# Boundary control of vibration in coupled nonlinear three dimensional marine risers

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## Abstract

This paper presents a design of boundary controllers implemented at the top end for global stabilization of a marine riser in three dimensional space under environmental loadings. Based on the energy approach, nonlinear partial differential equations of motion including bending-bending and longitudinal-bending couplings for the risers are derived. These couplings cause mutual effects between the three independent directions in the riser's motions and make it difficult to minimize its vibrations. The Lyapunov direct method is employed to design the boundary controller. It is shown that the proposed boundary controllers can effectively reduce the riser's vibration. Stability analysis of the closed-loop system is performed using the Lyapunov direct method. Numerical simulations illustrate the results.

## Introduction

In offshore petroleum production, marine risers are crucial in transporting petroleum products from wellheads to floating rigs, containing drill strings and carrying mud in drilling operations. The marine riser is subject to environmental loading (waves, wind, and ocean currents), vortex induced vibrations and rig drifts, tension the from rig heave motion. These phenomena can reduce risers' lifespan and lead to an interruption of offshore operations in some cases. Furthermore, the riser's slender body due to the high length-to-diameter ratio make its controlling and maintaining a challenging engineering task.

For the dynamic analysis purpose, the marine riser is considered as a distributed system which is modelled by a set of partial differential equations (PDE) and boundary conditions (Niedzwecki and Liagre, 2003). Dynamical systems governed by PDEs is difficult to control and receive a lot of attention. The most classical control strategy to the distributed systems was based on modal analysis (Balas, 1978; Cavallo and De Maria, 1999; Fung and Liao, 1995). The modal analysis was used to derive a truncated model of the given system. Only some critical modes of the infinite dimensional and distributed parameter systems were observed and controlled. However, the control quality is greatly affected by observation and control spill-over due to residual (uncontrolled) modes. In addition, the requirement of arranging distributed actuators and sensors poses many difficulties in bringing the modal analysis-based control into practice. It might be problematic to deploy distributed devices in some cases, such as when controlling a deep-water riser.

In order to overcome aforementioned drawbacks of modal analysis approach, a number of control methods are developed to deal with the original PDE systems of the infinite dimensional systems instead of their truncated model. In (Ge et al., 2001), the variable structure control was employed for regulating a flexible beam. The control design process was directly based on PDE governed equations of motion. However, it is difficult to generalize the design procedure to other flexible systems. An elegant boundary control design can be found in (Krstic et al., 2006a,b, 2007), the authors successfully established an integral transformation to convert a beam system into a target system whose dynamical responses are known. The main aim of this transformation were to find a proper gain kernel and then perform an inverse transformation, which can be a very complicated task due to the complexities of the given systems. Based on Lyapunov's direct method, various boundary controllers have been proposed for flexible string-like and beam-like systems. Boundary control with different string models is developed in (Shahruz and Narasimha, 1997; Shahruz and Kurmaji, 1997; Shahruz and Krishna, 1996; Kim and Jung, 2011), it is shown that with simple boundary feedbacks, exponential stability can be achieved. In (Fard and Sagatun, 2001; Queiroz et al., 2000), boundary control are used for stabilizing string and beam systems. Due to the systematic approach and the ease of implementation in practice, applications of boundary control in marine riser vibration suppression have received increasing attention. In (Do and Pan, 2008; How et al., 2009; Ge et al., 2010; He et al., 2011), boundary controllers were proposed for controlling vibration of marine risers based on Lyapunov direct method. In (Do and Pan, 2008), riser-actuator dynamics were taken into account, whereas, in (He et al., 2011) a marine riser with vessel dynamics were considered. An interesting work on controlling marine risers was presented in (Do and Pan, 2009), where a boundary controller for a coupled system consisting of a three-dimensional riser and boundary actuators was developed. The riser model is suitable for a class of flexible risers since the riser is modeled as a rod-type system and not a beam-type system. In the aforementioned references, coupled dynamics such as bending-bending and longitudinal-bending effects were not entirely considered, the riser motions were restricted in one plane. The ignorance of coupling can directly deteriorate the performance of the controlled system. Therefore, it is necessary to include couplings in the riser dynamics in the control design process.

In this paper, a global stabilization problem for three dimensional flexible marine risers under environmental disturbances is investigated. Motion of the riser is described by a set

of PDEs and boundary conditions derived by the energy approach. The riser dynamics possess high nonlinearities due to system couplings. The couplings show the direct effects between motions in three directions, and lead to a more complex control design process than the one carried out in (Do and Pan, 2008). Based on Lyapunov's direct method, a boundary controller at the top end of the riser is designed. The proof of existence, uniqueness, and convergence of the solutions of the closed-loop system is provided. The proposed boundary controller in this paper guarantees that when there are no environmental disturbances, the riser is globally exponentially stabilized at its equilibrium position and that when the disturbances are presented, the riser is stabilized at the neighborhood of its equilibrium position.
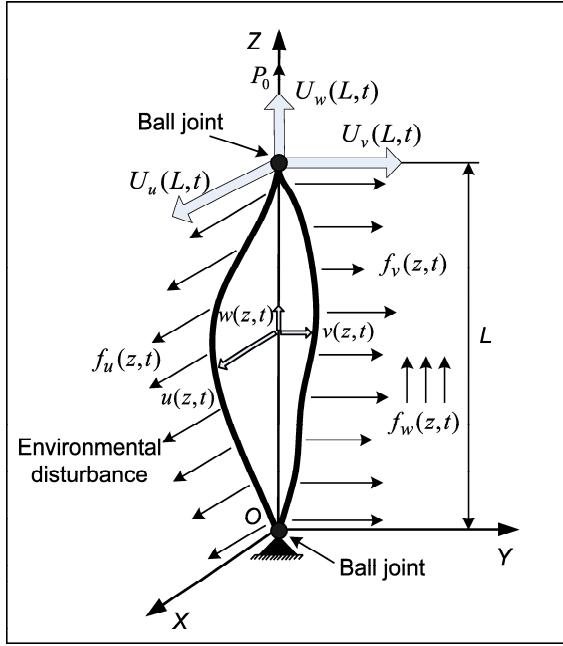
# Mathematical model



Figure 1: Riser coordinates.

The equations of motion of the riser derived in this section are subject to the following assumption:

**Assumption 1**

1. The riser can be modeled as a beam like structure because of its high length-to-diameter ratio.

2. Plane sections remain plane after deformation, i.e. warping is neglected.

3. The riser is locally stiff, i.e. cross-sections do not deform and the Poisson effect is neglected.

4. The riser material is homogeneous, isotropic and linearly elastic, i.e. it obeys Hookes's law.

5. Torsional and distributed moments induced by environmental disturbances are neglected.

6. The riser deforms in three dimensional space.

7. Ball joints are placed at the both ends of the riser, i.e. there is no bending at the both ends.

8. Environmental disturbances are bounded.

**Remark 0.1** *Items (1-4) imply that the riser will be modeled as a Bernoulli beam rather than a Timoshenko beam, and that the riser's extension is small. Bernoulli-Euler models are adequate for modeling the low-frequency responses of beams. Item (5) indicates that fluid/gas transportation risers, rather than drilling risers, are considered and that moments induced by asymmetrically relative flow due to vortex shedding are ignored. Item (7) and (8) always hold in practice.*

The kinetic energy of the riser is given by

$$T = \frac{m_0}{2}\int_0^L \Big[\Big(\frac{\partial u(z,t)}{\partial t}\Big)^2 + \Big(\frac{\partial v(z,t)}{\partial t}\Big)^2 + \Big(\frac{\partial w(z,t)}{\partial t}\Big)^2\Big]\mathrm{d}z, \quad (1)$$

where $u(z,t), v(z,t)$ are transverse displacements in the $X$ and $Y$ directions, respectively, and $w(z,t)$ is longitudinal displacement in the $Z$ direction. $L$ is the length of the riser, $m_0 = \rho A$ is the oscillating mass of the riser per unit length, $A$ is the riser's cross section area, $\rho$ is the mass density of the riser. It is assumed that the riser under consideration is subject to a constant tension $P_0$. The potential energy of the riser can be expressed as follows

$$\begin{aligned} P = &\frac{EI}{2}\int_0^L \Big[\Big(\frac{\partial^2 u(z,t)}{\partial z^2}\Big)^2 + \Big(\frac{\partial^2 v(z,t)}{\partial z^2}\Big)^2\Big]\mathrm{d}z \\ &+ \Big(\frac{\partial v(z,t)}{\partial z}\Big)^2\Big]\mathrm{d}z + \frac{EA}{2}\int_0^L \times\Big[\frac{\partial w(z,t)}{\partial z} \\ &+ \frac{1}{2}\Big(\frac{\partial u(z,t)}{\partial z}\Big)^2 + \frac{1}{2}\Big(\frac{\partial v(z,t)}{\partial z}\Big)^2\Big]^2 \mathrm{d}z, \quad (2)\end{aligned}$$

where $E$ is the Young's modulus, $I$ is the second moment of the riser's cross section area. The first part in the potential energy expression quantifies bending, the second part is due to tension force and the third term is strain energy of the riser. The work done by environmental disturbances acting on the riser is given by

$$\begin{aligned} W_f = &\int_0^L f_u(z,t)u(z,t)\mathrm{d}z + \int_0^L f_v(z,t)v(z,t)\mathrm{d}z \\ &+ \int_0^L f_w(z,t)w(z,t)\mathrm{d}z, \quad (3)\end{aligned}$$

where $f_u(z,t)$, $f_v(z,t)$ and $f_w(z,t)$ are the hydrodynamics forces acting on the riser in $X$, $Y$ and $Z$ directions, respectively. The work done by structure damping is represented as

$$\begin{aligned} W_d = &-d_1\int_0^L u_t(z,t)u(z,t)\mathrm{d}z - d_2\int_0^L v_t(z,t)v(z,t)\mathrm{d}z \\ &- d_3\int_0^L w_t(z,t)w(z,t)\mathrm{d}z, \quad (4)\end{aligned}$$

where $d_1$, $d_2$ and $d_3$ are the damping coefficients in $X$, $Y$ and $Z$ directions. The work done by active boundary actuators is

$$W_m = U_u(L,t)u(L,t) + U_v(L,t)v(L,t) + U_w(L,t)w(L,t), \quad (5)$$

where $U_u(L,t)$, $U_v(L,t)$ and $U_w(L,t)$ are the boundary control forces. The total work done on the system is presented as

$$\begin{aligned} W = &\int_0^L f_u(z,t)u(z,t)\mathrm{d}z + \int_0^L f_v(z,t)v(z,t)\mathrm{d}z \\ &+ \int_0^L f_w(z,t)w(z,t)\mathrm{d}z - d_1\int_0^L u_t(z,t)u(z,t)\mathrm{d}z \\ &- d_2\int_0^L v_t(z,t)v(z,t)\mathrm{d}z - d_3\int_0^L w_t(z,t)w(z,t)\mathrm{d}z \\ &+ U_u(L,t)u(L,t) + U_v(L,t)v(L,t) + U_w(L,t)w(L,t). \quad (6)\end{aligned}$$

The extended Hamilton's principle is given by

$$\int_{t_1}^{t_2} \delta(T - P + W)\mathrm{d}t = 0. \tag{7}$$

From this point onward, the arguments $(z, t)$ is omitted whenever it is not confusing. For the riser under consideration, ball joints placed at both ends imply that there is no bending at both ends, see Figure 1. In addition, the lower end is fixed. With aforementioned configuration, expanding (7) results in the following equations of motion

$$- m_0 u_{tt} - EI u_{zzzz} + P_0 u_{zz} + \frac{3EA}{2} u_z^2 u_{zz} + EA w_{zz} u_z$$
$$+ EA w_z u_{zz} + \frac{EA}{2} u_{zz} v_z^2 + EA v_z v_{zz} u_z - d_1 u_t + f_u = 0,$$
$$- m_0 v_{tt} - EI v_{zzzz} + P_0 v_{zz} + \frac{3EA}{2} v_z^2 v_{zz} + EA w_{zz} v_z$$
$$+ EA w_z v_{zz} + \frac{EA}{2} v_{zz} u_z^2 + EA u_z u_{zz} v_z - d_1 v_t + f_v = 0,$$
$$- m_0 w_{tt} - EA w_{zz} + EA u_z u_{zz} + EA v_z v_{zz} - d_3 w_t + f_w = 0,$$
$$- EI u_{zzz}(L, t) + P_0 u_z(L, t) + \frac{EA}{2} u_z^3(L, t)$$
$$+ EA w_z(L, t) u_z(L, t) + \frac{EA}{2} u_z(L, t) v_z^2(L, t) = U_u(L, t),$$
$$- EI v_{zzz}(L, t) + P_0 v_z(L, t) + \frac{EA}{2} v_z^3(L, t)$$
$$+ EA w_z(L, t) v_z(L, t) + \frac{EA}{2} v_z(L, t) u_z^2(L, t) = U_v(L, t),$$
$$EA w_z(L, t) + \frac{EA}{2} u_z^2(L, t) + \frac{EA}{2} v_z^2(L, t) = U_w(L, t),$$
$$u_{zz}(L, t) = v_{zz}(L, t) = u_{zz}(0, t) = v_{zz}(0, t) = 0,$$
$$u(0, t) = v(0, t) = w(0, t) = 0, \tag{8}$$

where the following notations $\frac{\partial(\bullet)}{\partial z} = (\bullet)_z$, $\frac{\partial^2(\bullet)}{\partial z^2} = (\bullet)_{zz}$, $\frac{\partial^3(\bullet)}{\partial z^3} = (\bullet)_{zzz}$, $\frac{\partial^4(\bullet)}{\partial z^4} = (\bullet)_{zzzz}$, $\frac{\partial(\bullet)}{\partial t} = (\bullet)_t$, and $\frac{\partial^2(\bullet)}{\partial t^2} = (\bullet)_{tt}$ have been used.

# Boundary control design

Subject to Assumption 1, design the boundary control forces $U_u(L, t)$, $U_v(L, t)$, and $U_w(L, t)$ from information at the top end of the riser for the riser system (8) to stabilize the riser at the initial state, and

1. in the case where disturbances $f_u$, $f_v$, and $f_w$ are ignored, $|u(z, t)|, |v(z, t)|, |w(z, t)|, \int_0^L u_z(z, t)\mathrm{d}z,$ $\int_0^L v_z(z, t)\mathrm{d}s, \int_0^L w_z(z, t)\mathrm{d}z, \int_0^L u_t(z, t)\mathrm{d}z,$ $\int_0^L v_t(z, t)\mathrm{d}z, \int_0^L w_t(z, t)\mathrm{d}z, \int_0^L u_{zz}\mathrm{d}z,$ and $\int_0^L v_{zz}\mathrm{d}z$ exponentially converge to zero $\forall z \in [0, L]$ and $\forall t \geq t_0$,

2. in the case where disturbances $f_u$, $f_v$, and $f_w$ are present, $|u(z, t)|, |v(z, t)|, |w(z, t)|, \int_0^L u_z(z, t)\mathrm{d}z,$ $\int_0^L v_z(z, t)\mathrm{d}s, \int_0^L u_t(z, t)\mathrm{d}z, \int_0^L w_z(z, t)\mathrm{d}z,$ $\int_0^L v_t(z, t)\mathrm{d}z, \int_0^L w_t(z, t)\mathrm{d}z, \int_0^L u_{zz}\mathrm{d}z,$ and $\int_0^L v_{zz}\mathrm{d}z$ exponentially converge to positive constants $\forall z \in [0, L]$ and $\forall t \geq t_0$.

Consider the following Lyapunov candidate function

$$V = \frac{m_0}{2} \int_0^L (u_t^2 + v_t^2 + w_t^2)\mathrm{d}z + \frac{P_0}{2} \int_0^L (u_z^2 + v_z^2)\mathrm{d}z$$
$$+ \frac{EA}{2} \int_0^L w_z^2\mathrm{d}z + \frac{EA}{2} \int_0^L \left(w_z + \frac{u_z^2}{2} + \frac{v_z^2}{2}\right)^2\mathrm{d}z$$
$$+ \frac{EI}{2} \int_0^L (u_{zz}^2 + v_{zz}^2)\mathrm{d}z + \rho_1 \int_0^L u u_t\mathrm{d}z$$
$$+ \rho_2 \int_0^L v v_t\mathrm{d}z + \rho_3 \int_0^L w w_t\mathrm{d}z + \left(k_1 + \frac{k_2 \rho_1}{m_0}\right) u^2(L)$$
$$+ \left(k_3 + \frac{k_4 \rho_2}{m_0}\right) v^2(L) + \left(k_5 + \frac{k_6 \rho_3}{m_0}\right) w^2(L). \tag{9}$$

A calculation shows that

$$V \geq \left(\frac{m_0}{2} - \frac{\rho_1}{\gamma_1}\right) \int_0^L u_t^2\mathrm{d}z + \left(\frac{m_0}{2} - \frac{\rho_2}{\gamma_2}\right) \int_0^L v_t^2\mathrm{d}z$$
$$+ \left(\frac{m_0}{2} - \frac{\rho_3}{\gamma_3}\right) \int_0^L w_t^2\mathrm{d}z + \left(\frac{P_0}{2} - 4L^2 \gamma_1 \rho_1\right) \int_0^L u_z^2\mathrm{d}z$$
$$+ \left(\frac{P_0}{2} - 4L^2 \gamma_2 \rho_2\right) \int_0^L v_z^2\mathrm{d}z + \left(\frac{P_0}{2} - 4L^2 \gamma_3 \rho_3\right) \int_0^L w_z^2\mathrm{d}z$$
$$+ \frac{EA}{2} \int_0^L \left(w_z + \frac{u_z^2}{2} + \frac{v_z^2}{2}\right)^2\mathrm{d}z + \frac{EI}{2} \int_0^L \left(u_{zz}^2 + v_{zz}^2\right)\mathrm{d}z$$
$$+ \frac{1}{2}\left(k_1 + \frac{k_2 \rho_1}{m_0}\right) u^2(L) + \frac{1}{2}\left(k_3 + \frac{k_4 \rho_2}{m_0}\right) v^2(L)$$
$$+ \left(k_5 + \frac{k_6 \rho_3}{m_0}\right) w^2(L), \tag{10}$$

and

$$V \leq \left(\frac{m_0}{2} + \frac{\rho_1}{\gamma_1}\right) \int_0^L u_t^2\mathrm{d}z + \left(\frac{m_0}{2} + \frac{\rho_2}{\gamma_2}\right) \int_0^L v_t^2\mathrm{d}z$$
$$+ \left(\frac{m_0}{2} + \frac{\rho_3}{\gamma_3}\right) \int_0^L w_t^2\mathrm{d}z + \left(\frac{P_0}{2} + 4L^2 \gamma_1 \rho_1\right) \int_0^L u_z^2\mathrm{d}z$$
$$+ \left(\frac{P_0}{2} + 4L^2 \gamma_2 \rho_2\right) \int_0^L v_z^2\mathrm{d}z + \left(\frac{P_0}{2} + 4L^2 \gamma_3 \rho_3\right) \int_0^L w_z^2\mathrm{d}z$$
$$+ \frac{EA}{2} \int_0^L \left(w_z + \frac{u_z^2}{2} + \frac{v_z^2}{2}\right)^2\mathrm{d}z + \frac{EI}{2} \int_0^L \left(u_{zz}^2 + v_{zz}^2\right)\mathrm{d}z$$
$$+ \frac{1}{2}\left(k_1 + \frac{k_2 \rho_1}{m_0}\right) u^2(L) + \frac{1}{2}\left(k_3 + \frac{k_4 \rho_2}{m_0}\right) v^2(L)$$
$$+ \left(k_5 + \frac{k_6 \rho_3}{m_0}\right) w^2(L). \tag{11}$$

We select $\rho_1, \rho_2, \rho_3, \gamma_1, \gamma_2,$ and $\gamma_3$ such that:

$$\frac{m_0}{2} - \frac{\rho_1}{\gamma_1} = c_1, \quad \frac{m_0}{2} - \frac{\rho_2}{\gamma_2} = c_2, \quad \frac{m_0}{2} - \frac{\rho_3}{\gamma_3} = c_3,$$
$$\frac{P_0}{2} - 4L^2 \gamma_1 \rho_1 = c_4, \quad \frac{P_0}{2} - 4L^2 \gamma_2 \rho_2 = c_5,$$
$$\frac{P_0}{2} - 4L^2 \gamma_3 \rho_3 = c_6. \tag{12}$$

where $c_i$, for $i = 1 \ldots 6$, are strictly positive constants. Equation (12) shows that the Lyapunov candidate $V$ is a proper function of $\int_0^L u_t^2\mathrm{d}z, \int_0^L v_t^2\mathrm{d}z, \int_0^L w_t^2\mathrm{d}z, \int_0^L u_z^2\mathrm{d}z, \int_0^L v_z^2\mathrm{d}z,$ $\int_0^L w_z^2\mathrm{d}z, \int_0^L u_{zz}^2\mathrm{d}z,$ and $\int_0^L v_{zz}^2\mathrm{d}z$. Differentiating (9) along the solutions of the equations of motion (8) results in

$$\dot{V} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6 + \Delta_7, \tag{13}$$

where

$$\Delta_1 = \int_0^L u_t\Big(-EIu_{zzzz}+P_0u_{zz}+\frac{3EA}{2}u_z^2u_{zz}+EAw_{zz}u_z$$
$$+EAw_zu_{zz}+\frac{EA}{2}u_{zz}v_z^2+EAv_zv_{zz}u_z-d_1u_t+f_u\Big)\mathrm{d}z, \tag{14}$$

$$\Delta_2 = \int_0^L v_t\Big(-EIv_{zzzz}+P_0v_{zz}+\frac{3EA}{2}v_z^2v_{zz}+EAw_{zz}v_z$$
$$+EAw_zv_{zz}+\frac{EA}{2}v_{zz}u_z^2+EAu_zu_{zz}v_z-d_2v_t+f_v\Big)\mathrm{d}z, \tag{15}$$

$$\Delta_3 = \int_0^L w_t\Big(EAw_{zz}+EAu_zu_{zz}+EAv_zv_{zz}-d_3w_t$$
$$+f_w\Big)\mathrm{d}z, \tag{16}$$

$$\Delta_4 = \frac{P_0}{2}\int_0^L(u_z^2+v_z^2)\mathrm{d}z+\frac{EA}{2}\int_0^L\Big(w_z+\frac{u_z^2}{2}+\frac{v_z^2}{2}\Big)^2\mathrm{d}z$$
$$+\frac{EI}{2}\int_0^L(u_{zz}^2+v_{zz}^2)\mathrm{d}z+\Big(k_1+\frac{k_2\rho_1}{m_0}\Big)u^2(L)$$
$$+\Big(k_3+\frac{k_4\rho_2}{m_0}\Big)v^2(L)+\Big(k_5+\frac{k_6\rho_3}{m_0}\Big)w^2(L), \tag{17}$$

$$\Delta_5 = \rho_1\int_0^L u_t^2\mathrm{d}z+\frac{\rho_1}{m_0}\int_0^L u\Big(-EIu_{zzzz}+P_0u_{zz}$$
$$+\frac{3EA}{2}u_z^2u_{zz}+EAw_{zz}u_z+EAw_zu_{zz}$$
$$+\frac{EA}{2}u_{zz}v_z^2+EAv_zv_{zz}u_z-d_1u_t+f_u\Big)\mathrm{d}z, \tag{18}$$

$$\Delta_6 = \rho_2\int_0^L v_t^2\mathrm{d}z+\frac{\rho_2}{m_0}\int_0^L v\Big(-EIv_{zzzz}+P_0v_{zz}$$
$$+\frac{3EA}{2}v_z^2v_{zz}+EAw_{zz}v_z+EAw_zv_{zz}$$
$$+\frac{EA}{2}v_{zz}u_z^2+EAu_zu_{zz}v_z-d_2v_t+f_v\Big)\mathrm{d}z, \tag{19}$$

$$\Delta_7 = \rho_3\int_0^L w_t^2\mathrm{d}z+\frac{\rho_3}{m_0}\int_0^L w\Big(EAw_{zz}$$
$$+EAu_zU_{zz}+EAv_zv_{zz}-d_3w_t+f_w\Big)\mathrm{d}z. \tag{20}$$

Integrating (14), (15), (16), (18), (19), and (20) by parts then substituting the result into (13) and using boundary conditions specified in (8) give

$$\dot{V} = \Big(u_t(L)+\frac{\rho_1}{m_0}u(L)\Big)\Big(-EIu_{zzz}(L)+P_0u_z(L)$$
$$+\frac{EA}{2}u_z^3(L)+EAw_z(L)u_z(L)+\frac{EA}{2}u_z(L)v_z^2(L)\Big)$$
$$+\Big(v_t(L)+\frac{\rho_2}{m_0}v(L)\Big)\Big(-EIv_{zzz}(L)+P_0v_z(L)$$
$$+\frac{EA}{2}v_z^3(L)+EAw_z(L)v_z(L)+\frac{EA}{2}v_z(L)v_z^2(L)\Big)$$
$$+\Big(w_t(L)+\frac{\rho_3}{m_0}w(L)\Big)\Big(EAw_z(L)+\frac{EA}{2}u_z^2(L)$$
$$+\frac{EA}{2}v_z^2(L)\Big)-\big(d_1-\rho_1\big)\int_0^L u_t^2\mathrm{d}z-\big(d_2-\rho_2\big)\int_0^L v_t^2\mathrm{d}z$$
$$-\big(d_3-\rho_3\big)\int_0^L w_t^2\mathrm{d}z-\frac{\rho_1EI}{m_0}\int_0^L u_{zz}^2\mathrm{d}z-\frac{\rho_2EI}{m_0}\int_0^L v_{zz}^2\mathrm{d}z$$
$$-\frac{\rho_1P_0}{m_0}\int_0^L u_z^2\mathrm{d}z-\frac{\rho_2P_0}{m_0}\int_0^L v_z^2\mathrm{d}z-\frac{\rho_1EA}{2m_0}\int_0^L u_z^4\mathrm{d}z$$
$$-\frac{\rho_2EA}{2m_0}\int_0^L v_z^4\mathrm{d}z-\frac{EA}{m_0}\Big(\rho_1+\frac{\rho_3}{2}\Big)\int_0^L u_z^2w_z\mathrm{d}z$$
$$-\frac{EA}{2m_0}(\rho_1+\rho_2)\int_0^L u_z^2v_z^2\mathrm{d}z-\frac{EA}{m_0}\Big(\rho_2+\frac{\rho_3}{2}\Big)\int_0^L v_z^2w_z\mathrm{d}z$$
$$-\frac{\rho_1d_1}{m_0}\int_0^L uu_t\mathrm{d}z+\frac{\rho_1}{m_0}\int_0^L uf_u\mathrm{d}z-\frac{\rho_2d_2}{m_0}\int_0^L vv_t\mathrm{d}z$$
$$+\frac{\rho_2}{m_0}\int_0^L vf_v\mathrm{d}z-\frac{\rho_3EA}{m_0}\int_0^L w_z^2\mathrm{d}z-\frac{\rho_3d_3}{m_0}\int_0^L ww_t\mathrm{d}z$$
$$+\int_0^L v_tf_v\mathrm{d}z+\int_0^L u_tf_u\mathrm{d}z+\int_0^L w_tf_w\mathrm{d}z+\frac{\rho_3}{m_0}\int_0^L wf_w\mathrm{d}z$$
$$+\Big(k_1+\frac{k_2\rho_1}{m_0}\Big)u(L)u_t(L)+\Big(k_3+\frac{k_4\rho_2}{m_0}\Big)v(L)v_t(L)$$
$$+\Big(k_5+\frac{k_6\rho_3}{m_0}\Big)w(L)w_t(L). \tag{21}$$

Since $-EIu_{zzz}(L,t)+P_0u_z(L,t)+\frac{EA}{2}u_z^3(L,t)+EAw_z(L,t)u_z(L,t)+\frac{EA}{2}u_z(L,t)v_z^2(L,t)=U_u(L,t)$, $-EIv_{zzz}(L,t)+P_0v_z(L,t)+\frac{EA}{2}v_z^3(L,t)+EAw_z(L,t)v_z(L,t)+\frac{EA}{2}v_z(L,t)u_z^2(L,t)=U_v(L,t)$, and $EAw_z(L,t)+\frac{EA}{2}u_z^2(L,t)+\frac{EA}{2}v_z^2(L,t)=U_w(L,t)$, the boundary control can be selected as follows,

$$U_u = -k_1u(L)-k_2u_t(L), \tag{22}$$

$$U_v = -k_3v(L)-k_4v_t(L), \tag{23}$$

$$U_w = -k_5w(L)-k_6w_t(L), \tag{24}$$

where coefficients $k_i$, for $i=1\ldots6$, are strictly positive constants. Substituting the controls (22), (23), and (24) into (21)

gives

$$\dot{V} \leq -\frac{k_1\rho_1}{m_0}u^2(L) - k_2u_t^2(L) - \frac{k_3\rho_2}{m_0}v^2(L) - k_4v_t^2(L)$$

$$-\frac{k_5\rho_3}{m_0}w^2(L) - k_6w_t^2(L) - \left(d_1 - \rho_1 - \frac{d_1\rho_1}{\gamma_4 m_0}\right)\int_0^L u_t^2 dz$$

$$-\left(d_2 - \rho_2 - \frac{d_2\rho_2}{\gamma_5 m_0}\right)\int_0^L v_t^2 dz - \left(d_3 - \rho_3 - \frac{d_3\rho_3}{\gamma_6 m_0}\right)\int_0^L w_t^2 dz$$

$$-\frac{\rho_1 EI}{m_0}\int_0^L u_{zz}^2 dz - \frac{\rho_2 EI}{m_0}\int_0^L v_{zz}^2 dz - \left(\frac{\rho_1 P_0}{m_0} - \frac{4L^2 d_1\rho_1\gamma_4}{m_0}\right)$$

$$\times\int_0^L u_z^2 dz - \left(\frac{\rho_2 P_0}{m_0} - \frac{4L^2 d_2\rho_2\gamma_5}{m_0}\right)\int_0^L v_z^2 dz - \left(\frac{\rho_3 EA}{m_0} - \frac{4L^2 d_3\rho_3\gamma_6}{m_0}\right)$$

$$\times\int_0^L w_z^2 dz - \frac{\rho_1 EA}{2m_0}\int_0^L u_z^4 dz - \frac{\rho_2 EA}{2m_0}\int_0^L v_z^4 dz - \frac{EA}{m_0}\left(\rho_1 + \frac{\rho_3}{2}\right)$$

$$\times\int_0^L u^2 w_z dz - \frac{EA}{2m_0}(\rho_1 + \rho_2)\int_0^L u_z^2 v_z^2 dz - \frac{EA}{m_0}\left(\rho_2 + \frac{\rho_3}{2}\right)$$

$$\times\int_0^L v_z^2 w_z dz - \frac{\rho_1 d_1}{m_0}\int_0^L uu_t dz + \frac{\rho_1}{m_0}\int_0^L uf_u dz - \frac{\rho_2 d_2}{m_0}\int_0^L vv_t dz$$

$$+\frac{\rho_2}{m_0}\int_0^L vf_v dz - \frac{\rho_3 d_3}{m_0}\int_0^L ww_t dz + \int_0^L v_t f_v dz + \int_0^L u_t f_u dz$$

$$+\int_0^L w_t f_w dz + \frac{\rho_3}{m_0}\int_0^L wf_w dz. \quad (25)$$

From (25), the designed parameters are selected such that

$$d_1 - \rho_1 - \frac{d_1\rho_1}{\gamma_4 m_0} = c_7, \quad d_2 - \rho_2 - \frac{d_2\rho_2}{\gamma_5 m_0} = c_8,$$

$$d_3 - \rho_3 - \frac{d_3\rho_3}{\gamma_5 m_0} = c_9, \frac{\rho_1 P_0}{m_0} - \frac{4L^2 d_1\rho_1\gamma_4}{m_0} = c_{10},$$

$$\frac{\rho_2 P_0}{m_0} - \frac{4L^2 d_2\rho_2\gamma_5}{m_0} = c_{11}, \frac{\rho_3 P_0}{m_0} - \frac{4L^2 d_3\rho_3\gamma_6}{m_0} = c_{12}, \quad (26)$$

where $c_1$, for $i = 7\dots11$, are strictly positive constants. Using the upper bound of $V$ specified in (11), (25) can be expressed as

$$\dot{V} \leq -\frac{k_1\rho_1}{m_0}u^2(L) - k_2u_t^2(L) - \frac{k_3\rho_2}{m_0}v^2(L) - k_4v_t^2(L)$$

$$-\frac{k_5\rho_3}{m_0}w^2(L) - k_6w_t^2(L) - cV + \frac{\rho_1}{m_0}\int_0^L uf_u dz$$

$$+\frac{\rho_2}{m_0}\int_0^L vf_v dz + \int_0^L v_t f_v dz + \int_0^L u_t f_u dz$$

$$+\int_0^L w_t f_w dz + \frac{\rho_3}{m_0}\int_0^L wf_w dz, \quad (27)$$

where

$$c = \frac{\min\left\{c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, \beta_1, \beta_2\right\}}{\max\left\{\frac{m_0}{2} + \frac{\rho_1}{\gamma_1}, \frac{m_0}{2} + \frac{\rho_2}{\gamma_2}, \frac{m_0}{2} + \frac{\rho_3}{\gamma_3}, \beta_2\right\}}, \quad (28)$$

where

$$\beta_1 = \left\{\frac{\rho_1 EI}{m_0}, \frac{\rho_2 EI}{m_0}, \frac{\rho_1 EA}{m_0}, \frac{\rho_2 EA}{m_0}, \frac{EA}{2m_0}(\rho_1 + \rho_2)\right\},$$

$$\beta_1 = \left\{\frac{EA}{m_0}\left(\rho_1 + \frac{\rho_3}{2}\right), \frac{EA}{m_0}\left(\rho_2 + \frac{\rho_3}{2}\right), k_1\frac{\rho_1}{m_0}, k_3\frac{\rho_2}{m_0}, k_5\frac{\rho_3}{m_0}\right\},$$

$$\beta_3 = \left\{\frac{P_0}{2} + 4L^2\gamma_1\rho_1, \frac{P_0}{2} + 4L^2\gamma_2\rho_2, \frac{EA}{2}, \frac{EI}{2}\right\},$$

$$\beta_4 = \left\{\frac{1}{2}\left(k_1 + \frac{k_2\rho_1}{m_0}\right), \frac{1}{2}\left(k_3 + \frac{k_4\rho_2}{m_0}\right), \frac{1}{2}\left(k_5 + \frac{k_6\rho_3}{m_0}\right)\right\}. \quad (29)$$

The main outcome of this paper is stated in the following theorem whose proof is omitted due to space limitation, see (Nguyen et al., 2012) for the proof.

**Theorem 0.1** *Under* Assumption 1, *the control inputs $U_u$, $U_v$, and $U_w$ given in* (22), (23), *and* (24) *solve the control objective provided that the design constants $\rho_1$, $\rho_2$, and $\rho_3$ are chosen such that the conditions specified in* (12) *and* (26) *hold. In particular, the solutions of the closed-loop system consisting of* (8), (22), (23), *and* (24) *exist and are unique. Moreover, when the external distributed disturbances $f_u$, $f_v$, and $f_w$ are zero, all the terms $|u(z,t)|$, $|v(z,t)|$, $|w(z,t)|$, $\int_0^L u_z(z,t)dz$, $\int_0^L v_z(z,t)ds$, $\int_0^L w_z(z,t)ds$, $\int_0^L u_t(z,t)dz$, $\int_0^L v_t(z,t)dz$, $\int_0^L w_t(z,t)dz$, $\int_0^L u_{zz}dz$, and $\int_0^L v_{zz}dz$ exponentially converge to zero $\forall z \in [0,L]$ and $\forall t \geq t_0$, and when the external distributed disturbances $f_u$, $f_v$, and $f_w$ are different from zero but bounded, all the terms $|u(z,t)|$, $|v(z,t)|$, $|w(z,t)|$, $\int_0^L u_z(z,t)dz$, $\int_0^L v_z(z,t)ds$, $\int_0^L w_z(z,t)ds$, $\int_0^L u_t(z,t)dz$, $\int_0^L v_t(z,t)dz$, $\int_0^L w_t(z,t)dz$, and $\int_0^L u_{zz}dz$ exponentially converge to some small positive constants $\forall z \in [0,L]$ and $\forall t \geq t_0$.*

# Numerical simulation

The effectiveness of the proposed control is illustrated through numerical simulations. The parameters of the marine riser system taken from (Do and Pan, 2008) are given in Table 1.

Table 1: The marine riser system parameters.

| Nomenclature | Description | Value |
|---|---|---|
| $L$ | Length | 1000 m |
| $D$ | Diameter | 0.2 m |
| $\rho$ | Density | 8200kg/m$^3$ |
| $E$ | Young's modulus | $2 \times 10^8$kg/m$^2$ |
| $P_0$ | Tension | $1.1 \times 10^4$N |
| $d_1$ | $X$-direction damping | 40s/m$^2$ |
| $d_2$ | $Y$-direction damping | 40s/m$^2$ |
| $d_3$ | $Z$-direction damping | 20s/m$^2$ |

The distributed disturbances are given as

$$f_u = C_M \frac{\rho_w \pi D^2 u_{1t}(z,t)}{4} + C_D \frac{\rho_w D}{2}\sqrt{\frac{8}{\pi}}\sigma_u(z,t)u_1(z,t), \quad (30)$$

$$f_v = C_M \frac{\rho_w \pi D^2 u_{2t}(z,t)}{4} + C_D \frac{\rho_w D}{2}\sqrt{\frac{8}{\pi}}\sigma_v(z,t)u_2(z,t), \quad (31)$$

$$f_w = \frac{-0.1z}{L}, \quad (32)$$

where $\rho_w = 1024$kg/m$^3$ is the water density, $C_D = 1.2$ is the drag coefficient, and $\sigma_u(z,t)$ and $\sigma_v(z,t)$ are the root-mean-square of the water particle velocities in the $X$ and $Y$ directions, respectively. The water particle velocities in the $X$ and $Y$ directions are $u_1(z,t) = \frac{0.5z}{L}$m/s and $u_2(z,t) = \frac{z}{L}$m/s, respectively. The control gains are selected to be $k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 320$. The initial conditions at $t_0 = 0$ are $u(z,t_0) = v(z,t_0) = 0 = w(z,t_0)$ and $u_t(z,t_0) = v_t(z,t_0) = w_t(z,t_0) = 0$. Simulations are carried out over 400 seconds without control ($k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 0$) and with the proposed control.

The transverse displacements in uncontrolled and controlled cases are plotted in Figures 2, and 3, respectively. Figure 3
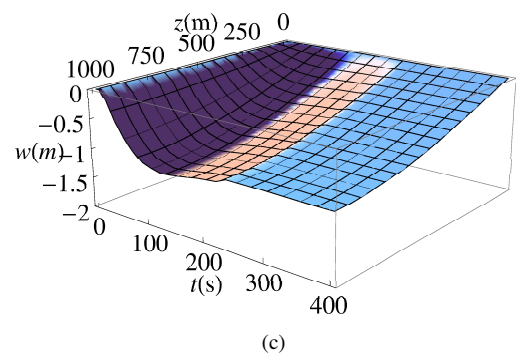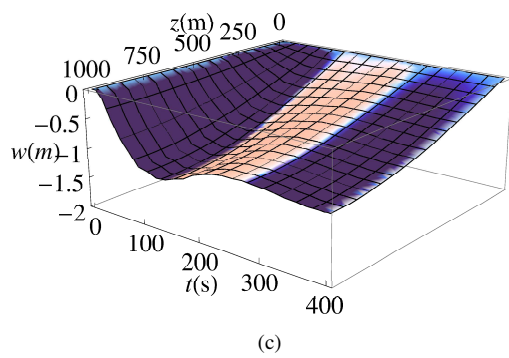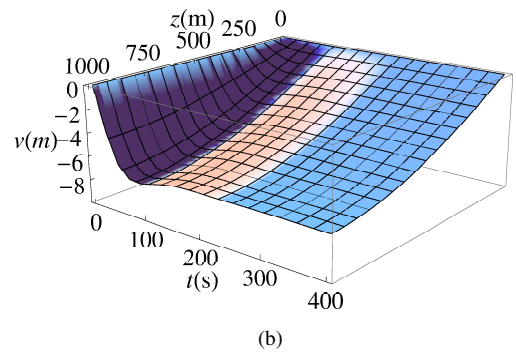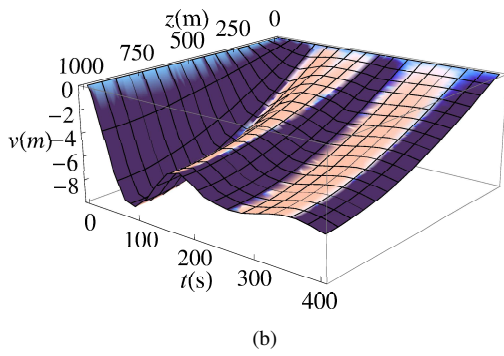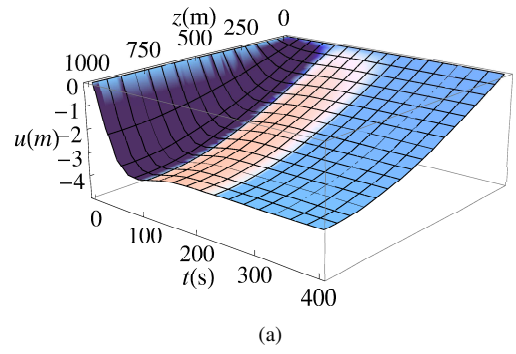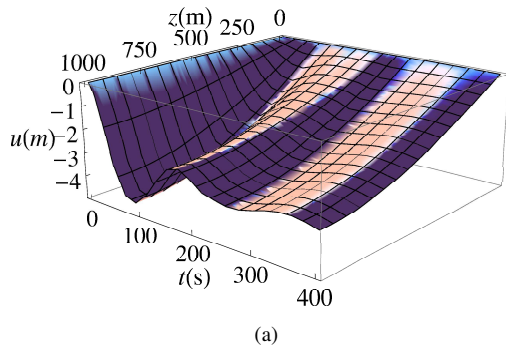
Figure 2: The riser's displacements with control: (a) $u(z,t)$, (b) $v(z,t)$, and (c) $w(z,t)$.



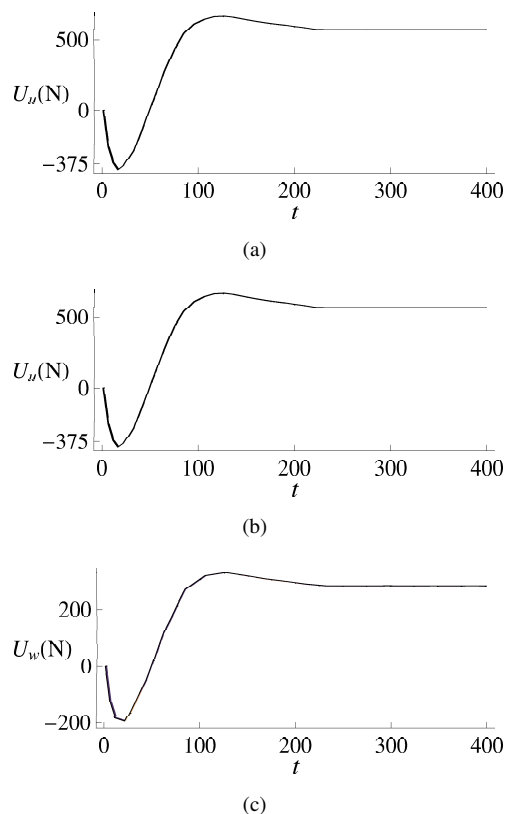Figure 3: The riser's displacements without control: (a) $u(z,t)$, (b) $v(z,t)$, and (c) $w(z,t)$.

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Figure 4: Control forces: (a) $U_u(L, t)$ (b) $U_v(L, t)$, and (c) $U_w(L, t)$.

shows that transverse and longitudinal displacements are reduced significantly when the proposed control is applied. From the numerical simulations, it can be observed that when the system is fully controlled, transverse displacements in both the $X$ and $Y$ directions are effectively reduced. Figure 4 indicates that control forces are in an implementable range in practice.

# Conclusion

The equations of motion which indicate strong nonlinear couplings for a marine riser system in three dimensions were derived using the extended Hamilton's principle. Subsequently, the Lyapunov direct method was employed to design the boundary controller applied at the top end of the riser. The designed controller's ability to stabilize the riser at its equilibrium position was proved analytically and illustrated numerically. The main contributions of this paper are the introduction of the Lyapunov function candidate (9) and the riser model in three dimensions with nonlinear couplings (8). An extension of this work is to include torsion to the riser dynamics.

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