A SUPERLINEARLY CONVERGENT METHOD FOR A CLASS OF COMPLEMENTARITY PROBLEMS WITH NON-LIPSCHITZIAN FUNCTIONS*

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Abstract. We consider a class of complementarity problems involving functions which are not Lipschitz continuous. In this paper we reformulate this class of non-Lipschitzian complementarity problems into a Lipschitzian complementarity problem. Then we propose an inexact smoothing Newton method to solve this Lipschitzian complementarity problem. We prove that our proposed method converges quadratically and globally under a mild condition. Numerical results show that this method is promising. This method can solve these kinds of complementarity problems with one million variables in reasonable time on a PC with 1 GB of RAM.

1. Introduction. Let $\Re^n_+ = \{x \in \Re^n : x \geq 0\}$, and define $F : \Re^n_+ \to \Re^n$ by

$$F(x) = Mx + Bx^p + q,$$

where M is an $n \times n$ matrix, B is an $n \times n$ diagonal matrix with positive diagonals, $x^p = [x_1^p, \ldots, x_n^p]^T$, $p \in (0, 1)$, and q is a vector in $\Re^n$. Because $p \in (0, 1)$, the function F is not Lipschitz continuous at a vector which has zero component. In this paper we consider the following nonlinear complementarity problem (NCP) and denote it by NCP(F). Find $x \in \Re^n_+$ such that

$$(1.1) \qquad x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0.$$

Such a non-Lipschitzian NCP arises from the reaction and diffusion problems [1, 4], which can be modeled as free boundary problems.

Example 1. [1, 4] Let Ω be a bounded open set in $\Re^2$ with Lipschitz boundary $\partial\Omega$. Given two positive numbers λ and $p \in (0, 1)$, consider the following free boundary problem:

$$\begin{aligned}
-\triangle u + \lambda u^p &= 0 \quad \text{in } \Omega_+, \\
u &= 0 \quad \text{in } \Omega_0, \\
u = |\nabla u| &= 0 \quad \text{on } \Gamma, \\
u &= 1 \quad \text{on } \partial\Omega,
\end{aligned}$$

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where $\Omega_+ = \{z \in \Omega : u(z) > 0\}$, $\Omega_0 = \{z \in \Omega : u(z) = 0\}$, and $\Gamma = \partial\Omega_0 = \partial\Omega_+ \cap \Omega$ are unknown, and $\triangle u = \frac{\partial^2 u}{\partial z_1^2} + \frac{\partial^2 u}{\partial z_2^2}$. Using finite element approximation or finite difference approximation, we obtain an NCP (1.1). Note that in (1.1), F is defined only on $\Re_+^n$. For the NCP obtained from Example 1, F has no definition on $\Re^n \backslash \Re_+^n$.

Over the last decade, a number of methods for solving the NCP have been developed (see [9, Chapters 9 and 11]). However, most of these efficient methods require the involved function F to be Lipschitz continuous. For instance, smoothing Newton methods [6, 14] and semismooth Newton methods [8] assume that F is continuously differentiable. For these methods, in order to get the Q-quadratic convergence property, it is assumed that F' is Lipschitz continuous. Without the Lipschitzian continuity of F, it seems hard to find a fast convergent method to solve the NCP(F).

Recently, Chen [5] presented a smoothing Newton method for solving the following system of equations with non-Lipschitzian functions by reformulating it as a system of equations with locally Lipschitzian functions.

(1.2)
$$S(x) := Mx + Bf(x) + q = 0,$$

where for $i = 1, 2, \ldots, n$,

$$f_i(x) = f_i(x_i) = \begin{cases} x_i^p, & x_i \geq 0, \\ 0, & x_i < 0. \end{cases}$$

Under the condition that M is a symmetric positive definite matrix, it is proved that the smoothing Newton method [5] converges superlinearly and globally. For a general class of non-Lipschitzian complementarity problems, a late research work was given by Alefeld and Chen [1], in which a regularized projection method has been proposed for solving this general class of non-Lipschitzian complementarity problems. This projection method has global and linear convergence properties under the condition that M is an H-matrix with positive diagonals.

An $n \times n$ matrix $A = (a_{ij})$ is called a positive definite matrix if for any $x \in \Re^n$ with $x \neq 0$, $x^T Ax > 0$. The matrix A is called a P-matrix if, for every $x \in \Re^n$ with $x \neq 0$, it holds that

$$\max_i x_i[Ax]_i > 0.$$

The matrix A is called a P_0-matrix if, for every $x \in \Re^n$ with $x \neq 0$, there is an index $i_0 = i_0(x)$ with

$$x_{i_0} \neq 0 \quad \text{and} \quad x_{i_0}[Ax]_{i_0} \geq 0.$$

Clearly a positive definite matrix is a P-matrix, and a P-matrix is a P_0-matrix. A is called an M-matrix [12] if (i) $a_{ii} > 0$ for $i = 1, 2, \ldots, n$, (ii) $a_{ij} \leq 0 (i \neq j)$ for $i, j = 1, 2, \ldots, n$, and (iii) A is nonsingular and $A^{-1} \geq 0$. Here $A^{-1} \geq 0$ means that each entry of A^{-1} is nonnegative. A is called an H-matrix if its comparison matrix $\tilde{A} = (\tilde{a}_{ij})$ is an M-matrix, where

$$\tilde{a}_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases} \quad i, j = 1, 2, \ldots, n.$$

An H-matrix with positive diagonals is a P-matrix.

The non-Lipschitzian NCP (1.1) and the non-Lipschitzian system (1.2) have a number of applications on some very important problems in physics and finance, such

as the reaction-diffusion problems [3, 4], the nonlinear parabolic complementarity problem [21], and European and American option valuation [2, 22]. In order to find a numerical solution for these problems, we can use a discretization method such as the central difference, the piecewise linear finite element, or a finite volume method [21, 22] to reduce these problems into (1.1) or (1.2). Then a numerical solution of these problems is obtained by solving (1.1) or (1.2). However, the resulting problem (1.1) or (1.2) is usually very large.

In this paper we propose a superlinearly convergent method to solve the NCP(F) (1.1). In section 2 we give some equivalent formulations for the NCP(F). In particular, by using the techniques in [5], we reformulate the NCP(F) as a nonlinear complementarity problem involving functions which are Lipschitz continuous. This is a crucial step in the development of a Newton-type method with superlinear convergence property for the NCP(F). In addition, under the condition that M is a P_0-matrix, we show that the NCP(F) has a unique solution and the reformulated system enjoys a desirable nonsingularity property. In section 3 we present a superlinearly convergent algorithm for solving the NCP(F), which is an inexact version of the smoothing methods proposed in [14]. We show that our proposed method has global and quadratic convergence properties under the condition that M is a P_0-matrix. This condition is weaker than the ones used in [1, 5]. Our numerical results reported in section 4 show that our proposed method can produce an approximate solution with high accuracy for large-scale problems. We conclude the paper with some remarks in section 5.

We conclude this section with some notation and terminology. For a continuously differentiable function $D : \Re^n \to \Re^n$, we denote the Jacobian of D at $x \in \Re^n$ by $D'(x)$. Let $D : \Re^n \to \Re^n$ be a locally Lipschitzian vector function. By Rademacher's theorem, D is differentiable almost everywhere. Let U_D denote the set of points where D is differentiable. Then the B-subdifferential of D at $x \in \Re^n$ is defined by

$$(1.3) \qquad \partial_B D(x) = \left\{ \lim_{\substack{x^k \to x \\ x^k \in U_D}} D'(x^k) \right\},$$

while Clarke's generalized Jacobian of D at x is defined by

$$(1.4) \qquad \partial D(x) = \mathrm{conv}\partial_B D(x)$$

(see [7, 15]). D is called *semismooth* at x if D is directionally differentiable at x and for all $V \in \partial D(x + h)$ and $h \to 0$,

$$(1.5) \qquad D'(x; h) = Vh + o(\|h\|);$$

D is called *strongly semismooth* at x if D is semismooth at x and for all $V \in \partial D(x+h)$ and $h \to 0$,

$$(1.6) \qquad D'(x; h) = Vh + O(\|h\|^2).$$

For $x \in \Re^n$, the 2-norm is denoted by $\|x\|$. For a convex set $X \subset \Re^n$, $\Pi_X(\cdot)$ is the Euclidean projection operator onto X, which is defined as follows:

$$\Pi_X(x) = \underset{y \in X}{\mathrm{argmin}} \|x - y\|, \qquad x \in \Re^n.$$

For a vector $x \in \Re^n$, $\mathrm{diag}(x)$ denotes the $n \times n$ diagonal matrix generated by x. For an $n \times n$ diagonal matrix B, B_i denotes the ith diagonal of B. Let $\Re_+ = \{t \in \Re : t \geq 0\}$ and $\Re_{++} = \{t \in \Re : t > 0\}$. Finally, we use $t \downarrow 0^+$ to denote the case that a positive scalar t tends to 0.

2. Some equivalent formulations. In this section, we give some equivalent formulations for the NCP(F). In particular, we reformulate the NCP(F) as an NCP involving functions which are Lipschitz continuous, which is a vital step in the development of a superlinearly convergent method for the NCP(F). The idea of this reformulation is same as the one in [5]. It is proved that the smoothing Newton method [5] converges superlinearly and globally if M is a symmetric positive definite matrix. In this paper, we will show our proposed method converges quadratically and globally under the condition that M is a P_0-matrix, which is a nice property of our method.

For a nonnegative vector $x \in \Re^n$ and $p \in (0,1)$, let $y = [x_1^p, x_2^p, \ldots, x_n^p]^T$ and $y^{\frac{1}{p}} = [y_1^{\frac{1}{p}}, \ldots, y_n^{\frac{1}{p}}]^T$. Define $W : \Re_+^n \to \Re^n$ by

$$(2.1) \qquad W(y) = My^{\frac{1}{p}} + By + q.$$

Then the NCP(F) becomes the following complementarity problem:

$$(2.2) \qquad y^{\frac{1}{p}} \geq 0, \quad W(y) \geq 0, \quad (y^{\frac{1}{p}})^T W(y) = 0.$$

Let NCP(W) be the following NCP:

$$(2.3) \qquad y \geq 0, \quad W(y) \geq 0, \quad y^T W(y) = 0.$$

Clearly, (2.2) and (2.3) are two equivalent problems. Hence, we have the following proposition.

PROPOSITION 2.1. *NCP(W) is a Lipschitzian NCP. If y is a solution of the NCP(W), then $x = y^{\frac{1}{p}}$ is a solution of the NCP(F). Conversely, if x is a solution of the NCP(F), then $y = [x_1^p, x_2^p, \ldots, x_n^p]^T$ is a solution of the NCP(W).*

For the NCP(W), we have the following proposition. The proof of this proposition will be given later.

PROPOSITION 2.2. *If M is a P_0-matrix, then the NCP(W) has a unique solution.*

From Propositions 2.1 and 2.2, we obtain the following proposition.

PROPOSITION 2.3. *If M is a P_0-matrix, then the non-Lipschitzian NCP(F) (1.1) has a unique solution.*

Let $X = \Re_+^n$. In order to prove that Proposition 2.2 holds, we need the following results.

LEMMA 2.1. *If M is a P_0-matrix, then the function W is a P-function on X; i.e., for any $x, y \in X$ with $x \neq y$, it holds that*

$$\max_{1 \leq i \leq n} (x_i - y_i)[W_i(x) - W_i(y)] > 0.$$

Proof. For any $x, y \in X$ with $x \neq y$, since M is a P_0-matrix, there exists an index i such that

$$x_i \neq y_i \quad \text{and} \quad \left(x_i^{1/p} - y_i^{1/p}\right) \left[M\left(x^{1/p} - y^{1/p}\right)\right]_i \geq 0.$$

For two nonnegative numbers x_i and y_i with $x_i \neq y_i$, we have $x_i^{1/p} - y_i^{1/p} > 0(< 0)$ if and only if $x_i - y_i > (< 0)$. Thus,

$$(x_i - y_i) \left[M\left(x^{1/p} - y^{1/p}\right)\right]_i \geq 0.$$

Hence,

$$(x_i - y_i)[W(x) - W(y)]_i = (x_i - y_i)\left[M\left(x^{1/p} - y^{1/p}\right)\right]_i + B_i(x_i - y_i)^2$$

$$\geq B_i(x_i - y_i)^2 > 0.$$

Therefore, we have

$$\max_{1 \leq i \leq n} (x_i - y_i)[W(x) - W(y)]_i > 0.$$

This completes the proof. $\square$

LEMMA 2.2. *If W is a P-function on X, then the NCP(W) has at most one solution.*

Proof. See Proposition 3.5.10 [9]. $\square$

LEMMA 2.3. *If there exists a vector $y^{\mathrm{ref}} \in X$ such that the set*

$$L'_< = \left\{y \in X : W_i(y)(y_i - y_i^{\mathrm{ref}}) < 0 \text{ if } y_i \neq y_i^{\mathrm{ref}} \text{ for all } i = 1, 2, \ldots, n\right\}$$

is bounded, then the NCP(W) has a solution.

Proof. See Proposition 3.5.1 [9]. $\square$

Now we give the proof of Proposition 2.2 as follows.

Proof of Proposition 2.2. Let $y^{\mathrm{ref}} = 0$ and

$$L'_0 = \{y \in X : W_i(y)y_i < 0 \quad \text{if } y_i \neq 0 \text{ for all } i = 1, 2, \ldots, n\}.$$

From Lemmas 2.1–2.3, if L'_0 is bounded, then Proposition 2.2 holds. Now we prove that L'_0 is bounded. Suppose this is false. Then there exists a sequence $\{y^k \in L'_0\}$ such that $\|y^k\| \to \infty$. Define the index set J by $J := \{i | \{y_i^k\}$ is unbounded, $i = 1, 2, \ldots, n\}$. Then $J \neq \emptyset$ because otherwise the sequence $\{y^k \in L'_0\}$ is bounded. We suppose that

$$(2.4) \qquad y_i^k \to +\infty \text{ as } k \to \infty \quad \text{for any } i \in J.$$

Otherwise, we can take a subsequence $\{y_i^{k_i}\}$ such that $y_i^{k_i} \to +\infty$ as $k_i \to \infty$. For each k, let

$$(2.5) \qquad \bar{y}_i^k = \begin{cases} y_i^k & \text{if } i \notin J, \\ 0 & \text{if } i \in J, \end{cases} \quad i = 1, 2, \ldots, n.$$

Clearly the sequence $\{\bar{y}^k\} \subset X$ is bounded and for each k,

$$y_i^k - \bar{y}_i^k = \begin{cases} 0 & \text{if } i \notin J, \\ y_i^k & \text{if } i \in J, \end{cases} \quad i = 1, 2, \ldots, n,$$

and

$$(y_i^k)^{1/p} - (\bar{y}_i^k)^{1/p} = \begin{cases} 0 & \text{if } i \notin J, \\ (y_i^k)^{1/p} & \text{if } i \in J, \end{cases} \quad i = 1, 2, \ldots, n.$$

For each k, since M is a P_0-matrix, there exists, for the two vectors $(y^k)^{1/p}, (\bar{y}^k)^{1/p} \in X$, an index $i_k \in J$ such that

$$\left[(y_{i_k}^k)^{1/p} - (\bar{y}_{i_k}^k)^{1/p}\right]\left[M\left((y^k)^{1/p} - (\bar{y}^k)^{1/p}\right)\right]_{i_k} \geq 0.$$

Since $\bar{y}_{i_k}^k = 0$, we have

$$y_{i_k}^k \left[M \left((y^k)^{1/p} - (\bar{y}^k)^{1/p} \right) \right]_{i_k} \geq 0.$$

Hence,

$$y_{i_k}^k \left[W(y^k) - W(\bar{y}^k) \right]_{i_k} = y_{i_k}^k \left[M \left((y^k)^{1/p} - (\bar{y}^k)^{1/p} \right) \right]_{i_k} + B_{i_k}(y_{i_k}^k)^2$$

$$\geq B_{i_k}(y_{i_k}^k)^2.$$

Thus, we obtain

$$y_{i_k}^k W_{i_k}(y^k) \geq B_{i_k}(y_{i_k}^k)^2 + y_{i_k}^k W_{i_k}(\bar{y}^k).$$

Since J has only a finite number of elements, by taking a subsequence if necessary, we may assume that for each k, $i_k = i \in J$. So we have

$$(2.6) \qquad\qquad y_i^k W_i(y^k) \geq B_i(y_i^k)^2 + y_i^k W_i(\bar{y}^k).$$

Since $\{\bar{y}^k\} \subset X$ is bounded, the sequence $\{W_i(\bar{y}^k)\}$ is bounded. From (2.4), for $i \in J$, $y_i^k \to +\infty$ as $k \to \infty$. Hence, from (2.6), we have

$$y_i^k W_i(y^k) \geq B_i(y_i^k)^2 + y_i^k W_i(\bar{y}^k) \to +\infty \quad \text{as } k \to \infty.$$

This is a contradiction to $y^k \in L_0'$; i.e., $y_i^k W_i(y^k) < 0$. In view of this contradiction, we conclude that L_0' is bounded. ☐

The NCP(W) can be reformulated into some semismooth equations. Based on these equations, we can develop a superlinearly convergent method for NCP(W). It is well known that solving the NCP(W) is equivalent to finding a solution of the Robinson's normal equation

$$(2.7) \qquad\qquad E(z) := W(\Pi_X(z)) + z - \Pi_X(z) = 0$$

in the sense that if $z^* \in \Re^n$ is a solution of (2.7), then $y^* := \Pi_X(z^*)$ is a solution of the NCP(W), and conversely if y^* is a solution of the NCP(W), then $z^* := y^* - W(y^*)$ is a solution of (2.7) [17]. Here, for any $z \in \Re^n$, $\Pi_X(z)$ is the Euclidean projection of z onto X.

Let ϕ be the Chen–Harker–Kanzow–Smale (CHKS) smoothing NCP function [13], which is defined as

$$(2.8) \qquad\qquad \phi(t, w) = \frac{\sqrt{w^2 + 4t^2} + w}{2}, \quad (t, w) \in \Re^2.$$

For any $t \neq 0$, by simple computation, we have

$$(2.9) \qquad \frac{\partial \phi(t, w)}{\partial t} = \frac{2t}{\sqrt{w^2 + 4t^2}} \quad \text{and} \quad \frac{\partial \phi(t, w)}{\partial w} = \frac{1}{2} + \frac{w}{2\sqrt{w^2 + 4t^2}}.$$

Define $\phi(t, z) = [\phi(t, z_1), \phi(t, z_2), \ldots, \phi(t, z_n)]^T$. Then, from [14], we have the following proposition.

PROPOSITION 2.4. $\phi(t, z)$ *has the following properties:*

(i) *For any given* $t \neq 0$, $\phi(t, z)$ *is continuously differentiable.*

(ii) $\phi(t, z) \in \text{int} X$ *for any given* $t > 0$.

(iii) $\phi(t, z)$ *is a strongly semismooth function on* $\Re^{n+1}$.

Define $H : \Re^{n+1} \to \Re^{n+1}$ by

$$(2.10) \qquad H(t, z) := \left(\begin{array}{c} t \\ G(t, z) \end{array} \right),$$

where

$$G(t, z) = W(\phi(t, z)) + z - \phi(t, z).$$

Then H is an augmented smoothing function of E defined in (2.7). Clearly, H is a locally Lipschitzian function, and H is continuously differentiable at any $(t, z) \in \Re^{n+1}$ with $t \neq 0$. From Proposition 2.4 (iii), $\phi(t, z)$ is a strongly semismooth function. Hence, W is a strongly semismooth function as well. Since H is a composite function of the two strongly semismooth functions W and ϕ, by Theorem 19 [10], we have the following proposition.

PROPOSITION 2.5. H *is a strongly semismooth function on* $\Re^{n+1}$. *Moreover,* $H(t^*, z^*) = 0$ *if and only if* $t^* = 0$ *and* $E(z^*) = 0$.

From Proposition 2.5, solving the system of nonlinear equations (2.7) is equivalent to finding a solution of $H(t, z) = 0$. Hence, we have the following proposition.

PROPOSITION 2.6. *Suppose that* (t^*, z^*) *is a solution of the system* $H(t, z) = 0$. *Then* $y^* = \Pi_X(z^*)$ *is a solution of the NCP(W), and* $x^* = (\Pi_X(z^*))^{\frac{1}{p}}$ *is a solution of the NCP(F).*

Let

$$D_1 = \left[\frac{1}{p}(\phi(t, z_i))^{\frac{1}{p}-1} \partial \phi(t, z_i)/\partial t, i = 1, 2, \ldots, n \right]^T,$$

$$D_2 = [\partial \phi(t, z_i)/\partial t, i = 1, 2, \ldots, n]^T,$$

$$D_3 = \text{diag}\left(\frac{1}{p}(\phi(t, z_i))^{\frac{1}{p}-1} \partial \phi(t, z_i)/\partial z_i, i = 1, 2, \ldots, n \right),$$

$$D_4 = \text{diag}(\partial \phi(t, z_i)/\partial z_i, i = 1, 2, \ldots, n).$$

For any $t \neq 0$, by simple computation, we have

$$(2.11) \qquad H'(t, z) := \left(\begin{array}{cc} 1 & 0 \\ G'_t(t, z) & G'_z(t, z) \end{array} \right),$$

where

$$(2.12) \qquad G'_t(t, z) = MD_1 + BD_2 - D_2$$

and

$$(2.13) \qquad G'_z(t, z) = MD_3 + BD_4 + I_n - D_4.$$

Here, I_n is the $n \times n$ identity matrix.

LEMMA 2.4 (see [16]). *Suppose that an* $n \times n$ *matrix* A *is a* P_0-*matrix and* C *and* O *are two diagonal matrices satisfying* $C_i O_i \geq 0$ *and* $O_i \neq 0$ *for each* $i = 1, 2, \ldots, n$. *Then,* $AC + O$ *is nonsingular.*

PROPOSITION 2.7. *For any $(t, z) \in \Re^{n+1}$ with $t \neq 0$, if M is a P_0-matrix, then $H'(t, z)$ is nonsingular.*

Proof. For any $(t, z) \in \Re^{n+1}$ with $t \neq 0$, from (2.8) and (2.9), we have

$$\phi(t, z_i) > 0 \text{ and } \partial\phi(t, z_i)/\partial z_i \in (0, 1), \qquad i = 1, 2, \ldots, n.$$

By Lemma 2.4, if M is a P_0-matrix, then $G'_z(t, z)$ is nonsingular. So $H'(t, z)$ is nonsingular. □

3. A superlinearly convergent algorithm. In this section, we present a Newton-type method to solve the system $H(t, z) = 0$, where H is defined in (2.10). We show that this method has global and superlinear convergence properties if M is a P_0-matrix. Using this method, we can generate a sequence $\{x^k\} \subset X$ such that x^k converges to the unique solution x^* of the NCP(F) if M is a P_0-matrix.

Choose $\bar{t} \in \Re_{++}$ and $\gamma \in (0, 1)$ such that $\gamma\bar{t} < 0.5$. Define the merit function $\psi : \Re^{n+1} \to \Re_+$ by

$$\psi(t, z) := \|H(t, z)\|^2,$$

and define $\beta : \Re^{n+1} \to \Re_+$ by

$$\beta(t, z) := \gamma \min\{1, \psi(t, z)\}.$$

Our proposed algorithm is stated as follows.

ALGORITHM 3.1.

Step 0. *Choose constants $\delta \in (0, 1)$ and $\sigma \in (0, \frac{1}{2})$. Let $t^0 := \bar{t}$, $z^0 \in \Re^n$ be an arbitrary point and $k := 0$.*

Step 1. *If $H(t^k, z^k) = 0$ then stop. Otherwise, let $\beta_k := \beta(t^k, z^k)$.*

Step 2. *Let*

$$\Delta t^k = -t^k + \beta_k \bar{t}.$$

Compute Δz^k by solving the following linear system of equations:

$$(3.1) \ G'_z(t^k, z^k)\Delta z^k = -G(t^k, z^k) - G'_t(t^k, z^k)\Delta t^k + R_k, \quad \|R_k\| \le \beta_k \bar{t}.$$

Step 3. *Let l_k be the smallest nonnegative integer l satisfying*

$$(3.2) \qquad \psi(t^k + \delta^l \Delta t^k, z^k + \delta^l \Delta z^k) \le [1 - 2\sigma(1 - 2\gamma\bar{t})\delta^l]\psi(t^k, z^k).$$

Define $t^{k+1} := t^k + \delta^{l_k}\Delta t^k$ and $z^{k+1} := z^k + \delta^{l_k}\Delta z^k$.

Step 4. *Replace k by $k + 1$ and go to Step 1.*

Remark 1. (a) Algorithm 3.1 can be regarded as an inexact version of the smoothing Newton methods proposed in [14]. In [14], Δz^k is an exact solution of the following linear system of equations:

$$(3.3) \qquad G'_z(t^k, z^k)\Delta z^k = -G(t^k, z^k) - G'_t(t^k, z^k)\Delta t^k.$$

In our proposed method, we require the linear system (3.3) to be solved inexactly. When the linear system (3.3) is large, the computation of an exact solution requires a lot of computer time and memory. In such a situation, one is forced to compute only an approximate solution by an iterative method.

More recently, in a paper by Gao and Sun [11], an inexact smoothing Newton method has been proposed for solving the least squares covariance matrix problem.

The numerical results reported in [11] show this method is very efficient for solving the least squares covariance matrix problem with simple constraints.

(b) Let $d^k = [\Delta t^k, (\Delta z^k)^T]^T \in \Re^{n+1}$ and $r^k = [\beta_k \bar{t}, R_k^T]^T \in \Re^{n+1}$. From Step 2 of Algorithm 3.1, we have

$$H(t^k, z^k) + H'(t^k, z^k)d^k = r^k \text{ and } \|r^k\| \leq 2\beta_k \bar{t}.$$

In [14], d^k is obtained by solving the following system exactly:

$$H(t^k, z^k) + H'(t^k, z^k)d^k = r_e^k,$$

where $r_e^k = [\beta_k \bar{t}, 0]^T \in \Re^{n+1}$ and $\|r_e^k\| = \beta_k \bar{t}$.

(c) Algorithm 3.1 has global and superlinear properties under the condition that M is a P_0-matrix. For Example 1, the assumption that M is a P_0-matrix can be guaranteed by a proper discretization method such as the central difference, the piecewise linear finite element, or a finite volume method. See [18].

In the following, we will give the convergence results for our proposed method. Note that Lemmas 3.1–3.3 are modifications of the corresponding results in [14].

LEMMA 3.1. *Suppose that for some $(\tilde{t}, \tilde{z}) \in \Re_{++} \times \Re^n$, $H'(\tilde{t}, \tilde{z})$ is nonsingular. Then, there exist a closed neighborhood $\mathcal{N}(\tilde{t}, \tilde{z})$ of $(\tilde{t}, \tilde{z})$ and a positive number $\bar{\alpha} \in (0, 1]$ such that for any $(t, z) \in \mathcal{N}(\tilde{t}, \tilde{z})$ and all $\alpha \in [0, \bar{\alpha}]$, it holds that $t \in \Re_{++}$, $H'(t, z)$ is invertible, and*

(3.4)
$$\psi(t + \alpha\Delta t, z + \alpha\Delta z) \leq [1 - 2\sigma(1 - 2\gamma\bar{t})\alpha]\psi(t, z),$$

where $\Delta t = -t + \beta(t, z)\bar{t}$ and Δz is a solution of the following linear system of equations:

$$G'_z(t, z)\Delta z = -G(t, z) - G'_t(t, z)\Delta t + R, \quad \|R\| \leq \beta(t, z)\bar{t}.$$

Proof. Since $H'(\tilde{t}, \tilde{z})$ is invertible and $\tilde{t} \in \Re_{++}$, there exists a closed neighborhood $\mathcal{N}(\tilde{t}, \tilde{z})$ of $(\tilde{t}, \tilde{z})$ such that for any $(t, z) \in \mathcal{N}(\tilde{t}, \tilde{z})$, we have $t \in \Re_{++}^n$, and $H'(t, z)$ is invertible. For any $(t, z) \in \mathcal{N}(\tilde{t}, \tilde{z})$, let $\Delta t = -t + \beta(t, z)\bar{t}$, and let Δz be a solution of the following linear system of equations:

$$G'_z(t, z)\Delta z = -G(t, z) - G'_t(t, z)\Delta t + R, \quad \|R\| \leq \beta(t, z)\bar{t}.$$

Let $d = [\Delta t, (\Delta z)^T]^T \in \Re^{n+1}$ and $r = [\beta(t, z)\bar{t}, R^T]^T \in \Re^{n+1}$. Then, we have

(3.5)
$$H(t, z) + H'(t, z)d = r, \quad \|r\| \leq 2\beta(t, z)\bar{t}.$$

For any $\alpha \in [0, 1]$, define

(3.6)
$$g_{(t,z)}(\alpha) = H(t + \alpha\Delta t, z + \alpha\Delta z) - H(t, z) - \alpha H'(t, z)d.$$

It follows from the mean value theorem that

$$g_{(t,z)}(\alpha) = \alpha \int_0^1 [H'(t + \theta\alpha\Delta t, z + \theta\alpha\Delta z) - H'(t, z)]d \, d\theta.$$

Since $H'(\cdot)$ is uniformly continuous on $\mathcal{N}(\tilde{t}, \tilde{z})$ and $d \to \tilde{d} = [\Delta\tilde{t}, (\Delta\tilde{z})^T]^T$ as $(t, z) \to (\tilde{t}, \tilde{z})$, it follows that for all $(t, z) \in \mathcal{N}(\tilde{t}, \tilde{z})$,

$$\lim_{\alpha \downarrow 0} \|g_{(t,z)}(\alpha)\|/\alpha = 0.$$

Then, from (3.5), (3.6), and the facts that $\beta(t, z) \leq \gamma[\psi(t, z)]^{\frac{1}{2}}$ and $\|r\| \leq 2\beta(t, z)\bar{t}$, for all $\alpha \in [0, 1]$ and all $(t, z) \in \mathcal{N}(\tilde{t}, \tilde{z})$, we have

$$\psi(t + \alpha \Delta t, z + \alpha \Delta z)$$

$$= \|H(t + \alpha \Delta t, z + \alpha \Delta z)\|^2$$

$$= \|H(t, z) + \alpha H'(t, z)d + g_{(t,z)}(\alpha)\|^2$$

$$= \|(1 - \alpha)H(t, z) + \alpha r + g_z(\alpha)\|^2$$

$$\leq (1 - \alpha)^2 \psi(z) + 2(1 - \alpha)\alpha\|H(t, z)\|\|r\| + o(\alpha) + O(\alpha^2)$$

$$\leq (1 - \alpha)^2 \psi(z) + 4\alpha\|H(t, z)\|\beta(t, z)\bar{t} + o(\alpha) + O(\alpha^2)$$

$$\leq (1 - \alpha)^2 \psi(z) + 4\alpha\gamma\bar{t}\psi(z) + o(\alpha)$$

$$\leq (1 - 2\alpha)\psi(z) + 4\alpha\gamma\bar{t}\psi(z) + o(\alpha)$$

$$= [1 - 2(1 - 2\gamma\bar{t})\alpha]\psi(z) + o(\alpha)$$

$$(3.7) \qquad \leq [1 - 2\rho(1 - 2\gamma\bar{t})\alpha]\psi(z) + o(\alpha).$$

Thus, by virtue of (3.7), we can find a positive number $\bar{\alpha} \in (0, 1]$ such that for all $\alpha \in [0, \bar{\alpha}]$ and all $(t, z) \in \mathcal{N}(\tilde{t}, \tilde{z})$, (3.4) holds. □

LEMMA 3.2. *Suppose that M is a P_0-matrix. Then, Algorithm 3.1 is well defined at the kth iteration and for any $k \geq 0$. Furthermore,*

$$(3.8) \qquad 0 < t^{k+1} \leq t^k \leq \bar{t}$$

and

$$(3.9) \qquad t^k \geq \beta(t^k, z^k)\bar{t}.$$

Proof. It follows from Proposition 2.7 and Lemma 3.1 that Algorithm 3.1 is well defined at the kth iteration. By the same argument as that given in the proof of Proposition 16 [14], we have (3.8) and (3.9). □

LEMMA 3.3. *Suppose that M is a P_0-matrix. Then, an infinite sequence $\{(t^k, z^k)\}$ is generated by Algorithm 3.1, and each accumulation point $(\tilde{t}, \tilde{z})$ of $\{(t^k, z^k)\}$ is a solution of $H(t, z) = 0$.*

Proof. From Lemma 3.2 and Proposition 2.7, it follows that an infinite sequence $\{(t^k, z^k)\}$ is generated such that $t^k \geq \beta_k\bar{t}$ for all $k \geq 0$. From Algorithm 3.1, $\psi(t^{k+1}, z^{k+1}) < \psi(t^k, z^k)$ for all $k \geq 0$. Let $\psi_k = \psi(t^k, z^k)$. Hence, the two sequences $\{\psi_k\}$ and $\{\beta_k\}$ are monotonically decreasing. Since $\psi_k, \beta_k > 0$ ($k \geq 0$), there exist $\tilde{\psi}, \tilde{\beta} \geq 0$ such that $\psi_k \to \tilde{\psi}$ and $\beta_k \to \tilde{\beta}$ as $k \to \infty$. If $\tilde{\psi} = 0$ and $\{(t^k, z^k)\}$ has an accumulation point $(\tilde{t}, \tilde{z})$, then, from the continuity of $\psi(\cdot)$ and $\beta(\cdot)$, we obtain $\psi(\tilde{t}, \tilde{z}) = 0$ and $\beta(\tilde{t}, \tilde{z}) = 0$. Thus, we obtain the desired result. Suppose that $\tilde{\psi} > 0$ and $(\tilde{t}, \tilde{z}) \in \Re^{n+1}$ is an accumulation point of $\{(t^k, z^k)\}$. By taking a subsequence, if necessary, we may assume that $\{(t^k, z^k)\}$ converges to $(\tilde{t}, \tilde{z})$. Let $\tilde{\psi} = \psi(\tilde{t}, \tilde{z})$ and $\tilde{\beta} = \beta(\tilde{t}, \tilde{z})$. It is easy to see that $\tilde{t} \geq \tilde{\beta}\bar{t} > 0$. Then, from Proposition 2.7, $H'(\tilde{t}, \tilde{z})$ exists and is invertible. Hence, by Lemma 3.3, there exist a closed neighborhood

$\mathcal{N}(\tilde{t}, \tilde{z})$ of $(\tilde{t}, \tilde{z})$ and a positive number $\bar{\alpha} \in (0, 1]$ such that for any $(t, z) \in \mathcal{N}(\tilde{t}, \tilde{z})$ and all $\alpha \in [0, \bar{\alpha}]$, we have $t \in \Re_{++}$, $H'(t, z)$ is invertible, and

$$\psi(t + \alpha \Delta t, z + \alpha \Delta z) \le [1 - 2\sigma(1 - 2\gamma \bar{t})\alpha]\psi(t, z),$$

where $\Delta t = -t + \beta(t, z)\bar{t}$ and Δz is a solution of the following linear system of equations:

$$G_z'(t, z)\Delta z = -G(t, z) - G_t'(t, z)\Delta t + R, \quad \|R\| \le \beta(t, z)\bar{t}.$$

Therefore, for a nonnegative integer l such that $\delta^l \in (0, \bar{\alpha}]$, we have

$$\psi(t^k + \delta^l \Delta t^k, z^k + \delta^l \Delta z^k) \le [1 - 2\sigma(1 - 2\gamma \bar{t})\delta^l]\psi(t^k, z^k)$$

for all sufficiently large k. Then, for every sufficiently large k, $l_k \le l$ and hence $\delta^{l_k} \ge \delta^l$. Thus,

$$\psi(t^{k+1}, z^{k+1}) \le [1 - 2\sigma(1 - 2\gamma \bar{t})\delta^{l_k}]\psi(t^k, z^k) \le [1 - 2\sigma(1 - 2\gamma \bar{t})\delta^l]\psi(t^k, z^k)$$

for all sufficiently large k. This contradicts the fact that the sequence $\{\psi_k\}$ converges to $\tilde{\psi} > 0$. So, we complete the proof. $\quad\square$

PROPOSITION 3.1. *Suppose that M is a P_0-matrix. Then, the level set*

$$L(t^0, z^0) = \{(t, z) \in \Re^{n+1} | \ \psi(t, z) \le \psi(t^0, z^0)\}$$

is bounded.

Proof. On the contrary, suppose that there exists a sequence $\{(t^k, z^k) \in \Re^{n+1}\}$ such that $(t^k, z^k) \in L(t^0, z^0)$ and $\|(t^k, z^k)\| \to \infty$. Since $\{t^k\}$ is bounded, $\|z^k\| \to \infty$. It is easy to prove that for $i = 1, 2, \ldots, n$,

(3.10) $\qquad |\max(0, z_i^k)| \to \infty \Longrightarrow |z_i^k| \to \infty \ \text{ and } \ |z_i^k - \max(0, z_i^k)| \to 0.$

From the definition of ϕ, we have

(3.11) $\qquad |\phi(t^k, z_i^k) - \max(0, z_i^k)| \le t^k, \quad i = 1, 2, \ldots, n.$

Define the index set J by $J := \{i| \ \{\phi(t^k, z_i^k)\} \text{ is unbounded, } i = 1, 2, \ldots, n\}$. Then, $J \ne \emptyset$, because otherwise $\|G(t^k, z^k)\| = \|W(\phi(t^k, z^k)) + z^k - \phi(t^k, z^k)\| \to \infty$. For each k, let

$$\bar{\phi}_i^k = \begin{cases} \phi(t^k, z_i^k) & \text{if } i \notin J, \\ 0 & \text{if } i \in J, \end{cases} \quad i = 1, 2, \ldots, n.$$

Let $\bar{\phi}^k = [\bar{\phi}_i^k, i = 1, 2, \ldots, n]^T$. Then, $\{\|\bar{\phi}^k\|\}$ is bounded. Since M is a P_0-matrix, for each k, there exists an $i_k \in J$ such that

$$\left[(\phi(t^k, z_{i_k}^k))^{1/p} - (\bar{\phi}_{i_k}^k)^{1/p}\right] \left[M\left(\phi(t^k, z^k)\right)^{1/p} - M(\bar{\phi}^k)^{1/p}\right]_{i_k} \ge 0.$$

Because $\bar{\phi}_{i_k}^k = 0$, we have

$$\phi(t^k, z_{i_k}^k) \left[M\left(\phi(t^k, z^k)\right)^{1/p} - M(\bar{\phi}^k)^{1/p}\right]_{i_k} \ge 0.$$

Therefore,

$$
\begin{aligned}
\phi(t^k, z_{i_k}^k)&[W(\phi(t^k, z^k)) - W(\bar\phi^k)]_{i_k} \\
&= \phi(t^k, z_{i_k}^k) \left[M\left(\phi(t^k, z^k)\right)^{\frac{1}{p}} - M(\bar\phi^k)^{\frac{1}{p}} \right]_{i_k} + B_{i_k} \left[\phi(t^k, z_{i_k}^k) \right]^2 \\
&\geq B_{i_k} [\phi(t^k, z_{i_k}^k)]^2.
\end{aligned}
$$

So,

$$
\left[W\left(\phi(t^k, z^k)\right) - W(\bar\phi^k) \right]_{i_k} \geq B_{i_k} \phi(t^k, z_{i_k}^k).
$$

Since $B_{i_k} > 0$ and $\phi(t^k, z_{i_k}^k) \to \infty$ as $k \to \infty$, we have $[W(\phi(t^k, z^k)) - W(\bar\phi^k)]_{i_k} \to \infty$ as $k \to \infty$. Note that $\{\|W(\bar\phi^k)\|\}$ is bounded. It follows that for each k, there exists at least one $i_k \in J$ such that

$$
|W_{i_k}(\phi(t^k, z^k))| \to \infty.
$$

Since J has only a finite number of elements, by taking a subsequence if necessary, we may assume that there exists an $i \in J$ such that

$$
|W_i(\phi(t^k, z^k))| \to \infty.
$$

Thus, by (3.11) and the definition of J, there exists at least one $i \in J$ such that

$$
|W_i(\phi(t^k, z^k))|, |\phi(t^k, z_i^k)|, |\max(0, z_i^k)| \to \infty.
$$

Hence, by (3.10), (3.11), and the boundedness of $\{t^k\}$, it follows that for such $i \in J$, $|z_i^k - \phi(t^k, z_i^k)|$ is bounded. Then, it is clear that for such an $i \in J$, $\{|G_i(t^k, z^k)|\}$ is unbounded. Because $\|H(t^k, z^k)\| \geq |G_i(t^k, z^k)|$, $\{\|H(t^k, z^k)\|\}$ is unbounded. This is a contradiction which shows $L(t^0, z^0)$ is bounded. $\quad\square$

From Proposition 3.1 and Lemma 3.3, we have the following theorem.

THEOREM 3.1. *Suppose that M is a P_0-matrix. Then an infinite sequence $\{(t^k, z^k)\}$ is generated by Algorithm 3.1, and there exists an accumulation point $(\tilde{t}, \tilde{z})$ of $\{(t^k, z^k)\}$ such that $(\tilde{t}, \tilde{z})$ is a solution of $H(t, z) = 0$.*

Define

(3.12) $$ A(0, z^*) = \{\lim H'(t^k, z^k) : t^k \downarrow 0^+ \text{ and } z^k \to z^*\}. $$

Clearly, by the definition given in (1.3), $A(0, z^*) \subseteq \partial_B H(0, z^*)$.

LEMMA 3.4. *Suppose that all $V \in A(0, z^*)$ are nonsingular. Then, there is a neighborhood $N(0, z^*)$ of $(0, z^*)$ and a constant C such that for any $(t, z) \in N(0, z^*)$ with $t \neq 0$, $H'(t, z)$ is nonsingular and*

$$
\|(H'(t, z))^{-1}\| \leq C.
$$

Proof. If the conclusion is not true, then there is a sequence $\{(t^k, z^k)\}$ with all $t^k \neq 0$ such that $(t^k, z^k) \to (0, z^*)$, and either all $H'(t^k, z^k)$ are singular or $\|(H'(t^k, z^k))^{-1}\| \to +\infty$. Since H is locally Lipschitzian, ∂H is bounded in a neighborhood of $(0, z^*)$. By passing to a subsequence, if necessary, we may assume that $H'(t^k, z^k) \to V$. Thus, V must be singular. This is a contradiction to the assumption of the lemma. This completes the proof. $\quad\square$

PROPOSITION 3.2. *If M is a P_0-matrix, then all $V \in A(0, z^*)$ are nonsingular.*

Proof. For any $(t^k, z^k) \in \Re^{n+1}$ with $t^k \neq 0$, from (2.8) and (2.9), we have

$$\phi(t^k, z_i^k) > 0 \quad \text{and} \quad \partial\phi(t^k, z_i^k)/\partial z_i^k \in (0, 1), i = 1, 2, \ldots, n.$$

Thus, for any $V \in A(0, z^*)$, V can be written as

$$V = \begin{pmatrix} 1 & 0 \\ G_t'(0, z^*) & G_z'(0, z^*) \end{pmatrix},$$

where

$$G_t'(0, z^*) = MD_1^* + (B - I_n)D_2^*, \quad G_z'(0, z^*) = MD_3^* + BD_4^* + I_n - D_4^*,$$

and

$$D_1 = \left[\frac{1}{p} |z_i^*|^{\frac{1}{p}-1} d_i^*, i = 1, \ldots, n \right]^T,$$

$$D_2 = [d_i^*, i = 1, \ldots, n]^T, \quad d_i^* \in [0, 1], \quad i = 1, \ldots, n,$$

$$D_3 = \text{diag}\left(\frac{1}{p} |z_i^*|^{\frac{1}{p}-1} e_i^*, i = 1, \ldots, n \right),$$

$$D_4 = \text{diag}(e_i^*, \quad i = 1, \ldots, n), \quad e_i^* \in [0, 1], \quad i = 1, \ldots, n.$$

By Lemma 2.4, if M is a P_0-matrix, then $G_z'(0, z^*)$ is nonsingular. Thus, V is nonsingular. This completes the proof. $\square$

THEOREM 3.2. *Suppose that M is a P_0-matrix and $(0, z^*)$ is an accumulation point of the infinite sequence $\{(t^k, z^k)\}$ generated by Algorithm 3.1. Then, the whole sequence $\{z^k\}$ converges to z^* and the convergence is Q-quadratic, i.e.,*

$$(3.13) \qquad \|(t^{k+1}, z^{k+1}) - (0, z^*)\| = O(\|(t^k, z^k) - (0, z^*)\|^2).$$

Proof. By Lemma 3.4 and Propositions 2.5 and 3.2, the conclusion of the theorem follows from similar arguments as those given in the proof of Theorem 23 [14]. $\square$

From Propositions 2.3 and 2.6 and Theorems 3.1 and 3.2, we have the following.

THEOREM 3.3. *Suppose that M is a P_0-matrix. Then, an infinite sequence $\{(t^k, z^k)\}$ is generated by Algorithm 3.1. For each k, let $x^k = (\Pi_X(z^k))^{\frac{1}{p}}$. Then, the sequence $\{x^k\}$ converges to the unique solution x^* of the NCP(F).*

4. Numerical experiments. In this section, we report our numerical experiments for testing the efficiency of Algorithm 3.1. At each iteration of Algorithm 3.1, the main task is to solve the linear system (3.1). The matrix $G_z'(t^k, z^k)$ in the system (3.1) is not symmetric, and from (2.13) and Lemma 2.4 it is nonsingular when M is a P_0-matrix. Therefore, it is natural to choose the BiCGStab [20] solver to solve the linear system (3.1), and from [20, 19] this BiCGStab method converges when $G_z'(t^k, z^k)$ is nonsingular.

Algorithm 3.1 was implemented in MATLAB and was run on a PC (Intel 3.20 GHz with 0.99 GB of RAM) for the following free boundary problem [1, 4].

Example 2. Let $\Omega = (0, 1) \times (0, 1)$ and $p \in (0, 1)$. We consider

$$\begin{aligned} -\triangle u + \frac{9}{(1-p)^2} u^p &= f(z) &&\text{in } \Omega_+, \\ u &= 0 &&\text{in } \Omega_0, \\ u = |\nabla u| &= 0 &&\text{on } \Gamma, \\ u &= g(z) &&\text{on } \partial\Omega, \end{aligned}$$

where $\Omega_+ = \{z \in \Omega : u(z) > 0\}$, $\Omega_0 = \{z \in \Omega : u(z) = 0\}$, and $\Gamma = \partial \Omega_0 = \partial \Omega_+ \cap \Omega$ are unknown. Let $r^2 = z_1^2 + z_2^2$. We choose

$$f(z) = \frac{9}{r(1-p)^2} \left(\frac{3r-1}{2}\right)^{\frac{2p}{1-p}} \max\left(0, r - \frac{1}{3}\right)$$

and

$$g(z) = \left(\frac{3r-1}{2}\right)^{\frac{2}{1-p}} \max\left(0, r - \frac{1}{3}\right), \quad z \in \partial\Omega.$$

Using the five-point finite difference approximation, we obtain an NCP with $F(x) = Mx + Bx^p + q$ for $x \in \Re_+^n$, and q is a vector in $\Re^n$. Here, the components of x are the approximations to the exact solution $u(z)$ at the grid points of Ω, and M is a H-matrix.

Throughout the computational experiments, the following parameter values are used:

$$\bar{t} = 0.002, \quad \gamma = 0.5, \quad \delta = 0.5, \quad \sigma = 0.0005, \quad \text{and} \quad z^0 = 0.$$

The maximum number of iterations is set as 2000, and the maximum number of the line search is set as 20. At each iteration of Algorithm 3.1, the linear system (3.1) was solved by the bicgstab solver in MATLAB. We stopped the iteration when

$$\|\min(x^k, F(x^k))\|_\infty \leq 10^{-7}.$$

First, we show the efficiency of the bicgstab solver. For Example 2, we use different discretizations $N \in \mathcal{N}$. Note that the dimension of the corresponding complementarity problem is $n = N^2$. We report our numerical results in Table 4.1. In this table, Ite denotes the number of iterations of Algorithm 3.1, $\|\min(x, F)\|_\infty$ denotes the value of $\|\min(x^k, F(x^k))\|_\infty$ at the final iteration, and A.N-BiCGStab denotes the average number of the iterations of the bicgstab solver. From this table, we can see the bicgstab solver can solve the linear system (3.1) efficiently and the average number of the iterations is less than $\sqrt{n} = N$, where n is the size of the test problem. This is a nice result for the bicgstab solver.

Next, we will compare our proposed method with the regularized projection method (RPM) proposed in [1] for the non-Lipschitzian NCP (1.1). Throughout the computational experiments, for the RPM, we use

$$\omega = 1, \quad \epsilon_k = 2^{-k}, \quad k = 0, 2, 4, \ldots, 30.$$

For both Algorithm 3.1 and the RPM, the stopping criterion is

$$\|\min(x^k, F(x^k))\|_\infty \leq 10^{-7}.$$

The numerical results obtained are summarized in Tables 4.2 and 4.3. In these two tables, CPU(s) denotes the total computer time in seconds used to solve the problem. For the cases $N = 90, 120, 150$, and 180, we set $p = 0.2, 0.3, \ldots, 0.9$. For the cases $N = 200, 400, 600, 800$, and 1000, we let $p = 0.6, 0.7, 0.8, 0.9$. Moreover, for the case $N = 1000$, the maximum CPU-time set for each algorithm is 10 hours. So "> 10 hours" in Table 4.3 indicates that the algorithm cannot give a solution within 10 hours.

The results reported in Tables 4.2 and 4.3 show that both Algorithm 3.1 and the RPM perform well for these test problems. In particular, we can see that Algorithm 3.1 can produce an approximate solution with high accuracy for these test problems and also can solve very large problems.

TABLE 4.1
Numerical results of Algorithm 3.1.

Problem		Algorithm 3.1		
N	p	Ite	$\|\min(x, F)\|_\infty$	A.N–BiCGStab
10	0.70	22	4.48e−008	7.6
10	0.90	10	9.29e−008	4.5
50	0.70	25	5.51e−008	44.5
50	0.90	13	2.17e−008	21.0
100	0.70	25	4.92e−008	82.2
100	0.90	13	3.82e−008	43.8
300	0.70	27	6.22e−008	207.2
300	0.90	13	8.79e−008	116.2
500	0.70	28	7.00e−008	330.8
500	0.90	15	5.47e−008	183.7
700	0.70	29	4.43e−008	446.4
700	0.90	14	1.76e−008	298.0
900	0.70	29	5.93e−008	542.6
900	0.90	16	3.04e−008	320.8

TABLE 4.2
Numerical results of Algorithm 3.1 *and the* RPM.

Problem		Algorithm 3.1			RPM [1]	
N	p	Ite	$\|\min(x, F)\|_\infty$	CPU(s)	$\|\min(x, F)\|_\infty$	CPU(s)
90	0.2	462	3.53e−008	1615.36	8.46e−008	175.50
90	0.3	202	6.44e−008	611.78	7.22e−008	175.80
90	0.4	74	3.96e−008	172.86	6.55e−008	178.80
90	0.5	1204	9.97e−008	2136.56	6.16e−008	154.80
90	0.6	36	7.82e−008	85.80	5.99e−008	176.58
90	0.7	25	4.45e−008	39.83	5.96e−008	176.42
90	0.8	17	5.45e−008	21.36	5.96e−008	146.56
90	0.9	13	3.62e−008	10.00	6.44e−008	46.45
120	0.2	677	5.17e−008	5448.44	**5.30e−005**	513.36
120	0.3	286	9.39e−008	2107.55	9.88e−008	345.02
120	0.4	86	8.64e−008	447.50	2.83e−008	342.73
120	0.5	174	9.93e−008	1145.81	8.12e−008	281.80
120	0.6	38	6.85e−008	213.92	6.60e−008	320.56
120	0.7	25	6.76e−008	85.53	6.03e−008	326.73
120	0.8	17	5.74e−008	49.58	5.96e−008	317.05
120	0.9	13	4.10e−008	21.81	6.44e−008	133.53
150	0.2	908	5.39e−008	14451.89	**9.39e−004**	854.31
150	0.3	416	7.43e−008	5734.89	**3.57e−004**	853.31
150	0.4	117	9.89e−008	1304.67	**4.28e−005**	855.77
150	0.5	394	9.95e−008	4618.00	3.48e−008	531.94
150	0.6	37	9.43e−008	377.16	9.56e−008	536.20
150	0.7	26	4.33e−008	176.50	6.63e−008	529.78
150	0.8	17	6.12e−008	93.56	5.99e−008	532.02
150	0.9	13	4.38e−008	40.58	6.45e−008	334.05
180	0.2	1158	3.65e−008	32184.38	**5.28e−003**	1158.42
180	0.3	523	9.35e−008	13253.72	**3.15e−003**	1161.84
180	0.4	142	5.66e−008	2796.27	**1.14e−003**	1190.44
180	0.5	284	9.95e−008	6182.42	**7.20e−005**	912.89
180	0.6	38	8.73e−008	708.69	2.98e−008	891.89
180	0.7	26	6.71e−008	317.06	8.64e−008	731.09
180	0.8	17	6.41e−008	168.64	6.18e−008	747.06
180	0.9	13	4.80e−008	71.17	6.46e−008	572.41

TABLE 4.3
Numerical results of Algorithm 3.1 *and the* RPM.

Problem		Algorithm 3.1			RPM [1]	
N	p	Ite	$\|\min(x,F)\|_\infty$	CPU(s)	$\|\min(x,F)\|_\infty$	CPU(s)
200	0.6	38	8.55e−008	956.75	2.98e−008	1410.88
200	0.7	26	8.49e−008	463.53	3.31e−008	985.39
200	0.8	17	6.56e−008	245.14	6.51e−008	929.34
200	0.9	13	5.45e−008	95.98	6.47e−008	771.55
400	0.6	42	7.52e−008	6820.59	**2.78e−002**	8170.84
400	0.7	28	4.51e−008	3911.89	**8.63e−003**	8255.02
400	0.8	17	7.30e−008	2075.92	2.52e−007	8241.44
400	0.9	14	9.65e−008	886.25	7.48e−008	5103.84
600	0.6	49	9.99e−008	19831.94	**3.87e−002**	19970.98
600	0.7	29	4.32e−008	11135.61	**3.41e−002**	19898.55
600	0.8	17	7.61e−008	6240.20	**6.68e−003**	19962.70
600	0.9	14	4.89e−008	3267.95	7.39e−008	14321.89
800	0.6	44	9.03e−008	28488.84	**3.33e−002**	36137.73
800	0.7	29	5.04e−008	20648.20	**3.39e−002**	36204.59
800	0.8	17	8.87e−008	11810.73	**2.58e−002**	36222.39
800	0.9	14	6.72e−008	7048.72	5.00e−007	36145.81
1000	0.6		> 10 hours		> 10 hours	
1000	0.7	29	7.80e−008	32103.77	> 10 hours	
1000	0.8	18	2.79e−008	23479.89	> 10 hours	
1000	0.9	14	2.03e−008	14017.41	> 10 hours	

5. Conclusions. In this paper, we proposed an inexact smoothing Newton method for solving a class of non-Lipschitzian complementarity problems defined in (1.1). This method has some nice features. (1) It converges quadratically and globally under the condition that M is a P_0-matrix. This condition is weaker than the ones used in [1, 5]. (2) Under the condition that M is a P_0-matrix, the reformulated system $H(t, z) = 0$, where H is defined in (2.10), has a unique solution and also enjoys a nonsingularity property. This is vital to apply the BiCGStab iterative solver to get an approximate solution of the resulting linear system (3.1). (3) Numerical results reported in this paper show that this method can produce an approximate solution with high accuracy for large-scale problems.

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