Provoking contingent moments in mathematics: knowledge for powerful teaching at the horizon

Chris Hurst

School of Education, Curtin University, Perth, Australia

Dr Chris Hurst

C.Hurst@curtin.edu.au

School of Education

Curtin University

GPO Box U1987

Perth 6845

Provoking contingent moments: Knowledge for 'powerful teaching' at the horizon

Structured abstract

Background Teacher knowledge continues to be a topic of debate in Australasia and in other parts of the world. There have been many attempts by mathematics educators and researchers to define the knowledge needed by teachers to teach mathematics effectively. A plethora of terms, such as mathematical content knowledge, pedagogical content knowledge, horizon content knowledge and specialised content knowledge have been used to describe aspects of such knowledge.

Purpose This paper proposes a model for teacher knowledge in mathematics that embraces and develops aspects of earlier models. It focuses on the notions of contingent knowledge and the connectedness of 'big ideas' of mathematics to enact what is described as 'powerful teaching'. It involves the teacher's ability to set up and provoke contingent moments to extend children's mathematical horizons. The model proposed here considers the various cognitive and affective components and domains that teachers may require to enact 'powerful teaching'. The intention is to validate the proposed model empirically during a future stage of research.

Sources of evidence Contingency is described in Rowland's Knowledge Quartet (2005) as the ability to respond to children's questions, misconceptions and actions and to be able to deviate from a teaching plan as needed. The notion of 'horizon content knowledge' (Ball et al.) is a key aspect of the proposed model and has provoked a discussion in this article about students' mathematical horizons and what these might comprise. Together with a deep mathematical content knowledge and a sensibility for students and their mathematical horizons, these ideas form the foundations of the proposed model.

Main argument It follows that a deeper level of knowledge might enable a teacher to respond better and to plan and anticipate contingent moments. By taking this further and considering teacher knowledge as 'dynamic', this paper suggests that instead of responding to contingent events, 'powerful teaching' is

about provoking contingent events. This necessarily requires a broad, connected content knowledge based on 'big mathematical ideas', a sound knowledge of pedagogies, and an understanding of common misconceptions, in order to be able to engineer contingent moments.

Conclusions In order to place genuine problem-solving at the heart of learning, this paper argues for the idea of planning for contingent events, provoking them, and 'setting them up'. The proposed model attempts to represent that process. It is anticipated that the new model will become the framework for an empirical research project, as it undergoes a validation process involving a sample of primary teachers.

Keywords: mathematical content knowledge; contingency; horizon content knowledge; powerful teaching.

Background and Context

International large-scale studies, such as the International Association for the Evaluation of Educational Achievement's (IEA) Trends in International Mathematics and Science Study (TIMSS) and the Organisation for Economic Co-operation and Development's (OECD) Programme for International Student Assessment (PISA), have focused attention not only on the mathematical achievements of children but also, by association, on the mathematical knowledge of their teachers. The rankings of various countries in terms of achievement have led to much comparison between, and speculation about, the school systems and teacher education programmes and structures in those countries (Coe, Aloisi, Higgins and Major 2014; Miao and Reynolds 2014). In Australia, the Teacher Education Ministerial Advisory Group (TEMAG) was established in 2014 to "provide advice to the federal Minister for Education on how teacher education programmes could be improved to better prepare new teachers with

the practical skills needed for the classroom" (Department of Education and Training 2015, 1).

However, discussion about the knowledge required by pre-service and in-service teachers to teach mathematics effectively is not new. It has been a focus for debate for decades, with many researchers developing models of what knowledge for teaching might look like (Shulman 1986; Ball, Thames and Phelps 2008; Hill, Ball and Schilling 2008). Some models, such as The Knowledge Quartet (Rowland, Huckstep and Thwaites 2005), have focused more specifically on mathematical content knowledge, while others have sought to represent that knowledge alongside knowledge of pedagogies, curriculum, and students (Department of Education, Western Australia 2013; Levenberg and Patkin 2014; Petrou and Goulding 2011).

Other educators and researchers have developed schemata for describing and organizing mathematical knowledge in terms of 'big ideas' (Charles 2005, 1), and 'concept knots' and 'knowledge packages' (Ma 1999, 78) while others have developed frameworks for considering how knowledge broadly develops (Daggett 2014). Charles (2005, 2) defined a 'big idea' as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole". (2005, 2). Ma (1999, 18) described 'knowledge packages' as a way of thinking in that teachers could "see mathematical topics group-by-group, rather than piece-by-piece" - for example, understanding the pieces of mathematical content that underpin a concept such as subtraction of whole numbers with regrouping. 'Concept knots' are described by Ma (1999, 115) as "another kind of key piece in a knowledge package" and she provided as an example the meaning of multiplication with fractions – it is a knot in the sense that it 'ties together' important concepts related to division by fractions. (Ma, 1999, 115). Related to these initiatives are ideas such as 'teacher

capacity', which is described as the capacity of teachers to utilise aspects of mathematical knowledge for teaching to enact reform, interpret what students do, and plan for teaching to progress student thinking (Zhang and Stephens, 2013, 489). The notion of teacher capacity links, too, with the 'knowledge quartet': in particular, the idea of 'contingency', (Rowland et al., 2005, 259), and this will be discussed shortly. Researchers have attempted, in various ways, to rationalize and clarify existing models or aspects of them (Goos 2013; Wilkie 2013) as well as consider the role played by attitudes and beliefs (Beswick, Callingham and Watson 2012; Beswick et al. 2011).

Such debates and discussions in the literature involve the use of many conceptual terms (sometimes overlapping and variously defined) that describe aspects of knowledge: pedagogical content knowledge, subject matter content knowledge, specialised content knowledge, horizon content knowledge, curricular content knowledge and so on (Shulman 1986; Ball et al. 2008). More recently, the allied notions of 'powerful teaching' and 'powerful learning' (Darling Hammond, 2016, 85) have been examined. Darling-Hammond discusses the importance of being able "to develop knowledge for teaching that can support more complex, strategic learning – a kind of teaching that goes far beyond giving a test, and giving a grade" (2016, 85). She notes that "Such powerful teaching and learning would require schools that value and evaluate serious intellectual performance" (2016, 85) and that current narrow measures of achievement limit our ability to learn about what constitutes powerful teaching. Previously, Darling-Hammond had described a 'need for powerful teaching' (2010, 3) and suggested that, amongst other things, teachers demonstrating this would "engage students in active learning and create intellectually ambitious tasks" (2010, 5).

Earlier, Brophy (2002, 12) had raised the notion of 'powerful ideas' which seems to align well with Charles' (2005, 1) discussion of 'big ideas'. Brophy does not

explicitly define powerful ideas, but discusses their connectedness as making them useful for attaining the essential classroom goals of understanding, appreciation and application (2002, 14). He situates powerful ideas as a key element of effective teaching by observing the following: "Content developed with these goals in mind is likely to be retained as meaningful learning that is internally coherent, well connected with other meaningful learning and accessible for application. This is most likely to occur when the content itself is structured around powerful ideas and the development of this content through classroom lessons and learning activities focuses on these ideas and their connections" (Brophy, 2002, 14).

Purpose

Given the currency of debate about teacher knowledge and the variety of models and schemata that exist for organizing teacher knowledge, it is helpful to rationalize and clarify the relationships between the models already developed. This article appraises a number of such models and proposes a new model. Although outside the scope of this paper, it is anticipated that this new model will become the framework for a research project that will seek to validate it empirically with a sample of primary teachers. This paper will focus on selected aspects of some of the models mentioned above – the notions of 'contingency knowledge' (Rowland et al., 2005, 259) and 'horizon content knowledge' (Ball et al., 2008, 403; Hill et al., 2008, 377). The paper will consider how they these two aspects, in conjunction with 'big idea thinking' (Charles 2005, 1), underpin and constitute 'powerful teaching' (Darling Hammond, 2016, 85).

A review of models of teacher knowledge in mathematics

The concept of contingency and contingent events

Rowland et al. (2005) describe contingency as an ability to respond to children's thoughts and actions and also to deviate from a teaching plan or agenda as required. It is also suggested that "greater knowledge will lead to fewer surprises when teaching since such knowledge enables the teacher to anticipate and plan for a greater number of pupil responses" (Rowland, Turner, Huckstep, and Thwaites 2009, 31). Rowland and Zazkis (2013, 139) later describe it as "the 'opposite' [their emphasis] of planning – to situations that are not planned and that have the potential to take a teacher outside of their planned route through the lesson".

This links to another idea that is considered central to this paper: that a teacher's knowledge is dynamic. This is supported by Rowland, Thwaites and Jared (2015, 88) where it is observed that "readiness for predictable errors and misconceptions is part of pedagogical content knowledge". Rowland et al. (2015, 89) also draw attention to the importance of giving pre-service teachers a feel for "the contingent disturbances that they will experience in the classroom". The authors express their interest in the unpredictability (i.e. 'contingency') of situations "in which this [teacher] knowledge of mathematics is activated and applied" (Rowland et al., 2015, 75). They suggest that "some aspects of the unknown can be anticipated by the knowledgeable teacher" (Rowland et al., 2015, 76) and "If teachers know in advance of some of the possible circumstances in which such events could arise, they might be better prepared to 'see them coming'" (2015, 77). This paper will suggest that it is beneficial to plan for contingent events, to provoke them, rather than only respond to them.

It is important to identify how knowledge is enacted in teaching, rather than the holding

of knowledge for teaching, as this may be where the nexus between contingency and powerful teaching exists. Rowland et al. (2015, 82) describe it as "reflection *in* action [referring to] teachers' monitoring and self-regulations of their actions *as* they perform them". Brophy (2002, 19) discussed the importance of 'thoughtful discourse' and the way in which "questions are planned to engage students in sustained discourse structured around powerful ideas". In addition, Brophy (2002, 19) comments that effective teachers "use questions to stimulate students to process and reflect on content, recognize relationships among its key ideas, think critically about it, and use it in problem solving, decision making or other higher-order applications". Planning for and provoking contingent events can be an important part of this process.   Hence, contingent knowledge, in this paper, will be conceptualised as reactive and proactive. It will be fundamental to the proposed new model of teacher knowledge.

Horizon content knowledge and 'big ideas'

One compelling aspect of the model developed by Hill et al. (2008) and Ball et al. (2008) is the notion of 'horizon content knowledge'. This is described by Ball et al. in the following way: "Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (2008, 403). The authors continue by explaining that knowledge of the 'mathematical horizon' is about teachers identifying the mathematics that children in a particular year level need to know but also knowing about the mathematics that precedes and follows it. This implies that teachers need to exercise 'big idea thinking' (Charles, 2005, 1) with the myriad links and connections within and between concepts informing their assessment of children and their planning. Such a proposition seems to be supported by Rowland and Zazkis (2013), who note that a breadth and depth of content knowledge is required for

effective teaching. Without specifically using the term 'horizon content knowledge', they state that knowing the curriculum would be sufficient if teaching did not present any unforeseen situations: but, since it does, "mathematical knowledge beyond the immediate curricular prescription is beneficial and demonstrably essential" (Rowland and Zazkis, 2013, 138). Here, they are also alluding to the idea of contingency knowledge and dealing with contingent moments.

Ball et al. (2008) note uncertainty in terms of where the notion of horizon content knowledge (HCK) fits within their model, explaining that "we are not sure whether this category is part of subject matter knowledge or whether it may run across the other categories" (2008, 403). Subsequently, Ball and Bass (2009, 5) described horizon knowledge as more than content knowledge, as it "supports a kind of awareness, sensibility, disposition that informs, orients, and culturally frames instructional practice" . . . and that it includes . . . "awareness of the mathematical horizon". The notion that a teacher requires a deep understanding of mathematical concepts in order to cater for the needs of all students in a given class is supported by Ball and Bass. It is given further credence by Siemon, Bleckley and Neal, who cite the results from the Scaffolding Numeracy in the Middle Years Project, observing that there is up to "an 8 year range in achievement at each year level" (2012, 23) and that as well as having a strong grasp of 'big ideas', "teachers need to know how they are connected and how they might be acquired over time" (2012, 24). The idea that a teacher must have a broad 'horizon content knowledge' is, then, suggested to be critical.

This article acknowledges the importance of horizon content knowledge (HCK) and agrees with Ball and Bass (2009) that it is much more than being an aspect of content knowledge. In doing so, horizon content knowledge and the idea of the 'mathematical

horizon', are positioned as separate, but closely related ideas which are generally enacted simultaneously.

Ball and Bass (2009, 8) make the point that horizon content knowledge involves 'key mathematical practices' such as 'questioning' and 'proving'. It is, then, worth viewing horizon content knowledge (or at least part of it) as being about the 'processes of learning' as opposed to purely being about content knowledge. Indeed, it could be said that it is the application of processes, or learning proficiencies such as reasoning, problem solving, justifying, elaborating, and hypothesising that enable students to develop a deep understanding of concepts. The Australian Curriculum: Mathematics (Australian Curriculum Assessment and Reporting Authority, 2013) highlights four proficiencies of fluency, understanding, reasoning and problem solving as vehicles by which teachers, having identified the particular mathematics to be known by students, facilitate their required learning. In the same vein, it could be argued that teachers provoke contingent moments by posing carefully phrased questions and setting up appropriate tasks based on their horizon content knowledge.

Knowledge as process, as well as product, is another representation of the horizon content knowledge /mathematical horizon relationship mentioned earlier. It is akin to the idea of teaching children to be mathematicians, rather than just teaching them the mathematics. This is how they will move beyond their existing horizons and it is the generation of contingent moments that provides the context for doing that. Ruthven (2011, p. 10) cites the work of Thurston (1994) in saying that "effective mathematical communication involves some degree of interactivation and mathematisation of the knowledge at stake". That is, knowledge as content is not sufficient on its own and needs to undergo "a process of reconstruction by means of active negotiation between the participants in classroom mathematical activity"

(Ruthven, 2011, p. 10). It needs to be an overt reflective process and Ruthven (2011, p. 11) argues that "taking on a teaching role involves a recasting of intrapersonal metacognition into interpersonal activity and dialogue". It is suggested that the proficiencies of reasoning and problem solving are examples of knowledge processes that are vehicles for development of deep understanding.

It is suggested here that Rowland's notion of contingency is linked to horizon content knowledge and that, together with 'big idea thinking', they may constitute 'powerful teaching'. Rowland, Thwaites and Jared (2015, p. 78) identify three situations or events that are 'triggers of contingent moments'. Two of them are described as being initiated by a student, such as "an unanticipated response to a stimulus within the lesson" and initiated by the teacher where "a consequence of some teacher insight caused them to reassess their lesson image" (Rowland et al., 2015, p. 78). The second, in particular, seems to be directly linked to horizon content knowledge. The third trigger is provided by the availability, unavailability, or use or a resource or artefact during a lesson. Rowland et al. (2015, p. 76) also provide an example of knowledge as process in reflecting that contingent moments may "unsettle and disturb the teacher [but] offer the teacher an opportunity to learn".

Mathematical horizons and horizon knowledge

The idea of the 'mathematical horizon' can be considered in a number of ways. We need to consider to what extent it is the teacher's mathematical horizon that is important or whether it is more important to focus on the mathematical horizons of the students being taught. After all, educators are teaching to students' mathematical futures (Ball and Bass 2009; Zazkis and Mamolo 2011). Hence, the mathematical horizon beyond the classroom also needs to be considered. It is important to consider whether teachers are

well enough equipped to provide students with the understanding, mindsets, and knowledge to prepare them for their mathematical futures. Ball and Bass (2009) also observed that horizon knowledge is more than just content: some of it is about teaching practices and some of it is about values. The preparation of students for progressing to the mathematical horizon is, then, more than detailed content knowledge; it is also the development of reflective attitudes, advanced problem-solving skills, and the ability to reason, justify, elaborate, question and hypothesize. It follows that teachers' horizon knowledge must consist of those same mindsets and abilities if they are to be effective in assisting their students to reach and cope with their mathematical horizons. The two ideas – horizon knowledge and knowledge of students' horizons – are considered as being separate - yet, because they are closely linked, they need to be considered together. For example, knowing about the mathematical horizon entails teachers being aware of the horizons that are applicable to their students but using their horizon content knowledge to help them to reach their immediate horizons and move beyond.

Part of the teacher's role can be conceptualised as recognising where the student's mathematical horizon is and positioning the student to be able to move beyond it. Irrespective of where that horizon may be, a teacher is better placed to help a student move forward if s/he has a deep understanding of the links and connections within a particular concept and between it and other concepts. In their discussion of 'big ideas', Hurst and Hurrell (2014, 1) refer to this as knowing the 'micro content' or the 'little ideas' that contribute to the 'big ideas'. For example, to teach the concept of place value, a teacher needs to understand the precursor knowledge that provides the foundation. There are two key aspects which are embedded in early number and pre-number work and a teacher needs to understand exactly how they provide the foundation for place value. One is the notion of grouping, derived from sorting and

classifying, subitizing, and later, flexible grouping. This inherently links to the idea of the 'ten group', which is the basis for place value, and flexible partitioning, which enables us to operate with numbers. The second is the number string, derived from counting. When combined with the idea of grouping, and the repeating pattern in naming numbers, we have the essence of place value. It is that depth of content knowledge that enables the teacher to identify a student's misconceptions and use appropriate pedagogies to develop a student's understanding of the particular concept at hand. Such horizon content knowledge also allows teachers to set appropriate tasks and ask questions to provoke errors and unmask misconceptions (Ball 2014). The position of the mathematical horizon will be different for different people; it will, simultaneously, be near and far. Deep content knowledge empowers teachers both to deal with and to provoke contingent moments in order to help their students generate what is, for them, new knowledge which lies beyond their current horizon. In other words, it is argued that horizon content knowledge is analogous to 'big idea thinking' and the links and connections within and between the 'big ideas' of mathematics.

In the same way that a mathematical horizon may be near or distant, contingency can be manifest in two ways. It can be serendipitous when a situation arises in the classroom and needs to be dealt with immediately. It can also be longer term where teachers may be able to position themselves to see the overview of the situation and plan the next stage of teaching. In any case, it is characterised by the feeling of 'What do I do when I don't know what to do?'. Horizon knowledge is brought to bear in such cases. Whatever the situation, contingent knowledge depends on the intimate knowledge of the mathematics and of the students in question, which enables the teacher to deal effectively with the situation at hand. It is this contingency itself which

can initiate and drive learning, problem solving, and the development of conceptual understanding.

Sensibility, awareness and 'big idea thinking'

The use of the term 'sensibility' by Ball (1993) and Ball and Bass (2009) is interesting as they describe it as an 'awareness' that informs practice. The implication here is that it would enable teachers to deal with contingent moments. It is described as "a kind of peripheral vision or awareness of the mathematical horizon" (Ball and Bass 2009, 5) and teaching is enhanced "when teachers have mathematical perspective on what lies in all directions, behind as well as ahead, for their pupils" (Ball and Bass, 2009, 11). The term 'sensibility' was also used by Askew (2008) in his discussion of teacher knowledge. He is critical of how mathematical content knowledge is reduced to lists of specific pointers that he terms "death by a thousand bullet points" (Askew 2008, 21). Rather, Askew calls for "a mathematical sensibility . . . that would enable them to deal with existing curricula but also be open to change" (2008, 22). His notion of 'sensibility' is akin to having a feel for the 'big ideas' of mathematics and being able to learn more about different aspects of mathematics, as connections become obvious. Most importantly, it means that teachers are potentially better able to teach more effectively. Teachers who have such 'sensibility' are likely to make mathematical connections explicit for their students.

Connected knowledge, contingency and 'powerful teaching'

In considering Rowland's notion of contingency (or contingent knowledge), the links between it and 'powerful teaching', as well as 'big idea thinking' and horizon content knowledge, become clear.

In a recent representation of the Knowledge Quartet (Rowland, Huckstep and Thwaites 2005), Rowland (2015) suggested that the components of transformation and connection, which effectively equate to 'big idea thinking' with its links and connections, underpin the teacher's ability to deal with contingent events. Horizon content knowledge enables teachers to have a peripheral view of the content needed by children prior to and following any particular stage of learning (Ball and Bass 2009). Hence, it can be said that horizon content knowledge and 'big ideas' are inextricably linked, or even could be considered as one and the same. Also, it is the proposition of this paper that contingency knowledge and the ability to deal with contingent events are essential for powerful teaching. That is to say, powerful teaching equates to exploring the learning potential of unexpected (or contingent) events in the classroom. Rowland et al. (2015, 88) cite Ball and Bass (2000) in saying that "readiness for predictable errors and misconceptions is part of pedagogical content knowledge [and] that teachers can even be prepared for the unpredictable uncertainties".

A model for provoking contingent events: a proposal

I argue here that the notion of contingency can be taken further with the idea that really powerful teaching is about teachers actually setting up contingent events. It is more than taking advantage of the teachable moment or the serendipitous opportunity: it is about engineering such opportunities. Put another way, it could be akin to providing provocation and what could be thought of as mild 'discomfort' in the form of mathematical challenges. This, indeed, requires powerful knowledge that is rich and connected. It also invokes the learning processes and proficiencies that can be described as being two parts of a duality that is horizon knowledge. The remainder of this paper will describe how a model is proposed which is based on this notion and which

incorporates and adapts ideas from other models. It is conceptualised in terms of a series of stages.

Stage 1: Hierarchy and sensibility

At this stage, two elements of the new model can be described as follows. One is the hierarchical nature of content knowledge. The other is the idea of 'sensibility' for students' mathematical horizons: assessing them effectively, and incorporating the proficiencies and processes of learning. Foundation knowledge (Rowland et al. 2005) or 'common content knowledge' (Ball et al. 2008, 403) could be considered to be similar and at the lower end of a hierarchy. Rowland (2015) seems to suggest such a hierarchy, whereas Ball et al. (2008, 403) include common content knowledge as one of three equal elements of the domain of Subject Matter Knowledge, the others being 'horizon content knowledge' and 'specialised content knowledge'. Using Rowland et al.'s (2005) notion of the Knowledge Quartet as the basis, the representation of mathematical knowledge proposed at this stage incorporates Ma's (1999, 115) notion of 'concept knots' (i.e. mathematical ideas that tie concepts together) at the next level. Ma's (1999, 120) idea of 'profound understanding of fundamental mathematics' (PUFM) which she described as "understanding of fundamental mathematics [that is] deep, broad, and thorough", and the notion of 'big idea thinking' (Charles, 2005, 1) are seen as belonging to the top of the hierarchy and the keys to teachers enacting their contingency knowledge and provoking contingent moments.

'Sensibility' is seen here as being an awareness of students' mathematical horizons. This can be derived from knowledge of students themselves and astute diagnostic assessment. The latter relies heavily on a teacher's understanding of big mathematical ideas and the pieces of micro-content that contribute to them (Hurst and Hurrell, 2014, 1). It also incorporates an awareness of the processes and proficiencies

for learning mathematics. Along with deep content knowledge, this sensibility enables teachers to deal with and provoke contingent moments to help their students move to and beyond their mathematical horizons.

Stage 2: Preparing for contingent moments

This discussion could be considered in terms of what constitutes the necessary resources for powerful teaching. First, it is the considerable extent of deep, rich, connected knowledge of the big ideas of mathematics, which can enable the teacher to understand and diagnose students' learning in terms of the mathematics they know and need to know. Secondly, teachers need to understand students' mathematical horizons and devise a path for helping them reach and move beyond those horizons. It is their rich content knowledge that will help enable them to devise that path. Thirdly, teachers require a mindset that helps them to provoke the contingent moments, provide the challenges, and invoke the key mathematical proficiencies and processes such as reasoning, justifying, hypothesizing and problem solving that are critical conduits for children progressing to their mathematical horizons and equipping them to move beyond the horizons.

It has already been suggested here that contingency knowledge is both reactive and proactive. An example of that relates to task design where a teacher seeks to address a particular obstacle to children's learning. Such obstacles may arise from the context of a lesson or they may be known to the teacher in the form of a common misconception or misunderstanding held by one or more students. In order to do that effectively, teachers need to develop deep and specific understanding of concepts akin to the connectedness of 'big idea thinking'. Common misconceptions in mathematical learning have been well documented (Ryan and Williams, 1999; Swan, 2003) and

teachers can prepare by looking beyond the immediate horizon, anticipating, and enacting some 'contingency knowledge'. This is proactive planning: teachers will anticipate that these potential obstacles will be there.

Swan (2003) provides some clear examples of typical misconceptions for which teachers can prepare. For instance, when asked to complete decimal number sequences such as 0.2, 0.4, 0.6, __, __, __, many children will provide an answer of 0.8, 0.10, 0.12 (Swan 2003, p. 118). If a teacher is aware of the possibility of such answers, s/he can prepare appropriate teaching strategies and tasks. Ryan and Williams (1999) also refer to a common misconception about decimals in that many children think that 0.15 is bigger than 0.2, on the basis of their previous knowledge of whole numbers. They pose the question "How can teachers find carefully crafted tasks that provide enough (but not too much cognitive conflict so that children establish correct reasoning for the new number domain?"(Ryan and Williams, 1999, p. 148). The answer lies in the depth of their content knowledge to underpin their capacity to plan contingent moments. It is therefore suggested that contingency knowledge may have a third dimension, planned contingent moments, in addition to contingency 'in the moment' and 'deferred contingency' which is enacted at a later time.

Stage 3: Inquiry, learning proficiencies and connected knowledge

Makar, Arthur and Ben-Zvi (2015, 1108) studied argumentation-based inquiry and observed that "inquiry problems are complex, so the entire pathway from beginning to end is not necessarily visible [and] as movement is made (or not) towards a viable solution, evidence provides a focal point for judging ideas, progress, and next steps within the problem context". This was even more so when problems were ill-structured and the route to the solution requires negotiation and careful consideration. It could be

said that 'ill-structured problems' provide 'contingent moments', once again, by design on the part of the teacher. This further strengthens the position of contingent knowledge and 'contingent teaching' as being proactive in nature and not simply reactive.

It is suggested that what has been considered previously as a single entity – horizon content knowledge – may be better considered as a duality consisting of rich, strongly connected content knowledge, and a deep understanding of mathematical processes, structures and proficiencies. When coupled with an understanding of children's mathematical horizons, they can be the tools of a 'powerful teacher' in provoking contingent moments and fostering deep learning. Rowland and Zazkis (2013) seem to support the proposition of a dual structure, noting that contingent situations pose both a 'mathematical problem' (knowledge of content) and a 'pedagogical problem' (knowledge of processes and proficiencies) for teachers (Rowland and Zazkis 2013, 151). They conclude that "a teacher's responses to problematic contingent moments that arise in teaching mathematics are fundamentally dependent on their mathematical knowledge, which prompts and guides pedagogical implementation" (Rowland and Zazkis 2013, 151).

Stage 4: Moving from the 'inner' to the 'outer' horizon

Ball and Bass (2009, 12) draw attention to the importance of a teacher having a "mathematical perspective on what lies in all directions, behind as well as forward, for their pupils" - that is, knowledge of what precedes and follows particular stages of learning. This is supported by Zazkis and Mamolo (2011). They also re-conceptualized the idea of horizon knowledge, based on Husserl's notions of 'inner and outer horizons' (citing Follesdal, 2003). Essentially, they considered the 'inner horizon' to be "specific features of the object itself and includes the attributes of the object that lie in the

periphery of our focus" (Zazkiz and Mamolo 2011, 9). The 'outer horizon comprises "not the particular features *of* the object . . . but rather features that are *connected* to the object and that embed it in a greater structure" (Zazkis and Mamolo 2011, 9-10). If we interpret this in terms of knowledge, the 'inner horizon' might consist of the specific identifying attributes or content that immediately characterises a concept or idea (or as they term it, 'an object'). The 'outer horizon' similarly connects that concept or idea to other concepts or ideas.

A clear link with 'big idea thinking' (Charles, 2005, 1) can helpfully be made here, which can be demonstrated by using place value (certainly a 'big idea') as an example. The 'inner horizon' might consist of ideas such as knowing the names of the places, knowing the cyclic pattern for reading and writing the numbers, knowing that there is a ten times multiplicative relationship that exists between the places, and knowing that zero can be a place holder. The 'outer horizon' would consist of ideas such as the use of 'trading up and down' to facilitate addition and subtraction of larger numbers, using standard place value partitioning and the distributive property to facilitate learning an algorithm for multiplication, and the use of the standard metric system of measurement units. These 'outer' ideas are inextricably connected to the essence of place value and are underpinned by it. Zazkis and Mamolo (2011) note that there may be some aspects of the 'mathematical horizon' that may not be within reach of students. For instance, "a teacher's knowledge at the mathematical horizon includes features in both the inner and outer horizons . . . while only some of those features are accessible to students" (Zazkis and Mamolo 2011, 10).

Stage 5: 'Big idea thinking' – the enabler of 'powerful teaching'

The notions of 'big idea thinking', 'horizon knowledge', 'powerful teaching', and

'contingency/contingent teaching' are, appear closely linked. Obviously, some aspects of knowledge and understanding are going to be more distant for students and for some students more than others. If 'big idea thinking' is employed, the connections within the 'inner horizon' and between it and the 'outer horizon' can be made accessible. Powerful teachers can engineer contingent moments that challenge their students, provoke learning, and make the connections and links explicit. Zazkis and Mamolo (2011, 13) summarise it aptly in their concluding comments about what they consider knowledge at the mathematical horizon to be, that it is: "advanced mathematical knowledge in terms of concepts (inner horizon), connections between concepts (outer horizon), and major disciplinary ideas and structures (outer horizon) applied to ideas in the elementary school or secondary school curriculum". This is broadly supported by Chick and Stacey (2013) in their discussion of how teachers deal with contingent moments. They note that without strong and specific content knowledge, "a mathematically appropriate solution to the teaching problem is unlikely. Moreover, the greater a teacher's conceptual fluency, the more likely it is that a suitable solution will come to mind" (Chick and Stacey 2013, 135). Here, I propose that 'big idea thinking' is the 'enabler' that allows teachers to move between the 'inner' and 'outer' horizons and, as such, it is the 'enabler' that allows teachers to set up the contingent moments that are the essence of powerful teaching.

'Horizon content knowledge' could be also considered as the knowledge that teachers create from their own practice. Hence, inevitably, it must be dynamic as opposed to static, as has been discussed earlier. Zazkis and Mamolo (2011) suggest that one requirement for developing horizon knowledge is engagement in learning of mathematics, though it is not the sole requirement. Rowland and Zazkis (2013, 150) say that 'syntactic knowledge' (Shulman, 1986) "is most likely to be acquired by doing

mathematics, by posing and solving problems, by 'signing up' to the world of mathematics". Again, there is a link here to the importance of disposition which is likely to underpin a teacher's decision to undertake such involvement. Chick and Stacey (2013, 122) also note the dynamic nature of teacher knowledge in citing Hodgen (2007) who discussed the extent to which such knowledge is abstract and theoretical "or whether it is tacit craft knowledge" – that is, it is informed by and developed through their practice.

Stage 6: Frameworks of teacher knowledge

Research about teacher knowledge over the last three years has provided further impetus to consider a new approach to knowledge for teaching mathematics. The research has typically been based on aspects of existing models such as those mentioned at the beginning of this paper. Researchers have generally attempted to re-conceptualize previous efforts to define teacher knowledge. For example, Beswick, Callingham and Watson (2012) acknowledged the importance of the work of Shulman (1986) and others in developing frameworks for teacher knowledge but noted that the "apparently multi-faceted nature of teachers' knowledge for teaching mathematics . . . has complicated efforts to establish clear links between it and students' mathematical achievement" (Beswick et al. 2012, 131). However, in putting the case for their conceptualization of knowledge for teaching mathematics as a 'single underlying variable' called 'teacher knowledge', they advocated further analysis and development of earlier models such as those proposed by Shulman (1986) and Ball and her colleagues (Ball et al. 2008; Hill et al. 2008). Within the single construct of teacher knowledge, Beswick et al. proposed four hierarchical levels: 'personal numeracy' (1), 'pedagogical awareness' (2), pedagogical content knowledge awareness' (3), and 'pedagogical content knowledge

consolidation'(4). Their 'uni-dimensional' construct of 'teacher knowledge' could also be considered in terms of the 'inner' and 'outer' horizons. Specifically, Levels 1 and 2 would be more likely to be in the realm of pre-service and novice teachers and Levels 3 and 4 would be more likely to characterise the teaching of experienced teachers.

Goos (2013) approached the issue of teacher knowledge by proposing a bipartite model consisting of pedagogical content knowledge (PCK) and mathematical content knowledge (MCK). Her findings suggested that "MCK and PCK work together and may develop in tandem with each other . . . yet PCK cannot exist without a foundation in MCK" (Goos 2013, 981). Goos' work connects with that of Beswick et al. (2012) in that the pedagogical content knowledge she describes relates closely to Levels 3 and 4 of their hierarchy. If the views of Goos (2013) and Beswick et al. (2012) are combined with what has already been presented, the proposed model can be extended. The most effective or most powerful teacher would be likely be one whose content knowledge is deep and whose pedagogical awareness is strong, enabling him/her to set up and deal with contingent moments.

Wilkie (2014) found, as did Goos (2013), that 'specialised content knowledge' (SCK), a term used by Ball and her colleagues, did not necessarily lead to strong levels of 'knowledge of content and teaching' (KCT) or 'knowledge of content and students' (KCS) (also terms coined by Ball and colleagues). For example, Wilkie (2014, 420) explained that teachers may have had high levels of content knowledge but did not necessarily understand how children learned or were able to "apply it to appropriate activities for teaching it". Further, she suggested that teachers whose content knowledge was procedure-based found it difficult to interpret students' mistakes and respond effectively to them. These findings indicate that there are factors other than content knowledge that come into play. It also suggests that such knowledge may need to be

better organized and held, which, again, has given impetus to the development of a proposed new model on the basis of what has been described earlier in this paper.

While it is generally accepted that content knowledge alone is not sufficient, Chick and Stacey (2013) have noted that the more conceptually fluent a teacher, the more likely it is that s/he will be able to address the learning needs of children. They describe such fluency in terms of Ma's (1999) notion of 'profound understanding of fundamental mathematics' (PUFM) and state that "Apparently small mathematical details of activities can affect learning. For this reason, development of excellent tasks demands substantial expertise" (Chick and Stacey 2013, 134). This links with the notion of 'specialised content knowledge' (Ball et al. 2008) as well as the concept of 'big ideas' that are based on deep and connected knowledge. The dual structure, proposed earlier for horizon knowledge, is also given credence by Chick and Stacey (2013) when they cited Polya's (1962) notion of how knowledge consists of 'information and know-how'. This seems similar to what Beswick et al. (2012) and Goos (2013) described as mathematical content knowledge and pedagogical content knowledge, which, in terms of the proposed structure for horizon knowledge, equate to 'content' (information) and 'processes' (know-how).

Zhang and Stephens (2013) developed a construct for 'teacher capacity' to analyse effectiveness of a sample of Australian and Chinese teachers. They state their preference for moving away from knowledge alone being the predictor of teacher capacity and that it is more important to see "how teachers utilise that knowledge in their practical teaching" (Zhang and Stephens 2013, 488). The critical element of their construct is 'design of teaching', which they describe as "teachers' capacity to design teaching in order to move students' thinking forward and to respond to specific examples of student's thinking in the light of official curriculum documents" (Zhang

and Stephens 2013, 489). 'Design of teaching' is informed by three other criteria: knowledge of mathematics, understanding of students' mathematical thinking, and interpretation of curriculum. This resonates with the earlier discussion about knowledge of students' mathematical horizons, and 'design for teaching' and could easily be seen as an analogy for 'contingency', given the comment about moving students' thinking forward. All of these ideas have greatly informed the development of the new model for teacher knowledge.

Stage 7: Attitudes, beliefs and dispositions

The construct for teacher capacity developed by Zhang and Stephens (2013) depicts 'dispositions, attitudes, beliefs, and values' as having a large impact on teacher effectiveness. The importance of teacher affect has also been noted by Chick and Stacey (2013) who state that "successful teachers must have the capacity to apply the knowledge they *do* have *during* [their emphasis] teaching" (Chick and Stacey 2013, 122). The inference here is that even if the capacity for effective action is there, the development and enacting of it in practice depends on a teacher's attitudes and beliefs. Rowland and Zazkis refer to a combination of "knowledge and inclination" (2013, 150), while findings from the research by Beswick et al. (2012) validated teacher beliefs as a contributor to their uni-dimensional construct for teacher knowledge.

The importance of affect cannot be underestimated in terms of its role in determining the effectiveness of a teacher. Therefore, the final version of the new model would have an affective domain as underpinning the idea of powerful teaching, in addition to the development of the other three domains.

Proposing a new model

Some clarifications about terminology

In the above discussion, I have attempted to build a case for a new model of teacher knowledge for the teaching of mathematics. I have argued that, from the analysis of existing models such as those reviewed in this paper, there appear to be two central ideas – mathematical content knowledge and pedagogical knowledge. I have not used the terms 'curricular content knowledge' or 'knowledge of curriculum' because modern curricula are often little more than syllabuses, the contents of which are easily encompassed in the term 'mathematical knowledge'. If a wider view of curriculum is taken, then elements such as learning processes, and knowledge of students and contexts, are included elsewhere in the proposed model (i.e., 'sensibility') as they are seen as critical elements of it.

At the centre, I place the two ideas which I believe provide the greatest and most positive impact on student learning – contingency and powerful teaching. The rationale for this is outlined in the previous discussion. As noted, a teacher's ability to enact contingent knowledge and to provoke contingent moments is the essence of powerful teaching and is, effectively, at the centre of mathematical problem solving and learning. 'Big idea thinking', which has been described as a key element, contributes greatly to a teacher's knowledge base and is seen as the 'enabler' for the generation of contingent moments. It sits within the 'mathematics' domain. The other three domains seen as having impact on the quality of teaching are 'pedagogies', 'sensibility (about students, contexts and processes)', and 'affect, attitudes, beliefs and dispositions'.

In this article, I have attempted to clarify what might be meant by 'horizon content knowledge' (Ball et al. (2008, 403) and based on Shulman's (1986) notion of 'vertical knowledge'. The conclusion reached is that it is not a single entity but a multi-

faceted one. It seems that teachers need two things: first, a knowledge of students' mathematical horizons, that is, where each student is situated in terms of his/her mathematical understanding, and, second, horizon knowledge. The latter part is considered as a duality consisting of connected content knowledge based on the 'big ideas' of mathematics and also a sensibility about mathematical proficiencies and processes that can be invoked to help children reach their mathematical horizons and move beyond them.

In the proposed model, these elements appear in all three of the domains – mathematics, sensibility and pedagogies. Knowledge of students' mathematical horizons is seen as being a part of 'sensibility', as it is dependent upon a teacher's awareness of effective diagnostic assessment, which is of course informed by deep and connected content knowledge. The first part of the duality is, indeed, that content knowledge and fits within the domain of 'mathematics'. The second part involves an awareness of proficiencies and processes which, again, are seen as a part of the 'sensibility' domain but also as belonging to the domain of 'pedagogies' in that the choice of appropriate pedagogies is a critical element of effective teaching. Therefore, I suggest that the notion of 'horizon knowledge' is quite varied and is difficult to situate in just one position in any model of knowledge for teaching. However, if a simpler version of the new model is considered, horizon knowledge would encompass the three domains of mathematics, sensibility, and pedagogies. Each domain is linked to 'mathematical content knowledge', 'knowledge of students and content', and 'pedagogical content knowledge' respectively.

Conclusion

In this article, I have attempted to make a case for proposing a new model to represent knowledge for teaching mathematics and to conceptualize elements and domains that constitute the 'act of teaching'. The latter has purposely been placed at the centre and termed as 'design for powerful teaching', the main component of which is the ability of the teacher to provoke and deal with contingent moments. In developing the new model, other models developed by Ball et al. (2008), Rowland et al. (2005), and Shulman (1986) have been rationalized and recent research by Beswick et al. (2012) and Goos (2013) has been incorporated. The new model has also been informed by the work of Askew (2008) and Zhang and Stephens (2013).

It is proposed that the four domains of knowledge of mathematics, knowledge of pedagogies, sensibility about students and process, and affect and beliefs all contribute to a teacher's capacity to enact powerful teaching in the form of contingent moments. It is suggested that it is this capacity that lies at the heart of effective mathematical activity.

A research project is currently being established to conduct an evidence-based exploration of the proposed model in order for it to undergo a validation process. The intention is to conduct extensive research with a large cohort of primary school teachers of different levels of expertise and experience. Data will be generated from a range of sources, including classroom observations, content analysis of planning documents, interviews, and questionnaires. The new model will form the conceptual framework for the research.

References

Askew, Mike. 2008. "Mathematical discipline knowledge requirements for prospective primary teachers, and the structure and teaching approaches of programs

designed to develop that knowledge". In *Knowledge and Beliefs in Mathematics Teaching and Teaching Development, Volume 1,* edited by Peter Sullivan and Terry Wood, 13-35. Rotterdam: Sense Publishers.

Australian Curriculum Assessment and Reporting Authority (ACARA). 2013. The Australian Curriculum: Mathematics. v. 8.2. http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=1

Ball, Deborah L. 1993. "With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics". *Elementary School Journal* 93 (4): 373-397.

Ball, Deborah L. 2014. "Developing teachers' mathematical knowledge for teaching". Presentation for the University of South Australia (via web conference), Adelaide, South Australia, June 16, 2014.

Ball, Deborah. L., Mark Thames, and Hyman Bass. 2008. "Content knowledge for teaching: What makes it special?" *Journal of Teacher Education,* 59(5): 389-407. http://jte.sagepub.com.dbgw.lis.curtin.edu.au/content/59/5/389.full.pdf+html

Ball, D. L., & Bass, H. 2000. "Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics". In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83–104). London: Ablex.

Ball, Deborah L., Hyman Bass. 2009. "With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures". Paper presented at the 2009 Curtis Center Mathematics and Teaching Conference, University of California at Los Angeles (UCLA) Department of Mathematics, Los Angeles, CA, February 21, 2009. http://wwwpersonal.umich.edu/~dball/presentations/022109_UCLA.pdf

Ball, Deborah L., and Hyman Bass. 2009. "With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures". Paper presented at the 43rd Jahrestagung der Gesellschaft für Didaktik der Mathematik, Oldenburg, Germany. http://www.mathematik.tu-dortmund.de/ieem/cms/media/BzMU/BzMU2009/Beitraege/Hauptvortraege/BALL_Deborah_BASS_Hyman_2009_Horizon.pdf

Beswick, Kim, Rosemary Callingham, and Jane Watson. 2012. "The nature and development of middle school mathematics teachers' knowledge". *Journal for Mathematics Teacher Education,* 15:131–157. DOI 10.1007/s10857-011-9177-9

Brophy, Jere. 2002. "Teaching". In H. Walberg (Ed.). *Educational Practices Series – 1.* Brussels: International Academy of Education and International Bureau of Education. http://www.ibe.unesco.org/fileadmin/user_upload/archive/Publications/educatio nalpracticesseriespdf/prac01e.pdf

Charles, Randal I. 2005. Big ideas and understandings as the foundation for early and middle school mathematics. *NCSM Journal of Educational Leadership, 8(*1), 9–24.

Chick, Helen, and Kaye Stacey. 2013. "Teachers of Mathematics as Problem-Solving Applied Mathematicians". *Canadian Journal of Science, Mathematics and Technology Education* 13(2): 121-136, DOI: 10.1080/14926156.2013.784829

Coe, Robert, Cesare Aloisi, Steve Higgins, and Lee E. Major. 2014. "What makes great teaching? Review of the underpinning research". http://www.suttontrust.com/researcharchive/great-teaching/

Daggett, Willard (Bill).2014. "The rigor/relevance framework". http://www.leadered.com/our-philosophy/rigor-relevance-framework.php

Darling-Hammond, Linda. 2010. "Teacher knowledge and student learning: Making the connection". Presentation to the Queensland Curriculum and Assessment Authority. http://www.qcaa.qld.edu.au/downloads/events/linda_darling_hammond.pdf

Darling-Hammond, Linda. 2016. "Research on teaching and teacher education and nits influences on policy and practice". *Educational Researcher* 45(2):83-91. DOI: 10.3102/0013189X16639597 http://edr.sagepub.com/content/45/2.toc

Department of Education and Training. 2015. "Teacher education ministerial advisory group: Terms of reference". http://www.studentsfirst.gov.au/terms-reference

Goos, Merrilyn. 2013. "Knowledge for teaching secondary school mathematics: what counts?' *International Journal of Mathematical Education in Science and Technology* 44(7): 972-983, DOI: 10.1080/0020739X.2013.826387

Hill, Heather, Deborah L. Ball, and Stephen G. Schilling. 2008. "Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-

specific knowledge of students". *Journal for Research in Mathematics Education* 39(4): 372-400. http://www.jstor.org.dbgw.lis.curtin.edu.au/stable/40539304?seq=1#page_scan_tab_contents

Hodgen, Jeremy. 2007. "The situated nature of mathematics teacher knowledge". Paper presented at the Nuffield seminar series on Mathematics Knowledge in Teaching, Cambridge, England. http://www.maths-ed.org.uk/mkit/Hodgen%20Nuffield%20Jan%202007.pdf

Hurst, Chris, and Derek Hurrell. 2014. "Developing the big ideas of number". *International Journal of Educational Studies in Mathematics* 1(2): 1-18. ISSN: 2148 – 5984. Retrieved from: http://www.ijesim.com/arsiv/volume-1-issue-2-december-2014

International Association for the Evaluation of Educational Achievement, (IEA). 2011. "Trends in international mathematics and science study, (TIMSS)". 2011. http://www.iea.nl/completed_studies.html

Levenberg, Ilana, and Dorit Patkin. 2014. "The contribution of in-service training programs to the professional development of mathematics teachers". *International Journal of Learning, Teaching and Educational Research* 7(1): 93-104. http://ijlter.org/index.php/ijlter/article/view/140

Ma, Liping. 1999. "*Knowing and teaching elementary mathematics*". Mahwah, NJ: Lawrence Erlbaum Associates.

Makar, Katie, Arthur and Ben-Zvi, Dani. 2015. Scaffolding norms of argumentation-based inquiry in a primary mathematics classroom. *ZDM*, *47* 7: 1107-1120. DOI: 10 1007/s11858-015-0732-1

Miao, Zhenzhen, and David Reynolds. 2014. How China teaches children maths so well. http://theconversation.com/how-china-teaches-children-maths-so-well-32052

Organisation for Economic Co-operation and Development, (OECD). 2012. "Programme for international Student assessment (PISA) http://www.oecd.org/pisa/

Petrou, Marilena., and Maria Goulding. 2011. "Conceptualising teachers' mathematical knowledge in teaching". In *Mathematical knowledge in teaching,* edited by Tim Rowland and Kenneth Ruthven, 9-25. Dordrecht: Springer.

Rowland, Tim. 2015. "Powerful teaching". Keynote address presented to Conversations on Knowledge for Teaching Conference, Launceston, Tasmania. February 11-13, 2015. http://conversationsonkft.weebly.com/powerful-teaching.html

Rowland, Tim, Peter Huckstep, and Anne Thwaites. 2005. "Elementary teachers' mathematics subject knowledge: The Knowledge Quartet and the case of Naomi". *Journal of Mathematics Teacher Education* 8(3): 255–281. http://download.springer.com.dbgw.lis.curtin.edu.au/static/pdf/288/art%253A10.1007%252Fs10857-005-0853-5.pdf?originUrl=http%3A%2F%2Flink.springer.com%2Farticle%2F10.1007%2Fs10857-005-0853-5&token2=exp=1473129502~acl=%2Fstatic%2Fpdf%2F288%2Fart%25253A10.1007%25252Fs10857-005-0853-5.pdf%3ForiginUrl%3Dhttp%253A%252F%252Flink.springer.com%252Farticle%252F10.1007%252Fs10857-005-0853-5*~hmac=bb832ca37346e8f22bb8ebb9cc0e8ad2892e2badb35b4b4f72eec076a398e256

Rowland, Tim, Faye Turner, Anne Thwaites, and Peter Huckstep. 2009. *Developing primary mathematics teaching.* London: Sage.

Rowland, Tim, and Rina Zazkis. 2013. "Contingency in the mathematics classroom: Opportunities taken and opportunities missed". *Canadian Journal of Science, Mathematics and Technology Education* 13(2): 121–136. http://www.tandfonline.com.dbgw.lis.curtin.edu.au/doi/full/10.1080/14926156.2013.784825#.VVvumbGQ-70

Rowland, Tim, Anne Thwaites, and Libby Jared. 2015. "Triggers of contingency in mathematics teaching". *Research in Mathematics Education* 17(2), pp. 74-91

Ruthven, K. 2011. "Conceptualising mathematical knowledge in teaching". In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching*. (Springer, Dordrecht Heidelberg London New York).

Ryan, Julie, and Julian Williams. 1999. "Children's mathematical understanding as a work in progress: Learning from errors and misconceptions. In I Thompson

(Ed.). *Issues in Teaching Numeracy in Primary Schools. (2nd ed.)* Berkshire: Open University Press..

Shulman, Lee. 1986. "Those who understand: Knowledge growth in teaching". *Educational Researcher* 15(4): 4-14. DOI: 10.3102/0013189X015002004

Siemon, Dianne, John Bleckley, J. and Denise Neal, D. 2012. Working with the Big Ideas in Number and the Australian Curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen & D. Siemon, (Eds.). *Engaging the Australian National Curriculum: Mathematics – Perspectives from the Field.* Online Publication: Mathematics Education Research Group of Australasia pp. 19-45.

Swan, Malcolm. 2003. "Making sense of mathematics". In I Thompson (Ed.). *Enhancing Primary Mathematics Teaching*. Berkshire: Open University Press.

Wilkie, Karina J. 2014. "Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra". *Journal for Mathematics Teacher Education* 17*:* 397–428 DOI 10.1007/s10857-013-9251-6

Willis, Sue, and Kaye Treacy. 2004. *First steps in mathematics: Overview*. Port Melbourne: Harcourt Education.

Zazkis, Rina, and Ami Mamolo. 2011. "Reconceptualising knowledge at the mathematical horizon". *For the Learning of Mathematics* 31(2): 8-13.

Zhang, Qinqiong, and Max Stephens. 2013. "Utilising a construct of teacher capacity to examine national curriculum reform in mathematics". *Mathematics Education Research Journal* 25: (481-502)