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An Optimal Machine Maintenance Problem with Probabilistic State Constraints

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Abstract

We consider a machine that is maintained via two types of maintenance action: (i) continuous (minor) maintenance that curbs natural degradation of the machine; and (ii) overhaul (major) maintenance that takes place at certain discrete time points and significantly improves the condition of the machine. We introduce an impulsive stochastic differential equation to model the condition of the machine over the time horizon. The problem we investigate is to choose the continuous maintenance rate and the overhaul maintenance times to minimize the total cost of operating and maintaining the machine, where probabilistic state constraints are imposed to ensure that the machine's state and output meet minimum acceptable levels with high probability. This impulsive stochastic optimal control problem is first transformed into a deterministic optimal control problem with state jumps and continuous inequality constraints. We then show that this equivalent problem can be solved using a combination of the control parameterization technique, the time-scaling transformation, and the constraint transcription method. Finally, we illustrate our approach by solving a numerical example.

Keywords:
Maintenance scheduling, impulsive system, optimal control, nonlinear optimization

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1. Introduction

In any industrial setting, maintenance is paramount to ensuring reliable machine operation. Although maintenance operations are sometimes costly and onerous, ignoring maintenance will increase the likelihood of machine failure, thus potentially leading to major disruptions in production at some future time. Hence, effective maintenance policies are essential for production planning purposes.

Maintenance policies in the literature can be classified into three types: (i) failure-based, where the maintenance is performed after failure of the machine; (ii) time-based, where maintenance is scheduled at fixed times; and (iii) preventive-based, where maintenance is scheduled depending on the machine's state. An importance part of preventive maintenance is the modeling of the machine's deterioration process. The condition-based maintenance approach that we propose in this paper models the state of the machine using a stochastic process, which allows for the random noise and disturbances present in any real-life system. Maintenance overhauls are scheduled so that there is a sufficiently high probability that the machine's state will always be above a minimum acceptable level.

Previous research on machine maintenance has typically studied the performance of different maintenance and production policies separately, even though these two activities are intrinsically linked (Wang (2013), Fitouhi and Nourelfath (2012)). However, in recent years, more researchers have begun to design production and maintenance policies simultaneously (Widyadana and Wee (2012), Pan et al. (2010), Chen (2009)). Graves and Lee (1999) considered the simultaneous optimization of production together with preventive maintenance and scheduling decisions, but their model allows just one maintenance activity during the planning horizon. Similarly, the maintenance scheduling model studied by Hsu et al. (2013) also allowed for just one maintenance activity throughout the planning horizon. Qi et al. (1999) allowed for multiple maintenance activities, but neglected the downside risk of not performing maintenance. Cassady and Kutanoglu (2005) improved on the studies in Graves and Lee (1999) and Qi et al. (1999) by explicitly incorporating the risk of not performing maintenance. However, they assumed that the duration between maintenance times is constant, and that preventive maintenance restores the machine to an 'as good as new' condition, which is not always a realistic assumption.

Batun and Azizoğlu (2009) considered a model involving multiple maintenance activities of known start times and durations, where several non-resumable production jobs need to be scheduled optimally. Sbihi and Varnier (2008) improved on this model by relaxing the assumption of fixed maintenance time intervals, and imposing limits on the machine's maximum continuous working time. Pan et al. (2010) considered machine degradation and variable maintenance times. However, as with Cassady and Kutanoglu (2005), they assumed that preventive maintenance activities are able to restore the machine to an 'as good as new' condition. Fitouhi and Nourelfath (2012) make the same assumption, but allowed for the machine's failure rate to increase with time. Wang (2013), too, allowed for a time-dependent failure rate by using a Weibull distribution to model the time to failure of the machine.

This paper improves on existing models by allowing flexible maintenance time intervals while ensuring the probability of machine failure is below a minimum specified value. The optimal interval lengths between maintenance overhauls are decision variables, which must be chosen to ensure that the likelihood of machine failure is small and that a given production target is met. We formulate this problem as a special type of stochastic impulsive optimal control problem, where the state impulses are due to overhaul maintenance activities occurring at a set of discrete time points.

Impulsive optimal control problems arise in many applications (Lin et al. (2014)). Li et al. (2009) present an impulsive optimal control model for designing optimal trajectories of horizontal oil wells. Loxton et al. (2009) also apply impulsive optimal control methodologies to determine the optimal switching instants for a switched-capacitor DC/DC power converter. In another paper, Wu and Teo (2006) consider the optimization of a general impulsive system that can be used as a model for many real-life systems such as robots, locomotives, hybrid power generators and biochemical reactors. However, none of these papers consider impulsive optimal control models for machine maintenance scheduling. This paper represents the first attempt at applying optimal control techniques for impulsive systems in the machine maintenance area.

The remainder of the paper is organized as follows. In Section 2, we formulate the maintenance scheduling problem as an impulsive optimal control problem with stochastic disturbances. Then, in Section 3, we discuss a transformation technique for converting the stochastic problem into a deterministic problem. In Sections 4 and 5, we apply the time-scaling transformation and the constraint transcription technique to solve this deterministic problem. A numerical example is provided in Section 6. Section 7 concludes the paper.

2. Problem formulation

Let $x(t)$ denote the state of the machine at time t, and let $y(t)$ denote the total output produced by the machine up to time t. The machine's state and output are governed by the following system of stochastic differential equations:

$$dx(t) = (u(t) - k_1)x(t)dt + k_2 dw(t), \tag{1}$$
$$dy(t) = k_3 x(t)dt, \tag{2}$$

where $u(t)$ denotes the continuous maintenance rate; $w(t)$ denotes the standard Brownian motion with mean 0 and covariance given by

$$\mathrm{Cov}\{\, w(t_1), w(t_2)\,\} = \min\{t_1, t_2\}; \tag{3}$$

and k_1, k_2 and k_3 are given constants representing, respectively, the machine's natural degradation rate, the propensity for random fluctuations in the machine's condition, and the extent to which the machine's state influences production. We impose the following bound constraints on the continuous maintenance rate:

$$0 \leq u(t) \leq ak_1, \quad t \geq 0, \tag{4}$$

where $a \in (0, 1)$ is a given constant.

The initial state of the machine and the initial production level are given by

$$x(0) = x^* + \delta_0, \tag{5}$$
$$y(0) = 0, \tag{6}$$

where δ_0 is a normal random variable with mean 0 and variance k_4. Note that $x(t) \approx x^*$ indicates that the machine is operating in an almost perfect condition.

Let N be the number of overhauls. Furthermore, let τ_i denote the time of the ith overhaul, with τ_{N+1} referring to the final time (the time at which the machine is replaced). We impose the following constraints:

$$\tau_i - \tau_{i-1} \geq \rho, \;\; i = 1, \ldots, N+1,$$

where $\rho > 0$ denotes the minimum duration between any two consecutive overhauls, and

$$\tau_{N+1} \geq t_{\min}.$$

The first constraint ensures that overhauls do not happen too frequently. The second constraint ensures that the final replacement time τ_{N+1} is greater than or equal to a pre-set minimum time $t_{\min}$.

We assume that the time required for each overhaul is negligible compared with the length of the time horizon. Hence, at each overhaul time, the machine's state improves instantaneously while the output level stays the same. This results in the following jump conditions:

$$x(\tau_i^+) = k_5 x(\tau_i^-) + \delta_i, \;\; i = 1, \ldots, N, \tag{7}$$

$$y(\tau_i^+) = y(\tau_i^-), \;\; i = 1, \ldots, N, \tag{8}$$

where k_5 is a positive constant and δ_i is a normal random variable with mean 0 and variance k_6. We assume throughout this paper that the Brownian motion $w(t)$ and the random variables δ_i, $i = 0, \ldots, N$, are mutually statistically independent.

There are two operational requirements that the machine needs to meet. First, there must be a high probability that the machine's state is always above a minimum acceptable level. This motivates the following probabilistic state constraint:

$$Pr\{\, x(t) \geq x_{\min} \,\} \geq p_1, \;\; t \in [0, \tau_{N+1}], \tag{9}$$

where $x_{\min}$ is the minimum acceptable level of the machine's state and p_1 is a given probability level.

Second, the machine's output over the entire time horizon must be such that there is a high probability that the accumulated output level is greater than or equal to a specified minimum level. This requirement can be formulated as follows:

$$Pr\{\, y(\tau_{N+1}) \geq y_{\min} \,\} \geq p_2, \tag{10}$$

where $y_{\min}$ denotes the minimum output level and p_2 is a given probability level. Note that constraint (10) is only imposed at the final time, while constraint (9) is imposed at all times in the time horizon.

Let $\boldsymbol{\tau} = [\tau_1, \ldots, \tau_{N+1}]^\top$ denote the vector of overhaul times. Furthermore, let $\mathcal{T}$ be the set defined by

$$\mathcal{T} = \left\{ \boldsymbol{\tau} \in \mathbb{R}^{N+1} : \tau_i - \tau_{i-1} \geq \rho, \ i = 1, \ldots, N+1; \ \tau_{N+1} \geq t_{\min} \right\}.$$

A vector $\boldsymbol{\tau} \in \mathcal{T}$ is called an *admissible overhaul time vector*. We assume for simplicity that the continuous maintenance rate is constant between consecutive overhauls (however, our approach can be easily extended to the case where the continuous maintenance rate assumes several different constant levels between overhauls). For a given $\boldsymbol{\tau} \in \mathcal{T}$, any piecewise-constant function $u : [0, \infty) \to \mathbb{R}$ that is constant on the intervals $[\tau_{i-1}, \tau_i)$, $i = 1, \ldots, N+1$, and satisfies (4) is called an *admissible control*. Let $\mathcal{U}(\boldsymbol{\tau})$ be the class of all admissible controls corresponding to $\boldsymbol{\tau} \in \mathcal{T}$. Then any element $(\boldsymbol{\tau}, u) \in \mathcal{T} \times \mathcal{U}(\boldsymbol{\tau})$ is called an *admissible pair*. Any admissible pair satisfying constraints (9) and (10) is called a *feasible pair*.

Our goal is to determine an optimal maintenance policy. To measure optimality, we need an appropriate cost function. The cost function we choose consists of four components—the operating cost, the continuous maintenance cost, the overhaul cost, and the salvage value—and is expressed by

$$\begin{aligned} g_0(\boldsymbol{\tau}, u) = & \int_0^{\tau_{N+1}} \mathcal{E}\Big\{ \underbrace{\mathcal{L}_1(x(t))}_{\text{Operating cost}} + \underbrace{\mathcal{L}_2(u(t))}_{\substack{\text{Continuous} \\ \text{maintenance cost}}} \Big\} dt \\ & + \sum_{i=1}^{N} \mathcal{E}\Big\{ \underbrace{\Psi_1(x(\tau_i^-))}_{\text{Overhaul cost}} \Big\} - \mathcal{E}\Big\{ \underbrace{\Psi_2(x(\tau_{N+1}))}_{\text{Salvage value}} \Big\}, \end{aligned} \tag{11}$$

where $\mathcal{E}$ denotes the mathematical expectation.

We assume that the functions in (11) satisfy the following assumptions.

- $\mathcal{L}_1 \colon \mathbb{R} \to \mathbb{R}$ and $\Psi_2 \colon \mathbb{R} \to \mathbb{R}$ are quadratic.
- $\mathcal{L}_2 \colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable with respect to each of its arguments.
- $\Psi_1 \colon \mathbb{R} \to \mathbb{R}$ is linear.

The problem of determining the optimal continuous maintenance rate and the optimal overhaul times may now be formulated naturally as the following stochastic optimal control problem.

Problem ($\mathbf{P_0}$). *Given the stochastic system (1)-(2) with the initial conditions (5)-(6) and the jump conditions (7)-(8), find an admissible pair* $(\boldsymbol{\tau}, u) \in \mathcal{T} \times \mathcal{U}(\boldsymbol{\tau})$ *such that the cost function* $g_0(\boldsymbol{\tau}, u)$ *defined by (11) is minimized subject to the constraints (9) and (10).*

Problem (P_0) is a stochastic impulsive optimal control problem with probabilistic state constraints. This problem involves determining the times at which the major overhauls take place (the jump points) and at what level the continuous maintenance effort should be kept (the piecewise control function). Conventional optimal control techniques cannot deal with this problem directly because of the stochastic disturbance in the state equations (1)-(2). In the next section, we will develop a novel technique for transforming Problem (P_0) into an equivalent deterministic problem, which can be solved using the time-scaling transformation and the constraint transcription method.

3. Deterministic transformation

In this section, we transform Problem (P_0), a stochastic optimal control problem, into a new deterministic optimal control problem. First, define

$$\begin{aligned}
&\mu_x(t) = \mathcal{E}\big[x(t)\big], \quad \mu_y(t) = \mathcal{E}\big[y(t)\big],\\
&\sigma_{xx}(t) = \mathrm{Var}\big[x(t)\big], \quad \sigma_{yy}(t) = \mathrm{Var}\big[y(t)\big],\\
&\sigma_{xy}(t) = \sigma_{yx}(t) = \mathrm{Cov}\big\{x(t),\, y(t)\big\}.
\end{aligned}$$

Let $\mathbf{\Phi}\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ denote the principal solution matrix of the following homogeneous system:

$$\begin{aligned}
&\frac{\partial \mathbf{\Phi}(t,s)}{\partial t} = \begin{pmatrix} u(t) - k_1 & 0 \\ k_3 & 0 \end{pmatrix} \mathbf{\Phi}(t,s), \quad t > s,\\
&\mathbf{\Phi}(s,s) = \boldsymbol{I},
\end{aligned}$$

where

$$\mathbf{\Phi}(t,s) = \begin{pmatrix} \Phi_{11}(t,s) & \Phi_{12}(t,s) \\ \Phi_{21}(t,s) & \Phi_{22}(t,s) \end{pmatrix}.$$

Then it follows from the results in Adivar (2011) and Liu et al. (2009) that for each $i = 1, \ldots, N+1$, the solution of the stochastic impulsive system

(1)-(2) on (τ_{i-1}, τ_i) can be expressed as follows:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{\Phi}(t, \tau_{i-1}) \begin{pmatrix} x(\tau_{i-1}^+) \\ y(\tau_{i-1}^+) \end{pmatrix} + \int_{\tau_{i-1}}^{t} \mathbf{\Phi}(t, s) \begin{pmatrix} k_2 \\ 0 \end{pmatrix} dw(s).$$

This can be written as:

$$x(t) = \Phi_{11}(t, \tau_{i-1}) x(\tau_{i-1}^+) + \Phi_{12}(t, \tau_{i-1}) y(\tau_{i-1}^+) + \int_{\tau_{i-1}}^{t} k_2 \Phi_{11}(t, s)\, dw(s), \quad (12)$$

and

$$y(t) = \Phi_{21}(t, \tau_{i-1}) x(\tau_{i-1}^+) + \Phi_{22}(t, \tau_{i-1}) y(\tau_{i-1}^+) + \int_{\tau_{i-1}}^{t} k_2 \Phi_{21}(t, s)\, dw(s). \quad (13)$$

Taking the expectation of $x(t)$ and $y(t)$ gives

$$\mu_x(t) = \Phi_{11}(t, \tau_{i-1}) \mu_x(\tau_{i-1}^+) + \Phi_{12}(t, \tau_{i-1}) \mu_y(\tau_{i-1}^+), \quad (14)$$
$$\mu_y(t) = \Phi_{21}(t, \tau_{i-1}) \mu_x(\tau_{i-1}^+) + \Phi_{22}(t, \tau_{i-1}) \mu_y(\tau_{i-1}^+). \quad (15)$$

By differentiating (14)-(15) with respect to t, we obtain

$$\begin{aligned} \dot{\mu}_x(t) &= (u(t) - k_1)\Big(\Phi_{11}(t, \tau_{i-1}) \mu_x(\tau_{i-1}^+) + \Phi_{12}(t, \tau_{i-1}) \mu_y(\tau_{i-1}^+)\Big) \\ &= (u(t) - k_1) \mu_x(t), \end{aligned} \quad (16)$$

and

$$\begin{aligned} \dot{\mu}_y(t) &= k_3 \Big(\Phi_{11}(t, \tau_{i-1}) \mu_x(\tau_{i-1}^+) + \Phi_{12}(t, \tau_{i-1}) \mu_y(\tau_{i-1}^+)\Big) \\ &= k_3 \mu_x(t). \end{aligned} \quad (17)$$

Now, the variances are:

$$\begin{aligned} \sigma_{xx}(t) = &\ \Phi_{11}^2(t, \tau_{i-1}) \sigma_{xx}(\tau_{i-1}^+) + \Phi_{12}^2(t, \tau_{i-1}) \sigma_{yy}(\tau_{i-1}^+) \\ &+ 2\Phi_{11}(t, \tau_{i-1}) \Phi_{12}(t, \tau_{i-1}) \sigma_{xy}(\tau_{i-1}^+) + \int_{\tau_{i-1}}^{t} k_2^2 \Phi_{11}^2(t, s)\, ds, \end{aligned} \quad (18)$$

and

$$\sigma_{yy}(t) = \Phi_{21}^2(t,\tau_{i-1})\sigma_{xx}(\tau_{i-1}^+) + \Phi_{22}^2(t,\tau_{i-1})\sigma_{yy}(\tau_{i-1}^+)$$
$$+ 2\Phi_{21}(t,\tau_{i-1})\Phi_{22}(t,\tau_{i-1})\sigma_{xy}(\tau_{i-1}^+) + \int_{\tau_{i-1}}^{t} k_2^2\Phi_{21}^2(t,s)\,ds. \tag{19}$$

Furthermore, the covariance is:

$$\begin{aligned}\sigma_{xy}(t) &= \sigma_{yx}(t)\\ &= \Phi_{11}(t,\tau_{i-1})\Phi_{21}(t,\tau_{i-1})\sigma_{xx}(\tau_{i-1}^+)\\ &\quad + \big(\Phi_{11}(t,\tau_{i-1})\Phi_{22}(t,\tau_{i-1}) + \Phi_{12}(t,\tau_{i-1})\Phi_{21}(t,\tau_{i-1})\big)\sigma_{xy}(\tau_{i-1}^+)\\ &\quad + \Phi_{12}(t,\tau_{i-1})\Phi_{22}(t,\tau_{i-1})\sigma_{yy}(\tau_{i-1}^+)\\ &\quad + \int_{\tau_{i-1}}^{t} k_2^2\Phi_{11}(t,s)\Phi_{21}(t,s)\,ds. \end{aligned}\tag{20}$$

Differentiating (18)-(20) with respect to time, we obtain

$$\begin{aligned}\dot{\sigma}_{xx}(t) &= 2(u(t)-k_1)\Big(\Phi_{11}^2(t,\tau_{i-1})\sigma_{xx}(\tau_{i-1}^+) + \Phi_{12}^2(t,\tau_{i-1})\sigma_{yy}(\tau_{i-1}^+)\Big)\\ &\quad + 4(u(t)-k_1)\Phi_{11}(t,\tau_{i-1})\Phi_{12}(t,\tau_{i-1})\sigma_{xy}(\tau_{i-1}^+)\\ &\quad + 2(u(t)-k_1)\int_{\tau_{i-1}}^{t} k_2^2\Phi_{11}^2(t,s)\,ds + k_2^2\\ &= 2(u(t)-k_1)\sigma_{xx}(t) + k_2^2, \end{aligned}\tag{21}$$

$$\begin{aligned}\dot{\sigma}_{yy}(t) &= 2k_3\Phi_{11}(t,\tau_{i-1})\Phi_{21}(t,\tau_{i-1})\sigma_{xx}(\tau_{i-1}^+)\\ &\quad + 2k_3\Phi_{12}(t,\tau_{i-1})\Phi_{22}(t,\tau_{i-1})\sigma_{yy}(\tau_{i-1}^+)\\ &\quad + 2k_3\Big(\Phi_{11}(t,\tau_{i-1})\Phi_{22}(t,\tau_{i-1}) + \Phi_{12}(t,\tau_{i-1})\Phi_{21}(t,\tau_{i-1})\Big)\sigma_{xy}(\tau_{i-1}^+)\\ &\quad + 2k_3\int_{\tau_{i-1}}^{t} k_2^2\Phi_{11}(t,s)\Phi_{21}(t,s)\,ds\\ &= 2k_3\sigma_{xy}(t), \end{aligned}\tag{22}$$

and

$$
\begin{aligned}
\dot{\sigma}_{xy}(t) &= (u(t)-k_1)\Phi_{11}(t,\tau_{i-1})\Phi_{21}(t,\tau_{i-1})\sigma_{xx}(\tau_{i-1}^{+}) \\
&\quad + (u(t)-k_1)\Big(\Phi_{11}(t,\tau_{i-1})\Phi_{22}(t,\tau_{i-1}) \\
&\quad + \Phi_{12}(t,\tau_{i-1})\Phi_{21}(t,\tau_{i-1})\Big)\sigma_{xy}(\tau_{i-1}^{+}) \\
&\quad + (u(t)-k_1)\Phi_{12}(t,\tau_{i-1})\Phi_{22}(t,\tau_{i-1})\sigma_{yy}(\tau_{i-1}^{+}) \\
&\quad + (u(t)-k_1)\int_{\tau_{i-1}}^{t} k_2^2\Phi_{11}(t,s)\Phi_{21}(t,s)\,ds \\
&\quad + k_3\Big(\Phi_{11}^2(t,\tau_{i-1})\sigma_{xx}(\tau_{i-1}^{+}) + \Phi_{12}^2(t,\tau_{i-1})\sigma_{yy}(\tau_{i-1}^{+})\Big) \\
&\quad + k_3\Big(2\Phi_{11}(t,\tau_{i-1})\Phi_{12}(t,\tau_{i-1})\sigma_{xy}(\tau_{i-1}^{+}) + \int_{\tau_{i-1}}^{t} k_2^2\Phi_{11}^2(t,s)\,ds\Big) \\
&= (u(t)-k_1)\sigma_{xy}(t) + k_3\sigma_{xx}(t). \qquad (23)
\end{aligned}
$$

The mean, variance and covariance of the initial conditions (5)-(6) are:

$$\mu_x(0) = x^*, \quad \mu_y(0) = 0, \qquad (24)$$

$$\sigma_{xx}(0) = k_4, \quad \sigma_{yy}(0) = 0, \quad \sigma_{xy}(0) = \sigma_{yx}(0) = 0. \qquad (25)$$

At the overhaul times $t = \tau_i$, $i = 1, \ldots, N$, the mean, variance and covariance of the state jump conditions (7)-(8) are:

$$\mu_x(\tau_i^{+}) = k_5\mu_x(\tau_i^{-}), \quad \mu_y(\tau_i^{+}) = \mu_y(\tau_i^{-}), \qquad (26)$$

$$\sigma_{xx}(\tau_i^{+}) = k_5^2\sigma_{xx}(\tau_i^{-}) + k_6, \quad \sigma_{yy}(\tau_i^{+}) = \sigma_{yy}(\tau_i^{-}), \qquad (27)$$

$$\sigma_{xy}(\tau_i^{+}) = \sigma_{yx}(\tau_i^{+}) = k_5\sigma_{xy}(\tau_i^{-}). \qquad (28)$$

Equations (16)-(17) and (21)-(28) constitute a system of ordinary differential equations with jump conditions. This new system replaces the original stochastic system consisting of (1)-(2) and (5)-(8).

Since the state equations (1)-(2) and the jump conditions (7)-(8) are linear, $x(t)$ and $y(t)$ are normally distributed random variables (Liu et al. (2009)). Thus, the probabilistic state constraints (9) and (10) can be written as follows:

$$\int_{x_{\min}}^{\infty} \frac{1}{(2\pi\sigma_{xx}(t))^{1/2}} \exp\left\{\frac{-(\eta-\mu_x(t))^2}{2\sigma_{xx}(t)}\right\} d\eta \geq p_1, \quad t \in [0, \tau_{N+1}], \qquad (29)$$

$$\int_{y_{\min}}^{\infty} \frac{1}{(2\pi\sigma_{yy}(\tau_{N+1}))^{1/2}} \exp\left\{ \frac{-(\eta - \mu_y(\tau_{N+1}))^2}{2\sigma_{yy}(\tau_{N+1})} \right\} d\eta \geq p_2. \tag{30}$$

Constraint (29) is a continuous inequality constraint in terms of the new state variables μ_x and σ_{xx}, and constraint (30) is a terminal state constraint involving the new state variables μ_y and σ_{yy}.

As the functions $\mathcal{L}_1(\cdot)$ and $\Psi_2(\cdot)$ appearing in the cost function (11) are quadratic, we can express $\mathcal{E}\big[\mathcal{L}_1(x(t))\big]$ and $\mathcal{E}\big[\Psi_2(x(\tau_{N+1}))\big]$ in terms of the new state variables by replacing $\mathcal{E}\big[x(t)\big]$ and $\mathcal{E}\big[x(t)^2\big]$ with $\mu_x(t)$ and $\sigma_{xx}(t) + \mu_x^2(t)$, respectively. We denote the resulting functions by $\tilde{\mathcal{L}}_1(\mu_x(t),\, \sigma_{xx}(t))$ and $\tilde{\Psi}_2(\mu_x(\tau_{N+1}),\, \sigma_{xx}(\tau_{N+1}))$, respectively. Similarly, since Ψ_1 is linear, we have $\mathcal{E}\big[\Psi_1(x(\tau_i^-))\big] = \Psi_1(\mu_x(\tau_i^-))$. Thus, the cost function (11) can be written as

$$\begin{aligned} g_0(\boldsymbol{\tau}, u) = &\int_0^{\tau_{N+1}} \Big\{ \tilde{\mathcal{L}}_1(\mu_x(t),\, \sigma_{xx}(t)) + \mathcal{L}_2(u(t)) \Big\}\, dt \\ &+ \sum_{i=1}^{N} \Psi_1(\mu_x(\tau_i^-)) - \tilde{\Psi}_2(\mu_x(\tau_{N+1}),\, \sigma_{xx}(\tau_{N+1})). \end{aligned} \tag{31}$$

We are now able to state the transformed problem as follows.

Problem ($\mathbf{P_1}$). *Given the dynamic system (16)-(17) and (21)-(23) with the initial conditions (24)-(25) and the jump conditions (26)-(28), find an admissible pair* $(\boldsymbol{\tau}, u) \in \mathcal{T} \times \mathcal{U}(\boldsymbol{\tau})$ *such that the cost function (31) is minimized subject to constraints (29) and (30).*

4. Time-scaling transformation

In Problem (P_1), the machine's state experiences N instantaneous jumps during the time horizon. The times at which these jumps occur are actually decision variables to be optimized. It is well known that optimizing variable jump times in a nonlinear dynamic system is a difficult task from a computational point of view (Lin et al. (2013); Lin et al. (2014)). Thus, in this section, we apply the time-scaling technique in Teo et al. (1999) to map the variable jump times into fixed time points in a new time horizon.

We first introduce a new time variable s ranging from 0 to $N+1$. Our aim is to map the old time scale $t \in [0, \tau_{N+1}]$ into the new time scale $s \in [0, N+1]$ in a judicious manner so that the variable jump points are mapped into fixed jump points. This mapping is realized by the following differential equation:

$$\frac{dt(s)}{ds} = \tilde{v}(s) = \sum_{i=1}^{N+1} v_i \chi_{[i-1,i)}(s), \tag{32}$$

$$t(0) = 0, \tag{33}$$

where $v_i = \tau_i - \tau_{i-1} \geq \rho$ for each $i = 1, \ldots, N+1$, $v_1 + \cdots + v_{N+1} \geq t_{\min}$, and $\chi_{[i-1,i)}(s)$ is the indicator function defined by

$$\chi_{[i-1,i)}(s) = \begin{cases} 1, & \text{if } s \in [i-1, i), \\ 0, & \text{otherwise.} \end{cases}$$

Note that v_i denotes the time duration between the $(i-1)$th and ith jump times. We collect the duration parameters into a vector $\boldsymbol{v} = [v_1, \ldots, v_{N+1}]^\top \in \mathbb{R}^{N+1}$. Let

$$\mathcal{V} = \left\{ \boldsymbol{v} \in \mathbb{R}^{N+1} : \ v_i \geq \rho, \ i = 1, \ldots, N+1; \ v_1 + \cdots + v_{N+1} \geq t_{\min} \right\}.$$

A vector $\boldsymbol{v} \in \mathcal{V}$ is called an *admissible duration vector*. From (32)-(33), we have, for each $i = 1, \ldots, N+1$,

$$t(i) = t(0) + \int_0^i \tilde{v}(s) ds = t(0) + v_1 + \cdots + v_i = \tau_i.$$

This equation shows the relationship between the variable jump points $t = \tau_i$, $i = 1, \ldots, N+1$, and the fixed jump points $s = i$, $i = 1, \ldots, N+1$.

Define $\tilde{u}(s) = u(t(s))$. Recall that the continuous maintenance rate is constant between consecutive overhauls. Thus, the admissible controls are restricted to piecewise-constant functions that assume constant values between consecutive jump times. As a result, $\tilde{u}(s)$ can be expressed as

$$\tilde{u}(s) = \sum_{i=1}^{N+1} h_i \chi_{[i-1,i)}(s),$$

where h_i, $i = 1, \ldots, N+1$, are control heights to be optimized. In view of (4), these control heights must satisfy the following constraints:

$$0 \leq h_i \leq a k_1, \ \ i = 1, \ldots, N+1. \tag{34}$$

Let $\boldsymbol{h} = [h_1, \ldots, h_{N+1}]^\top \in \mathbb{R}^{N+1}$. Furthermore, define

$$\mathcal{H} = \left\{ \boldsymbol{h} \in \mathbb{R}^{N+1} : \ 0 \le h_i \le ak_1, \ i = 1, \ldots, N+1 \right\}.$$

A vector $\boldsymbol{h} \in \mathcal{H}$ is called an *admissible control parameter vector*. Furthermore, a pair $(\boldsymbol{h}, \boldsymbol{v}) \in \mathcal{H} \times \mathcal{V}$ is called an *admissible pair.*

Note that it is possible to apply more advanced control parameterization schemes in which the control can switch value between consecutive jump times, as well as at the jump times themselves (Lin et al. (2013)). However, for simplicity, we assume throughout this paper that the control switches coincide with the overhaul times.

Let

$$\begin{aligned}
&\tilde{\mu}_x(s) = \mu_x(t(s)), \quad \tilde{\mu}_y(s) = \mu_y(t(s)),\\
&\tilde{\sigma}_{xx}(s) = \sigma_{xx}(t(s)), \quad \tilde{\sigma}_{yy}(s) = \sigma_{yy}(t(s)), \quad \tilde{\sigma}_{xy}(s) = \tilde{\sigma}_{yx}(s) = \sigma_{xy}(t(s)).
\end{aligned}$$

Then the dynamics (16)-(17) and (21)-(23) are transformed into:

$$\frac{d\tilde{\mu}_x(s)}{ds} = \tilde{v}(s)(\tilde{u}(s) - k_1)\tilde{\mu}_x(s), \tag{35}$$

$$\frac{d\tilde{\mu}_y(s)}{ds} = k_3\tilde{v}(s)\tilde{\mu}_x(s), \tag{36}$$

$$\frac{d\tilde{\sigma}_{xx}(s)}{ds} = \tilde{v}(s)\Big(2(\tilde{u}(s) - k_1)\tilde{\sigma}_{xx}(s) + k_2^2\Big), \tag{37}$$

$$\frac{d\tilde{\sigma}_{yy}(s)}{ds} = 2k_3\tilde{v}(s)\tilde{\sigma}_{xy}(s), \tag{38}$$

$$\frac{d\tilde{\sigma}_{xy}(s)}{ds} = \frac{d\tilde{\sigma}_{yx}(s)}{ds} = \tilde{v}(s)\Big((\tilde{u}(s) - k_1)\tilde{\sigma}_{xy}(s) + k_3\tilde{\sigma}_{xx}(s)\Big). \tag{39}$$

Furthermore, the initial states become

$$\tilde{\mu}_x(0) = x^*, \quad \tilde{\mu}_y(0) = 0, \tag{40}$$

$$\tilde{\sigma}_{xx}(0) = k_4, \quad \tilde{\sigma}_{yy}(0) = 0, \quad \tilde{\sigma}_{xy}(0) = \tilde{\sigma}_{yx}(0) = 0. \tag{41}$$

For each $i = 1, \ldots, N$, the new jump conditions are

$$\tilde{\mu}_x(i^+) = k_5\tilde{\mu}_x(i^-), \quad \tilde{\mu}_y(i^+) = \tilde{\mu}_y(i^-), \tag{42}$$

$$\tilde{\sigma}_{xx}(i^+) = k_5^2\tilde{\sigma}_{xx}(i^-) + k_6, \quad \tilde{\sigma}_{yy}(i^+) = \tilde{\sigma}_{yy}(i^-), \tag{43}$$

$$\tilde{\sigma}_{xy}(i^+) = \tilde{\sigma}_{yx}(i^+) = k_5\tilde{\sigma}_{xy}(i^-). \tag{44}$$

The state constraints (29)-(30) become

$$\int_{x_{\min}}^{\infty} \frac{1}{(2\pi\tilde{\sigma}_{xx}(s))^{1/2}} \exp\left\{\frac{-(\eta - \tilde{\mu}_x(s))^2}{2\tilde{\sigma}_{xx}(s)}\right\} d\eta \geq p_1, \quad s \in [0, N+1], \tag{45}$$

$$\int_{y_{\min}}^{\infty} \frac{1}{(2\pi\tilde{\sigma}_{yy}(N+1))^{1/2}} \exp\left\{\frac{-(\eta - \tilde{\mu}_y(N+1))^2}{2\tilde{\sigma}_{yy}(N+1)}\right\} d\eta \geq p_2. \tag{46}$$

An admissible pair $(\boldsymbol{h}, \boldsymbol{v}) \in \mathcal{H} \times \mathcal{V}$ is said to be *feasible* if it satisfies the constraints (45) and (46).

After applying the time-scaling transformation, Problem (P_1) becomes Problem (P_2) defined below.

Problem ($\mathbf{P_2}$). *Given the dynamic system (35)-(39) with the initial conditions (40)-(41) and the jump conditions (42)-(44), find a pair* $(\boldsymbol{h}, \boldsymbol{v}) \in \mathcal{H} \times \mathcal{V}$ *such that the cost function*

$$\tilde{g}_0(\boldsymbol{h}, \boldsymbol{v}) = \int_0^{N+1} \tilde{v}(s)\Big\{\tilde{\mathcal{L}}_1(\tilde{\mu}_x(s),\, \tilde{\sigma}_{xx}(s)) + \mathcal{L}_2(\tilde{u}(s))\Big\}\, ds + \sum_{i=1}^{N} \Psi_1(\tilde{\mu}_x(i^-)) - \tilde{\Psi}_2(\tilde{\mu}_x(N+1),\, \tilde{\sigma}_{xx}(N+1)) \tag{47}$$

is minimized subject to constraints (45) and (46).

5. Solving Problem ($\mathbf{P_2}$)

Problem (P_2) is an impulsive optimal parameter selection problem with state constraints. To solve such problems using gradient-based optimization algorithms, we require the gradients of the cost and constraint functions. However, it is not possible to derive the gradient of the state constraint (45) because this constraint is imposed at an infinite number of time points. Thus, we will apply the constraint transcription technique (Jennings and Teo (1990)) to approximate (45) by a conventional constraint.

5.1. Constraint transcription

In the constraint transcription technique, we aim to construct an approximation of constraint (45) that can be handled using conventional optimization techniques.

Define

$$\varrho(\tilde{\mu}_x(s), \tilde{\sigma}_{xx}(s)) = \int_{x_{\min}}^{\infty} \frac{1}{(2\pi\tilde{\sigma}_{xx}(s))^{1/2}} \exp\left\{ \frac{-(\eta - \tilde{\mu}_x(s))^2}{2\tilde{\sigma}_{xx}(s)} \right\} d\eta - p_1.$$

Then constraint (45) is equivalent to the following equality constraint:

$$\int_0^{N+1} \min\left\{ \varrho(\tilde{\mu}_x(s), \tilde{\sigma}_{xx}(s)), 0 \right\} ds = 0. \tag{48}$$

However, because of the $\min\{\cdot, 0\}$ function, constraint (48) is non-smooth when $\varrho(\tilde{\mu}_x(s), \tilde{\sigma}_{xx}(s)) = 0$. Thus, we introduce a smooth approximation for $\min\{\cdot, 0\}$ defined as follows:

$$\varphi_\varepsilon(\alpha) = \begin{cases} \alpha, & \text{if } \alpha < -\varepsilon, \\ -(\alpha - \varepsilon)^2/4\varepsilon, & \text{if } -\varepsilon \leq \alpha \leq \varepsilon, \\ 0, & \text{if } \alpha > \varepsilon. \end{cases}$$

Here, ε is a small positive number that controls the accuracy of the approximation. Using φ_ε to approximate $\min\{\cdot, 0\}$, we obtain an approximation of (48) as follows:

$$\beta + \sum_{i=1}^{N+1} \int_{i-1}^{i} \varphi_\varepsilon(\varrho(\tilde{\mu}_x(s), \tilde{\sigma}_{xx}(s))) \, ds \geq 0, \tag{49}$$

where $\beta > 0$ is an adjustable parameter that controls the feasibility of the approximation with respect to the original constraint (45). The following theorem justifies the use of (49) as an approximation of the original continuous inequality constraint (45). The proof of this theorem can be found in Jennings and Teo (1990).

Theorem 5.1. *For each $\varepsilon > 0$, there exists a corresponding $\beta(\varepsilon) > 0$ such that for all β satisfying $0 < \beta < \beta(\varepsilon)$, constraint (49) implies constraint (45).*

Hence, constraint (45) is approximated by constraint (49), where $\varepsilon > 0$ and $\beta > 0$ are adjustable parameters. Note that constraint (49) constitutes a single restriction on the system, whereas constraint (45) is imposed at an infinite number of points. Now, Problem (P_2) is converted into the following approximate problem.

Problem ($\mathbf{P_3}$). *Given the dynamic system (35)-(39) with the initial conditions (40)-(41) and the jump conditions (42)-(44), find a pair $(\boldsymbol{h}, \boldsymbol{v}) \in \mathcal{H} \times \mathcal{V}$ such that the value of the cost function (47) is minimized subject to the constraints (46) and (49).*

5.2. Solving the approximate problem

Problem (P_3), the approximate problem, involves minimizing the cost function (47) subject to the constraints (46) and (49), which are both in canonical form (Teo et al. (1991)). We now need to determine the gradients of the cost and constraint functions in Problem (P_3) so that gradient-based optimization methods can be deployed to generate an optimal solution.

To begin, we collect all the state variables in Problem (P_3) into a vector $\tilde{\boldsymbol{\omega}}(s) = \big(\tilde{\mu}_x(s), \tilde{\mu}_y(s), \tilde{\sigma}_{xx}(s), \tilde{\sigma}_{yy}(s), \tilde{\sigma}_{xy}(s)\big)^\top$. Let $\boldsymbol{f}$ denote the vector of functions appearing on the right-hand side of the dynamics (35)-(39):

$$\boldsymbol{f}(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s)) = \begin{pmatrix} \tilde{v}(s)(\tilde{u}(s) - k_1)\tilde{\mu}_x(s) \\ k_3\tilde{v}(s)\tilde{\mu}_x(s) \\ \tilde{v}(s)(2(\tilde{u}(s) - k_1)\tilde{\sigma}_{xx}(s) + k_2^2) \\ 2k_3\tilde{v}(s)\tilde{\sigma}_{xy}(s) \\ \tilde{v}(s)(\tilde{u}(s) - k_1)\tilde{\sigma}_{xy}(s) + k_3\tilde{v}(s)\tilde{\sigma}_{xx}(s) \end{pmatrix}.$$

Furthermore, define

$$\begin{aligned} H_0(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s)) &= \tilde{v}(s)\tilde{\mathcal{L}}_1(\tilde{\mu}_x(s), \tilde{\sigma}_{xx}(s)) + \tilde{v}(s)\mathcal{L}_2(\tilde{u}(s)) \\ &\quad + \boldsymbol{\lambda}^0(s)^\top \boldsymbol{f}(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s)), \\ H_1(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s)) &= \varphi_\varepsilon(\varrho(\tilde{\mu}_x(s), \tilde{\sigma}_{xx}(s))) + \boldsymbol{\lambda}^1(s)^\top \boldsymbol{f}(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s)), \\ H_2(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s)) &= \boldsymbol{\lambda}^2(s)^\top \boldsymbol{f}(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s)), \end{aligned}$$

where $\boldsymbol{\lambda}^k(s)$ is called the kth *costate vector* and H_k is called the kth *Hamiltonian.* The costate vector $\boldsymbol{\lambda}^k(s)$ satisfies the following costate system:

$$\frac{d\boldsymbol{\lambda}^k(s)}{ds} = -\Big[\frac{\partial H_k(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s))}{\partial \tilde{\boldsymbol{\omega}}}\Big]^\top,$$

with jump conditions

$$\boldsymbol{\lambda}^k(i^+) - \boldsymbol{\lambda}^k(i^-) = \begin{cases} -\big[\, \frac{\partial \Psi_1(\tilde{\mu}_x(i^-))}{\partial \tilde{\boldsymbol{\omega}}} \,\big]^\top, & \text{if } k = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

and terminal conditions

$$\boldsymbol{\lambda}^0(N+1) = -\Big[\, \frac{\partial \tilde{\Psi}_2(\tilde{\mu}_x(N+1), \tilde{\sigma}_{xx}(N+1))}{\partial \tilde{\boldsymbol{\omega}}} \,\Big]^\top,$$
$$\boldsymbol{\lambda}^1(N+1) = \mathbf{0},$$
$$\boldsymbol{\lambda}^2(N+1) = \left[\frac{\partial}{\partial \tilde{\boldsymbol{\omega}}} \int_{y_{\min}}^{\infty} \frac{1}{(2\pi \tilde{\sigma}_{yy}(N+1))^{1/2}} \exp\left\{ \frac{-(\eta - \tilde{\mu}_y(N+1))^2}{2\tilde{\sigma}_{yy}(N+1)} \right\} d\eta \right]^\top.$$

We use $\tilde{g}_1(\boldsymbol{h}, \boldsymbol{v})$ and $\tilde{g}_2(\boldsymbol{h}, \boldsymbol{v})$ to denote, respectively, the left-hand sides of constraints (49) and (46). As before, $\tilde{g}_0(\boldsymbol{h}, \boldsymbol{v})$ denotes the cost function. We now present the following theorem regarding the gradients of the cost and constraint functions in Problem (P_3). The proof of this theorem is similar to the proof of Theorem 4.3 in Martin and Teo (1994).

Theorem 5.2. *For $k = 0, 1, 2$, the gradients of the function $g_k(\boldsymbol{h}, \boldsymbol{v})$ with respect to $\boldsymbol{h}$ and $\boldsymbol{v}$ are given by*

$$\frac{\partial \tilde{g}_k(\boldsymbol{h}, \boldsymbol{v})}{\partial h_i} = \int_{i-1}^{i} \frac{\partial H_k(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s))}{\partial h_i}\, ds, \quad i = 1, \ldots, N+1,$$
$$\frac{\partial \tilde{g}_k(\boldsymbol{h}, \boldsymbol{v})}{\partial v_i} = \int_{i-1}^{i} \frac{\partial H_k(\tilde{\boldsymbol{\omega}}(s), \tilde{u}(s), \tilde{v}(s))}{\partial v_i}\, ds. \quad i = 1, \ldots, N+1.$$

For any given pair $(\boldsymbol{h}, \boldsymbol{v})$, we can solve the dynamic system (35)-(39) forward in time, and then use the information obtained to calculate the values of the cost function and the constraint functions. Then, we can solve the costate systems backward in time, with the corresponding costates used in the gradient calculations according to the formulae in Theorem 5.2. This gradient computation method can be incorporated into any gradient-based optimization solver to solve Problem (P_3). In the next section, we use this approach to solve a numerical example.

6. Numerical example

In this section, we consider the stochastic machine maintenance problem for a brand-new machine costing \$10,000. The manager in charge of the machine plans to replace the machine after 20 overhauls (major maintenance). Meanwhile, the workers in the factory will perform continuous maintenance on the machine (minor maintenance) to ensure that it is kept in good working order. This stochastic machine maintenance problem is a special case of the problem defined in Section 2. The model parameters are given by

$$\begin{array}{lllll} k_1 = 1.35 \times 10^{-2}, & k_2 = 10^{-3}, & k_3 = 2.5, & k_4 = 10^{-4}, & k_5 = 1.18, \\ k_6 = 10^{-4}, & a = 0.1, & x^* = 1.0, & p_1 = 0.8, & p_2 = 0.8, \\ x_{\min} = 0.1, & y_{\min} = 500, & \rho = 15.0, & t_{\min} = 400. & \end{array}$$

The explicit forms for the functions in the cost (11) are given as follows:

$$\begin{array}{ll} \mathcal{L}_1(x(t)) = 2.5x^2(t) - 20x(t) + 40, & \mathcal{L}_2(u(t)) = \dfrac{40}{k_1} u(t), \\ \Psi_1(x(\tau_i^-)) = 1000 - 500x(\tau_i^-), & \Psi_2(x(\tau_{N+1})) = \dfrac{1}{5} x(\tau_{N+1}) \times 10000. \end{array}$$

Note that $N = 20$ is the number of overhaul times, and 10,000 is the original capital cost of the machine.

We apply the procedures described in Sections 3-5 to yield an approximate problem in the form of Problem (P_3). This approximate problem is solved using the optimal control software package MISER3 (Jennings et al. (2004)). The optimal value of the cost function obtained is $\tilde{g}_0 = 11{,}602.7281$. The optimal jump times (the overhaul times) and the optimal terminal time (replacement time) are given in Table 1, while the optimal continuous maintenance rates (minor maintenance) are shown in Table 2. Note that we are assuming the continuous maintenance rate takes a constant value between consecutive jump points.

The optimal trajectories of the state variables $\mu_x(t)$, $\mu_y(t)$, $\sigma_{xx}(t)$, $\sigma_{yy}(t)$ and $\sigma_{xy}(t)$ are shown in Figures 6.1-6.5, respectively.

Figure 6.1 shows the mean of the machine's state over the duration of 400 time periods. The mean starts off at 1, and gradually decreases with time. However, with each overhaul, the mean of the state of the machine is restored to a higher value, close to where it was at the previous overhaul.

Figure 6.2 shows the mean of the accumulated output, which gradually

Table 1: Optimal Jump Times

i	τ_i	i	τ_i	i	τ_i	i	τ_i	i	τ_i	i	τ_i	i	τ_i
1	15	4	60	7	105	10	150	13	195	16	240	19	285
2	30	5	75	8	120	11	165	14	210	17	255	20	300
3	45	6	90	9	135	12	180	15	225	18	270	21	400

Table 2: Optimal Continuous Maintenance Rate

Interval	$u(t)$	Interval	$u(t)$	Interval	$u(t)$
1	1.35×10^{-3}	8	4.6386×10^{-31}	15	6.9333×10^{-33}
2	1.35×10^{-3}	9	4.2867×10^{-31}	16	0
3	1.35×10^{-3}	10	0	17	0
4	1.35×10^{-3}	11	4.0135×10^{-31}	18	0
5	1.35×10^{-3}	12	0	19	0
6	1.35×10^{-3}	13	4.1610×10^{-32}	20	0
7	1.35×10^{-3}	14	7.7037×10^{-34}	21	0

increases over time.

Figure 6.3 shows the variance of the state of the machine. The variance changes with each overhaul performed, then gradually decreases after the last overhaul.

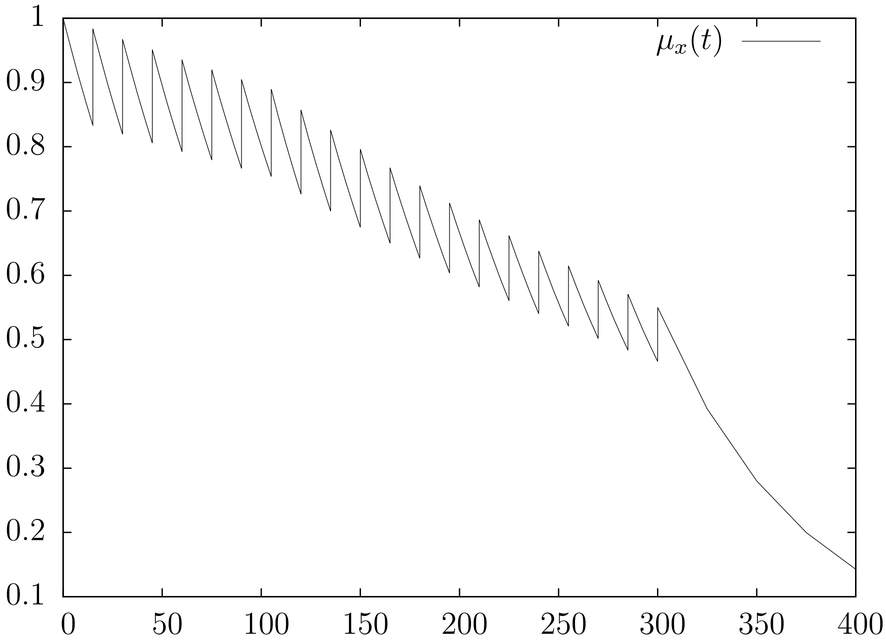
The variance of the output is shown in Figure 6.4, while Figure 6.5 shows the covariance of the output with the state of the machine.

To examine the performance of our optimal maintenance policy over a range of scenarios, 500 sample paths are simulated for the machine state (see Figure 6.6) and machine output (see Figure 6.7), respectively. The paths simulated are similar in shape to the paths of mean values as shown earlier.

7. Conclusion

In this paper, we have considered a single machine scheduling problem with continuous maintenance and scheduled major overhauls. The objective is to find the optimal maintenance scheduling policy to minimize total cost. The problem is formulated as a special type of stochastic impulsive optimal control problem, with flexible maintenance time intervals. The numerical example shows that the model performs well and the outputs are reasonable

Figure 6.1: $\mu_x(t)$



and realistic. Future research can extend the problem to allow for multiple machines and nonzero overhaul periods during which the machine undergoing maintenance is unproductive.

8. Acknowledgement

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M. Adivar. (2011). Principal Matrix Solutions and Variation of Parameters for Volterra Integro-Dynamic Equations On Time Scales. *Glasgow Mathematical Journal*, vol. 53, pp. 463-480.

S. Batun, and M. Azizoğlu. (2009). Single Machine Scheduling with Preventive Maintenances. *International Journal of Production Research*, vol. 47, pp. 2923-2929.

Figure 6.2: $\mu_y(t)$

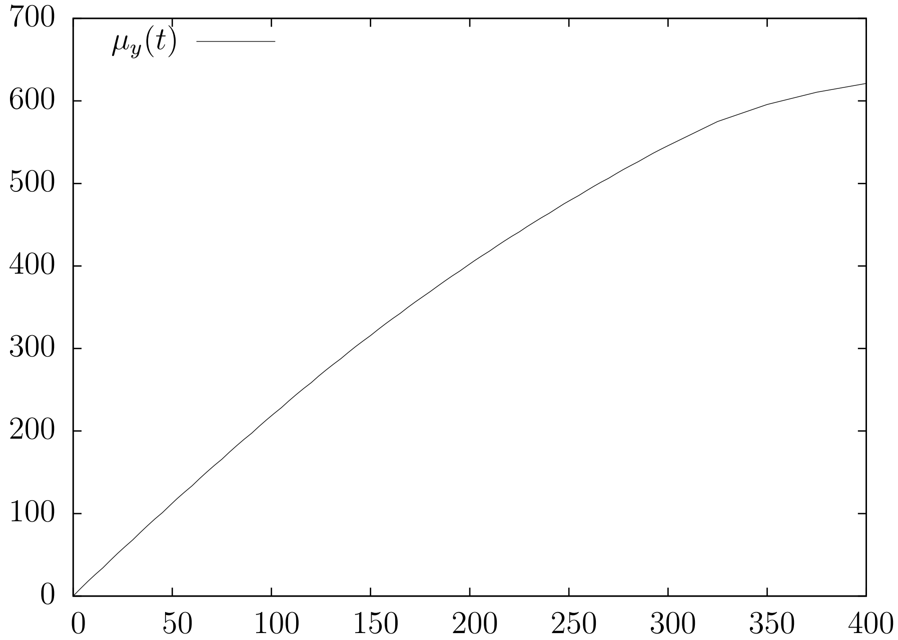
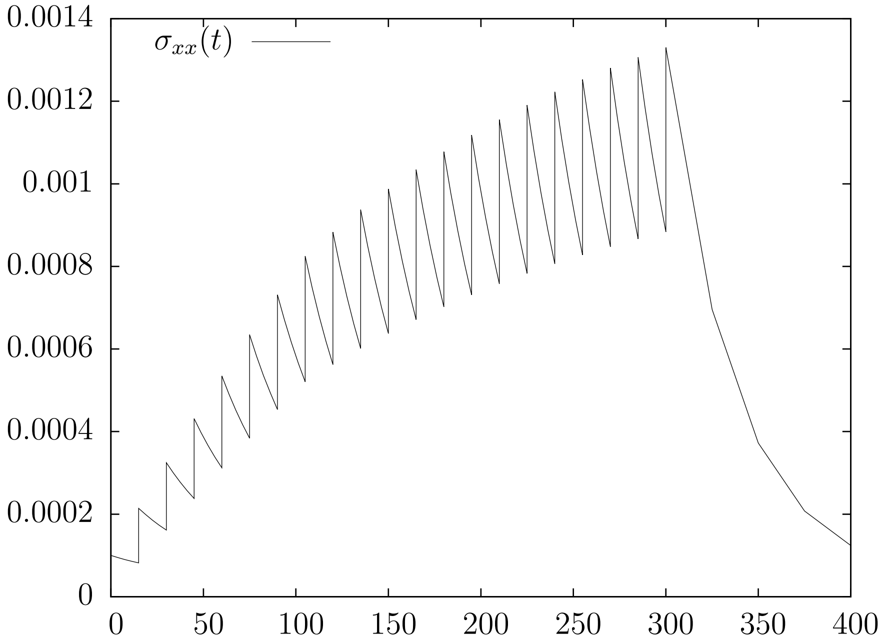


Figure 6.3: $\sigma_{xx}(t)$



C. R. Cassady, and E. Kutanoglu. (2005). Integrating Preventive Maintenance Planning and Production Scheduling for a Single Machine. *IEEE Transactions on Reliability*, vol. 54, pp. 304-309.

W. J. Chen. (2009). Minimizing Number of Tardy Jobs on a Single Machine Subject to Periodic Maintenance. *Omega*, vol. 37, pp. 591-599.

M.-C. Fitouhi, and M. Nourelfath. (2012). Integrating Noncyclical Preventive Maintenance Scheduling and Production Planning for a Single Machine. *International Journal of Production Economics*, vol. 136, pp. 344-351.

G. H. Graves, and C. Y. Lee. (1999). Scheduling Maintenance and Semiresumable Jobs on a Single Machine. *Naval Research Logistics*, vol. 46, pp. 845-863.

L. S. Jennings, M. E. Fisher, K. L. Teo, and C. J. Goh. (2004). *MISER3 Optimal Control Software: Theory and User Manual, version 3.0*. Australia: University of Western Australia.

Figure 6.4: $\sigma_{yy}(t)$

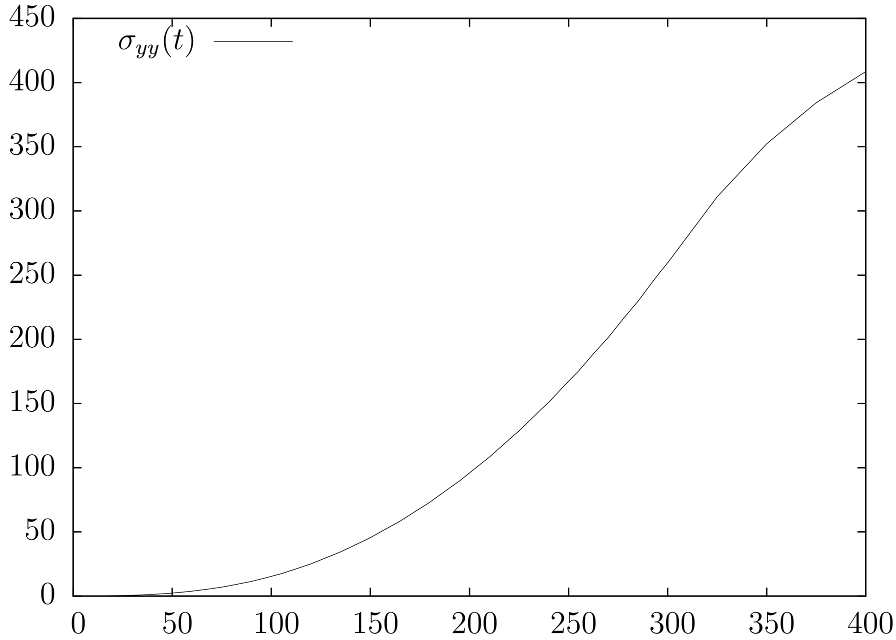


Figure 6.5: $\sigma_{xy}(t)$

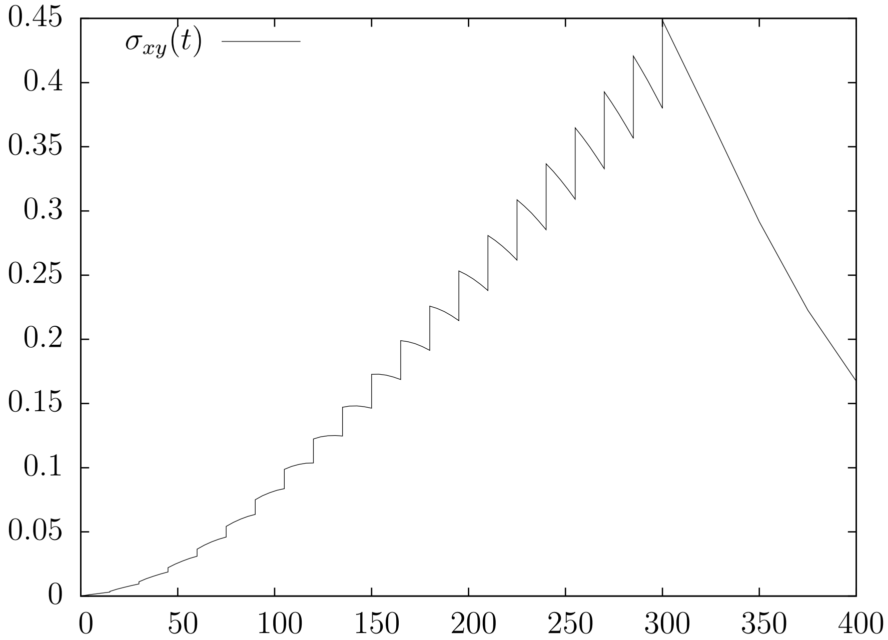


Figure 6.6: Simulation of $x(t)$ with 500 sample paths

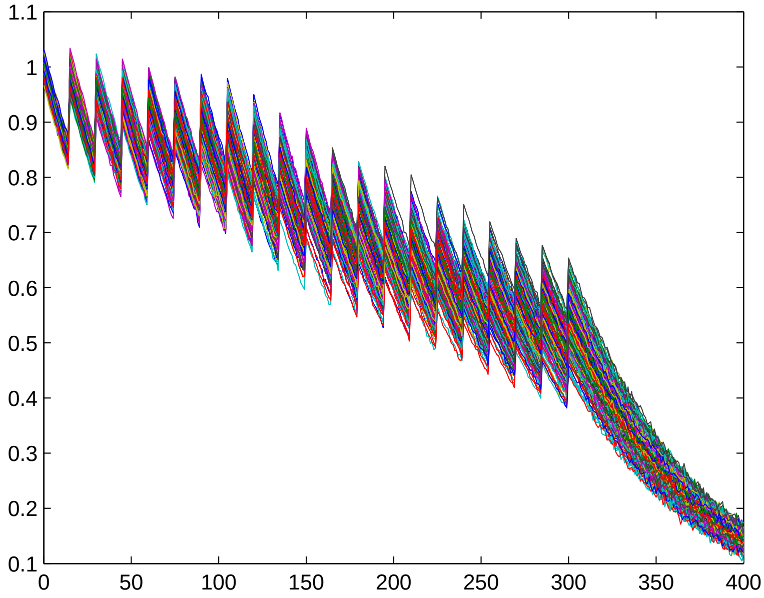
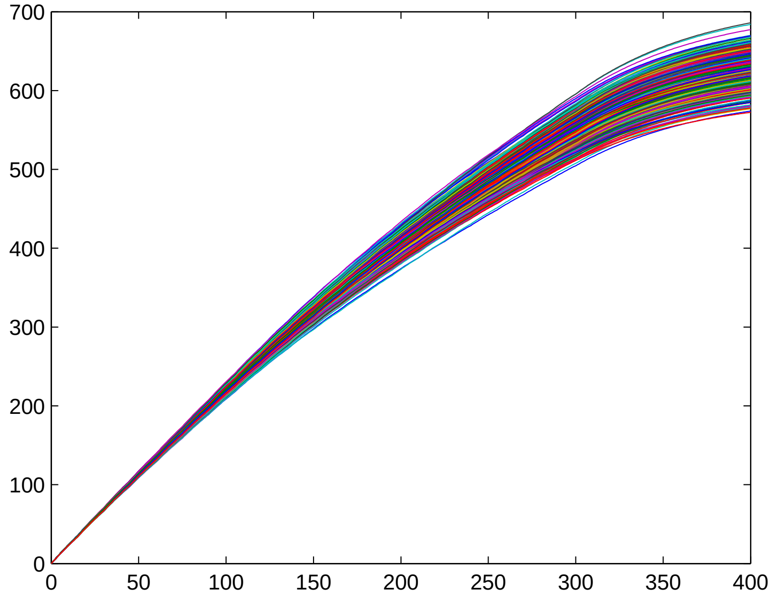


Figure 6.7: Simulation of $y(t)$ with 500 sample paths



C. J. Hsu, M. Ji, J. Y. Guo, and D. L. Yang. (2013). Unrelated Parallel-machine Scheduling Problems with Aging Effects and Deteriorating Maintenance Activities. *Information Sciences*, Available online 31 August 2013.

L. S. Jennings, and K. L. Teo. (1990). A Computational Algorithm for Functional Inequality Constrained Optimization Problems. *Automatica*, vol. 26, pp. 371-375.

A. Li, E. Feng, and L. Wang. (2009). Impulsive Optimal Control Model for the Trajectory of Horizontal Wells. *Journal of Computational and Applied Mathematics*, vol. 223, pp. 893-900.

Q. Lin, R. C. Loxton, and K. L. Teo. (2013). Optimal Control of Nonlinear Switched Systems: Computational Methods and Applications. *Journal of the Operations Research Society of China*, vol. 1, pp. 275-311.

Q. Lin, R. C. Loxton, and K. L. Teo. (2014). The Control Parameterization Method for Nonlinear Optimal Control: A Survey. *Journal of Industrial and Management Optimization*, vol. 10, pp. 275-309.

C. M. Liu, Z. G. Feng, and K. L. Teo. (2009). On a Class of Stochastic Impulsive Optimal Parameter Selection Problems. *International Journal of Innovative Computing, Information and Control*, vol. 5, pp. 1-7.

R. C. Loxton, K. L. Teo, V. Rehbock, and W. K. Ling. (2009). Optimal Switching Instants for a Switched-capacitor DC/DC Power Converter. *Automatica*, vol. 45, pp. 973-980.

R. Martin, and K. L. Teo. (1994). *Optimal Control of Drug Administration in Cancer Chemotherapy*. Singapore: World Scientific.

E. Pan, W. Liao, and L. Xi. (2010). Single-Machine-Based Production Scheduling Model Integrated Preventive Maintenance Planning. *The International Journal of Advanced Manufacturing Technology*, vol. 50, pp. 365-375.

X. Qi, T. Chen, and F. Tu. (1999). Scheduling the Maintenance on a Single Machine. *Journal of the Operational Research Society*, vol. 50, pp. 1071-1078.

M. Sbihi, and C. Varnier. (2008). Single-machine Scheduling with Periodic and Flexible Periodic Maintenance to Minimize Maximum Tardiness. *Computers & Industrial Engineering*, vol. 55, pp. 830-840.

K. L. Teo, C. J. Goh, and K. H. Wong. (1991). *A Verified Computational Approach to Optimal Control Problems*. Essex, England: Longman Scientific and Technical

K. L. Teo, L. S. Jennings, H. W. J. Lee, and V. Rehbock. (1999). The Control Parameterization Enhancing Transform for Constrained Optimal Control Problems. *Journal of Australian Mathematical Society*, vol. 40, pp. 314-335.

S. Wang. (2013). Bi-objective Optimisation for Integrated Scheduling of Single Machine with Setup Times and Preventive Maintenance Planning. *International Journal of Production Research*, vol. 51, pp. 3719-3733.

G. A. Widyadana, and H. M. Wee. (2012). Optimal Deteriorating Items Production Inventory Models with Random Machine Breakdown and Stochastic repair time. *Applied Mathematical Modelling*, vol. 35, pp. 3495-3508.

C. Z. Wu, and K. L. Teo. (2006). Global Impulsive Optimal Control Computation. *Journal of Industrial and Management Optimization*, vol. 2, pp. 435-450.