC flow equation with weight matrix $\mathbf{R}_1$, which will be shown in the proof of Theorem 4.2 in the sequel. This will guarantee that $\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t) > 0$. □

The above lemma confirms that the trajectory $\mathbf{W}(t)$ will be in $\mathcal{D}$ if its initial condition is as well. Observing that the inverse of $\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t)$ appears in the NIC flow (5), we are motivated to investigate the boundness of this factor along the trajectory $\mathbf{W}(t)$.

*Theorem 4.2:* Given any initial condition $\mathbf{W}_0 \in \mathcal{D}$, there exist two positive parameters $\alpha_1 > 0$ and $\alpha_2 > 0$, depending on the initial condition $\mathbf{W}_0$, such that

$$\alpha_1 I_r \leq \mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t) \leq \alpha_2 I_r, \quad t \geq 0.$$

*Proof:* Assume that $\mathbf{R} \geq \mathbf{0}$ and can be decomposed into

$$\mathbf{R} = \mathbf{U}^{\mathrm{T}}\begin{bmatrix}\mathbf{R}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}$$

where $\mathbf{U}$ is an orthogonal matrix, and $\mathbf{R}_1$ is a diagonal matrix with

$$\text{rank}\,\mathbf{R}_1 = \text{rank}\,R.$$

Then

$$\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t) = \mathbf{W}^{\mathrm{T}}(t)\mathbf{U}^{\mathrm{T}}\begin{bmatrix}\mathbf{R}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}UW(t).$$

Letting

$$\mathbf{U}\mathbf{W}(t) \triangleq \left[\mathbf{W}_1^{\mathrm{T}}(t), \mathbf{W}_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$$

one has

$$\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t) = \mathbf{W}_1^{\mathrm{T}}(t)\mathbf{R}_1\mathbf{W}_1(t)$$

where $\mathbf{W}_1(t)$ will satisfy the following ODE:

$$\dot{\mathbf{W}}_1(t) = \mathbf{R}_1\mathbf{W}_1(t)(\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{R}_1\mathbf{W}_1(t))^{-1} - \mathbf{W}_1(t).$$

According to Lemma 4.1, $\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{W}_1(t)$ is given by

$$\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{W}_1(t) = \mathbf{I}_r - e^{-2t}\mathbf{I}_r + \mathbf{W}_1(0)^{\mathrm{T}}\mathbf{W}_1(0). \quad (16)$$

With $\mathbf{R}_1 > 0$, one can deduce $\mathbf{W}_1(0)^{\mathrm{T}}\mathbf{W}_1(0) > 0$ from $\mathbf{W}_1(0)^{\mathrm{T}}\mathbf{R}_1\mathbf{W}_1(0) > 0$. Decompose $\mathbf{W}_1(0)^{\mathrm{T}}\mathbf{W}_1(0)$ into

$$\mathbf{W}_1(0)^{\mathrm{T}}\mathbf{W}_1(0) = U_0^{\mathrm{T}}\begin{bmatrix}w_1 & 0 & \cdots & 0\\ 0 & w_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & w_s\end{bmatrix}U_0$$

with $w_1 \geq w_2 \geq \cdots \geq w_s > 0$. Then, from (16), one can obtain

$$\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{W}_1(t) = \begin{bmatrix}q_1 & 0 & \cdots & 0\\ 0 & q_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & q_s\end{bmatrix}$$

with $q_i = 1 - e^{-2t} + w_i e^{-2t}, \quad i = 1, 2, \cdots, s$. Defining $w_\alpha \triangleq \min\{1, w_s\} \leq 1$ and $w_\beta \triangleq \max\{1, w_1\} \geq 1$, one can then prove

$$w_\alpha \mathbf{I}_r \leq \mathbf{W}_1(t)^{\mathrm{T}}\mathbf{W}_1(t) \leq w_\beta \mathbf{I}_r.$$

With $\alpha_1 \triangleq w_\alpha w_s$ and $\alpha_2 \triangleq w_\beta w_1$ depending on the eigenvalues of $\mathbf{W}_0^{\mathrm{T}}\mathbf{W}_0$, one can obtain the conclusion. □

From Lemma 4.2 and Theorem 4.2, one can see that if $\mathbf{W}_0 \in \mathcal{D}$, the trajectory $\mathbf{W}(t)$ will be kept in $\mathcal{D}$ where $\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t)$ is bounded. Recall that the convergence of the Oja flow has been solved in [11] and [21] and that the NIC flow has only one extra factor $\mathbf{W}^{\mathrm{T}}(t)\mathbf{R}\mathbf{W}(t)$ in the right side compared with the Oja flow; one hopes that the convergence analysis for the NIC flow can be done similarly. In order to solve this problem, we adopt the center manifold theory, which is a powerful tool in the stability analysis of nonlinear dynamic systems. This approach can give us some profound results on global convergence analysis. First, we derive the Hessian matrix associated with the NIC performance index.

*Lemma 4.5:* Let $\bar{\mathbf{W}}$ be a stationary point of $J_{\text{NIC}}(\mathbf{W})$. Then, the Hessian $H_{\bar{W}}$ of $J_{\text{NIC}}(\mathbf{W})$ at $\bar{\mathbf{W}}$ is given by

$$\mathbf{H}_{\bar{\mathbf{W}}}(\mathbf{X},\mathbf{Y}) = \text{tr}\Big[\mathbf{X}^{\mathrm{T}}\mathbf{Y} + \mathbf{X}^{\mathrm{T}}\bar{\mathbf{W}}\mathbf{Y}^{\mathrm{T}}\bar{\mathbf{W}} - \mathbf{X}^{\mathrm{T}}\left(\mathbf{I} - \bar{\mathbf{W}}\bar{\mathbf{W}}^{\mathrm{T}}\right)\mathbf{R}\mathbf{Y}\left(\bar{\mathbf{W}}^{\mathrm{T}}\mathbf{R}\bar{\mathbf{W}}\right)^{-1}\Big]$$

where $\mathbf{X}, \mathbf{Y} \in \mathcal{R}^{n\times r}$.

*Proof:* Regarding $J_{\rm NIC}(\mathbf{W})$ as a mapping from $\mathcal{N} \triangleq \{\mathbf{W} \mid \mathbf{W} \in \mathcal{R}^{n\times r}\}$ to $\mathbf{R}$, then for a fixed $\mathbf{W} \in \mathcal{D}$, from Definition 6.7 in the Appendix, the derivative of $J_{\rm NIC}(\mathbf{W})$ will be a mapping from $\mathbf{T_W}\mathcal{N}$ to $\mathbf{T}_{\mathbf{J}_{\rm NIC}(\mathbf{W})}\mathbf{R}$, where $\mathbf{T_W}\mathcal{N}$ denotes the tangent space of the function $\mathcal{N}$ at $\mathbf{W}$. Following this idea, it can be seen that $dJ_{\rm NIC}(\mathbf{W})$ will be a function of a new variable $\mathbf{X} \in \mathcal{R}^{n\times r}$. Fix $\mathbf{X} \in \mathcal{R}^{n\times r}$, and let $J_d(\mathbf{W}) = d\mathbf{J}_{\rm NIC(\mathbf{W})}(\mathbf{X})$. From Lemma 4.2, it follows that

$$J_d(\mathbf{W}) = \text{tr}\left\{\mathbf{X}^{\rm T}\left[\mathbf{W} - \mathbf{RW}(\mathbf{W}^{\rm T}\mathbf{RW})^{-1}\right]\right\}.$$

Therefore, we have

$$\begin{aligned}\mathbf{H_W}(\mathbf{X},\mathbf{Y}) =& d\mathbf{J}_{d,\mathbf{W}}(\mathbf{Y})\\ =&\text{tr}\big\{\mathbf{X}^{\rm T}\big[\mathbf{Y} - \mathbf{RY}(\mathbf{W}^{\rm T}\mathbf{RW})^{-1}\\ &+ \mathbf{RW}(\mathbf{W}^{\rm T}\mathbf{RW})^{-1}(\mathbf{Y}^{\rm T}\mathbf{RW} + \mathbf{W}^{\rm T}\mathbf{RY})\\ &\times(\mathbf{W}^{\rm T}\mathbf{RW})^{-1}\big]\big\}.\end{aligned}$$

The lemma immediately follows by substituting $\bar{\mathbf{W}}$ for $\mathbf{W}$ into the above and noting that

$$\mathbf{R}\bar{\mathbf{W}}\left(\bar{\mathbf{W}}^{\rm T}\mathbf{R}\bar{\mathbf{W}}\right)^{-1} = \bar{\mathbf{W}}.$$

□

It should be noted that the Hessian matrix $\mathbf{H_W}$ derived above is different in form from that obtained in [1]. The formula for the Hessian matrix $\mathbf{H_W}$ here is very explicit, which is effective for the NIC flow analysis. The same matrix was expressed by Kroneck product in [1], which is powerful for computation. Both of these two forms have their own advantages in dealing with different problems. Actually, they are same only with different appearances.

*Lemma 4.6:* The set

$$\mathcal{M} \triangleq \left\{\mathbf{W} \in \mathcal{R}^{n\times r} \mid \mathbf{W} = \mathbf{U}_r\mathbf{Q},\ \mathbf{Q}^{\rm T}\mathbf{Q} = \mathbf{I}\right\} \tag{17}$$

is a center manifold for the flow

$$\dot{\mathbf{W}} = -\nabla\mathbf{J}_{\rm NIC}(\mathbf{W}) = \mathbf{RW}(\mathbf{W}^{\rm T}\mathbf{RW})^{-1} - \mathbf{W} \tag{18}$$

at any point in the set where

$$\mathbf{U}_r \triangleq \{\mathbf{u}_1,\cdots,\mathbf{u}_r\}.$$

*Proof:* Obviously, $\mathcal{M}$ is a smooth manifold. One can show [18] that the tangent space for $\mathcal{M}$ at $\mathbf{W} \in \mathcal{M}$ is

$$T_{\mathbf{W}}\mathcal{M} = \left\{\mathbf{X} \in \mathcal{R}^{n\times r} \mid \mathbf{XW}^{\rm T} + \mathbf{WX}^{\rm T} = \mathbf{0}\right\}.$$

In addition, $\mathcal{M}$ is invariant under the flow (18) as any point in the set is an equilibrium point of (18). According to Definition 3.1, it remains to be shown that for any fixed $\mathbf{W} \in \mathcal{M}$, the Hessian $\mathbf{H_W}(\mathbf{X},\mathbf{X})$ vanishes if and only if $\mathbf{X} \in \mathbf{T_W}\mathcal{M}$. To do this, assume that $\mathbf{W} = \mathbf{U}_r\mathbf{Q}$ for some orthogonal $\mathbf{Q}$, and let

$$\begin{aligned}\mathbf{V} =& [\mathbf{u}_{r+1},\cdots,\mathbf{u}_n], \quad \mathbf{\Lambda}_1 = \text{diag}\{\lambda_1,\lambda_2,\cdots,\lambda_r\}\\ \mathbf{\Lambda}_2 =& \text{diag}\{\lambda_{r+1},\lambda_{r+2},\cdots,\lambda_n\}.\end{aligned}$$

Then, it is seen from Lemma 4.5 that

$$\begin{aligned}\mathbf{H_W}(\mathbf{X},\mathbf{X}) =& \text{tr}\big[\mathbf{X}^{\rm T}\mathbf{X} + \mathbf{X}^{\rm T}\mathbf{WX}^{\rm T}\mathbf{W}\\ &- \mathbf{X}^{\rm T}(I - \mathbf{WW}^{\rm T})\mathbf{RX}(\mathbf{W}^{\rm T}R\mathbf{W})^{-1}\big]\\ =& \text{tr}\big[\mathbf{X}^{\rm T}\mathbf{X} + \mathbf{X}^{\rm T}\mathbf{WX}^{\rm T}\mathbf{W}\\ &- \mathbf{X}^{\rm T}\mathbf{V}\Lambda_2\mathbf{V}^{\rm T}\mathbf{XQ}^{\rm T}\Lambda_1^{-1}\mathbf{Q}\big]\\ =& \text{tr}\big[(\mathbf{X}^{\rm T}\mathbf{WW}^{\rm T}\mathbf{X} + \mathbf{X}^{\rm T}\mathbf{WX}^{\rm T}\mathbf{W})\\ &+ \mathbf{X}^{\rm T}\mathbf{VV}^{\rm T}\mathbf{X} - \mathbf{X}^{\rm T}\mathbf{V}\Lambda_2\mathbf{V}^{\rm T}\mathbf{XQ}^{\rm T}\Lambda_1^{-1}\mathbf{Q}\big]\\ \geq& \text{tr}\Bigg[\mathbf{X}^{\rm T}\mathbf{W}(\mathbf{W}^{\rm T}\mathbf{X} + \mathbf{X}^{\rm T}\mathbf{W}) + \mathbf{X}^{\rm T}\mathbf{VV}^{\rm T}\mathbf{X}\\ &- \left(\frac{\lambda_{r+1}}{\lambda_r}\right)\mathbf{X}^{\rm T}\mathbf{VV}^{\rm T}\mathbf{X}\Bigg].\end{aligned}$$

Using the inequality that $\text{tr}(\mathbf{AA}^{\rm T} + \mathbf{A}^2) \geq 0$ for any square matrix $\mathbf{A}$ and $\lambda_{r+1}/\lambda_r < 1$, we infer from the above that $\mathbf{H_W}(\mathbf{X},\mathbf{X}) = 0$ if and only if the following conditions hold:

$$\text{tr}\left[\mathbf{X}^{\rm T}\mathbf{W}(\mathbf{W}^{\rm T}\mathbf{X} + \mathbf{X}^{\rm T}\mathbf{W})\right] = \mathbf{0} \text{ AND } \mathbf{X}^{\rm T}\mathbf{V} = 0.$$

Let

$$J_a(\mathbf{A}) \triangleq \text{tr}\left[\mathbf{A}(\mathbf{A}^{\rm T} + \mathbf{A})\right].$$

Then, $J_a(\mathbf{A}) = \mathbf{0}$ will be a minimum value, and therefore

$$\frac{\partial J_a(\mathbf{A})}{\partial\mathbf{A}} = 0, \ \Rightarrow\ \mathbf{A} + \mathbf{A}^{\rm T} = \mathbf{0}.$$

This indicates that

$$\mathbf{W}^{\rm T}\mathbf{X} + \mathbf{X}^{\rm T}\mathbf{W} = 0 \text{ AND } \mathbf{X}^{\rm T}\mathbf{V} = 0. \tag{19}$$

Let us prove that these conditions hold if and only if $\mathbf{X} \in T_{\mathbf{W}}\mathcal{M}$, i.e.,

$$\mathbf{XW}^{\rm T} + \mathbf{WX}^{\rm T} = \mathbf{0}. \tag{20}$$

In fact, due to

$$\mathbf{W}^{\rm T}\mathbf{W} = \mathbf{I}, \quad \mathbf{W}^{\rm T}\mathbf{V} = \mathbf{0}$$

the first equality in (19) can be obtained by multiplying (20) by $\mathbf{W}^{\rm T}$ from the left and by $\mathbf{W}$ from right, and the second one can be obtained by multiplying (20) by $\mathbf{V}$ from the right. Conversely, (20) follows from (19) because

$$\begin{aligned}&\begin{bmatrix}\mathbf{W}^{\rm T}\\ \mathbf{V}^{\rm T}\end{bmatrix}(\mathbf{XW}^{\rm T} + \mathbf{WX}^{\rm T})[\mathbf{W} \quad \mathbf{V}]\\ &= \begin{bmatrix}\mathbf{W}^{\rm T}\mathbf{X} + \mathbf{X}^{\rm T}\mathbf{W} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} = 0.\end{aligned}$$

It is thus concluded that $\mathbf{H_W}(\mathbf{X},\mathbf{X}) = 0$ if and only if $\mathbf{X} \in T_{\mathbf{W}}\mathcal{M}$. □

*Corollary 4.1:* There holds

$$\mathbf{H_W}(\mathbf{X},\mathbf{X}) \geq 0, \quad \forall\, \mathbf{W} \in \mathcal{M}.$$

*Theorem 4.3:* Consider the ODE (5) with an initial condition $\mathbf{W}(0) = \mathbf{W}_0 \in \mathcal{D}$. Then, the following statements hold:

i) The solution $\mathbf{W}(t)$ converges to some set of the form

$$\left\{\mathbf{W} = [\,\mathbf{u}_{i_1} \quad \cdots \quad \mathbf{u}_{i_r}\,]\,\mathbf{Q} \mid \mathbf{Q}^{\rm T}\mathbf{Q} = \mathbf{I}\right\}$$

where $1 \leq i_1 < \cdots < i_r \leq n$.

ii) If $\mathbf{W}(t)$ converges to the principal eigenspace (17), which has been proved to be a center manifold, then $\mathbf{W}(t)$ must converge to a point in this eigenspace as $t \to \infty$.

iii) If $\mathbf{W}(t)$ converges to some point in $\mathcal{M}$, then it will converge exponentially when it is near the center manifold.

*Proof:* The first statement follows from Lemma 4.2. To prove the second statement, assume that $\mathbf{W}(t)$ converges to $\mathcal{M}$. Then, there exists a sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = \infty$ such that $\mathbf{W}(t_k)$ converges to a point $\bar{\mathbf{W}} \in \mathcal{M}$. In view of Corollary 4.1 and Remark 3.2, there exists a homeomorphism $\phi: \mathbf{U} \to \mathcal{R}^{nr\times 1}$ with

$$\phi(\mathbf{W}) = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \quad \mathbf{x} \in \mathcal{R}^{n_0\times 1} \text{ AND } \mathbf{y} \in \mathcal{R}^{nr-n_0\times 1}$$

from a neighborhood $\mathbf{U} \subset \mathcal{R}^{n\times r}$ of $\bar{\mathbf{W}}$ onto a neighborhood of $\mathbf{0} \in \mathcal{R}^{nr\times 1}$ such that $\phi$ maps solutions of (18) starting within $U$ to solutions of

$$\begin{aligned} \dot{\mathbf{x}} =& \mathbf{h}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{R}^{n_0\times 1} \\ \dot{\mathbf{y}} =& -\mathbf{y}, \quad \mathbf{y} \in \mathcal{R}^{nr-n_0\times 1} \end{aligned}$$

where $n_0$ is the dimension of $\mathcal{M}$, and $\mathbf{h}(\mathbf{x})$ is equivalent to the dynamic flow on center manifold $\mathcal{M}$. Due to the fact that $\mathcal{M}$ is a set of equilibrium of the NIC flow, $\mathbf{h}(\mathbf{x}) = \mathbf{0}$. Now, let $m$ be such an integer that $\mathbf{W}(t_m) \in \mathbf{U}$, and put

$$\phi(\mathbf{W}(t)) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}, \quad t \geq t_m.$$

Since $\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}$ satisfies the above ODE for all $t \geq t_m$, it must converge to $\begin{bmatrix} \mathbf{x}(t_m) \\ 0 \end{bmatrix}$. Note that $m$ can be chosen to be so large that $\begin{bmatrix} \mathbf{x}(t_m) \\ 0 \end{bmatrix}$ is inside $\phi(\mathbf{U})$. Therefore, $\mathbf{W}(t)$ converges to $\phi^{-1}\left(\begin{bmatrix} \mathbf{x}(t_m) \\ 0 \end{bmatrix}\right)$ as $t \to \infty$. Obviously, $\bar{\mathbf{W}} = \phi^{-1}\left(\begin{bmatrix} \mathbf{x}(t_m) \\ 0 \end{bmatrix}\right)$ holds, which implies that $\mathbf{x}(t_m) = 0$. This completes the proof of part 2.

As for part 3, if $\mathbf{W}(t)$ converges at some point called $\mathbf{W}_a$ in $\mathcal{M}$, then when $t$ is large enough, $\mathbf{W}(t)$ will be near the center manifold $\mathcal{M}$. According to part two of Lemma 3.1 and Remark 3.1, there exists a neighborhood of $\mathbf{W}_a$ in which the convergence will be exponential. □

The importance of this theorem are threefold.

1) The NIC flow may converge to a set of the stationary points.
2) If $\mathbf{W}(t)$ converges to the principal eigenspace $\mathcal{M}$, then it must converge to one point instead of possible multiple points.
3) If $\mathbf{W}(t)$ converges to one point in the principal eigenspace, then it will converge exponentially when it reaches near enough to the center manifold, although the convergence rate is unknown accurately.

The results in [1] only tell us the fact that the NIC flow converges but without telling us whether it converges to a set or a point. Of course, we are more interested in the case of the NIC flow converging to a point. Thus, it is curious for us to know under what conditions $\mathbf{W}(t)$ will converge to the principle eigenspace. Theorem 4.3 did not answer this question directly. Next, we turn out our attention to this important question.

*Lemma 4.7:* With the same notation as in Theorem 4.3

$$\text{rank}\left[\mathbf{u}_i^{\mathrm{T}}\mathbf{W}_0\right] = 0 \quad \Longrightarrow \quad \text{rank}\left[\mathbf{u}_i^{\mathrm{T}}\mathbf{W}(t)\right] = 0,\ \forall\, t \geq 0$$

holds, where $\mathbf{u}_i$ is any eigenvector of $\mathcal{R}$.

*Proof:* Put $\mathbf{x}(t) = \mathbf{u}_i^{\mathrm{T}}\mathbf{W}(t)$. Then, it is obvious that

$$\dot{\mathbf{x}} = \mathbf{x}\left[\lambda_i(\mathbf{W}^{\mathrm{T}}\mathbf{R}\mathbf{W})^{-1} - \mathbf{I}\right] \text{ AND } \mathbf{x}(0) = 0.$$

By the uniqueness of solutions, it follows that $\mathbf{x}(t) = 0,\ \forall t \geq 0$. □

As a result of this lemma and the convergence theorem, the following corollary is obtained, which identifies a domain of attraction as well as a set of initial points that do not lead to the principal eigenspace.

*Corollary 4.2:* With the same notation as in Theorem 4.3, the solution $\mathbf{W}(t)$ will converge to a point in the principal eigenspace $\mathcal{M}$ if

$$\mathbf{W}_0^{\mathrm{T}}\left[\mathbf{u}_{r+1} \quad \cdots \quad \mathbf{u}_n\right] = 0.$$

Moreover, the solution $\mathbf{W}(t)$ cannot converge to the principal eigenspace $\mathcal{M}$ if $\mathbf{W}_0^{\mathrm{T}}\mathbf{u}_i = 0$ for some $i$ between 1 and $r$.

*Proof:* If

$$\mathbf{W}_0^{\mathrm{T}}\left[\mathbf{u}_{r+1} \quad \cdots \quad \mathbf{u}_n\right] = 0$$

then from Lemma 4.7

$$\mathbf{W}^{\mathrm{T}}(t)\left[\mathbf{u}_{r+1} \quad \cdots \quad \mathbf{u}_n\right] = 0 \tag{21}$$

and (21) indicates that $\mathbf{W}(\infty)$ will be in $\mathcal{M}$. By Theorem 4.3, $\mathbf{W}(t)$ will converge to a point in $\mathcal{M}$.

If $\mathbf{W}_0^{\mathrm{T}}\mathbf{u}_i = \mathbf{0}, 1 \leq i \leq r$, then $\mathbf{W}^{\mathrm{T}}(t)\mathbf{u}_i = \mathbf{0}$ according to Lemma 4.7. This implies that $\mathbf{W}^{\mathrm{T}}(\infty)\mathbf{u}_i = \mathbf{0}$, which tells us that the limit is not in $\mathcal{M}$. □

Although we cannot characterize the convergence domain completely, we specified some convergence and nonconvergence sets, which can be used effectively. Here, the word of convergence indicates that $\mathbf{W}(t)$ converges to the principal eigenspace. The attraction domain obtained in [1] includes the domain that may lead to converging to a point as well as to a set. The convergence domain given in Corollary 4.2 will lead to converging to a point. Therefore, it is easy to understand that the convergence domain obtained here is smaller than the attraction domain in [1].

Finally, it can be seen that if we scale the matrix $\mathbf{R}$ to $a\mathbf{R}$ with any $a \neq 0$, the trajectory $\mathbf{W}(t)$ of the NIC flow (5) will not change. This property is quite different from the Oja flow, in which the flow depends on the weight matrix $\mathbf{R}$ closely.

## V. Discussions and Conclusions

In this paper, we investigated some basic problems on the NIC flow, including the existence of its solution, convergence analysis, and convergence domain characterization. These properties are very important to the analysis and application of the NIC flow. Since a novel approach based on center manifold is adopted here, the results obtained here are more accurate compared with those in [1] in the following aspects.

i) The existence of the NIC flow is addressed in this paper.
ii) The convergence analysis is investigated; specifically, the convergence to a point rather than to a set is characterized.
iii) The convergence rate is proved to be exponential when the NIC flow reaches near enough to the center manifold.
iv) The convergence domain has been partly characterized.

Another interesting topic along this research direction is to investigate the convergence rate and convergence domain as [10] and [11] for the Oja flow. Regarding to the convergence domain, we propose the following conjecture.

**Conjecture:** $\mathbf{W}(t)$ converges to the principal eigenspace $\mathcal{M}$ if and only if $\mathbf{U}_r^{\mathrm{T}}\mathbf{W}_0$ is of full rank and $\mathbf{W}_0 \in \mathcal{D}$.

We believe that the convergence rate for the NIC flow can be estimated as the convergence analysis for the Oja flow in [21]. This is a topic of possible future research for us.

The NIC-based algorithms can be applied to many fields, as discussed in [1]. We believe that the NIC flow and its related algorithms will attract more attention in the future.

## APPENDIX

In order to understand the center manifold theory better, we will present some basic concepts on topological space and smooth manifolds in this Appendix.

### A. Topology Concepts

Letting $\mathbf{S}$ be a set, a *topology* on $\mathbf{S}$ is a collection of subsets of $\mathbf{S}$, called open sets, satisfying the following three axioms:

i) The union of any number of open sets is open.
ii) The intersection of any finite number of sets is open.
iii) The set $\mathbf{S}$ and empty set $\emptyset$ are open.

Conventionally, we denote a topology as topological space $\mathbf{S}$.

A *basis* for a topological space $\mathbf{S}$ is a collection of open sets, called basic open sets, with the following properties:

i) $\mathbf{S}$ is the union of basic open sets.
ii) A nonempty intersection of two basic open sets is an union of basic open sets.

A *neighborhood* of a point $p$ for a topological space $\mathbf{S}$ is any open set that contains $p$.

Next, we define the mapping between two topological spaces. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be topological spaces, where $\mathcal{F}$ is a mapping from $\mathbf{S}_1$ to $\mathbf{S}_2$

$$\mathcal{F}: \mathbf{S}_1 \to \mathbf{S}_2.$$

The mapping is called *continuous* if the inverse image of every open set of $\mathbf{S}_2$ is an open set of $\mathbf{S}_1$. The mapping is called *open* if the image of an open set of $\mathbf{S}_1$ is an open set of $\mathbf{S}_2$. The mapping is called an *homeomorphism* if it is a bijection and both continuous and open.

Finally, a topological space $\mathbf{S}$ is called a *Hausdorff* topological space if any two different points $p_1$ and $p_2$ have disjoint neighborhoods.

### B. Smooth Manifolds

*Definition 6.1:* A manifold $\mathcal{N}$ of dimension $n$ is a Hausdorff topological space with a countable basis such that for each $p \in \mathcal{N}$, there exists a homeomorphism $\phi$ mapping some open neighborhood of $p$ onto an open set in $\mathcal{R}^n$.

Usually, the homeomorphism mapping is reflected by a *coordinate chart* defined on a manifold $\mathcal{N}$. This *coordinate chart* is a pair $(U, \phi)$, where $U$ is an open set of $\mathcal{N}$, and $\phi$ is a homeomorphism of $U$ onto an open set of $\mathcal{R}^n$. Of course, on one manifold, there can be two different *coordinate charts*. Let $(U, \phi)$ and $(V, \varphi)$ be two coordinate charts on manifold $\mathcal{N}$ with $U \cap V \neq \emptyset$. Then, one can define

$$\varphi \circ \phi^{-1}\colon \phi(U \cap V) \to \varphi(U \cap V). \tag{22}$$

Roughly speaking, two coordinate charts $(U, \phi)$ and $(V, \varphi)$ are called smooth compatible if, whenever $U \cap V \neq \emptyset$, the coordinate transformation $\varphi \circ \phi^{-1}$ defined in (22) is smooth.

Now, it is time to define smooth manifolds.

*Definition 6.2:* A smooth manifold is a manifold equipped with a collection $\mathcal{A} = \{(V_i, \phi_i) : i \in I\}$ of pairwise smooth-compatible coordinate charts satisfying

$$\bigcup_{i \in I} U_i = \mathcal{N}.$$

On smooth manifolds, one can define smooth mappings as follows.

*Definition 6.3:* Let $\mathcal{N}$ and $\mathcal{M}$ be smooth manifolds. A mapping $F : \mathcal{N} \to \mathcal{M}$ is a smooth mapping if, for each $p \in \mathcal{N}$, there exist coordinate charts $(U, \phi)$ of $\mathcal{N}$ and $(V, \varphi)$ of $\mathcal{M}$, with $p \in U$ and $F(p) \in V$, such that the expression for $F$ in local coordinates is smooth.

The submanifolds are defined as follows.

*Definition 6.4:* Let $F : \mathcal{N} \to \mathcal{M}$ be a smooth mapping of manifolds.

i) $F$ is an *immersion* if $rank(F) = dim(\mathcal{N})$ for all $p \in \mathcal{N}$
ii) $F$ is a *univalent immersion* if $F$ is an immersion and injective.
iii) $F$ is an *embedding* if $F$ is an univalent immersion and the topology induced on $F(\mathcal{N})$ by the one of $\mathcal{N}$ coincides with the topology of $F(\mathcal{N})$ as a subset of $\mathcal{M}$.

*Definition 6.5:* The image $F(\mathcal{N})$ of a univalent immersion is called an immersed submanifold of $\mathcal{M}$. The image $F(\mathcal{N})$ of an embedding is called an embedded submanifold of $\mathcal{M}$.

By the use of word of submanifold, we indicate that one can equip $F(\mathcal{N})$ with a coordinate chart structure from a smooth manifold. A detailed description of the structure can be found in [17].

### C. Tangent Spaces

Let $\mathcal{N}$ be a smooth manifold of dimension $n$. A *tangent vector* at point $p \in \mathcal{N}$ can be defined as usual with detailed definition in [17]. Now, let us define tangent space.

*Definition 6.6:* Let $\mathcal{N}$ be a smooth manifold. The tangent space to $\mathcal{N}$ at $p$, which is denoted $T_p\mathcal{N}$, is the set of all tangent vectors at $p$.

Now, one can define the differential for a mapping between smooth manifolds.

*Definition 6.7:* Assume that $\mathcal{N}$ and $\mathcal{M}$ are two smooth manifolds. Let

$$F : \mathcal{N} \rightarrow \mathcal{M}$$

be a smooth mapping. Then, the differential of $F$ at $p \in \mathcal{N}$ is defined as a linear map

$$F_* : T_p\mathcal{N} \rightarrow T_{F(p)}\mathcal{M}$$

such that for $v \in T_p\mathcal{N}$ and $\eta \in C^\infty(F(p))$

$$F_*(v)(\eta) = v(\eta \circ F).$$

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