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GNSS Ambiguity Resolution with Constraints from Normal Equations

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Abstract: Carrier phase ambiguity resolution is the key to precise positioning with GNSS, therefore quite a few ambiguity resolution methods have been developed in the past two decades. In this paper, a new ambiguity searching algorithm by treating part of normal equations as constraints is developed. The process starts with the truncation of the terms with respect to the small eigenvalues from the normal equations of least squares estimation problem. The remaining normal equations are employed as the constraint equations for the efficient searching of integer ambiguities. In the case of short single baseline rapid GNSS positioning with double differenced phase measurements, there are only three real parameters of position to be estimated. Therefore three terms of the normal equations should be truncated off due to the fact that there is a large difference between the last three eigenvalues of the normal matrix of float solution and the others, and then the remaining ambiguities can be trivially solved with three independent ambiguities by means of the remaining normal equations. As a result, only three independent ambiguities are necessarily searched and the searching efficiency is dramatically enhanced. Moreover, a new indicator of minimizing the conditional number of the sub-square matrix of the remaining normal equations is introduced to select three independent ambiguities. Once the correct integer values of the selected three independent ambiguities are applied to solve the remaining ambiguities, the estimated real-valued solutions are very close to their integers, which can be applied as additional strong constraints to further improve the searching efficiency. Finally, two case studies, from real dual-frequency GPS data of about 10 km baseline and random simulations respectively, are carried out to demonstrate the efficiency of new algorithm. The results show that the new algorithm is efficient, especially for the scenarios of high-dimensional ambiguity parameters.

Keywords: GNSS; ambiguity resolution; fast positioning; ill-posed equation; singular value decomposition; normal equation.

Introduction

The precise positioning and navigation by using Global Navigation Satellite Systems (GNSS) must employ the phase measurements with correctly fixed ambiguities. The purpose of ambiguity resolution (AR) is to determine the integer cycle unknowns in phase measurements, which can lead to recover the millimeter precision of ranging measurements between a satellite and a receiver, making precise determination of the user coordinates and tropospheric and ionospheric delays if possible. We refer to some early publications, such as Frei and Beutler (1990), Hatch (1990), Euler and Landau (1992), Chen (1993), Teunissen (1994), Park et al. (1996), Xu (1998, 2001) and Shen and Li (2007), for the detailed arguments. In general, an AR process comprises the three procedures: (i) estimate the float (real-valued) ambiguities with a sufficient accuracy; (ii) search over the integer candidates with an efficient searching technique; and (iii) fix the float ambiguities to their integer values of being considered correct according to an appropriate criterion, for instance, ratio indicator. In this paper, we will confine ourselves to the second procedure, which is the most important for efficient AR.

In the fast relative positioning of single short baseline, it is crucial to find an appropriate algorithm to reliably determine the integer ambiguities only with a few epochs of phase measurements, even on the fly. Since the covariance matrix of the estimated float ambiguities is highly correlated, the decorrelation technique has been employed to make the covariance matrix minimally correlated, thus improving the efficiency of ambiguity searching. Although extensive studies have been made towards the ambiguity decorrelation (Teunissen 1994; Liu et al. 1999; Grafarend 2000; Xu 2001; Shen and Li 2007), numerical investigations by Xu (2001) and Lou and Grafarend (2003) have shown that all decorrelation algorithms cannot enhance the efficiency of AR in the case of high-dimensional ambiguity parameters.

In the kinematic GPS positioning, Hatch (1990) developed a so-called least squares ambiguity search technique to fix the ambiguities instantaneously, where the double differenced (DD) observables for each epoch are separated into two groups. In the primary group, there are three well-conditioned DD observables, and if their ambiguities are correctly fixed, the position can be trivially determined. For each integer candidate of three DD measurements in the primary group, the corresponding potential position can be uniquely determined. Furthermore, the remaining ambiguities can be individually achieved by substituting the solved potential position into the DD measurements of the secondary group. If the estimated ambiguities of the secondary group are not within their confident intervals previously determined by the code measurements, they would be directly rejected and thus their corresponding potential integer candidate of the primary group are excluded as well. Through this process, all potential integer

candidates of the primary group are validated and only correct one is reserved to determine the precise position (Hatch 1990). ARCE (Ambiguity Resolution using Constraint Equation) algorithm proposed by Park et al. (1996) is essentially identical to the Hatch's method, where the coordinate parameters are primarily eliminated epoch by epoch and the transformed equations for ambiguities are of rank defect with the number of three, which are used as the constraint equations in searching integer ambiguities.

In this paper, we will propose a new method using the constraints conveniently derived from the normal equations that are primarily obtained along with the float solution. The process starts with the truncation of the terms with respect to the smaller eigenvalues from the normal equations, and the remaining normal equations are of rank defect and thus employed as the constraint equations for efficiently searching the integer ambiguities. In the case of only three real coordinate parameters to be estimated, there is a large difference between the last three eigenvalues of the normal matrix for float solution and the others, therefore three terms of the normal equations with respect to the small eigenvalues should be truncated off. In new algorithm, only three independent ambiguities are necessary to be searched. Consequently the searching efficiency can be significantly enhanced. In addition, a new indicator of minimizing the conditional number of the sub-square matrix of the remaining normal equations is introduced to select three independent ambiguities. Once the correct integer values of the selected three independent ambiguities are substituted into the constraint equations to determine the remaining ambiguities, the estimated values are always rather close to their integers, which can be, in turn, applied as additional strong constraints to further improve the searching efficiency.

The rest of the paper is organized as follows. "Least Squares Ambiguity Search Technique" will briefly review the least squares ambiguity search technique from the point of view of the mixed integer least squares model, as was first set in this mathematical terminology by Xu et al. (1995) and Xu (1998, 2006). The new algorithm for AR by treating part of the normal equations as constraints is developed in "Ambiguity Resolution with Constraints from Normal Equations", and its difference from the currently existing related techniques as well as its benefits are discussed. In "Numerical Experiments and Analysis", two case studies, from real dual-frequency GPS data of about 10 km baseline and random simulations respectively, are carried out to demonstrate the performance and efficiency of the proposed algorithm. Finally, the research findings are given to conclude the paper.

Least Squares Ambiguity Search Technique

The unwanted real-valued parameters of biases, such as the satellite and receiver clock errors, are often first eliminated by using double difference technique, which is mathematically justified by the equivalence theorem in the

case of GPS (see e.g., Schaffrin and Grafarend 1986; Shen and Xu 2008; Shen et al. 2008). In addition, the DD ambiguities are specified to integer and the equivalence theorem holds true only for the real-valued parameters and cannot be used to eliminate the integer parameters (Xu 2006). Therefore, the DD model is preferred in the real GPS applications. The DD observational equation is described by

$$\lambda\Delta\varphi = \Delta\rho + \Delta\delta_{orb} + \Delta\delta_{tro} - \Delta\delta_{ion} + \lambda\Delta N + \Delta\varepsilon \tag{1}$$

where Δ denotes the double difference operation product. $\Delta\varphi$ is the DD measurement in cycle; $\Delta\rho$ is DD geometric distance from satellite to receiver antenna; $\Delta\delta_{orb}$ is DD satellite orbital error in meter; $\Delta\delta_{tro}$ is the DD tropospheric propagation delay with free of frequency influence in meter; $\Delta\delta_{ion}$ is the DD ionospheric delay with an inverse proportion to the squared frequency; ΔN is the DD integer ambiguity and λ is its corresponding wavelength; $\Delta\varepsilon$ is DD random noise. For a short baseline, all of the remaining systematic biases of Eq. (1), such as orbital error, tropospheric and ionospheric effects, can be basically ignored. If there are (m+1) satellites are simultaneously tracked, the linearized DD observation model for one epoch reads,

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \lambda\boldsymbol{z} + \boldsymbol{\varepsilon} \tag{2}$$

where $\boldsymbol{y}$ is an m-dimensional column vector of DD measurements; $\boldsymbol{x}$ is 3-dimensional column vector of coordinates; $\boldsymbol{z}$ is m-dimensional column vector of integer ambiguities, respectively. $\boldsymbol{A}$ is the $m\times3$ design matrix with full column rank. $\boldsymbol{\varepsilon}$ is an m-dimensional column vector of observation noises. It is noticed that the DD ionospheric delay would be still significant for short baseline in the case of severe sunspot activity and should be carefully considered for reliable AR and precise positioning (see e.g., Abdullah et al. 2009).

In the least squares ambiguity search technique, all DD observables of one epoch are separated into two groups, and three well-conditioned observables are classified into the primary group and the others into the secondary group, namely, $\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_P \\ 3\times1 \\ \boldsymbol{y}_S \\ (m-3)\times1 \end{bmatrix}$, $\boldsymbol{z} = \begin{bmatrix} \boldsymbol{z}_P \\ 3\times1 \\ \boldsymbol{z}_S \\ (m-3)\times1 \end{bmatrix}$ and $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_P \\ 3\times3 \\ \boldsymbol{A}_S \\ (m-3)\times3 \end{bmatrix}$. Here, the subscripts P and S denote the primary and secondary groups, respectively. For each potential integer candidate $\boldsymbol{z}_P$ in the primary group, the corresponding potential position $\hat{\boldsymbol{x}}_P$ is uniquely determined by

$$\hat{\boldsymbol{x}}_P = \boldsymbol{A}_P^{-1}\left(\boldsymbol{y}_P - \lambda\boldsymbol{z}_P\right) \tag{3}$$

By use of the sequential least squares adjustment, the remaining ambiguities of the secondary group with respect to the potential position solution $\hat{\boldsymbol{x}}_P$ is trivially solved by

$$\hat{\boldsymbol{z}}_S = \left(\boldsymbol{y}_S - \boldsymbol{A}_S\hat{\boldsymbol{x}}_P\right)/\lambda \tag{4}$$

In principle, all elements of the vector $\hat{\boldsymbol{z}}_S$ should be within their confident intervals that could be previously defined by the code measurements. If all the elements are really of a very small variance, they can be directly rounded off to their nearest integers, otherwise the float solution $\hat{\boldsymbol{z}}_S$ and its corresponding potential position $\hat{\boldsymbol{x}}_P$ as well as the potential integer candidate $\boldsymbol{z}_P$ will be excluded. By this way, all potential integer candidates in the primary group are validated via substituting them into Eqs. (3) and (4). If there are still more than two admissible integer ambiguity sets after the above validation, the ratio indicator will be further applied to pick out the final solution of the current epoch. The validation will be continuously performed over the next a few epochs until the final solution becomes reliably available. Furthermore, inserting Eq. (3) into Eq. (4), the direct relationship between $\boldsymbol{z}_P$ and $\boldsymbol{z}_S$ of two groups can be obtained, which is the essence of the ARCE algorithm (Park et al. 1996). Obviously, the ARCE algorithm is exactly equivalent to the least squares ambiguity search technique.

We would like to give some comments on the least squares ambiguity search process here. First of all, this method is, in general, suboptimal and thus we cannot guarantee to achieve the largest success probability of AR. Additionally, all formulae are derived on the basis of the single epoch observables because of the fact that the observation equations with carrier measurements are of rank defect with number of three only for the single epoch observation model. In other words, it is originally developed for kinematic positioning where the position is variable epoch by epoch. Admittedly, it can also be applied in the static scenario and the desired results could be obtained. However, the latent assumption that the different positions are defined for different epochs is utilized, which obviously loses the promising information that the unique position should be defined for all epochs in the case of static applications.

Ambiguity Resolution with Constraints from Normal Equations

Mathematical Model for Ambiguity Resolution with Constraints from Normal Equations

Without loss of the generality, the observation equations are newly symbolized to include the measurements of multiple epochs by

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} + \boldsymbol{\varepsilon} \tag{5}$$

where $\boldsymbol{x}$, $\boldsymbol{z}$, $\boldsymbol{A}$, $\boldsymbol{y}$ and $\boldsymbol{\varepsilon}$ have the same meanings as those of Eq. (2) except for multiple epochs; $\boldsymbol{B}$ is the newly introduced design matrix of full column rank for ambiguity parameters of multiple epochs. The Eq. (5) has been also known mathematically as the mixed integer linear model (see e.g., Xu et al. 1995; Xu 2006), and the least squares criterion is often applied to solve it by

$$\min: \Phi = \left(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{z}\right)^{\mathrm{T}} \boldsymbol{P} \left(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{z}\right) \tag{6}$$

Here $\boldsymbol{P}$ is the positive weight matrix of observation vector $\boldsymbol{y}$. The essential difference of Eq. (6) from the traditional purely real-valued least squares model is that the integer constraint is imposed to $\boldsymbol{z}$. According to the rigorous derivation of Xu et al. (1995), we can only differentiate Φ with respect to real-valued parameter $\boldsymbol{x}$ and equate the differential to zero since $\boldsymbol{z}$ is not continuous and cannot be differentiated. Therefore, the estimate of $\boldsymbol{x}$ can be expressed in terms of the estimate of $\boldsymbol{z}$. Then by substituting the estimate of $\boldsymbol{x}$ back into Eq. (6), the integer least squares problem is alternatively reduced to

$$\text{min: } \Phi=\left(\boldsymbol{z}-\hat{\boldsymbol{z}}_{LS}\right)^{T}\boldsymbol{Q}_{\hat{\boldsymbol{z}}}^{-1}\left(\boldsymbol{z}-\hat{\boldsymbol{z}}_{LS}\right) \tag{7}$$

where $\hat{\boldsymbol{z}}_{LS}$ is the least squares float ambiguity solution of the following normal equation,

$$\boldsymbol{N}\boldsymbol{z}=\boldsymbol{w} \tag{8a}$$

with

$$\begin{aligned}\boldsymbol{N}&=\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{B}\\ \boldsymbol{w}&=\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{l}\\ \boldsymbol{P}_1&=\boldsymbol{P}-\boldsymbol{P}\boldsymbol{A}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\end{aligned} \tag{8b}$$

The covariance matrix $\boldsymbol{Q}_{\hat{\boldsymbol{z}}}=\sigma^2\boldsymbol{N}^{-1}$ with σ^2 being the prior variance of unit weight. Since the discrete property of integer ambiguities, they are determined necessarily by searching technique based on the criterion (7).

In the fast GPS positioning, the normal matrix $\boldsymbol{N}$ is severely ill-conditioned, refer to e.g., Xu et al. (1999); Shen and Li (2007) and also see Fig. 2, which means that the covariance matrix $\boldsymbol{Q}_{\hat{\boldsymbol{z}}}$ is highly correlated and small noises in observation vector $\boldsymbol{y}$ will lead to large errors in $\hat{\boldsymbol{z}}_{LS}$. In the both static and kinematic GPS applications, the decorrelation technique is usually employed to make $\boldsymbol{Q}_{\hat{\boldsymbol{z}}}$ minimally correlated by conducting unimodular transformation on it (see e.g., Teunissen 1994; Xu et al. 1995; Liu et al. 1999; Grafarend 2000; Xu 2001, 2006). Nevertheless, the decorrelation cannot guarantee to improve the searching efficiency in the scenario of the high-dimensional integer searching as recognized by Xu (2001) and Lou and Grafarend (2003). In fact, the high-dimensional integer searching problem would become crucial when the number of integer unknowns becomes extremely large due to the use of multiple satellite systems and multiple frequency signals (Feng 2008; Li et al. 2009), with which we have to be confronted in the next few years. Therefore, we will attempt to develop an innovative method for more efficient AR comparing with the existing methods in the case of high-dimensional integer searching, which will bring some benefits to the applications of the future multiple frequency and satellite systems.

In the fast GPS positioning of single baseline model with either pure carrier measurements or carrier and code

measurements, there is often high correlation between three coordinate parameters and ambiguities, so that the normal equations are severely ill-posed. Even if the coordinate parameters are eliminated, this nature of high correlation will be, to a hair, translatablely imposed to the ambiguities and the reduced normal equations (8*a*) remains similarly ill-posed. In the single baseline solution, there is, in fact, a large difference between the last three eigenvalues of the coefficient matrix $\boldsymbol{N}$ of normal equations and the others, and there are three extremely small eigenvalues (see e.g., Fig. 3).

Applying Singular Value Decomposition (SVD) to the symmetric positive normal matrix $\boldsymbol{N}$ in Eq. (8*a*), we have

$$\boldsymbol{N} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{\mathrm{T}} \tag{9}$$

where $\boldsymbol{V}$ is an normalized orthogonal eigenvector matrix satisfied with $\boldsymbol{V}\boldsymbol{V}^{T} = \boldsymbol{V}^{T}\boldsymbol{V} = \boldsymbol{I}_{m}$. $\boldsymbol{I}_{m}$ denotes the *m*-dimensional identity matrix, and $\boldsymbol{\Lambda}$ is an *m*-dimensional diagonal matrix of eigenvalues of $\boldsymbol{N}$, whose non-zero diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_m$ are, without loss of the generality, assumed to be arranged in the decreasing order, i.e. $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_m > 0$. Substituting Eq. (9) into Eq. (8*a*) and considering the property of orthogonal matrix $\boldsymbol{V}$, the alternative form of the normal equations (8*a*) reads,

$$\boldsymbol{\Lambda}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{z} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{w} \tag{10}$$

For the derivation of the following context, the matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{V}$ are expressed with sub-matrices as follows,

$$\boldsymbol{\Lambda} = \begin{pmatrix} \underset{(m-3)\times(m-3)}{\boldsymbol{\Lambda}_1} & \\ & \underset{3\times 3}{\boldsymbol{\Lambda}_2} \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} \underset{(m-3)\times(m-3)}{\boldsymbol{V}_{11}} & \underset{(m-3)\times 3}{\boldsymbol{V}_{12}} \\ \underset{3\times(m-3)}{\boldsymbol{V}_{21}} & \underset{3\times 3}{\boldsymbol{V}_{22}} \end{pmatrix} \tag{11}$$

where $\boldsymbol{\Lambda}_2$ is the diagonal matrix comprising only three extremely small eigenvalues and diagonal matrix $\boldsymbol{\Lambda}_1$ all the other eigenvalues. Substituting Eq. (11) into Eq. (10), we obtain the following equations,

$$\begin{pmatrix} \boldsymbol{\Lambda}_1\boldsymbol{V}_{11}^{T} & \boldsymbol{\Lambda}_1\boldsymbol{V}_{21}^{T} \\ \boldsymbol{\Lambda}_2\boldsymbol{V}_{12}^{T} & \boldsymbol{\Lambda}_2\boldsymbol{V}_{22}^{T} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_D \\ \boldsymbol{z}_I \end{pmatrix} = \begin{pmatrix} \boldsymbol{V}_{11}^{T}\boldsymbol{w}_D + \boldsymbol{V}_{21}^{T}\boldsymbol{w}_I \\ \boldsymbol{V}_{12}^{T}\boldsymbol{w}_D + \boldsymbol{V}_{22}^{T}\boldsymbol{w}_I \end{pmatrix} \tag{12}$$

with $\boldsymbol{z} = \begin{pmatrix} \underset{(m-3)\times 1}{\boldsymbol{z}_D} \\ \underset{3\times 1}{\boldsymbol{z}_I} \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} \underset{(m-3)\times 1}{\boldsymbol{w}_D} \\ \underset{3\times 1}{\boldsymbol{w}_I} \end{pmatrix}$. Truncating off the last three normal equations of Eq. (12) with respect to three extremely small eigenvalues, the remaining (*m*-3) normal equations are further employed as the constraint equations among ambiguities to alleviate the burden of ambiguity searching. Multiplying $\boldsymbol{V}_{11}$ right to the remaining equations, the alternative form of constraint equations is obtained,

$$\begin{pmatrix} \boldsymbol{V}_{11}\boldsymbol{\Lambda}_1\boldsymbol{V}_{11}^{T} & \boldsymbol{V}_{11}\boldsymbol{\Lambda}_1\boldsymbol{V}_{21}^{T} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_D \\ \boldsymbol{z}_I \end{pmatrix} = \boldsymbol{V}_{11}\boldsymbol{V}_{11}^{T}\boldsymbol{w}_D + \boldsymbol{V}_{11}\boldsymbol{V}_{21}^{T}\boldsymbol{w}_I \tag{13}$$

Apparently, the constraint equations (13) are of rank defect with the number of three, therefore only three ambiguities

are independent. Some discussions on the constraint equations are expanded as follows. If one of three extremely small eigenvalues is grouped into the diagonal matrix $\boldsymbol{\Lambda}_1$, its corresponding normal equation is no longer truncated off but reserved. As a result, the number of remaining normal equations becomes (m-2), and then the remaining equations will be ill-posed and cannot be used as the constraint equations anymore, because the ambiguity solution is very sensitive to the observation noises in this ill-posed model. Oppositely, if the part of large eigenvalues is grouped into the diagonal matrix $\boldsymbol{\Lambda}_2$, more than three equations will be truncated off and the number of constraint equations is reduced. In general, the number of the independent ambiguities is equal to that of rank defect of constraint equations. In other words, the number of independent ambiguities will be increased with the decrease of the number of constraint equations. Therefore, the searching efficiency would be relatively somewhat faded when the part of large eigenvalues is truncated off. Fortunately, we can guarantee to separate all the terms with respect to the large eigenvalues into the matrix $\boldsymbol{\Lambda}_1$ and the others into the matrix $\boldsymbol{\Lambda}_2$ by means of SVD. Thus, the normal equations only with respect to the extremely small eigenvalues can be successfully separated and truncated off. In addition, the algorithm can be easily expanded to include more parameters, such as tropospheric and ionospheric biases, in the observation model (1). In that situation, we can also use our algorithm to efficiently determine and truncate off the terms with small eigenvalues.

Without loss of the generality, we assume that $\mathbf{z}_I$ consists of three independent ambiguities, and once they are correctly fixed to their integers, the remaining ambiguities $\mathbf{z}_D$ can be trivially solved as,

$$\hat{\boldsymbol{z}}_D = \left(\boldsymbol{V}_{11}\boldsymbol{\Lambda}_1\boldsymbol{V}_{11}^T\right)^{-1}\boldsymbol{V}_{11}\left(\boldsymbol{V}_{11}^T\boldsymbol{w}_D + \boldsymbol{V}_{21}^T\boldsymbol{w}_I - \boldsymbol{\Lambda}_1\boldsymbol{V}_{21}^T\boldsymbol{z}_I\right) \tag{14}$$

In principle, whatever three independent ambiguities are chosen, the corresponding matrix $\boldsymbol{V}_{11}\boldsymbol{\Lambda}_1\boldsymbol{V}_{11}^{\mathrm{T}}$ is well conditioned because three extremely small eigenvalues have been primarily excluded and their corresponding normal equations also have been truncated off. Thus, the solved float ambiguities $\hat{\boldsymbol{z}}_D$ are rather reliable and close to integers. Their integer solutions can be simply obtained by,

$$\boldsymbol{z}_D = \mathrm{round}\left(\hat{\boldsymbol{z}}_D\right) \tag{15}$$

where round(·) is rounding operator to map a real-valued number into its nearest integer. The key of the new algorithm is that all potential candidates are composed not of all ambiguities, but only of three independent ambiguities $\mathbf{z}_I$ around their least squares float solutions within their confident intervals. Therefore, the number of potential candidates can be dramatically reduced and then the searching efficiency is significantly enhanced. It is important to emphasize that the new method is also of sub-optimality similar to the least squares ambiguity searching technique since the rounding method is applied to fix the integers.

New Indicator for Efficient Ambiguity Searching

We have understood well that the better condition of the square matrix $\boldsymbol{V}_{11}\boldsymbol{\Lambda}_1\boldsymbol{V}_{11}^{\mathrm{T}}$ can derive the more precise float ambiguities $\boldsymbol{z}_D$ by Eq. (14), because observation noises in both $\boldsymbol{w}_D$ and $\boldsymbol{w}_I$ will be enlarged in the solved float ambiguities if the worse condition is assigned to $\boldsymbol{V}_{11}\boldsymbol{\Lambda}_1\boldsymbol{V}_{11}^{\mathrm{T}}$. Therefore, the choice of three independent ambiguities for well conditioned $\boldsymbol{V}_{11}\boldsymbol{\Lambda}_1\boldsymbol{V}_{11}^{\mathrm{T}}$ is very crucial. For this reason, an indicator named "ICOND" for determination of three independent ambiguities is defined to minimize the effects of observation noises on the estimator $\hat{\boldsymbol{z}}_{\mathrm{D}}$ as follows,

$$\mathrm{ICOND} = \mathrm{cond}\left(\boldsymbol{V}_{11}\boldsymbol{\Lambda}_1\boldsymbol{V}_{11}^{\mathrm{T}}\right) \tag{16}$$

where cond(·) is the operation product for calculating the conditional number of a matrix. The conditional numbers of all (m-3)×(m-3) sub-square matrices from the normal matrix $\boldsymbol{N}$ should be calculated, and three independent ambiguities vector $\boldsymbol{z}_I$, corresponding to minimal "ICOND", is accepted as the final three independent ambiguities.

Once the correct integer values of the selected three independent ambiguities based on the new indicator are used to determine the remaining ambiguities $\hat{\boldsymbol{z}}_D$, the estimated float ambiguities are, in principle, rather close to integers just because of the influence of the observation noise. In other words, any integer candidate $\boldsymbol{z}_I$ can obtain the unique estimate $\hat{\boldsymbol{z}}_D$ by Eq. (14) and the difference between the estimates and their nearest integers should be smaller than a fraction, for instance 0.1 cycles (see e.g. Fig. 3). If any of the difference is beyond the given fraction, this integer candidate should be directly rejected, which can be, in turn, applied as an additional constraint to further reduce searching burden.

The extensive experiments carried out by the authors show that this additional constraint is strong and can be used to exclude quite a few integer candidates, and thus significantly improve the searching efficiency. After filtering out the unacceptable candidates with this additional constraint, just several integer candidates are reserved even only one. If there are two or more reserved integer candidates, additional test, such as ratio statistic, should further be used to pass the final solution,

$$ratio = \frac{\Omega_{\mathrm{sec}}}{\Omega_{\mathrm{min}}} \geq r \tag{17}$$

where Ω_{min} and Ω_{sec} are the minimal and the second minimal statistics computed by Eq. (7). r is a threshold and always determined empirically more than theoretically. For more rigorous and efficient indicator, one can be referred to Han (1997).

We would also like to highlight the benefits of the new algorithm. First of all, all the formulae are derived just based

on the normal equations for AR, nothing else necessary, and in some senses it is suitable for both single epoch and multiple epoch solutions in the either static or kinematic models. Secondly, the new indicator "ICOND" can guarantee that the solved remaining ambiguities are rather close to their integers, and thus the additional strong constraint is introduced in the new algorithm for faster searching.

Numerical Experiments and Analysis

The flowchart of performance of the new algorithm is presented in Fig. 1. In the whole process, the inputs are just the normal equation information, i.e. normal matrix $\boldsymbol{N}$ and constant vector $\boldsymbol{w}$, for the both single epoch and multiple epoch solutions in either static or kinematic positioning applications, and no any other information necessary. In terms of the flowchart, the program was written in Matlab7.4 language for the postprocessing. In this section, two case studies, from real dual-frequency GPS data of about 10 km baseline and random simulations respectively, are carried out to demonstrate the successful performance of the new algorithm and its efficiency.

Case 1: Performance of New Algorithm for 10 km Baseline with Dual-frequency Data

The dual-frequency data of about 10 km baseline was collected by two dual-frequency Ashtech geodetic GPS receivers equipped with choke ring antennas for mitigating the effect of multipath. The sample interval is 5 seconds, and the cut off elevation angle of all observables is set to 15 degrees. Total 600 epochs are adopted in the experiment, and in the whole observation series, there are common 6 satellites above the cut off elevation and thus total 10 DD ambiguities at two frequencies. The DD ambiguities are primarily fixed to their integer values and then the precise baseline is also determined with all data, which serve as actual values in the whole experiments.

We study the rapid AR using 10 epoch data from 600 epochs for each performance. Starting with the first 10 epoch data, each new performance updates the last epoch with one new epoch data, there are total 591 sets. For each performance, one conditional number and 10 eigenvalues of the normal matrix $\boldsymbol{N}$ are calculated, and the results for all 591 performances are illustrated in Figs. 2 and 3. As Fig. 2 shown, the conditional numbers are about 10^6, and the normal equations for AR are severely ill-posed. Fig. 3 illustrates the fact that there is indeed a large difference with magnitude order of about 10^5 between the last three eigenvalues and the others, and the last three eigenvalues are extremely small. We can clearly see that all eigenvalues in the first seven subplots are between about $10^{-0.882}$ and $10^{-0.078}$, and all in the last three subplots between about $10^{-6.5}$ and $10^{-4.0}$. In addition, the eigenvalues in each subplot for all performances are rather stable with a slight variation.

After three independent ambiguities are chosen by means of the indicator defined by Eq. (16) for minimizing the effects of observation noises, the remaining ambiguities can be calculated by Eq. (14). In principle, if the correct

integer values of three independent ambiguities are substituted into Eq. (14), the computed remaining float ambiguities will be very close to their integers, and consequently their differences from their integers for all performances are very small as shown in Fig. 4. The differences for all remaining ambiguities are smaller than 0.1 cycles, and apparently the solved ambiguities can be successfully fixed. Oppositely, if one candidate of three independent ambiguities is used to determine the remaining ambiguities and the estimated ambiguities are biased from their nearest integers by 0.1 cycles, it can be directly excluded. Finally, the residuals of three coordinate components of the baseline are calculated using the fixed ambiguities. As Fig. 5 illustrated, all residuals are smaller than 3.5 cm, which in turn verifies the correctness of AR.

Case 2: Random Simulations for Assessment of Efficiency

The random simulation technique was firstly proposed by Xu (2001) to numerically compare the performance of different decorrelation methods, and it is employed to assess the efficiency of the proposed algorithm for the different dimensional integer parameters by comparing with LAMBDA method that is currently the most popularly used AR method.

The random simulations were implemented in this paper mainly according to Xu (2001) and Chang et al. (2005). The float ambiguity vector $\hat{\boldsymbol{z}}$ is primarily constructed as

$$\hat{\boldsymbol{z}} = 100 \times \mathrm{randn}(n,1) \tag{18}$$

where randn(n,1) is a MATLAB built-in function to generate a vector of n random elements that are normally distributed with mean zero and variance one. The normalized orthogonal matrix $\boldsymbol{V}$ is computed by factorization of a random square matrix, and the eigenvalues λ_i (i=1, 2, …, n) of the diagonal matrix $\boldsymbol{\Lambda}$ are the positive simulated random numbers. Then the simulated normal equations consisting of $\boldsymbol{N}$ and $\boldsymbol{w}$ can be trivially determined by $\boldsymbol{N} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{\mathrm{T}}$ and $\boldsymbol{w} = \boldsymbol{N}\hat{\boldsymbol{z}}$ with the added normally distributed errors. For more details on random simulation, the reader is referred to Xu (2001) and Chang et al. (2005).

It is crucial to apply a scale factor, for instance 10^{-5}, to the last three eigenvalues for obtaining three extremely small eigenvalues and thus a gap between them and the others. Furthermore, we should also notice that the ill-posed degree for the fast AR would be no longer so serious due to more measurements being simultaneously used but the number of coordinate parameters being still three in the future multiple GNSS system and multiple frequency applications, although the dimension of the integer unknowns is increased. Therefore, in order to demonstrate the full performance of the new algorithm for the future high dimensional AR, four scale factors, i.e., 10^{-2}, 10^{-3}, 10^{-4} and 10^{-5} are individually applied to describe the ill-posed degree of simulated normal equations.

Table 1. Average running time for different dimensions with scale factors of 10^{-2} and 10^{-4} (unit: seconds)

Scale factors	Dimensions						
	10	15	20	25	30	35	40
10^{-2}	0.084	0.122	0.272	0.705	1.554	3.289	6.807
10^{-4}	0.195	0.253	0.432	1.222	2.376	4.478	6.871

All of the computation schemes were performed with MATLAB7.4 programs on a Pentium D, 2×1.86GHz PC with 2GB memory running Windows XP professional 2002. We simulate the dimension of integer ambiguities as n=6,7,…,40 for each scale factor, and perform 20 runs for each dimension. The average running time for each dimension with different scale factor is presented in Fig. 6, where the column axis stands for the common logarithm of average running time in seconds. Subplots a, b, c and d illustrate the running time for the scale factors of 10^{-2}, 10^{-3}, 10^{-4} and 10^{-5}, respectively. Tables 1 present the comparison of average running time for some given dimensions with scale factors of 10^{-2} and 10^{-4}. In general, the more running time is needed when the dimension is higher and the scale factor smaller. It is important to notice that the running time is still modest and acceptable in high dimensional cases, e.g., the running time is just about 6.8 seconds for all 40 dimension experiments. Moreover, we confirm that the running time can never increase dramatically with the higher dimensional integer parameters because only three integer parameters are necessary to be searched in whatever situations. However, the running time is several hundred seconds for LAMBDA algorithm in the 40 dimension case due to the inefficient decorrelation processing, referring to Chang et al. (2005). To sum up, the new algorithm can be efficiently applied for integer ambiguity searching, especially in the high-dimensional cases.

Concluding Remarks

We have proposed to truncate the terms of the normal equations for AR with respect to three extremely small eigenvalues in the cases of three real coordinate parameters to be estimated, and then use the remaining normal equations as constraints in efficiently searching integer ambiguities. In this way only three independent ambiguities need to be searched and the number of searching integer candidates is dramatically reduced. In addition, the new indicator has been introduced to determine three independent ambiguities, and the conditional number of the sub-square matrix corresponding to the selected three independent ambiguities is minimum, which can guarantee that the computed remaining ambiguities are so close to their integers that they can be directly rounded off. Since the normal equations are always primarily formed along with the float ambiguity solution, the new algorithm is rather convenient and efficient to obtain the constraint equations, and in some senses suitable for both single epoch and multiple epoch solutions in the either static or kinematic models. The results from two case studies have demonstrated

that new algorithm is very efficient, especially for high-dimensional cases. It is worthy to point out that this approach is easy to be extended in the medium or long baseline cases, where more real parameters except for 3 coordinates need to be estimated, such as zenith tropospheric delay.

This research finding would become more important to the fast GNSS positioning situations using multiple satellite systems with multiple frequencies, such as modernized GPS, GALILEO, COMPASS and others, where the number of ambiguity parameters would be huge. In general, the new algorithm has shown computational advantages in dealing with high dimensional integer estimation, and would be a preferred approach for future GNSS AR.

Acknowledgements

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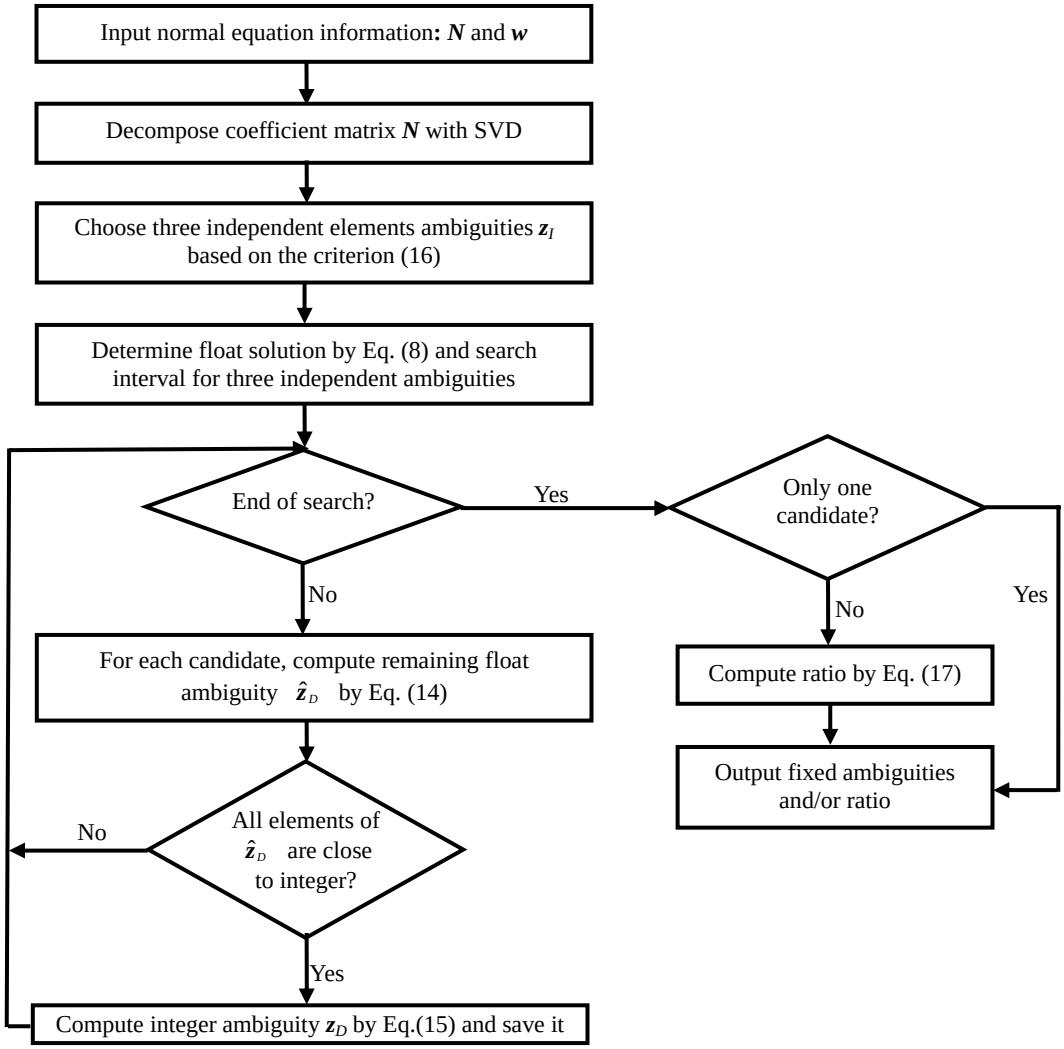
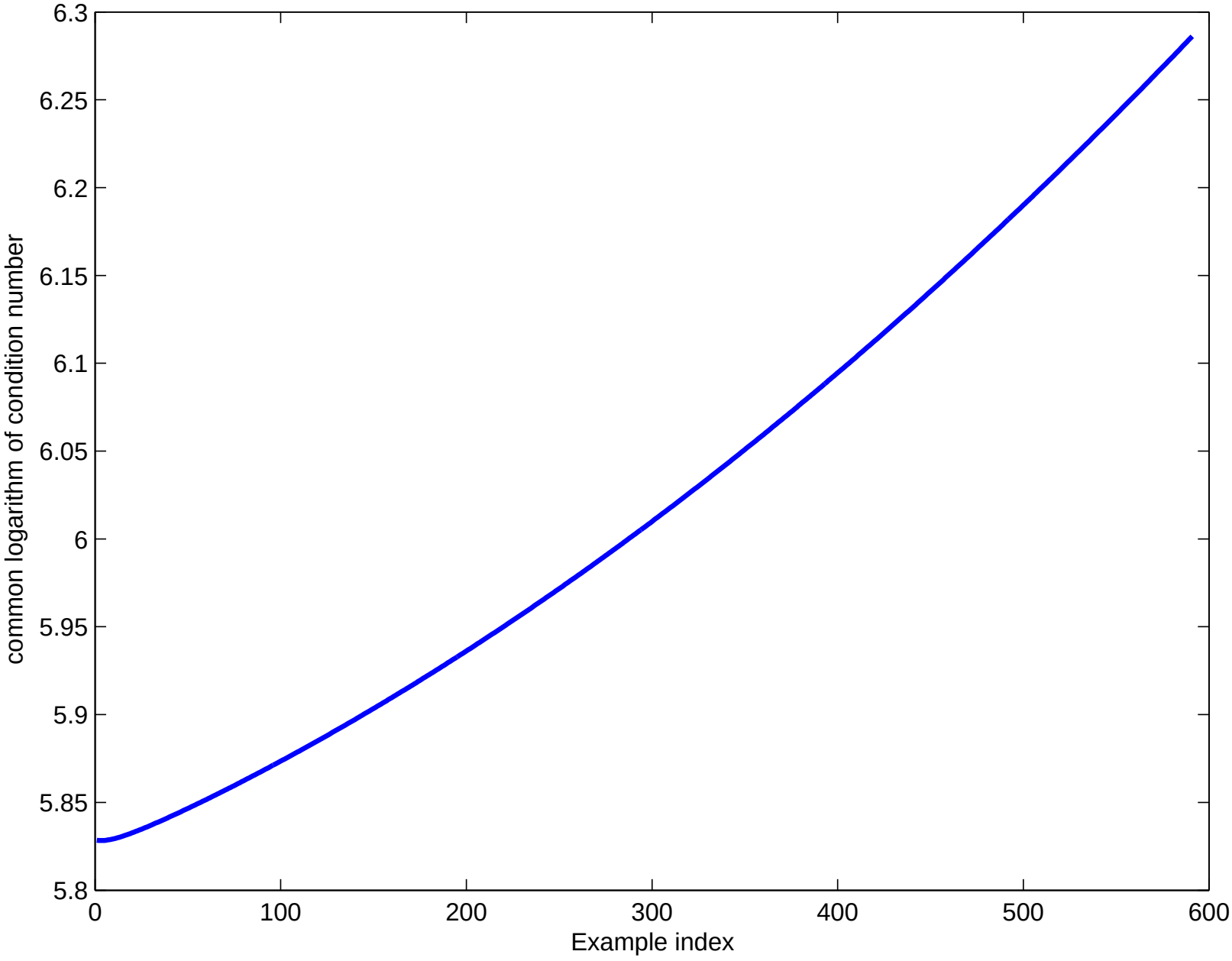
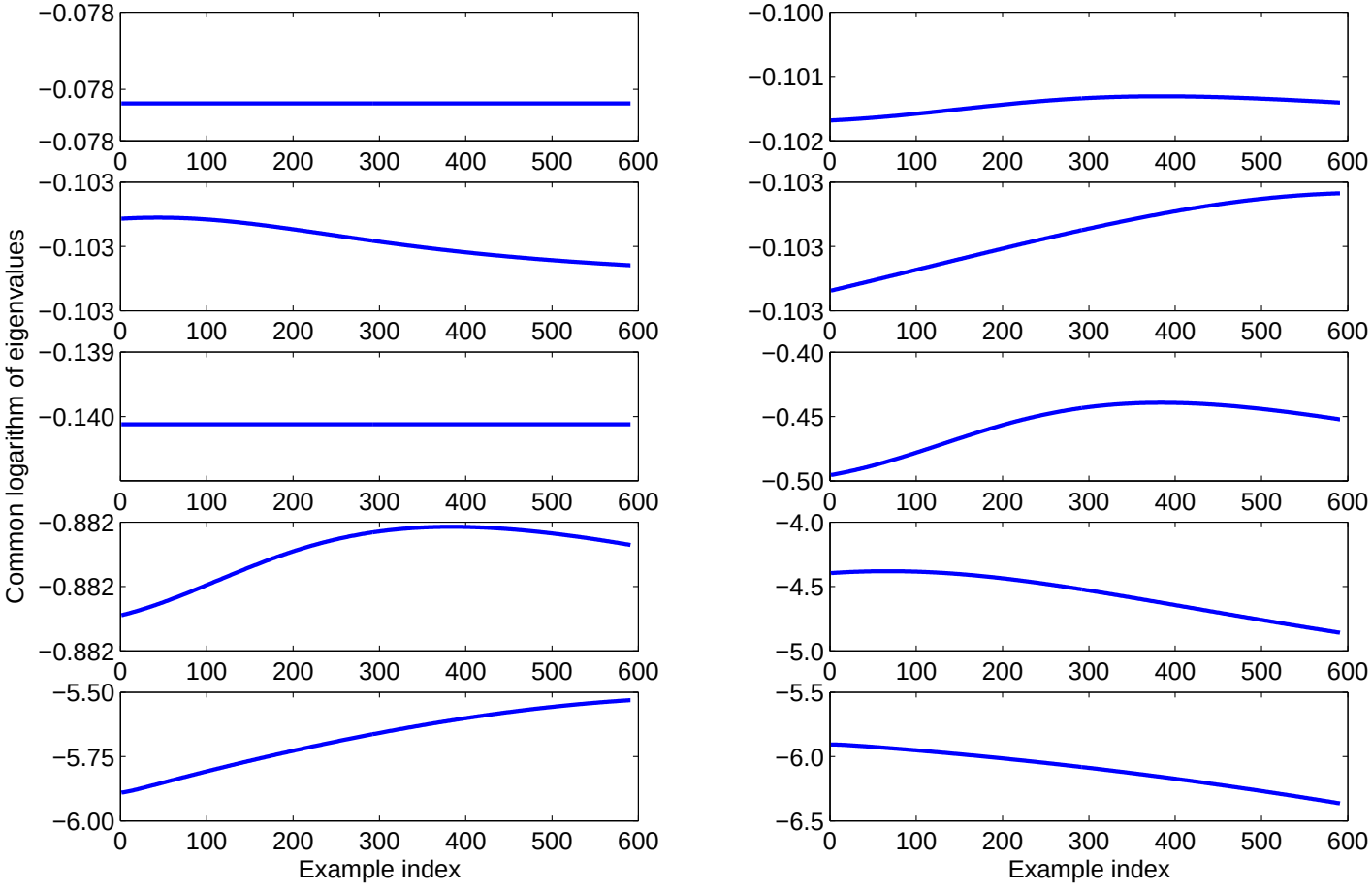
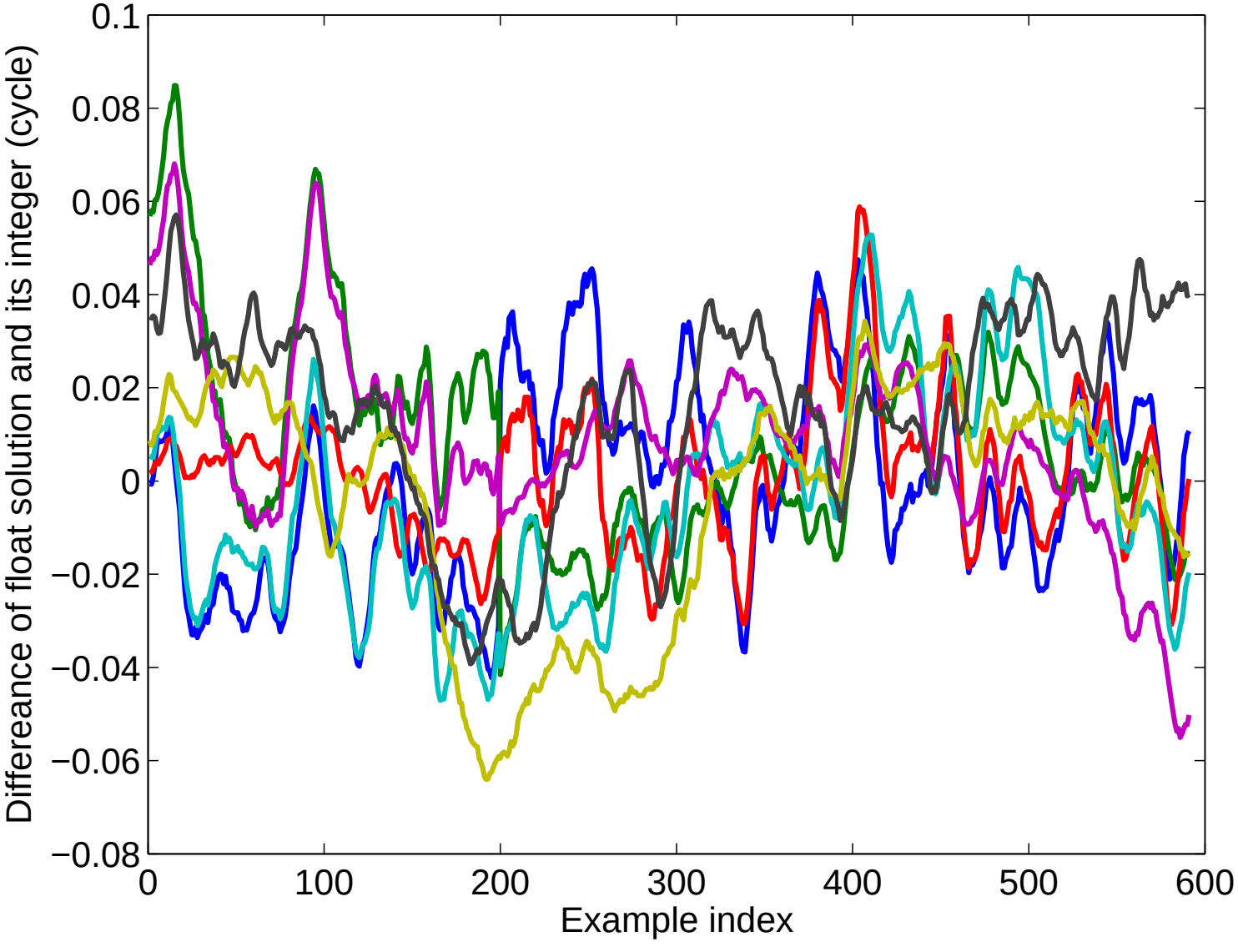
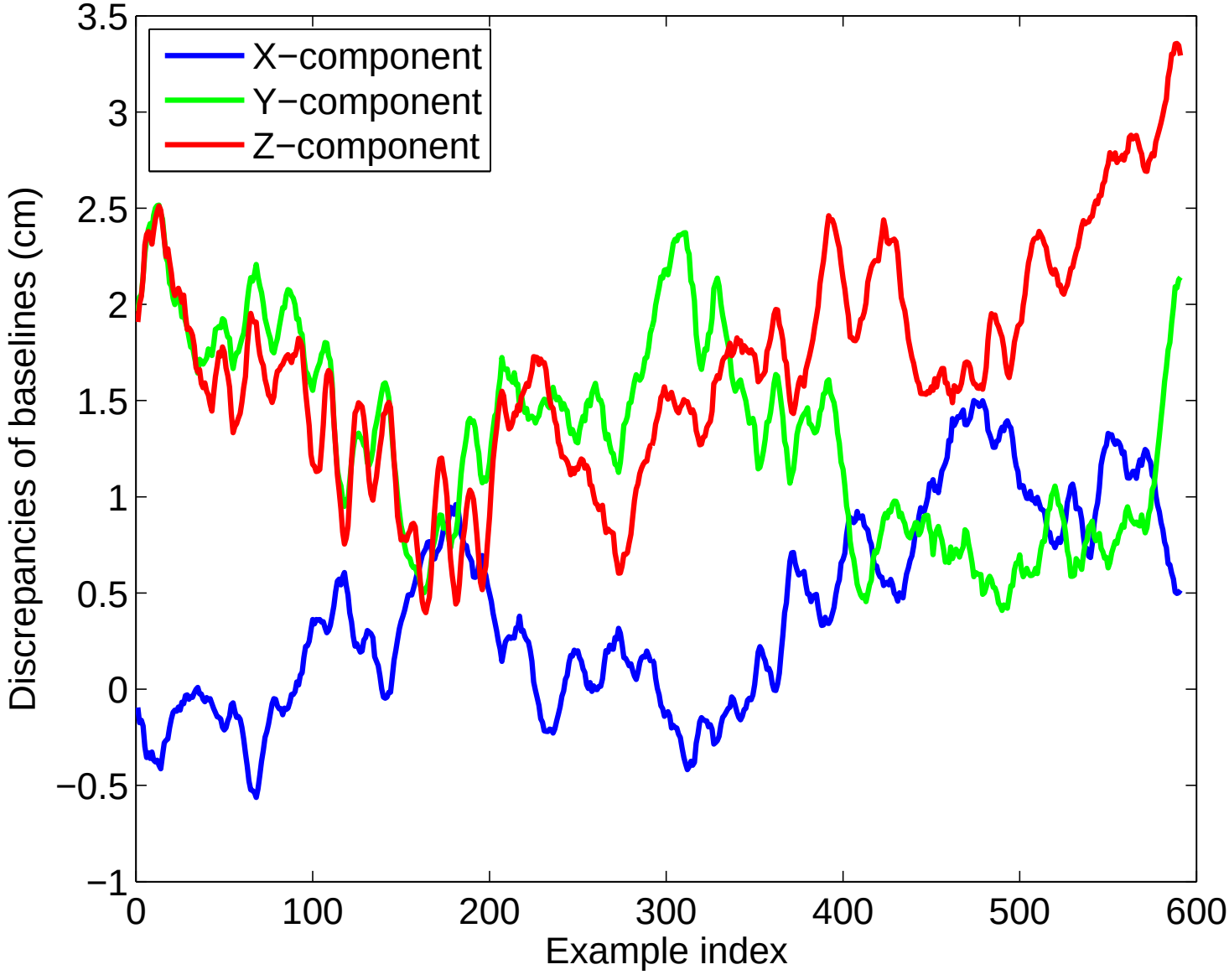
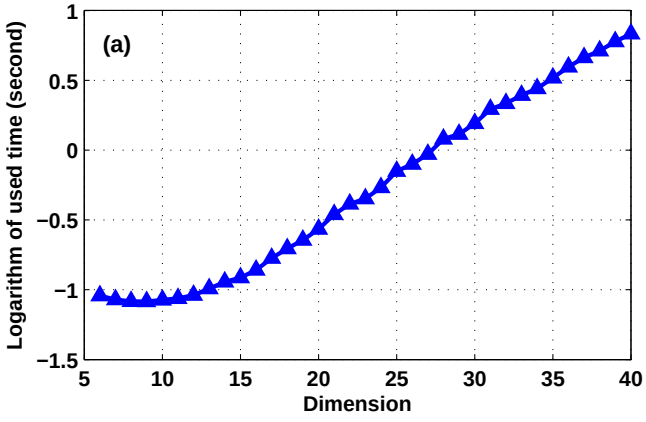

Input normal equation information: N and w
Decompose coefficient matrix N with SVD
Choose three independent elements ambiguities z_I based on the criterion (16)
Determine float solution by Eq. (8) and search interval for three independent ambiguities
End of search?
Yes
Only one candidate?
No
Yes
No
For each candidate, compute remaining float ambiguity $\hat{z}_D$ by Eq. (14)
Compute ratio by Eq. (17)
Output fixed ambiguities and/or ratio
All elements of $\hat{z}_D$ are close to integer?
No
Yes
Compute integer ambiguity z_D by Eq.(15) and save it

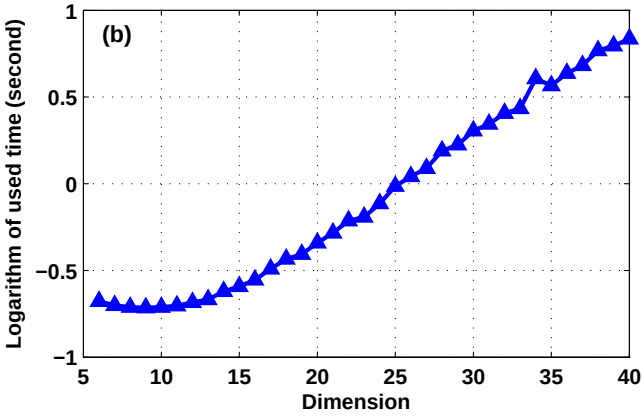

6.3
6.25
6.2
6.15
6.1
6.05
6
5.95
5.9
5.85
5.8
common logarithm of condition number
0
100
200
300
400
500
600
Example index

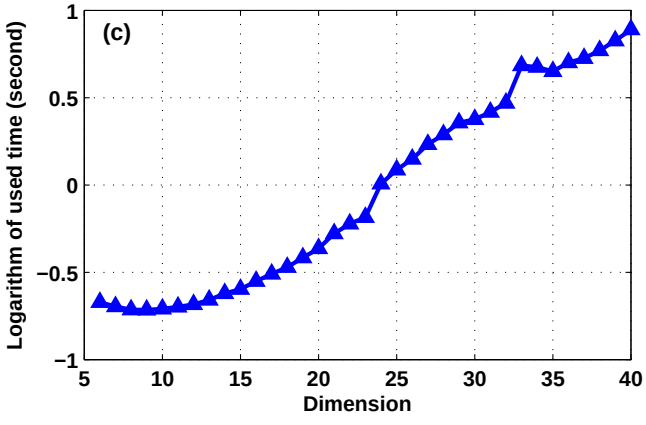

Common logarithm of eigenvalues
−0.078
−0.078
−0.078
−0.103
−0.103
−0.103
−0.139
−0.140
−0.882
−0.882
−0.882
−5.50
−5.75
−6.00
−0.100
−0.101
−0.102
−0.103
−0.103
−0.103
−0.40
−0.45
−0.50
−4.0
−4.5
−5.0
−5.5
−6.0
−6.5
0
100
200
300
400
500
600
Example index
Example index

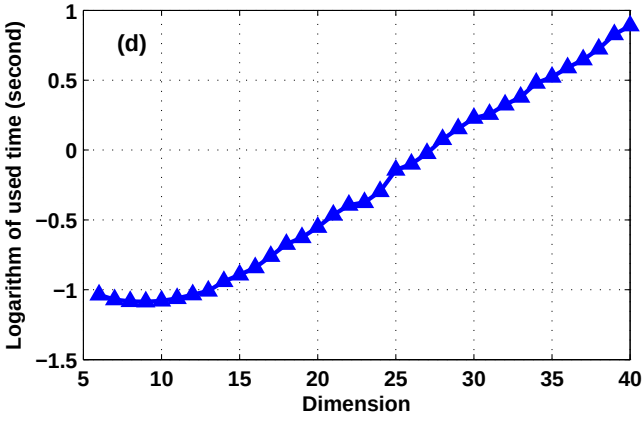

0.1
0.08
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
0
100
200
300
400
500
600
Example index
Differeance of float solution and its integer (cycle)


X−component
Y−component
Z−component
Discrepancies of baselines (cm)
Example index
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
0
100
200
300
400
500
600


(a)
Logarithm of used time (second)
Dimension
1
0.5
0
−0.5
−1
−1.5
5
10
15
20
25
30
35
40


(b)
Logarithm of used time (second)
Dimension
1
0.5
0
−0.5
−1
5
10
15
20
25
30
35
40


(c)
Logarithm of used time (second)
Dimension
1
0.5
0
−0.5
−1
5
10
15
20
25
30
35
40


(d)
Logarithm of used time (second)
Dimension
1
0.5
0
−0.5
−1
−1.5
5
10
15
20
25
30
35
40