School of Electrical Engineering, Computing and Mathematical Sciences

Linear Algebra Methods for the Control of Multidimensional Systems

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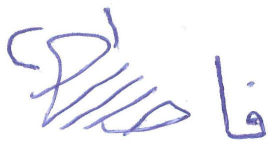
Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

Fatma H SH Mohamed



June 6, 2019

Abstract

The general purpose of this thesis is to develop a comprehensive theory of the geometric control for N-dimensional systems. In particular, the geometric approach will be extended from the two-dimensional setting to the general N-dimensional one. To this end, two possible representations of multidimensional systems will be considered: the Fornasini-Marchesini first order model and Fornasini-Marchesini second order model. For each model, the concept of structural invariance and the related subspaces, such as controlled invariant, output-nulling and self-bounded, as well as the respective dual notion of conditioned invariance and the input containing and self-hidden subspaces are studied and generalised to the N-dimensional case. Furthermore, the developed theory on output-nulling subspaces and stabilisation of controlled invariant subspaces will be used to address the disturbance decoupling problem and the model matching problem for N-dimensional Fornasini-Marchesini first and second order models. The obtained solutions will offer a methodology to address a variety of control and estimation problems in which the disturbance decoupling problem and the model matching one act as building blocks. Finally, with regards to the existence of solutions for linear shift-invariant two-dimensional Fornasini-Marchesini models, necessary and sufficient conditions will be provided. Importantly, it will be possible to check these conditions recursively by exploiting a suitable sequence of subspaces. After that, all the obtained results will be generalised to the N-dimensional case.

List of publications during PhD candidature

- Mohamed, F., Padula, F., Ntogramatzidis, L. (2017, September). Geometric conditions for the existence of solutions of singular multidimensional systems. In *Multidimensional (nD) Systems (nDS), 2017 10th International Workshop on* (pp. 1-6). IEEE.

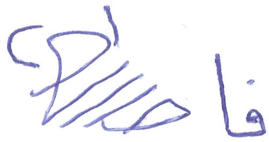
I acknowledge that my contribution in the above research (Geometric conditions for the existence of solutions of singular multidimensional systems) is the development of most of the proofs presented in this paper. More specifically, A/Prof Lorenzo Ntogramatzidis had the original idea of determining geometric conditions for the existence of solution of linear shift-invariant implicit first order Fornasini-Marchesini two-dimensional models over a finite region of the 2-D plane by incorporating the local states on a boundary region into a global state of finite extent. I applied this idea for the characterisation of the the admissible boundary conditions in a 2-D Fornasini-Marchesini singular model

$$E\,\mathbf{x}_{(i+1,j+1)} = A_1\,\mathbf{x}_{(i+1,j)} + A_2\,\mathbf{x}_{(i,j+1)} + B_1\,\mathbf{u}_{(i+1,j)} + B_2\,\mathbf{u}_{(i,j+1)},$$

and I have formulated two situations for the problem of the existence of solutions: in the case of arbitrary input functions, and in the case of assigned input functions. Dr. F. Padula and myself contributed to the problem formulation, by identifying the correct boundary conditions for this type of models and for the determination of a precise mathematical way to incorporate the local states on the boundaries into a single global state vector. The notational apparatus of Sections II and III, which in this work is crucial, was also developed by Dr. Padula and me. Additionally, I contributed to the development of the proofs of Theorem 2.1 and Theorem 2.2, under the guidance of my supervisor and co-supervisor. I also conducted an extensive literature review, and I was actively engaged in the writing of the abstract, introduction and concluding remarks.

The general idea is presented at the beginning of Chapter (5), and it is developed further in Section 5.1 (for 2-D systems) and in Section 5.2 (for general $N - D$ systems). Theorems 2.1 and 2.2 (and the relative proofs) are both in Section 5.1.

Fatma



Acknowledgements

Firstly, I would like to thank Allah who has granted me the patience to achieve this work. I would like to extend my thanks and gratitude to my late father, Hassan Shike. May God have mercy on him. My father had always been very encouraging and proud of me. How I wish for him to be able to see what I have achieved during this period. I would like also to thank all the staff members in the mathematics department of Curtin University, especially, my supervisor Lorenzo Ntogramatzidis and co-supervisor Fabrizio Padula, for their guidance, assistance and their patiences. I am also grateful to my family for their encouragement and support, especially my husband (Abdullah Mohamed), who was very helpful throughout this work. Finally, I would like to thank my children and friends for their constant support and also to thank everyone else who has helped me in one way or other.

Contents

Notation and Symbols

$\overset{\text{def}}{=}$	equal by definition
$\forall$	for all
$\exists$	there exists
$\in$	belonging to
$\subset$	contained in
$\subseteq$	contained in or equal to
$\supset$	containing
$\supseteq$	containing or equal to
$\cup$	union
$\cap$	intersection
$\setminus$	difference of sets with repetition count
$\times$	cartesian product
$\oplus$	direct sum
$\mathbb{N}$	the set of natural integers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}^-$	left half complex plane
$\mathbb{R}^n$	the set of n-ples of real numbers
$\mathbb{R}^{n\times m}$	the space of $n \times m$ real matrices
$\dim \mathcal{V}$	the dimension of subspace $\mathcal{V}$
$\mathcal{V}^\perp$	the orthogonal complement of subspace $\mathcal{V}$

I	the identity matrix
I_n	the $n \times n$ identity matrix
$0_{n\times m}$	the $n \times m$ null matrix
$A^\top$	the transpose of matrix A
A^{-1}	the inverse of A
$\mathrm{im}\ A$	the image of A
$\ker\ A$	the null-space of A
$\mathrm{tr}\, A$	the trace of A
$\mathrm{rank}\, A$	the rank of A
$\det\ A$	the determinant of A
$\mathrm{diag}\ A$	the block diagonal of A
$\mathrm{adj}\ A$	the adjoint of A
$\sigma(A)$	the spectrum of A

Given a subspace $\mathcal{Y}$ of $\mathbb{R}^n$, the symbol $M^{-1}\,\mathcal{Y}$ stands for the inverse image of $\mathcal{Y}$ with respect to the linear transformation M. The restriction of a mapping A to the A-invariant subspace $\mathcal{I}$ is written $A|_{\mathcal{I}}$; the eigenvalues of A restricted to $\mathcal{I}$ are denoted by $\sigma\,(A)|_{\mathcal{I}}$. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are A-invariant subspaces and $\mathcal{I}_1 \subseteq \mathcal{I}_2$, the mapping induced by A on the quotient space $\mathcal{I}_2/\mathcal{I}_1$ is denoted by $A|_{\mathcal{I}_2/\mathcal{I}_1}$.

CHAPTER 1

Introduction and Previous Work

Control theory mainly deals with dynamical systems with inputs and outputs. It is an interdisciplinary offshoot of applied mathematics, science and engineering. More precisely, control theory is concerned with the task of designing the inputs for such systems which cause the output to behave in a desired way, or according to some preassigned specifications. Typically, such input functions are expressed in a feedback form. A major goal of the control theory is to determine a control function that ensures the stability (or, in general, good performance) of the feedback system.

Multidimensional systems (N-D) are useful in control theory because they arise in the modelling of several systems: they are used in the discretisation of partial differential equations. They also arise in areas such as image processing and in all contexts where there is the need to model the system using more than one indeterminate (Marszalek, 1984, 1987; Kaczorek, 1985; F. Lewis, Marszalek, & Mertzios, 1990). Therefore, multidimensional system analysis has received extensive attention over the past few decades. However, there is still a plethora of modelling and control problems that have never been solved for multidimensional systems.

A first important difficulty in dealing with multidimensional systems is the presence of major differences between the algebraic structure of the multidimensional systems and the classical one-dimensional (1-D) systems. This is the reason behind the limited success obtained so far in the generalisation of well-established one-dimensional control and estimation techniques to an N-D setting. As mentioned above, N-D systems arise when modelling systems whose dynamics depend on spatial and temporal coordinates: the pixels in an image, the concentration of pollutants in a lake and the temperature of a metal rod are good examples of physical systems where it is natural to require the mathematical model to involve both spatial and temporal independent variables. Therefore, the mathematical description of a multidimensional system includes the use of several independent variables, one of which represents time (Bose, 1982). In addition, there have been other interesting contributions relating multi-dimensional systems theory with applications. For example, Fornasini (1991) discussed the approach based on Fornasini Marchesini model to river pollution modelling, Marszalek (1984) introduced the discretisation of PDE's which described the absorption of gas and water stream heating and Vomiero (1992) presented the diffusion process of a tracer into a blood vessel. Furthermore, the same result of the diffusion process of a tracer into a blood vessel is confermed by Valcher (1997). In a one-dimensional context, it is usually implicitly assumed that the independent variable is time. Moreover, there is only one first-order state space representation of a one-dimensional system. The state, in a one-dimensional system, is typically denoted by $\mathbf{x}$, and represents both the variable, which is updated in the equations of the model, and the so-called memory of the system. Therefore, to find the solution, we need to assign the vector $\mathbf{x}$ at a particular time instant.

Multidimensional systems are represented by different mathematical models which make them different from the classical one-dimensional system theory and they have been used for the investigation of various properties (Kurek, 1985). These include the Attasi model, introduced in Attasi (1976) and recently in Hinamoto and Fairman (1984), the classical Roesser model introduced in Roesser (1975) and two Fornasini-Marchesini models introduced in the two papers: Fornasini and Marchesini (1976) and Fornasini and Marchesini (1978). Further research has also been completed by Kung, Levy, Morf, and Kailath (1977), which discusses the methods to recast Fornasini-Marchesini models into Roesser models and vice-versa. The Roesser and Fornasini-Marchesini models are equivalent. Their drawback is the possibility of realising only two-dimensional (2-D) systems which have a quarter-plane causal structure. Considerable focus has been put on 2-D Fornasini-Marchesini and Roesser models as well as related definitions and properties of invariance because of their practical importance (Kaczorek, 1985). In these models, the concept of "state" needs to be generalised in a multidimensional framework: we need to distinguish between a "local state", which represents the variable which gets updated in the equations of the model, and a "global state", which represents the memory of the system, or, in other words, represents the variables that need to be assigned in the space of independent variables to iterate the equations of the model. The global state is infinite dimensional. The regions where the local state needs to be assigned to iterate the model are referred to as "separation sets".

When the only independent variable is time, it is natural to assume causality, because the concepts of past, present and future are well-defined. However, in the context of multidimensional systems, the presence of additional independent variables (which, as already mentioned, may be representative of spatial coordinates) makes the assumption of causality artificial. Indeed, there is no natural notion of past, present and future when spatial coordinates are involved.

An important area of control theory is the one referred to as "geometric control". This branch of control theory studies the properties of systems, their interactions and the techniques to control them in terms of subspaces of the state space. In the classic geometric control theory, controlled invariance is considered as a key concept. Controlled invariance for one-dimensional systems (1-D) was originally introduced by Basile and Marro (1969) and W. Wonham (1979). Loosely speaking, a controlled invariant subspace is a set of initial states for which it is possible to determine an input that keeps the whole state trajectory on that subspace. An alternative characterisation is to define a controlled invariant subspace both for continuous time systems $\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t)$ and discrete time systems $\mathbf{x}_{k+1} = A\,\mathbf{x}_k + B\,\mathbf{u}_k$, as the subspace $\mathcal{V}$ that satisfies the inclusion $A\,\mathcal{V} \subseteq \mathcal{V} + \text{im}\ B$ (Basile & Marro, 1969). A fundamental property of controlled invariance in the one-dimensional case is the fact that the input function that keeps the trajectory on $\mathcal{V}$ can always be expressed by a static state feedback $\mathbf{u} = F\,\mathbf{x}$. Different from the previous condition, this constitutes a closed-loop characterisation of controlled invariance, which has also often been used as the very definition of a controlled invariant subspace. Controlled invariance is a key ingredient in the solution of fundamental problems such as the disturbance decoupling, model matching and the non-interacting control problem (W. M. Wonham, 1974; Basile & Marro, 1992; Trentelman, Stoorvogel, & Hautus, 2001).
Another key notion is that of conditioned invariance. Conditioned invariant subspaces were also introduced in Basile and Marro (1969); they were defined by duality from controlled invariant subspaces and they found applications in problems such as unknown-input observation and fault detection.

However, when trying to adapt these notions to the multidimensional case (N-D), one immediately notices that the first issue is the absence of a single representation of a two-dimensional system (2-D) in terms of state variables, since, as aforementioned, the representation depends on the model that one considers. For this reason, there has been a significant stream of recent literature on the extension of controlled and conditioned invariance for 2-D systems, which originated in Conte and Perdon (1988) and Karamancioglu and Lewis (1990). Moreover, if we consider N-D case instead of the 2-D case these subspaces have not been defined.

The following dissertation will be primarily concerned with presenting suitable algebraic structures for 2-D systems that will be considered and generalised to the N-D case. These structures will enable the defining of state space models, controlled invariance and system theoretic properties for 2-D systems, such as the transfer function, which is considered the mathematical tool used for studying the input-output behaviour of a system. Contrary to 1-D system descriptions, various algebraic settings can be imposed for 2-D systems. Indeed, even if the two most frequently used state-space models for 2-D systems, namely the Fornasini-Marchesini and Roesser models, have been shown in Galkowski (1996) to be intrinsically equivalent, they are characterised by different types of boundary conditions.

The first model was proposed by Fornasini-Marchesini in Fornasini and Marchesini (1976). In this paper, the 2-D Fornasini-Marchesini update equation is defined as

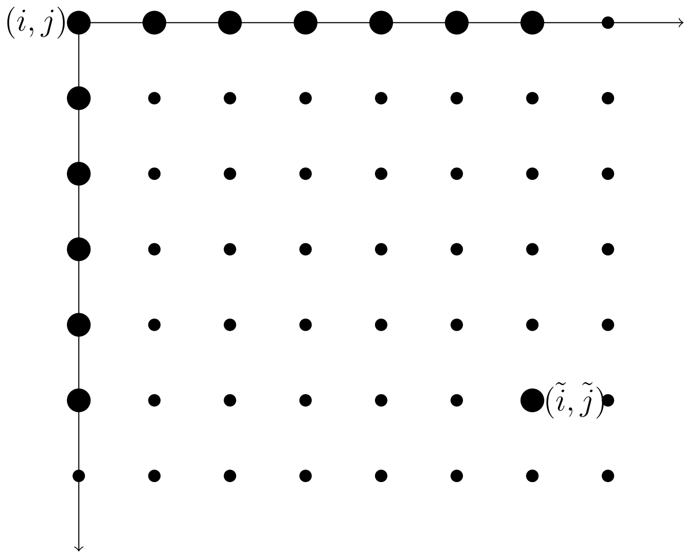


Figure 1.1

$$\begin{aligned} \mathbf{x}_{(i+1,j+1)} &= A_0\,\mathbf{x}_{(i,j)} + A_1\,\mathbf{x}_{(i+1,j)} + A_2\,\mathbf{x}_{(i,j+1)} + B\,\mathbf{u}_{(i,j)}, \\ \mathbf{y}_{(i,j)} &= C\,\mathbf{x}_{(i,j)}, \end{aligned} \tag{1.1}$$

where, for all admissible values of the integer indices i and j, the vector $\mathbf{x}_{(i,j)} \in \mathbb{R}^n$ denotes the local state and $\mathbf{u}_{(i,j)} \in \mathbb{R}^m$ is the control function. Here, $A_0, A_1, A_2 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times m}$. The mathematical model (1.1) is referred to as the 2-D Fornasini-Marchesini second-order state space model (or also first Fornasini-Marchesini model, because it was introduced first) (Fornasini & Marchesini, 1976). In addition, Fornasini and Marchesini (1980) set the situation when A_0 is identically zero.

For the considered model, it can be seen that the local states are defined as points in the plane. In the equation (1.1), the update state $\mathbf{x}_{(i+1,j+1)}$ depends on neighbours local states $\mathbf{x}_{(i,j)}, \mathbf{x}_{(i+1,j)}$ and $\mathbf{x}_{(i,j+1)}$. If an arbitrary point $(\tilde{i}, \tilde{j})$ is considered in the two-dimensional state space such that $(\tilde{i}, \tilde{j}) \geq (i, j)$, a local state $\mathbf{x}_{(\tilde{i},\tilde{j})}$, obtained by iterating backwards using the recursion (1.1), will not depend only on the state $\mathbf{x}_{(i,j)}$ but also on the local states $\mathbf{x}_{(k,0)}$, $k = 0, \ldots, \tilde{i}$ and $\mathbf{x}_{(0,\ell)}$, $\ell = 0, \ldots, \tilde{j}$ on the coordinate axis, as illustrated in Figure 1.1. These two lines of points provide the boundary conditions of the systems, which are referred to as a separation set.

Along with the boundary condition with assigning $\mathbf{x}_{(i,j)}$ for $(i, j) \in \mathfrak{Q}_0$ ($\mathfrak{Q}_0$ is the separation set), we present the sets

$$\mathfrak{Q}_i \stackrel{\text{def}}{=} (\{i\} \times \{j \in \mathbb{Z} \mid j \geq i\}) \cup (\{j \in \mathbb{Z} \mid j \geq i\} \times \{i\}). \tag{1.2}$$

Thus, as shown in Figure 1.2 a suitable set of boundary conditions for (1.1) is given for an arbitrary vector $\bar{\mathbf{x}}_{(i,j)}$ by

$$\mathbf{x}_{(i,j)} = \bar{\mathbf{x}}_{(i,j)} \qquad \text{for all}\,(i, j) \in \mathfrak{Q}_0, \tag{1.3}$$

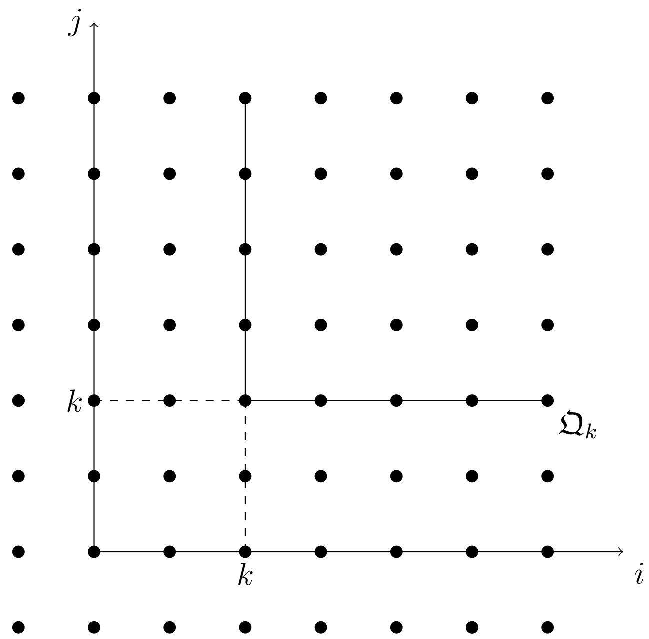


Figure 1.2

Ntogramatzidis (2012) was the first paper to extend and introduce a definition of controlled invariance for the 2-D second-order Fornasini-Marchesini model (1.1). This definition considers as a controlled invariant subspace $\mathcal{V}$ that satisfies the inclusion $A_i\,\mathcal{V} \subseteq \mathcal{V} + \mathrm{im}\ B,\ i \in \{0, 1, 2\}$. This definition is used also to solve the problems of output-nulling and disturbance decoupling, by leading to necessary and sufficient conditions (see Theorem 3.2 by Ntogramatzidis (2012)). From the structure of model (1.1), it is clear that the model is closed under the state feedback input $\mathbf{u}_{(i,j)} = F\,\mathbf{x}_{(i,j)}$. However, it is clear that this definition of controlled invariance alone does not automatically ensure the existence of a feedback matrix F, which keeps the state evolutions on the controlled invariant subspace $\mathcal{V}$ for the 2-D model (1.1) (Karamancioglu & Lewis, 1992). For this reason, the concept of 2-D controlled invariance of feedback type was introduced in Ntogramatzidis (2012). Here, a controlled invariant of feedback type is defined as a subspace $\mathcal{W}$ that fulfils the three conditions:

- $A_0\,\mathcal{W} \subseteq \mathcal{W} + \mathrm{im}\ B$;
- $A_1\,\mathcal{W} \subseteq \mathcal{W}$;
- $A_2\,\mathcal{W} \subseteq \mathcal{W}$.

2-D controlled invariant subspaces of feedback type are the subspaces where the trajectories of a 2-D system which are produced by static feedback controls lie.

This concept is instrumental in the solution of the classic disturbance decoupling problem without relying upon previously proposed conservative solutions (Ntogramatzidis, Cantoni, & Yang, 2008; Conte & Perdon, 1988). Additionally, Ntogramatzidis (2012) has addressed the problem of characterising the set of feedback inputs $\mathbf{u}_{(i,j)} = F\,\mathbf{x}_{(i,j)}$ which guarantee the existence of solutions of (1.1). the 2-D conditioned invariant subspace and 2-D conditioned invariant of output-injection type are the subspaces introduced as the dual concepts of controlled invariance and controlled invariance of feedback type, respectively, in a very natural way. Furthermore, most of the notions related to controlled and conditioned invariance, for example, self-boundedness, reachability and unobservability subspaces have been generalised to the 2-D counterpart (Ntogramatzidis, 2012).
A second model of 2-D Fornasini-Marchesini models in the classical form was given by Fornasini and Marchesini (1978):

$$
\begin{aligned}
\mathbf{x}_{(i,j)} &= A_1\,\mathbf{x}_{(i,j-1)} + A_2\,\mathbf{x}_{(i-1,j)} + B_1\,\mathbf{u}_{(i,j-1)} + B_2\,\mathbf{u}_{(i-1,j)}, \\
\mathbf{y}_{(i,j)} &= C\,\mathbf{x}_{(i,j)},
\end{aligned}
\tag{1.4}
$$

where, for all admissible values of the integer indices i and j, the vector $\mathbf{x}_{(i,j)} \in \mathbb{R}^n$ denotes the local state and $\mathbf{u}_{(i,j)} \in \mathbb{R}^m$ is the control function. Here, $A_1, A_2 \in \mathbb{R}^{n\times n}$, $B_1, B_2 \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. The model (1.4) is known in the literature as the 2-D Fornasini-Marchesini first-order state space models (or second Fornasini-Marchesini model) (Fornasini & Marchesini, 1978).
For this model, the separation sets, as shown in Figure 1.3 are defined in Ntogramatzidis et al. (2008) as

$$
\mathfrak{C}_k = \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \,|\, i+j=k\}, \qquad \text{for all } k \in \mathbb{Z}.
$$

Thus, boundary conditions for (1.4) is given for an arbitrary vector $\bar{\mathbf{x}}_{(i,j)}$ by a set:
$\mathbf{x}_{(i,j)} = \bar{\mathbf{x}}_{(i,j)} \in \mathbb{R}^n \qquad \text{for all}\,(i,j) \in \mathfrak{C}_{-1} \cup \mathfrak{C}_0.$

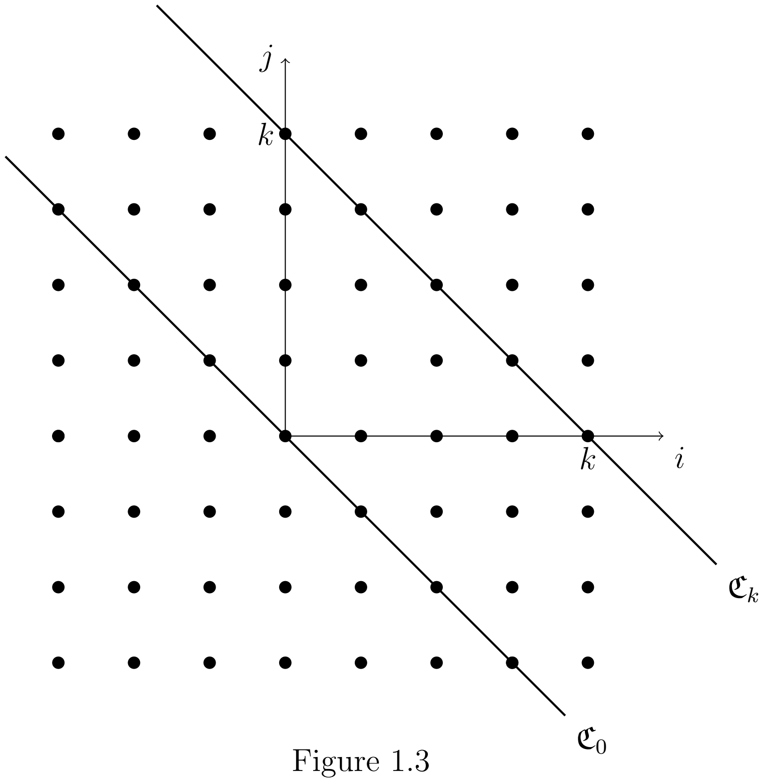


Figure 1.3

Despite the structural difference between the 2-D Fornasini-Marchesini first and second order models (1.1) and (1.4) respectively, these models are both capable of realising any strictly causal rational transfer function. Conte and Perdon (1988) provide a definition for controlled invariance for 2-D Fornasini Marchesini first-order models (1.4) as the subspace $\mathcal{V}$, such that there exists a control input that keeps the state trajectories on that subspace, if the boundary conditions are given on the same subspace. Equivalently, $\mathcal{V}$ satisfies this subspace inclusion $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \oplus \mathcal{V}) + \mathrm{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, and it is possible to express the control that keeps the states on $\mathcal{V}$ as a static feedback $\mathbf{u}_{(i,j)} = F\,\mathbf{x}_{(i,j)}$ (Conte & Perdon, 1988). Compared with the previous definition of controlled invariance, this definition can provide only conservative solutions of the control problems listed above (i.e., disturbance decoupling, model matching and estimation problems), in terms of sufficient conditions (Conte & Perdon, 1988). Moreover, these problems have been solved without stability constraints in Conte and Perdon (1988). However, in Ntogramatzidis et al. (2008), a new notion of stabilisability has been introduced based on a static feedback for this definition of invariant, in order to solve many of the control problems. In this case, the properties of the controlled invariant have been extended to the N-D case .

A further class of Fornasini-Marchesini models has been shown in Karamancioglu and Lewis (1992), and it is described by

$$\mathbf{x}_{(i,j)} = A_1\,\mathbf{x}_{(i,j-1)} + A_2\,\mathbf{x}_{(i-1,j)} + B\,\mathbf{u}_{(i,j)}. \tag{1.5}$$

With the subspace $\mathcal{V}$ of $\mathbb{R}^n$, the controlled invariant subspace is the loci of the state $\mathbf{x}_{(i,j)}$ that is controlled by $\mathbf{u}_{(i,j)}$. Geometrically, $\mathcal{V}$ is a controlled invariant subspace, if it satisfies the inclusion $A_1\mathcal{V} + A_2\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$. However, from the structure of the description model (1.5), the controlled invariant subspace $\mathcal{V}$ does not enjoy the feedback properties. There is another model type called the Roesser model. This model generalises the one-dimensional state space systems by introducing two-state sets instead of a one state set. These two states are a vertical and horizontal states that are propagated in the corresponding directions (Gopinath, Kar, & Bhatt, 2010). It is introduced by Roesser (1975) as

$$\begin{aligned}
\begin{bmatrix} \mathbf{x}^h_{(i+1,j)} \\ \mathbf{x}^v_{(i,j+1)} \end{bmatrix} &= \underbrace{\begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} \mathbf{x}^h_{(i,j)} \\ \mathbf{x}^v_{(i,j)} \end{bmatrix} + \underbrace{\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}}_{\hat{B}} \mathbf{u}_{(i,j)}, \\
\mathbf{y}_{(i,j)} &= \underbrace{\begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}}_{\hat{C}} \begin{bmatrix} \mathbf{x}^h_{(i+1,j)} \\ \mathbf{x}^v_{(i,j+1)} \end{bmatrix},
\end{aligned} \tag{1.6}$$

where $\mathbf{x}^h$ and $\mathbf{x}^v$ are respectively the horizontal and vertical states, so that the local state is the direct sum of theses two states (see also Kung et al. (1977)). It can be proven that the Fornasini-Marchesini models and Roesser model are equivalent by assuming the vector $\mathbf{x}_{(i,j)} = \begin{bmatrix} \mathbf{x}^h_{(i+1,j)} \\ \mathbf{x}^v_{(i,j+1)} \end{bmatrix}$ in (1.4) as a local state space, with

$$A_1 = \begin{bmatrix} 0 & 0 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}, A_2 = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ \hat{B}_2 \end{bmatrix}, B_2 = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}.$$

Moreover, Kurek (1985) presents another way of combining the two previous 2-D Fornasini-Marchesini first-order and second-order models:

$$\begin{aligned}
\mathbf{x}_{(i,j)} &= A_0\,\mathbf{x}_{(i-1,j-1)} + A_1\,\mathbf{x}_{(i,j-1)} + A_2\,\mathbf{x}_{(i-1,j)} + B_0\,\mathbf{x}_{(i-1,j-1)} + B_1\,\mathbf{u}_{(i,j-1)} + B_2\,\mathbf{u}_{(i-1,j)}, \\
\mathbf{y}_{(i,j)} &= C\,\mathbf{x}_{(i,j)} + D\,\mathbf{u}_{(i,j)}, .
\end{aligned}$$

This approach considers the first step of the generalisation of the 2-D Fornasini-Marchesini first and second order models to the N-D case. The extension of the 2-D Fornasini-Marchesini first-order and second-order models and Roesser model to the N-D models is apparent in other studies (Galkowski, 2001; Kaczorek, 1992; Matsushita, Saito, & Xu, 2013). Generalising the 2-D Fornasini-Marchesini first and second order models to the N-D case is not restrictive because as can be seen, the Roesser model can be recast into the mentioned models. Previous discussion involves models that are called "explicit" in the relevant literature: these are characterised in general by a recursive structure. A common way to obtain state space models which do not require a quarter-plane causal structure (non-recursive) for example, is extending the Roesser and Fornasini-Marchesini models into descriptor counterparts, by multiplying the update local state vector by a matrix which is not necessarily invertible or even nonsquare, to capture over or under-determined systems. In the 1-D case, descriptor systems have been studied widely in many monographs (Bernhard, 1982; Luenberger, 1977). Their two-dimensional models were introduced in Kaczorek (1988). With the interest that is growing in the singular systems, the N-D generalisation of the 2-D Fornasini-Marchesini first-order and second-order models and Roesser model have been carried out as follows (Kaczorek, 1992; F. L. Lewis, 1992; Matsushita et al., 2013; Kurek, 1989; Alpay & Dubi, 2003):

$$
\begin{aligned}
E\,\mathbf{x}_{i_1+1,i_2+1,\ldots,i_N+1} &= A_0\,\mathbf{x}_{i_1,i_2,\ldots,i_N} + \sum_{j=1}^{N} A_j\,\mathbf{x}_{i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_N} \\
&\quad + \sum_{1\leqslant j<k\leqslant N} A_{jk}\,\mathbf{x}_{i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_{k-1},i_k+1,i_{k+1},\ldots,i_N} + \ldots + \\
&\quad + \sum_{j=1}^{N} A_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_N}\,\mathbf{x}_{i_1+1,\ldots,i_{j-1}+1,i_j,i_{j+1}+1,\ldots,i_N+1} \\
&\quad + B_0\,\mathbf{u}_{i_1,i_2,\ldots,i_N} + \sum_{j=1}^{N} B_j\,\mathbf{u}_{i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_N} \\
&\quad + \sum_{1\leqslant j<k\leqslant N} A_{jk}\,\mathbf{x}_{i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_{k-1},i_k+1,i_{k+1},\ldots,i_N} + \ldots + \\
&\quad + \sum_{j=1}^{N} B_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_N}\,\mathbf{u}_{i_1+1,\ldots,i_{j-1}+1,i_j,i_{j+1}+1,\ldots,i_N+1} \\
\mathbf{y}_{i_1,i_2,\ldots,i_N} &= C\,\mathbf{x}_{i_1,i_2,\ldots,i_N},
\end{aligned}
\tag{1.7}
$$

which is considered as a generalisation of 2-D Fornasini-Marchesini second-order models (1.1). In this thesis, (1.1) has been adopted to extend the geometric approach of 2-D Fornasini-Marchesini second-order models to the N-D case (as presented in Ntogramatzidis (2012)). For the aims of this work, the general model (1.7) has been adopted. The reason for this choice is because this model can be used to capture the structure of the many different models that are considered in this thesis. For instance, when model (1.7) is used to deal with a multidimensional system that is not the descriptor, the matrix E can simply be considered to be the identity. Also, when we want the same model (1.7) to describe the N-D Fornasini-Marchesini second-order model to this end, the input is selected to appear only once. The descriptor N-D model that generalises the 2-D first-order Fornasini- Marchesini model (1.4) is

$$\begin{aligned} E\,\mathbf{x}_{(i_1,i_2,\dots,i_N)} = & A_1\,\mathbf{x}_{(i_1-1,i_2,\dots,i_N)} + \ldots + A_N\,\mathbf{x}_{(i_1,\dots,i_{N-1},i_N-1)} \\ & +B_1\,\mathbf{u}_{(i_1-1,i_2,\dots,i_N)} + \ldots + B_N\,\mathbf{u}_{(i_1,\dots,i_{N-1},i_N-1)}, \end{aligned} \tag{1.8}$$

(Alpay & Dubi, 2003; Matsushita et al., 2013). This model represents a true extension of the 2-D implicit model of Kaczorek (1988) because it can realise every possible purely dynamical multivariate transfer function with a positive quarter-plane causal structure. The geometric approach for this model has also been developed within this work with the matrix E to be the identity matrix. Among the contributions where a geometric approach has been developed for these models, we refer to Conte, Perdon, and Kaczorek (1991); Karamancioglu and Lewis (1990); Ntogramatzidis and Cantoni (2011) and the references therein. The difficulty related with the models of the descriptor form is the lack of a recursive structure (see F. L. Lewis (1992)). Other important issues arising for descriptor systems (one-dimensional and multidimensional) are the existence and uniqueness problems of the solutions. Obviously, existence problems depend on the size and shape of the boundary conditions of the dynamics. This thesis involves descriptor (first-order) Fornasini-Marchesini multidimensional models, with boundary conditions on the separation sets that were explained in Fornasini and Marchesini (1978) and generalised in Fornasini and Marchesini (1980). Articles generalising Fornasini-Marchesini models to an N-D case for $N > 2$ include Alpay and Dubi (2003) and Matsushita et al. (2013).

This thesis shows that existence conditions can be expressed on points (or even segments) of the local state on a certain separation set, in terms of suitable boundary conditions on a separation set that is assumed as the "initial" condition for the global state. In this way, a recursive structure can be obtained for these conditions, even if the structure of the model is not recursive. However, the sequence of subspaces increases in size as the local state progresses away from the separation set that is assumed to represent the origin for the global state.

Considering the previous discussion, three situations are addressed in this work. Firstly, a focus will be on the generalisation of what is normally referred in the literature as the 2-D second-order Fornasini-Marchesini model, when the matrices A_0, A_1, A_2 associated with the local states $\mathbf{x}_{(i,j)}$ and a single matrix B are related to the input function $\mathbf{u}_{(i,j)}$. This generalisation will be called the N-D second-order Fornasini-Marchesini model. After that, the situation is generalised with the model when the states and inputs appear equally, which is known as a 2-D first-order Fornasini-Marchesini model. Here, in this thesis, is called the N-D first-order Fornasini-Marchesini model. Finally, for the classical form of the model in Kaczorek (1988):

$$E\,\mathbf{x}_{(i,j)} = A_1\,\mathbf{x}_{(i,j-1)} + A_2\,\mathbf{x}_{(i-1,j)} + B_1\,\mathbf{u}_{(i,j-1)} + B_2\,\mathbf{u}_{(i-1,j)},$$

which is known as the implicit Fornasini-Marchesini first-order model, the problem of existence for the two situations (all possible input functions and given an appropriate control input) is addressed.

In the same vein, these results have extended to the general N-D setting (1.8). In this case, the complexity of the problem increases significantly, but a pattern can be recognised to express the way recursive existence conditions can be written for single points on separation sets, which are parallel to the one that we consider as the origin.

CHAPTER 2

Background material

The objective of this chapter is to briefly recall some of the foundations of the systems theory. We begin with the fundamental concepts of structural invariants in the one-dimensional (1-D) case for a linear transformation. For example, A-invariance, controlled invariance and all the related notions are presented with their properties and geometric approach. After that, the duality of these concepts is introduced. Finally, the structural invariants in the two-dimensional (2-D) case is established by defining the same concepts for the different models.

2.1 Structural invariants of (1-D) systems

The state update equation in continuous time is usually written in the form:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)), \tag{2.1}$$

and in discrete time, it is written as

$$\mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{u}(t)). \tag{2.2}$$

Also, the output equation for both is

$$\mathbf{y}\,(t) = g(\mathbf{x}(t), \mathbf{u}(t)), \tag{2.3}$$

where f and g are vectors whose components are $f_1, f_2, \ldots, f_n$ and $g_1, g_2, \ldots, g_p$. Moreover, $\mathcal{X}$ is a subset of $\mathbb{R}^n$, $\mathcal{U}$ is a subset of $\mathbb{R}^m$ and $\mathcal{Y}$ is a subset of $\mathbb{R}^p$, for suitable $n, m, p \in \mathbb{N}$. For all $t \in \mathbb{T}$ ($\mathbb{T}$ is $\mathbb{R}$ in the continuous time or $\mathbb{Z}$ in discrete time), the vector

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_n(t) \end{bmatrix}$$

is called the state vector, and the scalar functions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of time that solve (2.1) or (2.2) are called state variables. When t varies in $\mathbb{T}$, these variables describe a curve in $\mathcal{X}$ called the state trajectory. In the input vector

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}_1(t) \\ \mathbf{u}_2(t) \\ \vdots \\ \mathbf{u}_m(t) \end{bmatrix},$$

the scalar functions of time $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are the input functions, and the scalar functions $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p$ in the output vector

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{y}_1(t) \\ \mathbf{y}_2(t) \\ \vdots \\ \mathbf{y}_p(t) \end{bmatrix},$$

are the output functions. The sets $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are the state space, input space and output space, respectively. The equation (2.1) provides an explicit update. In fact, if the initial state $\mathbf{x}(0)$ is assigned and the input functions are given for all $t \geqslant 0$, then (2.1) delivers $\mathbf{x}(t)$ for all $t \geqslant 0$. In addition, if the functions $f_1, f_2, \ldots, f_n$ and $g_1, g_2, \ldots, g_p$ are linear, the dynamical systems (2.1), (2.2) and (2.3) are linear and if the same functions do not depend on time, then, the dynamical systems (2.1), (2.2) and (2.3) are time invariant, i.e., (2.1), (2.2) and (2.3) can be alternatively written in the matrix form as

$$\begin{cases} \dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) \\ \mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t) \end{cases}, \tag{2.4}$$

in continuous time, and

$$\begin{cases} \mathbf{x}(t+1) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) \\ \mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t) \end{cases}, \tag{2.5}$$

in discrete time. In both models (2.4) and (2.5), a matrix A is an $n \times n$ real-valued matrix, and is referred to as the system matrix, a matrix B is an $n \times m$ real-valued matrix, usually called the input matrix, a matrix C is a $p \times n$ real-valued matrix, and is often referred to as the output matrix and a matrix D is a $p \times m$ real-valued matrix, which is usually called the feedthrough matrix. If a system has $D = 0$, it is called strictly proper, purely dynamical or strictly causal. For the sake of brevity, systems (2.4) and (2.5) is identified by $\Sigma = (A, B, C, D)$. Models (2.4) and (2.5) can be written in compact form as

$$\begin{cases} \rho\,\mathbf{x}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) \\ \mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t) \end{cases}, \tag{2.6}$$

where the operator ρ denotes the time derivative in the continuous case, i.e., $\rho\,\mathbf{x}(t) = \dot{\mathbf{x}}(t)$, or to the unit time shift in the discrete case, i.e., $\rho\,\mathbf{x}(t) = \mathbf{x}(t+1)$.

2.1.1 A-invariance

A-invariant subspaces are loci of the trajectories generated by the autonomous dynamical system $\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t)$ in continuous time or $\mathbf{x}(t+1) = A\,\mathbf{x}(t)$ in discrete time. The following definition introduces A-invariance in terms of the well known geometric inclusion $A\,\mathcal{I} \subseteq \mathcal{I}$.

Definition 1. Let $\mathcal{I}$ be a subspace of $\mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$ be a matrix represent a linear map $\mathcal{A} : \mathcal{X} \longrightarrow \mathcal{X}$ with respect to a particular basis. $\mathcal{I}$ is an A-invariant if for all $\mathbf{x} \in \mathcal{I}$

$$A\,\mathbf{x} \in \mathcal{I} \qquad \textit{or} \qquad A\,\mathcal{I} \subseteq \mathcal{I} \tag{2.7}$$

Remark 2.1. The following are satisfied with an A-invariant $\mathcal{I}$:

- A subspace $\mathcal{I}$ with a basis matrix J is A-invariant, if and only if

$$\operatorname{im}(A\,J) \subseteq \operatorname{im} J. \tag{2.8}$$

- A subspace $\mathcal{I}$ with $\ker Q = \mathcal{I}$ is A-invariant, if and only if

$$\ker Q \subseteq \ker(Q\,A). \tag{2.9}$$

- It is clear that the origin $\{0\}$ and the state space $\mathcal{X}$ are both A-invariant, since $A\,\{0\} = \{0\} \subseteq \{0\}$ and $A\,\mathcal{X} \subseteq \mathcal{X}$.
- Consider two A-invariant subspaces $\mathcal{I}_1$ and $\mathcal{I}_2$. Their addition and intersection are A-invariant. Instead, by virtue of (2.7), we obtain (Basile & Marro, 1969)

$$A\,(\mathcal{I}_1 + \mathcal{I}_2) = A\,\mathcal{I}_1 + A\,\mathcal{I}_2 \subseteq \mathcal{I}_1 + \mathcal{I}_2,$$

  and by virtue of equation (2.7)

$$A\,(\mathcal{I}_1 \cap \mathcal{I}_2) \subseteq A\,\mathcal{I}_1 \cap A\,\mathcal{I}_2 \subseteq \mathcal{I}_1 \cap \mathcal{I}_2.$$

- The set of all A-invariant subspaces of $\mathcal{X}$ is denoted by $\mathfrak{G}_A(\mathcal{X})$. Figure (2.1) shows that this set is closed under subspace addition and intersection.

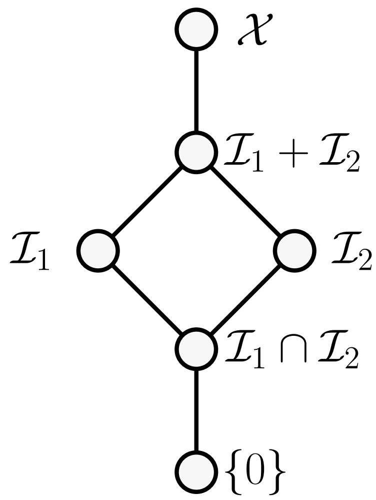


Figure 2.1: Lattice ($\mathfrak{G}_A(\mathcal{X})$).

Theorem 2.1. *A subspace $\mathcal{I}$ with a dimension r and basis matrix V is A-invariant, if and only if a matrix X exists such that*

$$A\,V = V\,X. \tag{2.10}$$

Proof: Let $\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_r \end{bmatrix}$ and $\begin{bmatrix} X_1 & X_2 & \dots & X_r \end{bmatrix}$ be the columns of V and X, respectively. Then, (2.10) is equivalent to

$$\begin{aligned} A\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_r \end{bmatrix} &= \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_r \end{bmatrix}\begin{bmatrix} X_1 & X_2 & \dots & X_r \end{bmatrix} \\ &= \begin{bmatrix} [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \dots\ \boldsymbol{v}_r]\,X_1 & [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \dots\ \boldsymbol{v}_r]\,X_2 & \dots & [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \dots\ \boldsymbol{v}_r]\,X_r \end{bmatrix}. \end{aligned}$$

This relation says that A transforms a basis vector of $\mathcal{I}$ into a vector of $\mathcal{I}$ as

$$A\,\boldsymbol{v}_i = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_r \end{bmatrix} X_i, \quad \forall i \in \{1,\dots,r\}, \tag{2.11}$$

which is equivalent to saying that $\mathcal{I}$ is A-invariant.

■

A Changes of basis with invariants

A change of basis is presented in Appendix A. Here, the relation between invariants and change of basis is investigated.
Let $\mathcal{I}$ with basis matrix V be a subspace of $\mathcal{X}$ and let J be another basis matrix for $\mathcal{I}$. Then, there exists an $n \times n$ real valued non-singular matrix T, such that $J = T^{-1}\,V$. Clearly, from (2.10) we derive

$$T^{-1}\,A\,T(T^{-1}\,V) = (T^{-1}\,V)\,X,$$

which is equivalent to

$$A'\,J = J\,X. \tag{2.12}$$

Theorem 2.2. *[Basile and Marro (1969)] Let $\mathcal{I}$ with dimension r be an A-invariant subspace. Then, there exists an $n \times n$ non-singular matrix T, such that the columns of $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ form a basis of $\mathcal{X}$, with $\mathcal{I} = \operatorname{im}(T_1)$, such that*

$$A' = T^{-1}\,A\,T = \begin{bmatrix} A'_{1,1} & A'_{1,2} \\ 0 & A'_{2,2} \end{bmatrix}, \tag{2.13}$$

where $A'_{1,1}$ of dimension $r \times r$ is the restriction of A to $\mathcal{I}$ in the new basis and $A'_{2,2}$ is an $(n-r) \times (n-r)$ matrix.

There is a correlation between the concept of invariant with the trajectories that are solutions to the differential and difference equations

$$\rho\mathbf{x}(t) = A\,\mathbf{x}(t), \tag{2.14}$$

that can be explained as follows.

Theorem 2.3. *Let $\mathcal{I}$ be a subspace of $\mathcal{X}$. The trajectories that solve system (2.14) are contained in $\mathcal{I}$, if and only if $\mathcal{I}$ is an A-invariant subspace.*

Proof: (**If**). Let $\mathcal{I}$ be an A-invariant. Consider $\mathbf{x}_0 \in \mathcal{I}$ and the change of coordinate matrix $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ such that $\operatorname{im} T_1 = \mathcal{I}$. The components of $\mathbf{x}$, with respect to the new basis are given by $T^{-1}\,\mathbf{x}$. We can write $T^{-1}\,\mathbf{x}_0$ as

$$\mathbf{x}'_0 \stackrel{\text{def}}{=} T^{-1}\,\mathbf{x}_0 = \begin{bmatrix} \mathbf{x}'_1(0) \\ 0 \end{bmatrix},$$

for some vector $\mathbf{x}'_1(0) \in \mathbb{R}^r$. Then, by defining

$$\begin{bmatrix} \mathbf{x}'_1(t) \\ \mathbf{x}'_2(t) \end{bmatrix} \stackrel{\text{def}}{=} T^{-1}\,\mathbf{x}(t),$$

the system (2.14) can be written as

$$T^{-1}\,\rho\mathbf{x}(t) = T^{-1}\,A\,\mathbf{x}(t),$$

which is equivalent to

$$T^{-1}\,\rho\mathbf{x}(t) = (T^{-1}\,A\,T)\,T^{-1}\,\mathbf{x}(t).$$

By Theorem 2.2, this gives

$$\begin{bmatrix} \rho\mathbf{x}'_1(t) \\ \rho\mathbf{x}'_2(t) \end{bmatrix} = \begin{bmatrix} A'_{1,1} & A'_{1,2} \\ 0 & A'_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{x}'_1(t) \\ \mathbf{x}'_2(t) \end{bmatrix},$$

in both continuous and discrete time. This leads to:

$$\begin{cases} \rho\mathbf{x}'_1(t) = A'_{1,1}\,\mathbf{x}'_1(t) + A'_{1,2}\,\mathbf{x}'_2(t), & \mathbf{x}'_1(0) = \mathbf{x}'_{1,0} \\ \rho\mathbf{x}'_2(t) = \qquad\qquad\quad A'_{2,2}\,\mathbf{x}'_2(t), & \mathbf{x}'_2(0) = 0 \end{cases}$$

in the continuous and discrete cases. In both cases, the solution of (2.14) is in the form $\mathbf{x}'(t) = \begin{bmatrix} \mathbf{x}'_1(t) \\ 0 \end{bmatrix}$ because for $\mathbf{x}'_2(t)$ and $\mathbf{x}'_2(t+1)$, the equations admits only the zero solution, which means that $\mathbf{x}'(t) \in \mathcal{I}$ for all $t \in \mathbb{T}$, as the external components of $\mathbf{x}'(t)$ with respect to $\mathcal{I}$, are identically zero.

(**Only if**). Assume that the trajectory is in $\mathcal{I}$. We prove that $\mathcal{I}$ is A-invariant. By contradiction, we suppose that $\mathcal{I}$ is not an A-invariant, so that there exists $\tilde{\mathbf{x}} \in \mathcal{I}$ such that $A\,\tilde{\mathbf{x}} \notin \mathcal{I}$. Then, choosing $\mathbf{x}_0 = \tilde{\mathbf{x}}$ in (2.14) yields $\dot{\mathbf{x}}(0) \notin \mathcal{I}$ in continuous time and $\mathbf{x}(1) \notin \mathcal{I}$ in the discrete case. In both cases, the solution of (2.14) is not contained in $\mathcal{I}$, thus contradicting the assumption. ■

From the previous result, in the absence of a control action (i.e., when the input function $\mathbf{u}$ is identically zero), all the trajectories remain on a subspace of the state space $\mathcal{X}$, if and only if it is A-invariant.

B A-invariant subspaces containing a subspace

Let $A \in \mathbb{R}^{n\times n}$ be a matrix that represents a linear map with respect to a certain basis in $\mathcal{X}$. Let $\mathcal{M}$ be a linear subspace of $\mathcal{X} = \mathbb{R}^n$. The set of all A-invariant subspaces containing the subspace $\mathcal{M}$ is denoted by

$$\mathfrak{G}_{A,\subseteq}(\mathcal{M}) \stackrel{\text{def}}{=} \{\mathcal{I} \text{ subspace of } \mathcal{X} \mid A\mathcal{I} \subseteq \mathcal{I} \text{ and } \mathcal{I} \supseteq \mathcal{M}\},$$

which is called the Grassmannian $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$, as mentioned in the Appendix. We notice that:

- If $\mathcal{M} = \mathcal{X}$, then $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$ only contains the element $\mathcal{X}$, while if $\mathcal{M} = \{0\}$, the set $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$ represents the whole set of A-invariant subspaces.

- Consider $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{G}_{A,\subseteq}(\mathcal{M})$. Their addition is in $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$, i.e., $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$ is closed under subspace addition. This is because we have proved that the addition of two A-invariant subspaces is A-invariant. From the fact that $\mathcal{I}_1$ and $\mathcal{I}_2$ both contain $\mathcal{M}$, their addition also contains $\mathcal{M}$. Moreover, the smallest subspace of $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$ that contains the subspaces $\mathcal{I}_1$ and $\mathcal{I}_2$ is the addition of them $\mathcal{I}_1 + \mathcal{I}_2$.

- Given $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{G}_{A,\subseteq}(\mathcal{M})$, then $\mathcal{I}_1 \cap \mathcal{I}_2 \in \mathfrak{G}_{A,\subseteq}(\mathcal{M})$, i.e., $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$ is closed under

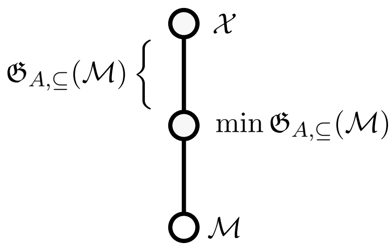


Figure 2.2: Lattice $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$.

subspace intersection. Indeed, we have already shown that the intersection of two A-invariant subspaces is A-invariant. In addition, if $\mathcal{I}_1$ and $\mathcal{I}_2$ both contain $\mathcal{M}$, then their intersection obviously contains $\mathcal{M}$, and the intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ is the largest element of $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$ contained in both $\mathcal{I}_1$ and $\mathcal{I}_2$.

- The set $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$ has a maximum and a minimum. The maximum is $\mathcal{X}$ and the minimum is the intersection of all the elements of $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$, that will be denoted with the symbol $\langle A \,|\, \mathcal{M} \rangle$. Then,

$$\langle A \,|\, \mathcal{M} \rangle = \bigcap_{\mathcal{I} \in \mathfrak{G}_{A,\subseteq}(\mathcal{M})} \mathcal{I}. \tag{2.15}$$

  It seems clear that the result of the latter is A-invariant and contains $\mathcal{M}$, because as mentioned above, $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$ is closed under intersection and as such, the intersection of elements which all contain $\mathcal{M}$ still contains $\mathcal{M}$.

Thus, the smallest among all elements of $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$ is defined by the subspace $\mathcal{I}_* \stackrel{\text{def}}{=} \langle A \,|\, \mathcal{M} \rangle$, which satisfies the following three properties:

(i) $A\,\mathcal{I}_* \subseteq \mathcal{I}_*$;

(ii) $\mathcal{I}_* \supseteq \mathcal{M}$;

(iii) for every A-invariant subspace $\mathcal{I}$ containing $\mathcal{M}$, there holds $\mathcal{I}_* \subseteq \mathcal{I}$.

Theorem 2.4. *The subspace $\mathcal{I}_* = \langle A \,|\, \mathcal{M} \rangle$ coincides with the last term of the sequence:*

$$\begin{cases} \mathcal{Z}_0 = \mathcal{M} \\ \mathcal{Z}_i = \mathcal{M} + A\,\mathcal{Z}_{i-1} \quad i \in \{1, \ldots, k\}, \end{cases} \tag{2.16}$$

where the value of $k \leq n-1$ is determined by condition $\mathcal{Z}_{k+1} = \mathcal{Z}_k$.

Proof: First, note that $\mathcal{Z}_i \supseteq \mathcal{Z}_{i-1}$ for all $i \in \{1, 2, \ldots, k\}$. In fact, instead of (2.16), consider the recursion expression:

$$\begin{cases} \mathcal{Z}'_0 = \mathcal{M} \\ \mathcal{Z}'_i = \mathcal{Z}'_{i-1} + A\,\mathcal{Z}'_{i-1} \quad i \in \{1, \ldots, k\}, \end{cases}$$

which defines a sequence of subspaces such that $\mathcal{Z}'_i \supseteq \mathcal{Z}'_{i-1}$ for all $i \in \{1, 2, \ldots, k\}$. We want to show that this sequence is equal to (2.16), i.e., that $\mathcal{Z}_i = \mathcal{Z}'_i$ for every $i \in \mathbb{N}$. We proceed by induction. Clearly, $\mathcal{Z}_0 = \mathcal{Z}'_0$. Let us assume that $\mathcal{Z}'_j = \mathcal{Z}_j$, for all $j \in \{1, 2, \ldots, i-1\}$, and we prove that $\mathcal{Z}'_i = \mathcal{Z}_i$. There holds:

$$
\begin{aligned}
\mathcal{Z}'_i &= \mathcal{Z}'_{i-1} + A\,\mathcal{Z}'_{i-1} \\
&= \mathcal{Z}_{i-1} + A\,\mathcal{Z}_{i-1} \\
&= \mathcal{M} + A\mathcal{Z}_{i-2} + A\mathcal{Z}_{i-1} = \mathcal{Z}_i,
\end{aligned}
$$

since $A\mathcal{Z}'_{i-2} \subseteq A\mathcal{Z}'_{i-1}$ implies $A\mathcal{Z}_{i-2} \subseteq A\mathcal{Z}_{i-1}$, in view of the inductive assumption. If $\mathcal{Z}_{k+1} = \mathcal{Z}_k$, also $\mathcal{Z}_j = \mathcal{Z}_k$ for all $j > k+1$ and $\mathcal{Z}_k$ is an A-invariant subspace containing $\mathcal{M}$. In fact, in such a case, $\mathcal{Z}_k = \mathcal{M} + A\,\mathcal{Z}_k$. Therefore, $\mathcal{M} \subseteq \mathcal{Z}_k$, $A\mathcal{Z}_k \subseteq \mathcal{Z}_k$. Since two subspaces that are subsequent in sequence (2.16) are coincident if and only if they have the same dimensions and the dimension of the first subspace is at least one, an A-invariant subspace is obtained in at most $n-1$ steps. We now show that the last subspace of the sequence is the minimal A-invariant subspace containing $\mathcal{M}$. Let $\mathcal{J}$ be another A-invariant containing $\mathcal{M}$. We prove, in particular, that every subspace of sequence (2.16) is contained in $\mathcal{J}$. Clearly, $\mathcal{Z}_0 = \mathcal{M} \subseteq \mathcal{J}$. Assume $\mathcal{J} \supseteq \mathcal{Z}_{i-1}$. We prove that $\mathcal{J} \supseteq \mathcal{Z}_i$. From $A\,\mathcal{J} \subseteq \mathcal{J}$ and $\mathcal{J} \supseteq \mathcal{M}$, we find $\mathcal{J} \supseteq \mathcal{M} + A\mathcal{J} \supseteq \mathcal{M} + A\mathcal{Z}_{i-1} = \mathcal{Z}_i$. ■

Sequence (2.16) delivers an A-invariant subspace, which is the minimum of $\mathfrak{G}_{A,\subseteq}(\mathcal{M})$.

C A-invariant subspaces contained in a subspace

Given $A \in \mathbb{R}^{n\times n}$ and a linear subspace $\mathcal{N}$ of $\mathcal{X} = \mathbb{R}^n$, the set of all A-invariant subspaces contained in the subspace $\mathcal{N}$ is denoted by

$$
\mathfrak{G}_{A,\supseteq}(\mathcal{N}) \overset{\text{def}}{=} \{\mathcal{I} \text{ subspace of } \mathcal{X} \;\mid\; A\mathcal{I} \subseteq \mathcal{I} \text{ and } \mathcal{I} \subseteq \mathcal{N}\}.
$$

This characterisation will play a fundamental role within the important context of observability and constructibility. Some of its properties are as follows:

- If $\mathcal{N} = \{0\}$, the set $\mathfrak{G}_{A,\supseteq}(\mathcal{N})$ only contains the element $\{0\}$, whereas if $\mathcal{N} = \mathcal{X}$, then $\mathfrak{G}_{A,\supseteq}(\mathcal{N})$, coincides with the set of A-invariant subspaces of $\mathcal{X}$.

- It is closed under subspace addition, i.e., given $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{G}_{A,\supseteq}(\mathcal{N})$, then $\mathcal{I}_1 + \mathcal{I}_2 \in \mathfrak{G}_{A,\supseteq}(\mathcal{N})$, since the addition of two A-invariant subspaces is A-invariant, and since $\mathcal{I}_1 \subseteq \mathcal{N}$ and $\mathcal{I}_2 \subseteq \mathcal{N}$ implies $(\mathcal{I}_1 + \mathcal{I}_2) \subseteq \mathcal{N}$. The addition $\mathcal{I}_1 + \mathcal{I}_2$ is the smallest element of $\mathfrak{G}_{A,\supseteq}(\mathcal{N})$, containing both $\mathcal{I}_1$ and $\mathcal{I}_2$.

- It is closed under subspace intersection, i.e., given $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{G}_{A,\supseteq}(\mathcal{N})$, then $\mathcal{I}_1 \cap \mathcal{I}_2 \in \mathfrak{G}_{A,\supseteq}(\mathcal{N})$, since the intersection of two A-invariant subspaces is A-invariant and $\mathcal{I}_1 \subseteq \mathcal{N}$ and $\mathcal{I}_2 \subseteq \mathcal{N}$ imply $(\mathcal{I}_1 \cap \mathcal{I}_2) \subseteq \mathcal{N}$. Moreover, the intersection $\mathcal{I}_1 \cap \mathcal{I}_2$

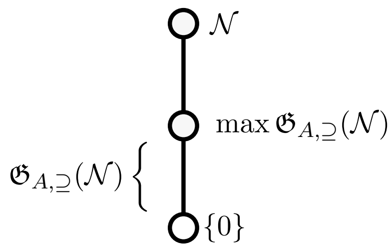


Figure 2.3: Lattice $\mathfrak{G}_{A,\supseteq}(\mathcal{N})$.

is the largest element of $\mathfrak{G}_{A,\supseteq}(\mathcal{N})$, contained in both $\mathcal{I}_1$ and $\mathcal{I}_2$.

- $\mathfrak{G}_{A,\supseteq}(\mathcal{N})$ has a maximum and minimum. As illistrated, the minimum is the origin, while the maximum is the addition of all the elements of $\mathfrak{G}_{A,\supseteq}(\mathcal{N})$, that will be denoted by the symbol $\langle \mathcal{N} \,|\, A \rangle$. Then,

$$\langle \mathcal{N} \,|\, A \rangle = \sum_{\mathcal{I} \in \mathfrak{G}_{A,\supseteq}(\mathcal{N})} \mathcal{I}. \tag{2.17}$$

Indeed, the sum on the right hand-side of equation (2.17) is clearly A-invariant and is contained in $\mathcal{N}$ because $\mathfrak{G}_{A,\supseteq}(\mathcal{N})$ is closed under addition and because the addition of subspaces which are all contained in the subspace $\mathcal{N}$ is still contained in the subspace $\mathcal{N}$. Thus, the largest among all elements of $\mathfrak{G}_{A,\supseteq}(\mathcal{N})$ is defined by the subspace $\mathcal{I}^* \stackrel{\text{def}}{=} \langle \mathcal{N} \,|\, A \rangle$ that satisfies the following three properties:

(i) $A\mathcal{I}^* \subseteq \mathcal{I}^*$;

(ii) $\mathcal{I}^* \subseteq \mathcal{N}$;

(iii) for every A-invariant subspace $\mathcal{I}$ contained in $\mathcal{N}$, there holds $\mathcal{I}^* \supseteq \mathcal{I}$.

Lemma 2.1. *The subspace $\mathcal{Z}$ is an A-invariant subspace containing $\mathcal{M}$, if and only if $\mathcal{Z}^\perp$ is an $A^\top$-invariant subspace contained in $\mathcal{M}^\perp$, i.e.,*

$$\mathcal{Z} \in \mathfrak{G}_{A,\subseteq}(\mathcal{M}) \quad \Leftrightarrow \quad \mathcal{Z}^\perp \in \mathfrak{G}_{A^\top,\supseteq}(\mathcal{M}^\perp).$$

Proof: There holds $\mathcal{Z} \in \mathfrak{G}_{A,\subseteq}(\mathcal{M})$, if and only if: **(i)** $A\,\mathcal{Z} \subseteq \mathcal{Z}$ and **(ii)** $\mathcal{Z} \subseteq \mathcal{M}$. On the other hand, **(i)** is equivalent to the inclusion $A^\top\,\mathcal{Z}^\perp \subseteq \mathcal{Z}^\perp$, while **(ii)** is equivalent to the inclusion $\mathcal{Z}^\perp \supseteq \mathcal{M}^\perp$. Therefore **(i)** and **(ii)** are equivalent to the fact that $\mathcal{Z}^\perp \in \mathfrak{G}_{A^\top,\supseteq}(\mathcal{M}^\perp)$. ■

Theorem 2.5. *Let $\mathcal{S}$ be a subspace of $\mathcal{X}$. The orthogonal complement of the smallest A-invariant subspace containing $\mathcal{S}$ is the largest $A^\top$-invariant subspace contained in $\mathcal{S}^\perp$, i.e., there holds:*

$$\langle A \,|\, \mathcal{S} \rangle^\perp = \langle \mathcal{S}^\perp \,|\, A^\top \rangle,$$

or, equivalently,

$$(\min \mathfrak{G}_{A,\subseteq}(\mathcal{S}))^{\perp} = \max \mathfrak{G}_{A^\top,\supseteq}(\mathcal{S}^{\perp}).$$

Proof: Let us denote by $\mathcal{I}_*$ the subspace $\langle A \,|\, \mathcal{S}\rangle$. Thus, $\mathcal{I}_*$ satisfies:

1a) $A\,\mathcal{I}_* \subseteq \mathcal{I}_*$;

2a) $\mathcal{I}_* \supseteq \mathcal{S}$;

3a) for any A-invariant subspace $\mathcal{I}$ containing $\mathcal{S}$, we have $\mathcal{I}_* \subseteq \mathcal{I}$, i.e., $\forall\, \mathcal{I} \in \mathfrak{G}_{A,\subseteq}(\mathcal{S})$, we have $\mathcal{I}_* \subseteq \mathcal{I}$.

Equivalently, $\mathcal{I}_*$ satisfies:

1b) $A^\top \mathcal{I}_*^{\perp} \subseteq \mathcal{I}_*^{\perp}$;

2b) $\mathcal{I}_*^{\perp} \subseteq \mathcal{S}^{\perp}$;

3b) $\forall\, \mathcal{I} \in \mathfrak{G}_{A^\top,\supseteq}(\mathcal{S}^{\perp})$, we have $\mathcal{I}_*^{\perp} \supseteq \mathcal{I}^{\perp}$.

In view of Lemma 2.1, we can write these conditions as:

1c) $A^\top \mathcal{I}_*^{\perp} \subseteq \mathcal{I}_*^{\perp}$;

2c) $\mathcal{I}_*^{\perp} \subseteq \mathcal{S}^{\perp}$;

3c) $\forall\, \mathcal{G} \in \mathfrak{G}_{A^\top,\supseteq}(\mathcal{S}^{\perp})$, we have $\mathcal{I}_*^{\perp} \supseteq \mathcal{G}$.

In other words, $\mathcal{I}_*^{\perp}$ is $A^\top$-invariant and it is contained in $\mathcal{S}^{\perp}$. Moreover, it is greater than any other $A^\top$-invariant subspace contained in $\mathcal{S}^{\perp}$. Hence, $\mathcal{I}_*^{\perp} = \max \mathfrak{G}_{A^\top,\supseteq}(\mathcal{S}^{\perp})$. ■

Theorem 2.6. *The subspace $\mathcal{I}^* = \langle \mathcal{N} \,|\, A\rangle$ coincides with the last term of the sequence:*

$$\begin{cases} \mathcal{Z}_0 = \mathcal{N} \\ \mathcal{Z}_i = \mathcal{N} \cap A^{-1}\,\mathcal{Z}_{i-1} \quad i \in \{1,\dots,k\}, \end{cases} \tag{2.18}$$

where the value of $k \le n-1$ is determined by condition $\mathcal{Z}_{k+1} = \mathcal{Z}_k$.

Proof: First, note that $\mathcal{Z}_i \supseteq \mathcal{Z}_{i-1}$ for all $i \in \{1,2,\dots,k\}$. In fact, instead of (2.18), consider the recursion:

$$\begin{cases} \mathcal{Z}_0' = \mathcal{N} \\ \mathcal{Z}_i' = \mathcal{Z}_{i-1}' \cap A^{-1}\,\mathcal{Z}_{i-1}' \quad i \in \{1,\dots,k\}, \end{cases}$$

which defines a sequence of subspaces such that $\mathcal{Z}_i' \subseteq \mathcal{Z}_{i-1}'$ for all $i \in \{1,2,\dots,k\}$. We prove by induction that this sequence is coincident to (2.18). Clearly, $\mathcal{Z}_0 = \mathcal{Z}_0'$; assume that $\mathcal{Z}'_j = \mathcal{Z}_j$ for all $j \in \{1, 2, \ldots, i-1\}$ and let us prove that $\mathcal{Z}'_i = \mathcal{Z}_i$. There holds:

$$\begin{aligned}\mathcal{Z}'_i &= \mathcal{Z}'_{i-1} \cap A^{-1}\,\mathcal{Z}'_{i-1} \\ &= \mathcal{Z}_{i-1} \cap A^{-1}\,\mathcal{Z}_{i-1} \\ &= \mathcal{N} \cap A^{-1}\,\mathcal{Z}_{i-2} \cap A^{-1}\,\mathcal{Z}_{i-1}.\end{aligned} \tag{2.19}$$

In view of the inductive assumption, $\mathcal{Z}'_{i-1} \subseteq \mathcal{Z}'_{i-2}$ implies that $\mathcal{Z}_{i-1} \subseteq \mathcal{Z}_{i-2}$, which in turn gives $A^{-1}\,\mathcal{Z}_{i-1} \subseteq A^{-1}\,\mathcal{Z}_{i-2}$, then, (2.19) becomes $\mathcal{Z}'_i = \mathcal{N} \cap A^{-1}\,\mathcal{Z}_{i-1} = \mathcal{Z}_i$.
If $\mathcal{Z}_{k+1} = \mathcal{Z}_k$, also $\mathcal{Z}_j = \mathcal{Z}_k$ for all $j > k+1$ and $\mathcal{Z}_k$ is an A-invariant subspace contained in $\mathcal{N}$. In fact, in such a case, $\mathcal{Z}_k = \mathcal{N} \cap A^{-1}\,\mathcal{Z}_k$. Therefore, $\mathcal{Z}_k \subseteq \mathcal{N}$ and $\mathcal{Z}_k \subseteq A^{-1}\mathcal{Z}_k$, is equivalent to $A\,\mathcal{Z}_k \subseteq \mathcal{Z}_k$. Since two subspaces that are subsequent in sequence (2.18) are coincident, if and only if they have the same dimensions and the dimension of the first subspace is at least one, then, an A-invariant subspace is obtained in at most $n-1$ steps. We now show that the last subspace of the sequence is the maximal A-invariant subspace contained in $\mathcal{N}$. Let $\mathcal{J}$ be another A-invariant subspace contained in $\mathcal{N}$, so that $A\,\mathcal{J} \subseteq \mathcal{J}$ and $\mathcal{J} \subseteq \mathcal{N}$. We want to show that every subspace $\mathcal{Z}_i$ of sequence (2.18) contains $\mathcal{J}$. So that, in particular, $\mathcal{Z}_{n-1} = \mathcal{I}^*$ is the maximum of all the A-invariant subspaces contained in $\mathcal{N}$. Clearly, $\mathcal{Z}_0 = \mathcal{N} \supseteq \mathcal{J}$. Assume $\mathcal{J} \subseteq \mathcal{Z}_{i-1}$. Then, we show that $\mathcal{J} \subseteq \mathcal{Z}_i$. From $A\,\mathcal{J} \subseteq \mathcal{J}$, it follows that $\mathcal{J} \subseteq A^{-1}\mathcal{J}$, which, together with $\mathcal{J} \subseteq \mathcal{N}$, yields $\mathcal{J} \subseteq \mathcal{N} \cap A^{-1}\mathcal{J} \subseteq \mathcal{N} \cap A^{-1}\mathcal{Z}_{i-1} = \mathcal{Z}_i$. ∎

2.1.2 Duality

Let $\Sigma = (A, B, C, D)$ be the continuous or discrete linear time invariant (LTI) system. The dual of $\Sigma = (A, B, C, D)$ is the system $\Sigma^T = (A^T, C^T, B^T, D^T)$, which can be written as

$$\Sigma : \begin{cases} \rho\,\mathbf{x}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) \\ \mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t) \end{cases} \qquad \Sigma^T : \begin{cases} \rho\,\tilde{\mathbf{x}}(t) = A^T\,\tilde{\mathbf{x}}(t) + C^T\,\tilde{\mathbf{u}}(t) \\ \tilde{\mathbf{y}}(t) = B^T\,\tilde{\mathbf{x}}(t) + D^T\,\tilde{\mathbf{u}}(t) \end{cases}.$$

The dual of the system has some properties that are given by:

- Inputs and outputs are exchanged in Σ and Σ^T, which means that if Σ has m inputs and p outputs, then Σ^T has p inputs and m outputs.

- $(\Sigma^T)^T = \Sigma$.

- The transfer function of the dual system $\Sigma^T = (A^T, C^T, B^T, D^T)$ is equal to the transpose of the transfer function of the system $\Sigma = (A, B, C, D)$, i.e., $G_{\Sigma^T}(\lambda) = [G_\Sigma(\lambda)]^T$.

2.1.3 State feedback control and output injection

Let Σ be a regular system that is governed by

$$\Sigma : \begin{cases} \rho\,\mathbf{x}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) \\ \mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t) \end{cases} . \tag{2.20}$$

Assume that a state feedback control input is

$$\mathbf{u}\,(t) = F\,\mathbf{x}\,(t) + Q\,\mathbf{v}\,(t), \tag{2.21}$$

where $F \in \mathbb{R}^{m\times n}$ is a state feedback matrix, $Q \in \mathbb{R}^{m\times m}$ is a nonsingular matrix, and $\mathbf{v}(t) \in \mathbb{R}^m$ is an additional input. From equations (2.20) and (2.21), the closed-loop system $\hat{\Sigma}$ is

$$\hat{\Sigma} : \begin{cases} \rho\,\mathbf{x}(t) = (A + B\,F)\,\mathbf{x}(t) + B\,Q\,\mathbf{v}(t) \\ \mathbf{y}(t) = (C + D\,F)\,\mathbf{x}(t) + D\,Q\,\mathbf{v}(t), \end{cases} \tag{2.22}$$

where $\rho\,\mathbf{x}(t) = (A + B\,F)\,\mathbf{x}(t)$ is the closed-loop equation and $A + B\,F$ is called the closed-loop matrix. The problem of stabilisation by state feedback is solvable, if is possible to calculate the matrix F, such that the closed-loop matrix $A + B\,F$ is asymptotically stable. An output injection is the dual aspect of the state feedback. Let an input of Σ (output injection) be

$$\mathbf{u}'(t) = G\,\mathbf{y}(t) = G\,C\,\mathbf{x}(t) + G\,D\,\mathbf{u}\,(t), \tag{2.23}$$

where $G \in \mathbb{R}^{m\times p}$ is called the output injection matrix. Then, Σ becomes

$$\Sigma : \begin{cases} \rho\,\mathbf{x}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + \mathbf{u}'(t) \\ \mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t). \end{cases} \tag{2.24}$$

By replacing (2.23) in (2.24), we obtain

$$\hat{\Sigma} : \begin{cases} \rho\,\mathbf{x}(t) = (A + G\,C)\,\mathbf{x}(t) + (B + G\,D)\,\mathbf{u}(t) \\ \mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t). \end{cases} \tag{2.25}$$

2.1.4 Controlled and Conditioned Invariance

In the classic geometric control theory, controlled invariance is considered as a fundamental notion. Controlled invariant subspaces are the spaces that contain the complete state evolution when the control is present. As explained in Basile and Marro (1969), a controlled invariant subspace is a subspace such that it is possible to control the system in a way that at all times the state is inside the subspace if the initial state is in the subspace. Controlled invariant subspaces are also referred to as (A, B)-controlled invariant or $A(\text{mod } B)$-controlled invariant subspaces. In defining controlled invariant subspaces, we will consider a linear system described as

$$\rho\mathbf{x}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t). \tag{2.26}$$

The sizes of matrices A and B are $n \times n$ and $n \times m$ respectively. If for any $\mathbf{x}(0) \in \mathcal{V}$, there is an input $\mathbf{u}(t)$, such that $\mathbf{x}(t) \in \mathcal{V}$ for all non-negative t, then $\mathcal{V} \subseteq \mathbb{R}^n$ is a controlled invariant (Basile & Marro, 1969).
From the the definition, we immediately obtain the following:

- $\mathcal{X}$ and $\{0\}$ are both controlled invariant subspaces.
- The addition of any number of controlled invariant subspaces is a controlled invariant subspace.
- The controlled invariant subspace is A-invariant subspace with $B = 0$.
- Any A-invariant subspace is an (A, B)-controlled invariant subspace.

Definition 2. Consider the state equation,

$$\rho\,\mathbf{x}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t). \tag{2.27}$$

A subspace $\mathcal{V}$ of the state space $\mathcal{X}$ is called a controlled invariant subspace, if for any initial state $\mathbf{x}_0$ of $\mathcal{V}$, there exist an input function $\mathbf{u}$, such that the state trajectory $\mathbf{x}_{\mathbf{u}}(t, \mathbf{x}_0)$, generated by the state equation remains in $\mathcal{V}$ for all $t \geq 0$.

We now provide alternative characterisations of controlled invariance.

Theorem 2.7. *[Trentelman et al. (2001)] Let $\mathcal{V}$ be a subspace of $\mathcal{X}$. The following are equivalent:*

(i) *$\mathcal{V}$ is an (A, B)-controlled invariant subspace;*

(ii) *$\mathcal{V}$ satisfies the subspace inclusion,*

$$A\,\mathcal{V} \subseteq \mathcal{V} + \text{im } B; \tag{2.28}$$

(iii) *there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that,*

$$(A + B\,F)\mathcal{V} \subseteq \mathcal{V}; \tag{2.29}$$

which means $\mathcal{V}$ is an $(A + B\,F)$-invariant.

Theorem 2.8. *[Basile and Marro (1992)] The following are equivalent if $\mathcal{V}$ is a subspace of a state space $\mathcal{X}$:*

(i) *$\mathcal{V}$ is (A,B)-controlled invariant;*

(ii) *there exist two matrices X and U such that*

$$A\,V = V\,X + B\,U, \tag{2.30}$$

where V is a basis matrix of $\mathcal{V}$;

(iii) *two matrices X and F exist such that*

$$(A + B\,F)\,V = V\,X. \tag{2.31}$$

Theorem 2.8 expresses an alternative geometric condition for the definition of controlled invariant subspaces in the matrix form.

Definition 3. [Basile and Marro (1969)] Let $\mathcal{V}$ be an (A,B)-controlled invariant subspace, the family of all feedback matrices $F \in \mathbb{R}^{m\times n}$ that satisfies the inclusion $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$ is denoted by $\mathfrak{F}(\mathcal{V})$ and F is called a controlled invariant friend of $\mathcal{V}$.

Theorem 2.9. *[Trentelman et al. (2001)] Let $\mathcal{V}$ be a controlled invariant subspace. Suppose that $F \in \mathfrak{F}(\mathcal{V})$ and let $L \in \mathbb{R}^{m\times m}$ be invertible and* $\operatorname{im} L = B^{-1}\,\mathcal{V}$. *For all initial state $\mathbf{x}_0 \in \mathcal{V}$ and input $\mathbf{u}$, the corresponding state trajectory $\mathbf{x}_{\mathbf{x}_0,\mathbf{u}}(t)$ remains in $\mathcal{V}$ for all $t \geq 0$ if and only if $\mathbf{u}$ has the form*

$$\mathbf{u}\,(t) = F\,\mathbf{x}\,(t) + L\,\mathbf{w}\,(t), \tag{2.32}$$

for some control input $\mathbf{w}$.

Lemma 2.2. *Let $\mathcal{V}$, with dimension r and basis matrix V, be a controlled invariant subspace of (2.27). Matrix F is a controlled invariant friend of $\mathcal{V}$ if $U = -F\,V$, where $U \in \mathbb{R}^{m\times r}$ is a solution of (2.30) for some matrix $X \in \mathbb{R}^{r\times r}$.*

Proof: Let F be a controlled invariant friend of $\mathcal{V}$. Therefore, $(A + B\,F)\,V = V\,\Psi$ for some $\Psi \in \mathbb{R}^{m\times r}$. Then, $A\,V = V\,X + B\,U$ with $X = \Psi$ and $U = -F\,V$. Conversely, let F be such that $U = -F\,V$, where U is a solution of (2.30) for a certain X. Then, $A\,V = V\,X + B\,U$ can be written as $(A + B\,F)\,V = V\,X$. Thus, Theorem 2.8 implies that F is a controlled invariant friend of $\mathcal{V}$. ■

The computation of all controlled invariant friends F of a controlled invariant subspace $\mathcal{V}$ in the inclusion (2.29) relies on equation (2.30) which can be obtained as follows:

$$A\,V = \begin{bmatrix} V & B \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix}, \tag{2.33}$$

which gives

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} V & B \end{bmatrix}^{\dagger} A\,V + \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} K_1, \tag{2.34}$$

where $\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$ is a full column-rank matrix such that:

$$\operatorname{im} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \ker \begin{bmatrix} V & B \end{bmatrix}$$

and K_1 is an arbitrary matrix of suitable size. In the same way, the set of solution of $U = -F\,V$ is given by

$$F = -U\,(V^{\top} V)^{-1} V^{\top} + K_2\,\Phi, \tag{2.35}$$

where Φ is a full column-rank matrix, such that $\ker \Phi = \mathcal{V}$ and K_2 is an arbitrary matrix of suitable size.
By indicating with the symbol F_{K_1,K_2}, any controlled invariant friends of $\mathcal{V}$ obtained from (2.34) and (2.35), the new coordinates of a closed-loop matrix $(A + B\,F_{K_1,K_2})$ can be written as

$$T^{-1}(A + B\,F_{K_1,K_2})\,T = \begin{bmatrix} M_{1,1}(K_1) & M_{1,2}(K_1, K_2) \\ 0 & M_{2,2}(K_2) \end{bmatrix}, \tag{2.36}$$

where $M_{1,1}(K_1)$ is related with $A + B\,F \mid \mathcal{V}$, whereas $M_{2,2}(K_2)$ is related with $A + B\,F \mid \mathcal{X}/\mathcal{V}$.

Definition 4. Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. A subspace $\mathcal{V}^*_{\mathcal{K}}$ is defined as the subspace of all initial conditions $\mathbf{x}_0 \in \mathcal{X}$, such that there exists a control function $\mathbf{u}$, which yields $\mathbf{x}_{\mathbf{x}_0,\mathbf{u}}(t) \in \mathcal{K}$ for all $t \geq 0$.

Theorem 2.10. *[Trentelman et al. (2001)] Consider the subspace $\mathcal{K}$ of $\mathcal{X}$. Then, the subspace $\mathcal{V}^*_{\mathcal{K}}$ is the largest controlled invariant subspace contained in $\mathcal{K}$, i.e.,*

(i) $\mathcal{V}^*_{\mathcal{K}}$ *is a (A, B)-controlled invariant subspace;*

(ii) $\mathcal{V}^*_{\mathcal{K}} \subseteq \mathcal{K}$*;*

(iii) *if there is another controlled invariant subspace $\mathcal{V}$ contained in $\mathcal{K}$, then, $\mathcal{V} \subseteq \mathcal{V}^*_{\mathcal{K}}$.*

Theorem 2.11. *[Basile and Marro (1969); W. M. Wonham and Morse (1970)] Consider a sequence of subspaces defined by*

$$\begin{cases} \mathcal{V}_0 &= \mathcal{K} \\ \mathcal{V}_{q+1} &= \mathcal{K} \cap A^{-1}(\mathcal{V}_q + \operatorname{im} B) \quad q \in \{1, \ldots, k\}. \end{cases} \tag{2.37}$$

Then, it follows that:

(i) $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq ...$;

(ii) *there exists* $k \leq \dim\ \mathcal{K}$*, such that* $\mathcal{V}_k = \mathcal{V}_{k+1}$*;*

(iii) $\mathcal{V}_k = \mathcal{V}_{k+t}$ *for all* $t \geq 0$ *if* $\mathcal{V}_k = \mathcal{V}_{k+1}$*, and then* $\mathcal{V}^\star_{\mathcal{K}} = \mathcal{V}_k$.

Definition 5. Consider system (2.20) with $D = 0$ (a strictly proper system). The output-nulling subspace $\mathcal{V}^*_{\ker\ C}$ is the subspace of initial states $\mathbf{x}_0 \in \mathcal{X}$, for which an input function $\mathbf{u}$ exists that maintains the output function $\mathbf{y}_{\mathbf{x}_0,\mathbf{u}}(t)$ at zero for all $t \geq 0$, i.e., the state trajectory $\mathbf{x}_{\mathbf{x}_0,\mathbf{u}}(t)$ must be contained in ker C. Correspondingly, $\mathcal{V}^*_{\ker\ C}$ is the largest (A, B)-controlled invariant subspace contained in ker C. For the sake of simplicity, $\mathcal{V}^*_{\ker\ C}$ will be denoted by $\mathcal{V}^*$.

Now, the notion of conditioned invariance will be introduced as the dual of controlled invariance. It leads to the definition of conditioned invariant subspaces, which is usually referred to as (C, A)-conditioned invariant subspaces.

Definition 6. [Basile and Marro (1969)] Consider a pair (A, C). A subspace $\mathcal{S}$ of $\mathcal{X}$ is said to be (C, A)-conditioned invariant if

$$A\,(\mathcal{S} \cap \ker\ C) \subseteq \mathcal{S}. \tag{2.38}$$

Remark 2.2. An (C, A)-conditioned invariant subspaces enjoy these following properties:

(i) Both $\mathcal{X}$ and $\{0\}$ are always both (C, A)-conditioned invariant subspaces;

(ii) when $C = 0$, the notion of (C, A)-invariance reduces to the notion of A-invariance;

(iii) the intersection of conditioned invariant subspaces is conditioned invariant. Indeed, given two (C, A)-conditioned invariant subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$, we find

$$A((\mathcal{S}_1 \cap \mathcal{S}_2) \cap \ker\ C) \subseteq A\,(\mathcal{S}_1 \cap \ker\ C) \cap A\,(\mathcal{S}_2 \cap \ker C) \subseteq \mathcal{S}_1 \cap \mathcal{S}_2,$$

which implies that $\mathcal{S}_1 \cap \mathcal{S}_2$ is itself (C, A)-conditioned invariant. However, the addition $\mathcal{S}_1 + \mathcal{S}_2$ is not, in general, (C, A)-conditioned invariant;

(iv) given any pair (C, A), any subspace $\mathcal{S}$ of $\mathcal{X}$ such that $\mathcal{S} \cap \ker\ C = 0$ is conditioned invariant, because in this case $A\,(\mathcal{S} \cap \ker\ C) = \{0\}$, and equation (2.38) is automatically satisfied. In particular, $(\ker\ C)^\perp = \text{im}\ C^\top$ is always conditioned invariant. Moreover, any subspace $\mathcal{S}$ of ker A is conditioned invariant, since in that case $A\,(\mathcal{S} \cap \ker\ C) \subseteq A\mathcal{S} \subseteq A \ker\ A = \{0\}$ and (2.38) is automatically satisfied. However, not all conditioned invariant subspaces $\mathcal{S}$ are such that $\mathcal{S} \cap \ker\ C = \{0\}$ or subspaces of ker A.

Theorem 2.12. *Let $\mathcal{S}$ be a subspace of $\mathcal{X}$. Then, $\mathcal{S}$ is (R, A)-conditioned invariant, if and only if $\mathcal{S}^\perp$ is $(A^\top, R^\top)$-controlled invariant.*

Proof: The subspace $\mathcal{S}$ is (R, A)-conditioned invariant, if and only if $A\,(\mathcal{S} \cap \ker\, R) \subseteq \mathcal{S}$. This inclusion is equivalent to $A^\top \mathcal{S}^\perp \subseteq (\mathcal{S} \cap \ker\, R)^\perp$. On the other hand, $(\mathcal{S} \cap \ker\, R)^\perp = \mathcal{S}^\perp + (\ker\, R)^\perp = \mathcal{S}^\perp + \mathrm{im}\, R^\top$. Thus, $A\,(\mathcal{S} \cap \ker\, R) \subseteq \mathcal{S}$ is equivalent to $A^\top \mathcal{S}^\perp \subseteq \mathcal{S}^\perp + \mathrm{im}\, R^\perp$, which by Theorem 2.7 is a necessary and sufficient condition for $\mathcal{S}^\perp$ to be $(A^\top, R^\top)$-conditioned invariant. ■

Theorem 2.13. *[Basile and Marro (1969)] Given a subspace $\mathcal{S}$ of $\mathcal{X}$, the following statements are equivalent:*

(i) *$\mathcal{S}$ is (C, A)-conditioned invariant;*

(ii) *$\mathcal{S}$ satisfies the inclusion:*

$$A\,(\mathcal{S} \cap \ker\, C) \subseteq \mathcal{S}; \tag{2.39}$$

(iii) *a matrix $G \in \mathbb{R}^{n \times p}$ exists such that $\mathcal{S}$ is $(A + G\,C)$-invariant, which is equivalent to*

$$(A + G\,C)\mathcal{S} \subseteq \mathcal{S}. \tag{2.40}$$

Theorem 2.14. *[Basile and Marro (1969)] The following statements are equivalent for a subspace $\mathcal{S}$ of $\mathcal{X}$ with dimension μ:*

(i) *$\mathcal{S}$ is (C, A)-conditioned invariant;*

(ii) *there exist two matrices $Z \in \mathbb{R}^{(n-\mu)\times(n-\mu)}$ and $L \in \mathbb{R}^{(n-\mu)\times p}$ such that*

$$Q\,A = Z\,Q + L\,C, \tag{2.41}$$

where Q is a matrix such that $\ker\, Q = \mathcal{S}$;

(iii) *two matrices $Z \in \mathbb{R}^{(n-\mu)\times(n-\mu)}$ and $G \in \mathbb{R}^{n \times p}$ exist such that*

$$Q\,(A + G\,C) = Z\,Q. \tag{2.42}$$

Definition 7. Let $\mathcal{S}$ be a (C, A)-conditioned invariant subspace $\mathcal{X}$. The output injection matrix $G \in \mathbb{R}^{n \times p}$ such that (2.40) holds, is called a conditioned invariant friend of $\mathcal{S}$. Thus, the set of all conditioned invariant friends of $\mathcal{S}$ is denoted by $\mathfrak{G}_{C,A}(S)$.

Corollary 2.1. *[Trentelman et al. (2001)] Let $G \in \mathbb{R}^{n \times p}$ and let $M \in \mathbb{R}^{p \times p}$ be invertible. Then, the subspace $\mathcal{S}$ is (C, A)-conditioned invariant subspace, if and only if it is $(M\,C, A + G\,C)$-conditioned invariant.*

As what has been established about the duality of controlled and conditioned invariant subspaces, this statement will be adapted to dualised the notion of the largest controlled invariant subspace contained in a subspace $\mathcal{K}$.

Definition 8. Let $\mathcal{H}$ be a subspace of $\mathcal{X}$. A subspace $\mathcal{S}^*_{\mathcal{H}}$ is the smallest (C, A)-conditioned invariant subspace containing $\mathcal{H}$.

To compute $\mathcal{S}^*_{\mathcal{H}}$, use the following result.

Theorem 2.15. *[Basile and Marro (1969)] Consider a sequence of subspaces defined by*

$$\begin{cases} \mathcal{S}_0 &= \mathcal{H} \\ \mathcal{S}_{q+1} &= \mathcal{H} + A\,(\mathcal{S}_q \cap \ker C) \quad q \in \{1, \ldots, h\}. \end{cases} \tag{2.43}$$

Then, it follows that:

(i) $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \ldots$*;*

(ii) *there exists* $h \leq n - \dim\ \mathcal{H}$*, such that* $\mathcal{S}_h = \mathcal{S}_{h+1}$*;*

(iii) *if* $\mathcal{S}_h = \mathcal{S}_{h+1}$*, then* $\mathcal{S}^*_{\mathcal{H}} = \mathcal{S}_h$*.*

At the end of the discussion, subspaces $\mathcal{V}^*_{\mathcal{K}}$ and $\mathcal{S}^*_{\mathcal{H}}$ are respectively, the largest of all the family of (A, B)-controlled invariant subspaces contained in $\mathcal{K}$ and the smallest of all the family of (C, A)-conditioned invariant subspaces containing $\mathcal{H}$, and they are dual of each other.

2.1.5 Self-boundedness controlled invariance and Self-hidden conditioned invariance

Definition 9. Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Let $\mathcal{V}$ be an (A, B)-controlled invariant subspace contained in $\mathcal{K}$. The subspace $\mathcal{V}$ is said to be a self-bounded (A, B)-controlled invariant with respect to $\mathcal{K}$ if, for all $\mathbf{x}_0 \in \mathcal{V}$, any control function $\mathbf{u}$ yielding $\mathbf{x}_{\mathbf{x}_0,\mathbf{u}(t)} \in \mathcal{K}$, for all $t \in \mathbb{T}$ is such that $\mathbf{x}_{\mathbf{x}_0,\mathbf{u}(t)} \in \mathcal{V}$ for all $t \in \mathbb{T}$.

Theorem 2.16. *Let* $\mathcal{K}$ *be a subspace of* $\mathcal{X}$*. Let* $\mathcal{V}$ *be an* (A, B)*-controlled invariant subspace contained in* $\mathcal{K}$*. The subspace* $\mathcal{V}$ *is said to be a self-bounded* (A, B)*-controlled invariant with respect to* $\mathcal{K}$*, if and only if* $\mathcal{V}^*_{\mathcal{K}} \cap \mathrm{im}\ B \subseteq \mathcal{V}$*.*

Proof: (**If**). Let $\mathbf{x}(0) \in \mathcal{V}$, and let $\mathbf{u}$ be an input such that $\mathbf{x}_{\mathbf{x}_0,\mathbf{u}(t)} \in \mathcal{K}$. Then, $\mathbf{x}_{\mathbf{x}_0,\mathbf{u}(t)} \in \mathcal{V}^*_{\mathcal{K}}$, which implies that $\mathbf{u}$ can be expressed as $\mathbf{u}(t) = F\mathbf{x}(t) + \mathbf{v}(t)$, where F is a controlled invariant friend of $\mathcal{V}^*_{\mathcal{K}}$ and $\mathbf{v}(t) \in B^{-1}\mathcal{V}^*_{\mathcal{K}}$ for all $t \in \mathbb{T}$. Since $\mathcal{V}^*_{\mathcal{K}} \cap \mathrm{im}\ B \subseteq$

$\mathcal{V} \subseteq \mathcal{V}^*_{\mathcal{K}}$, by Lemma 2.2, it follows that F is a controlled invariant friend of $\mathcal{V}$. The state equation can thus be written as

$$\rho\,\mathbf{x}(t) = (A + B\,F)\,\mathbf{x}(t) + B\,\mathbf{v}(t). \tag{2.44}$$

Hence, for all $t \in \mathbb{T}$, there holds $B\,\mathbf{v}(t) \in B(B^{-1}\mathcal{V}^*_{\mathcal{K}})$, so that, since F is a controlled invariant friend of $\mathcal{V}$ and $\mathcal{V} \supseteq \mathcal{V}^*_{\mathcal{K}} \cap \text{im } B$, it is found that $\mathbf{x}(t) \in \mathcal{V}$ for all $t \in \mathbb{T}$. (**Only if**). Let $\mathcal{V}$ be self-bounded with respect to $\mathcal{K}$, and let $\mathbf{x}(0) \in \mathcal{V}$. The set of control functions ensuring that the state trajectory is maintained in $\mathcal{K}$ can be written as $\mathbf{u}(t) = F\mathbf{x}(t) + \mathbf{v}(t)$, where F is a controlled invariant friend of $\mathcal{V}^*_{\mathcal{K}}$ and $\mathbf{v}(t) \in B^{-1}\mathcal{V}^*_{\mathcal{K}}$ for all $t \in \mathbb{T}$. From (2.44), it follows that the state trajectory lies on $\mathcal{V}$ only if $B\,\mathbf{v}(t) \in \mathcal{V}$, for all $t \in \mathbb{T}$, which implies that $\mathcal{V}^*_{\mathcal{K}} \cap \text{im } B \subseteq \mathcal{V}$. ■

Lemma 2.3. *[(Basile & Marro, 1969)] Let $\mathcal{K}$ be a subspace of the state space $\mathcal{X}$. Let $\mathcal{V}$ and $\tilde{\mathcal{V}}$ be two self-bounded (A, B)-controlled invariant, with respect to $\mathcal{K}$ and $\tilde{\mathcal{V}} \subseteq \mathcal{V}$. Then, any controlled invariant friend of $\mathcal{V}$ is also a controlled invariant friend of $\tilde{\mathcal{V}}$.*

Proof: Let F be a controlled invariant friend of $\mathcal{V}$. Since $\tilde{\mathcal{V}}$ is a self-bounded (A, B)-controlled invariant subspace, then, it is a (A, B)-controlled invariant subspace, so that $A\,\tilde{\mathcal{V}} \subseteq \tilde{\mathcal{V}} + \text{im } B$. By addition of this inclusion with $B\,F\,\tilde{\mathcal{V}} \subseteq \text{im } B$, we obtain $(A + B\,F)\,\tilde{\mathcal{V}} \subseteq \tilde{\mathcal{V}} + \text{im } B$. Since $\tilde{\mathcal{V}} \subseteq \mathcal{V}$, we find $(A + B\,F)\,\tilde{\mathcal{V}} \subseteq (A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$, which once intersected yields $(A + B\,F)\,\tilde{\mathcal{V}} \subseteq \mathcal{V} \cap (\tilde{\mathcal{V}} + \text{im } B) = \tilde{\mathcal{V}} + (\mathcal{V} \cap \text{im } B)$. Since $\tilde{\mathcal{V}} \supseteq \mathcal{V}^*_{\mathcal{K}} \cap \text{im } B \supseteq \mathcal{V} \cap \text{im } B$, we obtain $(A + B\,F)\,\tilde{\mathcal{V}} \subseteq \tilde{\mathcal{V}}$. Hence, F is a controlled invariant friend of $\tilde{\mathcal{V}}$.

Corollary 2.2. *[Basile and Marro (1969)] Let $\mathcal{K}$ be a subspace of the state space. Let F be a controlled invariant friend of $\mathcal{V}^*_{\mathcal{K}}$. Then, F is a controlled invariant friend of any self-bounded (A, B)-controlled invariant subspace, with respect to $\mathcal{K}$.*

Definition 10. Let $\mathcal{H}$ be a subspace of $\mathcal{X}$. Let $\mathcal{S}$ be a (C, A)-conditioned invariant subspace containing $\mathcal{H}$. The subspace $\mathcal{S}$ is said to be a self-hidden (C, A)-conditioned invariant subspace with respect to $\mathcal{H}$ if

$$\mathcal{S} \subseteq \mathcal{S}^*_{\mathcal{H}} + \ker C.$$

Remark 2.3. Consider matrix G that satisfies $(A + G\,C)\,\mathcal{S}^*_{\mathcal{H}} \subseteq \mathcal{S}^*_{\mathcal{H}}$. Any $\mathcal{S}$ self-hidden (C, A)-conditioned invariant, with respect to $\mathcal{H}$, satisfies $(A + G\,C)\,\mathcal{S} \subseteq \mathcal{S}$.

2.1.6 Reachability and observability subspaces in the discrete case

The concept of reachability is investigated in the situation of determining the set of states of the system, that can be brought from the origin, choosing a suitable control function.

Definition 11. Consider the discrete system (2.5). We denote by $R_{A,B}$, the matrix:

$$R_{A,B} \overset{\text{def}}{=} \begin{bmatrix} B & A\,B & \ldots & A^{n-2}\,B & A^{n-1}\,B \end{bmatrix}.$$

The $n \times n\,m$ matrix $R_{A,B}$ is called the reachability matrix in n steps of the pair (A, B).

Theorem 2.17. *[Basile and Marro (1969)] The reachable subspace $\mathfrak{R}$ from the origin in n steps of the system (2.5) is, for every $t > 0$, the image of the reachability matrix $R_{A,B}$, i.e.:*

$$\mathfrak{R} = \text{im } R_{A,B}.$$

Remark 2.4. The associated pair (A, B) of system (2.5) is said to be reachable or completely reachable in n steps if $\mathfrak{R} = \mathcal{X}$, i.e., if rank $R_{A,B} = n$.

Theorem 2.18. *[Basile and Marro (1969)] Given the pair (A, B). The reachable subspace $\mathfrak{R}$ from the origin of (A, B), which determines the family of the states that can be brought from the origin in bounded steps using a suitable input function, is the smallest A-invariant subspace containing the image of B, i.e.,*

$$\mathfrak{R} = \langle A \,|\, \text{im } B \rangle.$$

Definition 12. Let $\mathcal{V}$ be a controlled invariant subspace for (2.5). The reachability subspace $\mathfrak{R}_{\mathcal{V}}$ on $\mathcal{V}$ is the set of all points $\tilde{\mathbf{x}}$ of $\mathcal{V}$, such that there exist $T > 0$ and an input function $\mathbf{u}$ such that $\mathbf{x}_{0,\mathbf{u}}(T) = \tilde{\mathbf{x}}$ and $\mathbf{x}_{0,\mathbf{u}}(t) \in \mathcal{V}$, for all $t \in [0, T]$.

Basile and Marro (1969) shows that a reachability subspace $\mathfrak{R}_{\mathcal{V}}$ on $\mathcal{V}$ for (2.5) is given by

$$\mathfrak{R}_{\mathcal{V}} = \langle A + B\,F \,|\, \mathcal{V} \cap \text{im } B \rangle.$$

From this definition, is obvious that every reachability subspace is controlled invariant. Now, the notion of observability is related to how can be determined the state of a system at time $t = t_0$ based on the input and output functions of the system itself at $t > t_0$.

Definition 13. Consider the discrete system (2.5). We denote by $O_{C,A}$ the matrix:

$$O_{C,A} \overset{\text{def}}{=} \begin{bmatrix} C \\ C\,A \\ \vdots \\ C\,A^{n-1} \end{bmatrix}.$$

The $n\,p \times n$ matrix $O_{C,A}$ is called the observability matrix of the pair (C, A).

Theorem 2.19. *[Basile and Marro (1969)] The unobservable subspace $\mathcal{Q}$ of the system (2.5) is, for every $t > 0$, the kernel of the observability matrix $O_{C,A}$, i.e.,*

$$\mathcal{Q} = \ker\, O_{C,A}.$$

Remark 2.5. The system (2.5) is said to be completely observable, if $\mathcal{Q} = \{0\}$, i.e., if rank $O_{C,A} = n$.

Theorem 2.20. *[Basile and Marro (1969)] The unobservable subspace $\mathcal{Q}$ is the largest A-invariant subspace contained in the kernel of C.*

Indeed, duality is also achieved between the concepts of reachability and observability. By considering the system $\Sigma = (A, B, C, D)$ and its dual $\Sigma^T = (A^T, C^T, B^T, D^T)$. Then, Σ is completely reachable if Σ^T is completely observable and vice-versa (Basile & Marro, 1992).

2.1.7 Stability and stabilizability subspaces

Stability and stabilizability subspaces are two fundamental concepts in control theory which are used to solve many control problems.

Definition 14. The differential equation of system (2.14) is called asymptotically stable, if every solution tends to zero for $t \longrightarrow \infty$.

Corollary 2.3. *[Trentelman et al. (2001)] System (2.14) is asymptotically stable, if and only if*

$$\max\{\mathcal{R}e\lambda \mid \lambda \in \sigma(A)\} < 0.$$

Lemma 2.4. *[Kar and Singh (2003)] System (2.14) is asymptotically stable, if there exist $n \times n$ symmetric positive definite matrix P, such that:*

$$P\,A + A^T\,P < 0,$$

in continuous time and

$$A^T\,P\,A - P < 0,$$

in discrete time.

Definition 15. Let $\mathcal{I}$ be an A-invariant subspace and let $\mathbf{x}_0$ be the initial state of (2.14). We say that $\mathcal{I}$ is:

- internally stable, if for every $\mathbf{x}_0 \in \mathcal{I} \setminus \{0\}$, there holds $\mathbf{x}(t) \longrightarrow 0$ as $t \longrightarrow \infty$,
- externally stable, if for every $\mathbf{x}_0 \notin \mathcal{I}$, there holds $\mathbf{x}(t) \longrightarrow \mathcal{I}$ as $t \longrightarrow \infty$.

A more formal theorem is stated as follows:

Theorem 2.21. *[Trentelman et al. (2001)] Let $\mathcal{I}$ be an A-invariant subspace. Then,*

- *$\mathcal{I}$ is internally stable, if and only if $A'_{1,1}$ in (2.13) is asymptotically stable,*
- *$\mathcal{I}$ is externally stable, if and only if $A'_{2,2}$ in (2.13) is asymptotically stable.*

Corollary 2.4. *Let $\mathcal{V}$ be a controlled invariant subspace. Then,*

- *$\mathcal{V}$ is internally stabilisable, if K_1 in (2.34) exists, such that $M_{1,1}(K_1)$ in (2.36) is asymptotically stable,*
- *$\mathcal{V}$ is externally stabilisable, if K_2 in (2.35) exists, such that $M_{2,2}(K_2)$ in (2.36) is asymptotically stable.*

Definition 16. A subspace $\mathcal{V}$ of a state space $\mathcal{X}$ is called a stabilizability subspace if, for the initial state $\mathbf{x}_0$ on it, there exists a suitable control function $\mathbf{u}$, such that $\mathbf{x}_{\mathbf{x}_0,\mathbf{u}}(t) \in \mathcal{V}$, for all $t \in \mathbb{T}$ and $\mathbf{x}_{\mathbf{x}_0,\mathbf{u}}$ converges to zero as t converges to infinity.

This definition immediately shows that every stabilizability subspace is controlled invariant.

Theorem 2.22. *[Trentelman et al. (2001)] Consider system (2.27). Let $\mathcal{V}$, with a controlled invariant friend F, be a controlled invariant subspace of $\mathcal{X}$. Then,*

- *$\mathcal{V}$ is internally stabilisable, if and only if $\sigma(A + B\,F|\mathcal{V})$ is asymptotically stable,*
- *$\mathcal{V}$ is externally stabilisable, if and only if $\sigma(A+B\,F|\mathcal{X}/\mathcal{V})$ is asymptotically stable.*

2.1.8 Disturbance decoupling problem (DDP)

Consider the system:

$$\rho\,\mathbf{x}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + H\mathbf{w}(t), \tag{2.45}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t), \tag{2.46}$$

where $\mathbf{w}$ is a disturbance. Now, the problem of disturbance decoupling by state feedback has been solved by using a geometric method. There are two versions of the problem, the basic version which is a solution without stability. It is based on finding a control law $\mathbf{u}(t) = F\,\mathbf{x}(t)$, such that the output $\mathbf{y}(t)$ of the closed-loop system,

$$\rho\,\mathbf{x}(t) = (A + B\,F)\,\mathbf{x}(t) + H\mathbf{w}(t), \tag{2.47}$$

is not affected by the disturbance $\mathbf{w}$. Then, the same problem will be solved using the notion of stability, i.e., in addition to the previous requirement, the closed-loop matrix $(A + B\,F)$ of (2.47) has to be asymptotically stable.

Corollary 2.5. *[Basile and Marro (1969)] The disturbance decoupling problem is solvable with state feedback, if and only if* $\mathrm{im}\, H \subseteq \mathcal{V}^*$.

Corollary 2.6. *[Trentelman et al. (2001)] The disturbance decoupling problem with stability is solvable, if and only if* $\mathrm{im}\, H \subseteq \mathcal{V}^*$ *and* $\mathcal{V}^*$ *is internally and externally stabilisable.*

2.2 Structural invariants of 2-D systems

2.2.1 Overview

Discrete 2-D systems are discrete dynamical systems that evolve in the plane $\mathbb{Z} \times \mathbb{Z}$. Typically, but not necessarily, the points of the plan are partially ordered by the relation:

$$(i, j) \leq (h, k) \Longleftrightarrow i \leq h,\ j \leq k. \tag{2.48}$$

In this way, we can introduce the past and future of the point (h, k) in the plane $\mathbb{Z} \times \mathbb{Z}$.

Definition 17. The past of (i, j) is the set of all points given by

$$P_{(i,j)} = \{(h, k) \in \mathbb{Z} \times \mathbb{Z} \,|\, (i, j) \geq (h, k)\}, \tag{2.49}$$

and the future of the same point is the set:

$$F_{(i,j)} = \{(h, k) \in \mathbb{Z} \times \mathbb{Z} \,|\, (i, j) \leq (h, k)\}. \tag{2.50}$$

We now formalize the concept of future with respect to the separation set as the set of points for which we can compute the states of the system with a valid boundary condition. Note that there are other points that do not precede and follow the point (i, j). For instance, in Figure 2.4, the past of (i, j) and the future are the two discrete cones in the plane $\mathbb{Z} \times \mathbb{Z}$. The points, not contained in such cones, do not belong to the past or future of (i, j).

Definition 18. [Fornasini and Marchesini (1980)] A nonempty set $\mathfrak{C}$ in the plane $\mathbb{Z} \times \mathbb{Z}$, with the following properties, is called a separation set:

- if $i > h,\ j > k,\ (i, j)$ and (h, k) cannot belong simultaneously to $\mathfrak{C}$;
- if $(i, j) \in \mathfrak{C}$, then $\mathfrak{C}$ intersects the sets $\{(i-1, j), (i, j+1), (i-1, j+1)\}$ and $\{(i+1, j), (i, j-1), (i+1, j-1)\}$ and does not contain the set $\{(i+1, j), (i, j+1)\}$;
- for any (h, k) in $\mathbb{Z} \times \mathbb{Z}$, the relation $(i, j) \leq (h, k)$ cannot be satisfied by infinitely many elements (i, j) in $\mathfrak{C}$.

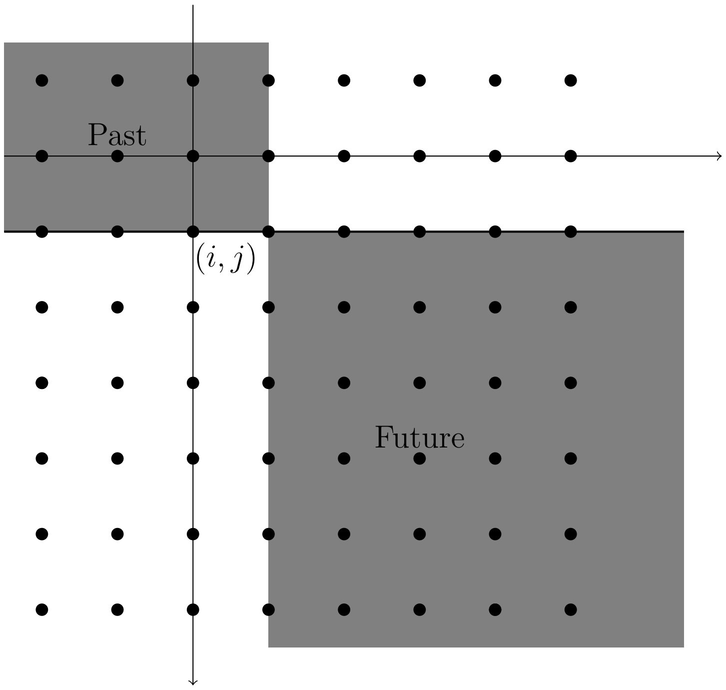


Figure 2.4

The following separation sets are a particular case of this definition (Gapinski, 1988):

$$\mathfrak{C}_k \stackrel{\text{def}}{=} \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, i + j = k\}, \qquad k \in \mathbb{Z} \tag{2.51}$$

$\mathfrak{C}_k$ divides the plane $\mathbb{Z} \times \mathbb{Z}$ into two regions: the past region of the separation set $\mathfrak{C}_k$ is the union of the cones that constitute the past of points $\mathfrak{C}_k$, which are sets of

$$P_{\mathfrak{C}_k} = \bigcup_{(i,j) \in \mathfrak{C}_k} P_{(i,j)} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, i + j \geqslant k\} \tag{2.52}$$

and the future region is the union of the cones that constitute the future of points $\mathfrak{C}_k$, given by

$$F_{\mathfrak{C}_k} = \bigcup_{(i,j) \in \mathfrak{C}_k} F_{(i,j)} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, i + j \leqslant k\}. \tag{2.53}$$

Figure 2.5 illustrates some examples of separation sets, past and future regions of each one in $\mathbb{Z} \times \mathbb{Z}$.

Note that there are many possibilities for choosing a separation set $\mathfrak{C}_k$.

Definition 19. A local state space $\mathcal{X}$ is a finite dimensional vector space, where the local state $\mathbf{x} \in \mathcal{X}$ is assigned to each point (i, j) of the plane, while a global state space

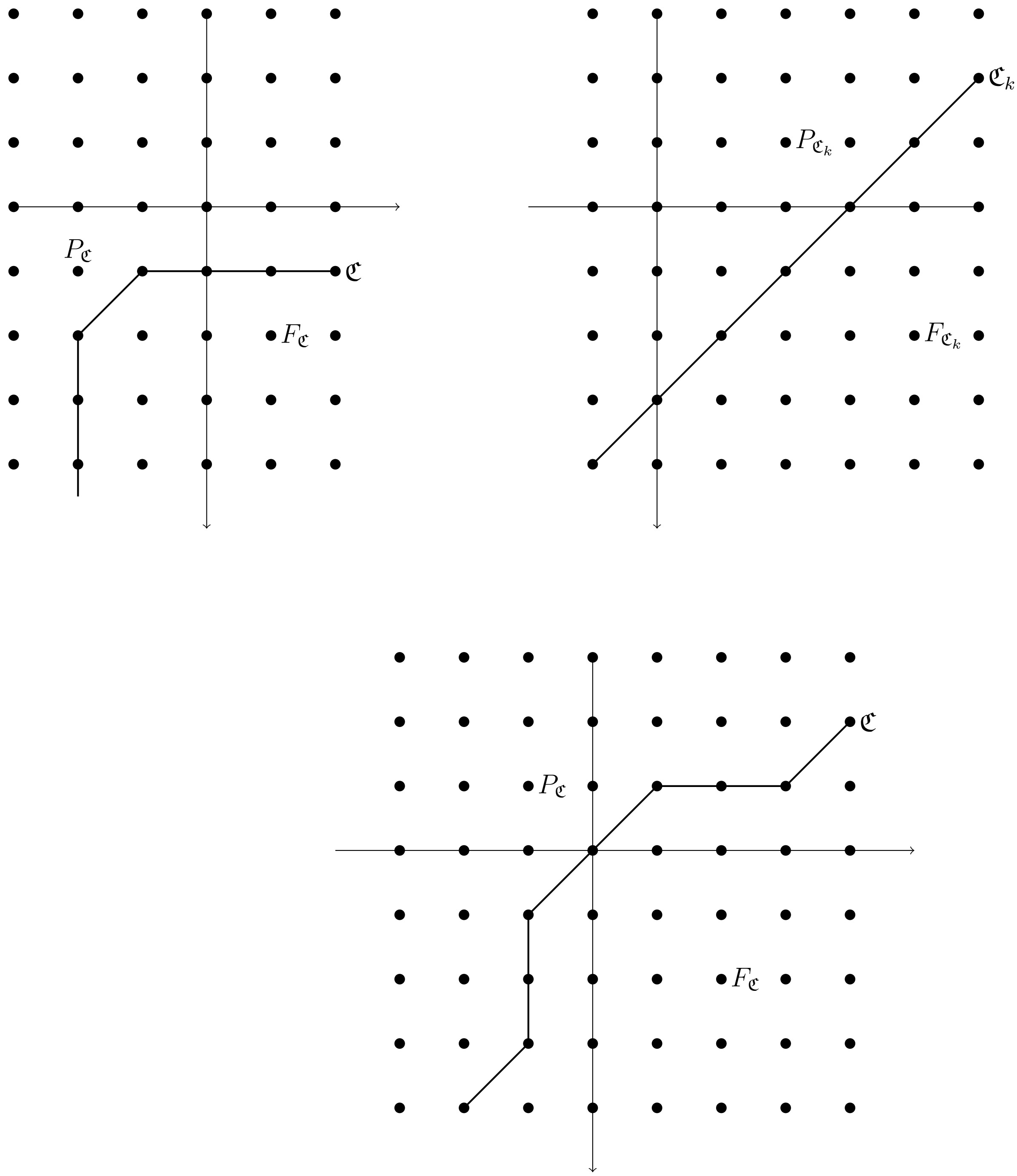

$P_{\mathfrak{c}}$
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Figure 2.5

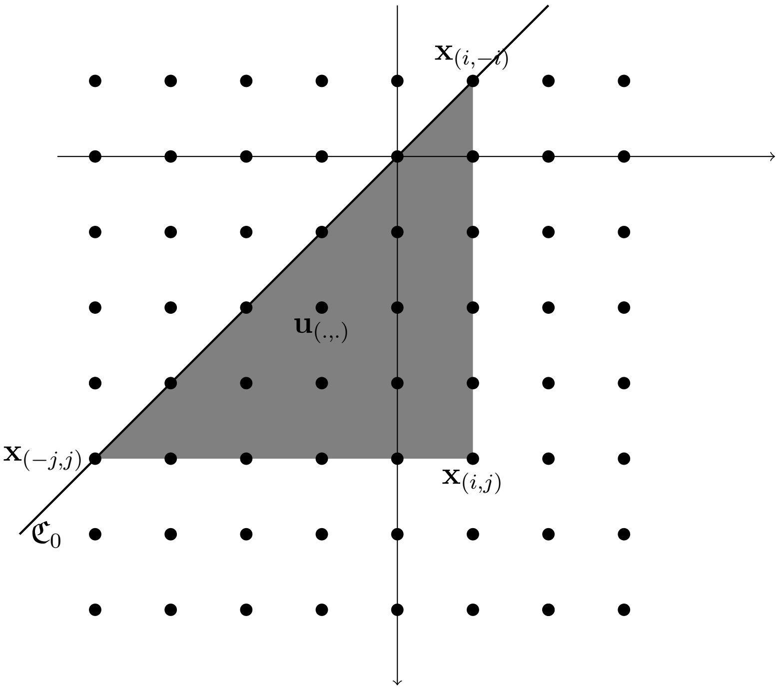

Figure 2.6

$\mathcal{X}_k$ is the set of points on the separation set $\mathfrak{C}_k$ and is defined by

$$\mathcal{X}_{\mathfrak{C}_k} = \{\mathbf{x}_{(i,j)} \in \mathcal{X} \,|\, (i,j) \in \mathfrak{C}_k\}. \tag{2.54}$$

The previous definition gives rise to a distinction between the local and global state. In the 1-D case, the present separates the past the from future, and it is given by a point (i.e. the boundary condition is on the state itself). In the 2-D case, the present will be given by an infinite dimension of vectors. In particular, in the case of the separation set $\mathfrak{C}_k$, the boundary condition is a line. For further clarification, in Figure 2.6, if the global state $\mathcal{X}_0 = \{\mathbf{x}_{(h,-h)} \,|\, h \in \mathbb{Z}\}$ is known, the local state at an arbitrary point (i,j) belonging to the future of $\mathfrak{C}_0$, can be calculated in a finite number of steps by the entrance into the future of $\mathfrak{C}_0$ with $\{\mathbf{u}_{(h,k)}, h+k \geq 0\}$.

2.2.2 2-D states models

In the 2-D state model that we consider, the present local state is $\mathbf{x}_{(i+1,j+1)}$, and it is determined by the two local states and two inputs values belonging to the past of $(i+1, j+1)$ which are placed at the two points that are also immediately in the past of $(i+1, j+1)$. In this case, the structure of the model is given by Fornasini-Marchesini first order model (Fornasini & Marchesini, 1978), which is described by

$$\mathbf{x}_{(i+1,j+1)} = A_1\mathbf{x}_{(i,j+1)} + A_2\mathbf{x}_{(i+1,j)} + B_1\mathbf{u}_{(i,j+1)} + B_2\mathbf{u}_{(i+1,j)}. \tag{2.55}$$

The output at the point (i, j), as in the 1-D case, is determined by the state and function of input at the same point:

$$\mathbf{y}_{(i,j)} = C\mathbf{x}_{(i,j)}, \tag{2.56}$$

where the vectors $\mathbf{x}, \mathbf{y}, \mathbf{u}$ belong, respectively, to $\mathbb{R}^n$, $\mathbb{R}^p$, and $\mathbb{R}^m$, and A_1, A_2, B_1, B_2, C have suitable dimensions. An appropriate set of boundary conditions of model (2.55)-(2.56) is given by the separation set (2.51) and the boundary conditions for the same model is defined by assigning the local state $\mathbf{x}_{(i,j)}$ for all $(i, j) \in \mathfrak{C}_0$ (Ntogramatzidis et al., 2008). The model is obviously linear and, as we mentioned, first order, because the quantities involved in the update are related only to the points that precede the points at which the state is calculated. In addition, in the rest of the chapter, we will concentrate our attention on model (2.55) because it is the most general model. Indeed, it can be used to describe the majority of other 2-D models which are presented in the literature.

- Fornasini-Marchesini second order model is introduced in Fornasini and Marchesini (1976, 1975) as:

$$\bar{\mathbf{x}}_{(i+1,j+1)} = \bar{A}_0\bar{\mathbf{x}}_{(i,j)} + \bar{A}_1\bar{\mathbf{x}}_{(i,j+1)} + \bar{A}_2\bar{\mathbf{x}}_{(i+1,j)} + \bar{B}\mathbf{u}_{(i,j)} \tag{2.57}$$

$$\mathbf{y}_{(i,j)} = \bar{C}\bar{\mathbf{x}}_{(i,j)}, \tag{2.58}$$

which is a special case of (2.55), when assuming the vector

$$\mathbf{x}_{(i,j)} = \begin{bmatrix} \bar{\mathbf{x}}_{(i,j)} \\ \bar{\mathbf{x}}_{(i,j-1)} \\ \mathbf{u}_{(i,j-1)} \end{bmatrix},$$

as a local state with the matrices:

$$A_1 = \begin{bmatrix} \bar{A}_1 & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \bar{A}_2 & \bar{A}_0 & \bar{B} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \bar{C} & 0 & 0 \end{bmatrix}.$$

Then, model (2.57) can be rewritten in form (2.55). An appropriate set of boundary conditions of model (2.57) is given by the separation set (2.51) and the boundary conditions for the same model is defined by assigning the local state $\mathbf{x}_{(i,j)}$ for all $(i, j) \in \mathfrak{C}_{-1} \cup \mathfrak{C}_0$ (Ntogramatzidis et al., 2008).

- Roesser's model (Roesser, 1975; Givone & Roesser, 1972, 1973) is described by

$$\begin{bmatrix} \mathbf{x}^h_{(i+1,j)} \\ \mathbf{x}^v_{(i,j+1)} \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} \mathbf{x}^h_{(i,j)} \\ \mathbf{x}^v_{(i,j)} \end{bmatrix} + \underbrace{\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}}_{\hat{B}} \mathbf{u}_{(i,j)}, \tag{2.59}$$

$$\mathbf{y}_{(i,j)} = \underbrace{\begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}}_{\hat{C}} \begin{bmatrix} \mathbf{x}^h_{(i+1,j)} \\ \mathbf{x}^v_{(i,j+1)} \end{bmatrix}, \tag{2.60}$$

where $\mathbf{x}^h$ and $\mathbf{x}^v$ are the horizontal and vertical states respectively, so that the local states is the direct addition of theses two states. Model (2.59-2.60) can clearly be recast in the form of (2.55-2.56) by assuming that the vector

$$\mathbf{x}_{(i,j)} = \begin{bmatrix} \mathbf{x}^h_{(i+1,j)} \\ \mathbf{x}^v_{(i,j+1)} \end{bmatrix}$$

as local state, with matrices:

$$A_1 = \begin{bmatrix} 0 & 0 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}, \; A_2 = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ \hat{B}_2 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}.$$

- The model of Attasi introduced in Attasi (1973) is a special case of model (2.57) with $\bar{A}_0 = -\bar{A}_1\bar{A}_2 = -\bar{A}_2\bar{A}_1$.

In the following, regarding the 2-D structure invariance, attention will be paid on the Fornasini-Marchesini first and second order models. The reason for this choice is that these models will provide a base for the generalisation into the N-D case.

2.2.3 Controlled invariance with related properties and their dual for models in the 2-D system

The geometric theory of the 2-D models described here parallels the one in the 1-D case, albeit with some modification.

Definition 20. [Ntogramatzidis et al. (2008)] Consider a 2-D FM model described by (2.55), where the input is identical to zero. A subspace $\mathcal{I}$ of local state space $\mathcal{X}$ is (A_1, A_2)-invariant, if the following inclusion holds:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \mathcal{I} \subseteq \mathcal{I} \times \mathcal{I}, \tag{2.61}$$

where A_1, A_2 are matrices representation of linear maps in particular bases.

Lemma 2.5. *Let $\mathcal{I}$ be a subspace of $\mathcal{X}$, and let J be a basis matrix of $\mathcal{I}$. Let r be the dimension of $\mathcal{I}$. Then, $\mathcal{I}$ is (A_1, A_2)-invariant, if and only if two matrices X_1, X_2 from $\mathbb{R}^{r\times r}$ exist such that:*

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} J = \begin{bmatrix} J & 0_{n\times r} \\ 0_{n\times r} & J \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{2.62}$$

Proof: It is clear that the relation of the subspace inclusion (2.61) can be expressed as in (2.62). ■

Definition 21. [Conte and Perdon (1988)] The subspace $\mathcal{V}$ of $\mathcal{X}$ is said to be of controlled invariant for model (2.55), if the following subspace inclusion holds:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \times \mathcal{V}) + \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Lemma 2.6. *[Ntogramatzidis et al. (2008)] Let $\mathcal{V}$ be a subspace of $\mathcal{X}$, and let V denote a basis matrix of $\mathcal{V}$. Let r be the dimension of $\mathcal{V}$. The following statements are equivalent:*

(i) *$\mathcal{V}$ is the controlled invariant for model (2.55);*

(ii) *there exist matrices $X \in \mathbb{R}^{2r\times r}$ and $\Omega \in \mathbb{R}^{m\times r}$, such that:*

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} V = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} X + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \Omega; \tag{2.63}$$

(iii) *there exists a matrix $F \in \mathbb{R}^{m\times n}$, such that:*

$$\begin{bmatrix} A_1 + B_1 F \\ A_2 + B_2 F \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \times \mathcal{V}; \tag{2.64}$$

(iv) *there exist matrices $F \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{R}^{2r\times r}$, such that:*

$$\begin{bmatrix} A_1 + B_1 F \\ A_2 + B_2 F \end{bmatrix} V = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} X.$$

Definition 22. [Ntogramatzidis (2012)] A subspace $\mathcal{V}$ of $\mathcal{X}$ is said to be a 2-D controlled invariant subspace for model (2.57), if it is at the same time (A_0, B)-, (A_1, B)-, and (A_2, B)-controlled invariant subspaces in the 1-D counterpart, which is equivalent to

$$A_i \mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B, \quad i \in \{0, 1, 2\}.$$

Theorem 2.23. *[Ntogramatzidis (2012)] Let $\mathcal{V}$ be a subspace of $\mathcal{X}$. For any $\mathcal{V}$-valued boundary condition, system (2.57) has a $\mathcal{V}$-valued solution, if and only if $\mathcal{V}$ is a 2-D controlled invariant subspace.*

Proof: See Theorem 3.2 in Ntogramatzidis (2012). ■

The family of all controlled invariant subspaces contained in ker C in the 2-D case for model (2.55) and the family of all controlled invariant subspaces contained in ker C in the 2-D case for model (2.57) are both closed under subspace addition. Then, the addition of the elements in each family is the maximum. This subspace is called the output-nulling subspace and it is denoted by $\mathcal{V}^*$. The following algorithms compute $\mathcal{V}^*$ for each model.

Proposition 2.1. *[Conte and Perdon (1988)] Consider model (2.55). $\mathcal{V}^*$ coincides with the last term of the sequence of subspaces:*

$$\begin{cases} \mathcal{V}_0 & = \mathcal{X} \\ \mathcal{V}_{q+1} & = \ker C \cap \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^{-1} (\mathcal{V}_q \times \mathcal{V}_q + \mathrm{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}), \quad q \in \{1, 2, \dots, k\}, \end{cases} \tag{2.65}$$

where the value of $k \leq n-1$ is determined by condition $\mathcal{V}_{k+1} = \mathcal{V}_k$.

Lemma 2.7. *[Ntogramatzidis (2012)] Consider model (2.57). $\mathcal{V}^*$ is the last term of the monotonically nonincreasing sequence:*

$$\begin{cases} \mathcal{V}_0 = \ker C, \\ \mathcal{V}_q = \bigcap_{j=0}^{2} A_j^{-1}(\mathcal{V}_{q-1} + \mathrm{im}\ B) \cap \ker C, \quad q \in \{1, 2, \dots, \kappa\}, \end{cases}$$

where the integer $\kappa \leq n-1$ is determined by the condition $\mathcal{V}_{\kappa+1} = \mathcal{V}_\kappa$, i.e., $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \mathcal{V}_2 \supset \ldots \supset \mathcal{V}_\kappa = \mathcal{V}_{\kappa+1} = \mathcal{V}^*$.

Definition 23. [Ntogramatzidis (2012)] A subspace $\mathcal{S}$ of $\mathcal{X}$ is said to be a 2-D conditioned invariant subspace for model (2.57), if it at the same time (C, A_0)-, (C, A_1)-, and (C, A_2)-conditioned invariant in the 1-D counterpart, which is equivalent to

$$A_i\,(\mathcal{S} \cap \ker\ C) \subseteq \mathcal{S}, \quad i \in \{0, 1, 2\}.$$

The concept of subspace introduced in the previous definition is the dual of the 2-D controlled invariant subspace. Moreover, the 2-D output-nulling subspace $\mathcal{V}$ has also a dual, named 2-D input-containing subspaces $\mathcal{S}$, which is a 2-D conditioned invariant subspace containing im B. The intersection of all these 2-D input-containing subspaces is denoted by $\mathcal{S}^*$ and dualizes the subspace $\mathcal{V}^*$: its computation can be carried out by dualizing Lemma 2.7 as follows.

Lemma 2.8. *[Ntogramatzidis (2012)] Consider the model (2.57). $\mathcal{S}^*$ is the last term of the monotonically nondecreasing sequence:*

$$\begin{cases} \mathcal{S}_0 = \text{im } B, \\ \mathcal{S}_q = \Sigma_{j=0}^{2} A_j(\mathcal{S}_{q-1} \cap \ker C) + \text{im } B, \quad q \in \{1, 2, \ldots, \kappa\}, \end{cases}$$

where the integer $\kappa \leq n-1$ is determined by the condition $\mathcal{S}_{\kappa+1} = \mathcal{S}_\kappa$, i.e., $\mathcal{S}^* = \mathcal{S}_\kappa$.

Definition 24. A subspace $\mathcal{W}$ is said to be a 2-D controlled invariant subspace of the feedback type for model (2.57) if $\mathcal{W}$ is $A_0\,\mathcal{W} \subseteq \mathcal{W} + \text{im } B$, and it satisfies both inclusions $A_1\,\mathcal{W} \subseteq \mathcal{W}$ and $A_2\,\mathcal{W} \subseteq \mathcal{W}$.

Theorem 2.24. *[Ntogramatzidis (2012)] A subspace $\mathcal{W}$ is said to be a 2-D controlled invariant subspace of the feedback type for (2.57), if and only if a static feedback control $\mathbf{u}_{(i,j)} = F\,\mathbf{x}_{(i,j)}$ exists such that for any $\mathcal{W}$-valued boundary condition, (2.57) has a $\mathcal{W}$-valued solution lies in $\mathcal{W}$.*

A 2-D controlled invariant subspace of the feedback type contained in ker C is an output-nulling subspace of the feedback type. The addition of all the families of these subspaces is denoted by $\mathcal{W}^*$, and it is computed by the following algorithm.

Lemma 2.9. *[Ntogramatzidis (2012)] Consider the model (2.57). $\mathcal{W}^*$ is the last term of the monotonically nonincreasing sequence:*

$$\begin{cases} \mathcal{W}_0 = \ker C, \\ \mathcal{W}_q = A_0^{-1}(\mathcal{W}_{q-1} + \text{im } B) \cap A_1^{-1}\mathcal{W}_{q-1} \cap A_2^{-1}\mathcal{W}_{q-1} \cap \ker C, \quad q \in \{1, 2, \ldots, \kappa\}, \end{cases}$$

where the condition $\mathcal{W}_{\kappa+1} = \mathcal{W}_\kappa$ determines the integer $\kappa \leq n-1$, which means that $\mathcal{W}^* = \mathcal{W}_\kappa$.

2.2.4 Friends and stabilisation for models in 2-systems

Definition 25. A matrix F, such that (2.64) in Lemma 2.6 holds, is called a controlled invariant friend of the controlled invariant subspace $\mathcal{V}$, and the family of all these matrices is denoted by $\mathfrak{F}(\mathcal{V})$.

Lemma 2.10. *[Ntogramatzidis et al. (2008)] Let $\mathcal{V}$, with dimension r and a basis matrix V, be a controlled invariant subspace for (2.55). A linear equation $\Omega = -F\,V$, where Ω of dimension $(m \times r)$ is a solution of (2.63) in Lemma 2.6 for some X of dimension $2r \times r$, has a solution for each $F = -\Omega\,(V^\top V)^{-1} V^\top + \Lambda$, from $\mathfrak{F}(\mathcal{V})$ where $\Lambda\,V = 0$.*

It follows that the change of basis $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$, with im $T_1 = \mathcal{V}$, is such that:

$$T^{-1}(A_i + B_i\,F)\,T = \begin{bmatrix} G_{i,11}(\Omega, \Lambda) & G_{i,12}(\Omega, \Lambda) \\ 0 & G_{i,22}(\Omega, \Lambda) \end{bmatrix}.$$

Lemma 2.11. *[Ntogramatzidis (2012)] Let $\mathcal{W}$, with a basis matrix W, be a 2-D controlled invariant subspace of the feedback type for (2.57). The family of matrices F that solve the equation $\Omega = -F\,W$, where Ω solves the equation $A_0\,W = W\,X_0 + B\,\Omega$ for some matrix X_0, is considered as the family of a feedback type controlled invariant friend of $\mathcal{W}$.*

The computation of all feedback type controlled invariant friends F of a 2-D controlled invariant subspace of the feedback type $\mathcal{W}$ comes from solving the equation $A_0\,W = W\,X_0 + B\,\Omega$ in the unknown X_0 and Ω as

$$\begin{bmatrix} X_0 \\ \Omega \end{bmatrix} = \begin{bmatrix} W & B \end{bmatrix}^{\dagger} A_0\,W + \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} K_1, \tag{2.66}$$

where $\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$ is a full column-rank matrix such that:

$$\operatorname{im} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \ker \begin{bmatrix} V & B \end{bmatrix}$$

and K_1 is an arbitrary matrix of suitable size. In the same way, the set of solution of $\Omega = -F\,W$ is given by

$$F = -\Omega(K_1)\,(W^{\top} W)^{-1} W^{\top} + K_2\,\Phi, \tag{2.67}$$

where Φ is a full column-rank matrix, such that ker $\Phi = \mathcal{V}$ and K_2 is an arbitrary matrix of a suitable size.
By indicating the symbol $F = F_{K_1,K_2}$ for all feedback type controlled invariant friends of $\mathcal{W}$, obtained from equations (2.66) and (2.67), the new coordinates of a closed-loop matrix $(A_0 + B\,F_{K_1,K_2})$ using the change of basis $T = \begin{bmatrix} W & W^c \end{bmatrix}$ can be written as

$$\begin{aligned} T^{-1}(A_0 + B\,F_{K_1,K_2})\,T &= \begin{bmatrix} L_1(K_1,K_2) & L_2(K_1,K_2) \\ 0 & L_3(K_2,K_2) \end{bmatrix}, \\ T^{-1}A_1\,T &= \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix},\; T^{-1}A_2\,T = \begin{bmatrix} N_1 & N_2 \\ 0 & N_3 \end{bmatrix}. \end{aligned}$$

Then, as shown in Basile and Marro (1992), as well as W. Wonham (1985) in the 1-D case, the matrix $L_1(K_1,K_2)$ does not depend on K_2. Likewise, the matrix $L_3(K_1,K_2)$ does not depend on K_1.

Definition 26. The system (2.55) is called asymptotically stable, if every solution tends to zero for $i + j \longrightarrow \infty$.

Proposition 2.2. *[Fornasini and Marchesini (1978)] The pair of matrices* (A_1, A_2) *is asymptotically stable, if and only if*

$$\det(I_n - A_1\, z_2 - A_2\, z_1) \neq 0,$$

for all z_1 *and* z_2 *in the unit bidisc* $\{(\zeta_1, \zeta_2) \in \mathbb{C} \times \mathbb{C} \mid |\zeta_1| \leq 1$ *and* $|\zeta_2| \leq 1\}$.

Lemma 2.12. *[Kar and Singh (2003)] The Linear Matrix Inequality condition for asymptotic stability of the pair of matrices* (A_1, A_2) *is that two symmetric positive definite matrices* P_1 *and* P_2 *exist such that*

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A_1^\top \\ A_2^\top \end{bmatrix} (P_1 + P_2) \begin{bmatrix} A_1 & A_2 \end{bmatrix} > 0.$$

Lemma 2.13. *[Ntogramatzidis et al. (2008)] Let* $\mathcal{I}$*,with dimension* r *and a basis matrix* J*, be an* (A_1, A_2)*-invariant subspace of* $\mathbb{R}^n$ *and equation (2.62) to hold with* X_1, X_2 *from* $\mathbb{R}^{r\times r}$*. Thus,* $\mathcal{I}$ *is internally stable, if and only if the pair of matrices* X_1, X_2 *is asymptotically stable.*

Definition 27. Let $\mathcal{V}$ be the controlled invariant subspace for the model (2.55). $\mathcal{V}$ is internally stabilisable, if a friend of $\mathcal{V}$ exists, such that the pair $(A_1 + B_1\, F, A_2 + B_2\, F)$ is asymptotically stable, i.e., $\mathcal{V}$ is an internally stable $(A_1 + B_1\, F, A_2 + B_2\, F)$-invariant subspace.

The definition of externally stabilisable of $\mathcal{V}$ comes from putting the term "externally", rather than "internally" in the previous definition .

Corollary 2.7. *[Ntogramatzidis (2012)] Let* $\mathcal{W} \subseteq \mathcal{X}$ *be a 2-D controlled invariant subspace of feedback type, so that*

- $\mathcal{W}$ *is internally stabilisable, if and only if* K_1 *in equation (2.66) exists such that* $(L_1(K_1), M_1, N_1)$ *in equation (2.68) is asymptotically stable;*
- $\mathcal{W}$ *is externally stabilisable if,* K_2 *in equation (2.67) exists such that* $(L_3(K_2), M_3, N_3)$ *in (2.68) is asymptotically stable.*

2.2.5 Disturbance decoupling problem

Given the 2-D FM first and second order models:

$$\begin{aligned} \mathbf{x}_{(i+1,j+1)} &= A_1\mathbf{x}_{(i,j+1)} + A_2\mathbf{x}_{(i+1,j)} + B_1\mathbf{u}_{(i,j+1)} \\ &+ B_2\mathbf{u}_{(i+1,j)} + H_1\mathbf{w}_{(i,j+1)} + H_2\mathbf{w}_{(i+1,j)}, \qquad (2.68) \\ \mathbf{y}_{(i,j)} &= C\mathbf{x}_{(i,j)}, \qquad (2.69) \end{aligned}$$

$$\begin{aligned} \mathbf{x}_{(i+1,j+1)} &= A_0\mathbf{x}_{(i,j)} + A_1\mathbf{x}_{(i,j+1)} \\ &\quad + A_2\mathbf{x}_{(i+1,j)} + B\mathbf{u}_{(i,j)} + H\mathbf{w}_{(i,j)}, \end{aligned} \tag{2.70}$$

$$\mathbf{y}_{(i,j)} = C\mathbf{x}_{(i,j)}, \tag{2.71}$$

where $\mathbf{w}_{(i,j)}$ is a disturbance which we want to decouple from the output $\mathbf{y}_{(i,j)}$ with a suitable control $\mathbf{u}_{(i,j)} = F\,\mathbf{x}_{(i,j)}$, and H, H_1 and H_2 are matrices of appropriate dimensions. The disturbance decoupling problem (DDP) is studied and solved for 2-D FM first and second order models (2.68-2.69) and (2.70-2.71) by finding conditions ensuring that a feedback law $\mathbf{u}_{(i,j)} = F\,\mathbf{x}_{(i,j)}$ exists, such that the disturbance $\mathbf{w}_{(i,j)}$ does not affect the output function $\mathbf{y}_{(i,j)}$ of the closed-loop system. The other decoupling problem is considered when the disturbance $\mathbf{w}_{(i,j)}$ is measurable. In this case, with the measurable disturbance decoupling problem (MDDP), a feedback law takes the form $\mathbf{u}_{(i,j)} = F\,\mathbf{x}_{(i,j)} + S\,\mathbf{w}_{(i,j)}$.

Proposition 2.3. *[Conte and Perdon (1988)] Consider system (2.68-2.69), and consider the largest output-nulling controlled invariant subspace $\mathcal{V}^*$ of system (2.55). Then,*

(i) *the (DDP) is solvable if*

$$\operatorname{im}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \subseteq \mathcal{V}^* \times \mathcal{V}^*,$$

(ii) *the (MSDP) is solvable if*

$$\operatorname{im}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \subseteq \mathcal{V}^* \times \mathcal{V}^* + \operatorname{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

The following theorem solving the (DDP) and (MSDP) with notion of stability, is considered as a special case of Theorem 5.1 in Ntogramatzidis et al. (2008), with feedthrough matrices, D and G, to be zero.

Proposition 2.4. *[Ntogramatzidis et al. (2008)] Consider system (2.68-2.69), and consider the largest output-nulling controlled invariant subspace $\mathcal{V}^*$ of system (2.55). Then,*

(i) *the (DDP) is solvable if*

- $\operatorname{im}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \subseteq \mathcal{V}^* \times \mathcal{V}^*,$
- *$\mathcal{V}^*$ is both internally and externally stabilisable.*

(ii) *the (MSDP) is solvable if*

- $\operatorname{im}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \subseteq \mathcal{V}^* \times \mathcal{V}^* + \operatorname{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$

- $\mathcal{V}^*$ *is both internally and externally stabilisable.*

When the conditions are satisfied, any output-nulling friend F of $\mathcal{V}^*$, which makes $\mathcal{V}^*$ internally and externally stabilisable, is considered a solution of the problem.

Proposition 2.5. *[Ntogramatzidis (2012)] Consider system (2.70-2.71), and consider the largest output-nulling subspace of feedback type $\mathcal{W}^*$ of system (2.57). Then,*

(i) *the (DDP) is solvable if and only if*

$$\operatorname{im} H \subseteq \mathcal{W}^*,$$

(ii) *the (MSDP) is solvable if and only if*

$$\operatorname{im} H \subseteq \mathcal{W}^* + \operatorname{im} B.$$

CHAPTER 3

Geometric approach for N-D second-order Fornasini-Marchesini state space models

3.1 Structural invariants for N-D model

In this chapter, we generalise a geometric approach of what is normally referred in the literature as Fornesini-Marchesini second order models, where we have A_0, A_1, A_2 and a single matrix B. In this case, this generalisation is called N-D second-order Fornesini-Marchesini models. The generalised a N-D model of a linear discrete system proposed by Kaczorek (1992) can be written as

$$\begin{aligned}
\mathbf{x}_{i_1+1,i_2+1,\ldots,i_N+1} &= A_0\,\mathbf{x}_{i_1,i_2,\ldots,i_N} + \sum_{j=1}^{N} A_j\,\mathbf{x}_{i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_N} \\
&\quad + \sum_{1\leqslant j<k\leqslant N} A_{jk}\,\mathbf{x}_{i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_{k-1},i_k+1,i_{k+1},\ldots,i_N} + \ldots \\
&\quad + \sum_{j=1}^{N} A_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_N}\,\mathbf{x}_{i_1+1,\ldots,i_{j-1}+1,i_j,i_{j+1}+1,\ldots,i_N+1} \\
&\quad + B\,\mathbf{u}_{i_1,i_2,\ldots,i_N} && (3.1)\\
\mathbf{y}_{i_1,i_2,\ldots,i_N} &= C\,\mathbf{x}_{i_1,i_2,\ldots,i_N}, && (3.2)
\end{aligned}$$

where, for all $i_1, i_2, ..., i_N \in \mathbb{Z}$, the vector $\mathbf{x}_{i_1,i_2,\ldots,i_N}$ is the local state which belongs to $\mathbb{R}^n$, the input $\mathbf{u}_{i_1,i_2,\ldots,i_N}$ and the output $\mathbf{y}_{i_1,i_2,\ldots,i_N}$ belong to $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively. The matrices $A_0, A_j, A_{jk}, \ldots, A_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_N}, B$ and C are real matrices with appropriate dimensions. We can also provide an alternative way of writing (3.1-3.2) in a more compact way, by defining $S_N = \{z \in \mathbb{N}^N \;|\; \|z\|_\infty \leq 1 \text{ and } \|z\|_2^2 \; < N\}$ as follows:

$$\mathbf{x}_{(i_1+1,i_2+1,\ldots,i_N+1)} = \sum_{\ell \in S_N} A_\ell \, \mathbf{x}_{(i_1,i_2,\ldots,i_N)+\ell} + B \, \mathbf{u}_{(i_1,i_2,\ldots,i_N)} \tag{3.3}$$

$$\mathbf{y}_{(i_1,i_2,\ldots,i_N)} = C \, \mathbf{x}_{(i_1,i_2,\ldots,i_N)}, \tag{3.4}$$

where the notation $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ denotes the local state and $\mathbf{u}_{(i_1,i_2,\ldots,i_N)}$ denotes the input function. A_ℓ, B and C are real matrices of suitable dimensions. In the model considered, $\ell_0 = \underbrace{(0, 0, \ldots, 0)}_{N\ times}$ is the zero element of S_N, ℓ_i for $i = \{1, \ldots, card(S_N)\}$ and the elements of this set are $\ell = (\ell(1), \ell(2), \ldots, \ell(N))$, such that $\ell(i)$ denotes the i-th element of ℓ from the left. For example, if $\ell = (1, 0, 0, 1)$, then $\ell(1) = 1$, $\ell(2) = 0$, $\ell(3) = 0$, and $\ell(4) = 1$. In the sequel, we will consider S_N as a set of all indexes in (3.3-3.4). For brevity, we denote system (3.3-3.4) by $\Sigma = (A_\ell, B, C)$.

For defining an appropriate boundary conditions for (3.3), we introduce the following sets

$$\begin{aligned}
\mathfrak{Q}_i \overset{\text{def}}{=} & \underbrace{(\{i\} \times \{j \in \mathbb{Z} \mid j \geq i\} \times \{j \in \mathbb{Z} \mid j \geq i\} \times \ldots \times \{j \in \mathbb{Z} \mid j \geq i\})}_{N\ times} \\
& \cup \underbrace{(\{j \in \mathbb{Z} \mid j \geq i\} \times \{i\} \times \{j \in \mathbb{Z} \mid j \geq i\} \times \ldots \times \{j \in \mathbb{Z} \mid j \geq i\})}_{N\ times} \cup \ldots \\
& \cup \underbrace{(\{j \in \mathbb{Z} \mid j \geq i\} \times \{j \in \mathbb{Z} \mid j \geq i\} \times \ldots \times \{j \in \mathbb{Z} \mid j \geq i\} \times \{i\})}_{N\ times}.
\end{aligned}$$

By assigning the local state $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ for $(i_1, i_2, ..., i_N) \in \mathfrak{Q}_0$, we obtain a suitable set of boundary conditions given for an arbitrary vector $\bar{\mathbf{x}}_{(i_1,i_2,\ldots,i_N)}$ by

$$\mathbf{x}_{(i_1,i_2,\ldots,i_N)} = \bar{\mathbf{x}}_{(i_1,i_2,\ldots,i_N)} \in \mathbb{R}^n \qquad \text{for all } (i_1, i_2, ..., i_N) \in \mathfrak{Q}_0. \tag{3.5}$$

In this case, by considering $\mathcal{W}$, any subspace of $\mathbb{R}^n$, we say that the boundary condition is a $\mathcal{W}$-valued boundary condition of (3.3), if $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ belongs to $\mathcal{W}$ for all $(i_1, i_2, ..., i_N)$ $\in \mathfrak{Q}_0$. Equivalently, if $\mathbf{x}_{(i_1,i_2,\ldots,i_N)} \in \mathcal{W}$ for all $(i_1, i_2, ..., i_N) \in \mathfrak{Q}_i$, then, (3.3) also has a $\mathcal{W}$-valued solution.

3.1.1 N-D Invariant subspaces

Let us now consider (3.3) with $B = 0$, i.e.,

$$\mathbf{x}_{(i_1+1,i_2+1,\ldots,i_N+1)} = \sum_{\ell \in S_N} A_\ell \, \mathbf{x}_{(i_1,i_2,\ldots,i_N)+\ell}. \tag{3.6}$$

We begin our investigation by introducing the notion of invariance for this model.

Definition 28. A subspace $\mathcal{I} \subseteq \mathbb{R}^n$ is called an invariant for (3.6), if

$$A_\ell \mathcal{I} \subseteq \mathcal{I}, \qquad \forall\, \ell \in S_N. \tag{3.7}$$

where A_ℓ, for all $\ell \in S_N$ are matrices representation of linear maps in particular bases.

The origin $\{0\}$ and $\mathcal{X}$ are both obviously N-D invariant. In addition, the intersection and addition of N-D invariant subspaces is N-D invariant. Indeed, given two N-D invariant subspaces, $\mathcal{I}_1$ and $\mathcal{I}_2$, by virtue of (3.7), we find

$$A_\ell\,(\mathcal{I}_1 \cap \mathcal{I}_2) \;\subseteq\; A_\ell\,\mathcal{I}_1 \cap A_\ell\,\mathcal{I}_2 \subseteq \mathcal{I}_1 \cap \mathcal{I}_2,$$

and also by virtue of (3.7):

$$A_\ell\,(\mathcal{I}_1 + \mathcal{I}_2) \;=\; A_\ell\,\mathcal{I}_1 + A_\ell\,\mathcal{I}_2 \subseteq \mathcal{I}_1 + \mathcal{I}_2.$$

Hence, the family of all N-D invariant subspaces of $\mathcal{X}$ under subspace addition and intersection is closed, so that the addition of all invariant subspaces of this family gives the maximum element which is $\mathcal{X}$. The minimum is the intersection of these invariant subspaces, which is $\{0\}$. In addition, from (3.7), we obtain the equal condition: that is equivalent to (3.7)

$$\sum_{\ell \in S_N} A_\ell \mathcal{I} \subseteq \mathcal{I}. \tag{3.8}$$

The connection between the geometric definition of invariance in Definition 28 and the solutions of (3.6) is given by the following Lemma.

Lemma 3.1. *A subspace $\mathcal{I}$ of $\mathbb{R}^n$ is invariant for (3.6), if and only if for any $\mathcal{I}$-valued boundary conditions, the model (3.6) admits an $\mathcal{I}$-valued solution.*

Proof: Let $\mathcal{I}$ be invariant for (3.6). Consider the arbitrary $\mathcal{I}$-value boundary conditions. We want to show that for all $\ell \in S_N$, the solution $\mathbf{x}_\ell$ of (3.6) is such that $\mathbf{x}_\ell \in \mathcal{I}$. By virtue of (3.8), we can write $\mathbf{x}_{(1,1,\ldots,1)}$ as

$$\mathbf{x}_{(1,1,\ldots,1)} = \sum_{\ell \in S_N} A_\ell\, \mathbf{x}_\ell.$$

For all $\ell \in S_N$, we have $A_\ell \mathbf{x}_\ell \in \mathcal{I}$. Since $\mathbf{x}_\ell \in \mathcal{I}$, the right hand side is in $\mathcal{I}$, therefore, $\mathbf{x}_{(1,1,\ldots,1)} \in \mathcal{I}$. This process can be repeated for all vectors of the $\mathcal{I}$-valued boundary condition. $\mathcal{I}$-valued solution for equation (3.6) is constructed by involving only vectors from $\mathcal{I}$ for $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}_j$. By continuing in this manner recursively, a solution can be obtained for $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}_j$, for $j \geqslant 0$. Now we show the converse. For all $\mathcal{I}$-value boundary conditions, the corresponding solutions $\mathbf{x}_\ell$ of equation (3.6) are all contained in $\mathcal{I}$, which means that $\mathcal{I}$ is invariant. Suppose that $\mathcal{I}$ is not invariant, so that there exists $\mathbf{x}_\ell \in \mathcal{I}$ for all $\ell \in S_N$, for which $\sum_{\ell\in S_N} A_\ell\,\mathbf{x}_\ell \notin \mathcal{I}$. Obviously, for $\mathbf{x}_{(1,1,\dots,1)} \in \mathcal{I}$, a solution of equation (3.6) cannot be exist. As such, with arbitrary $\mathcal{I}$-valued boundary conditions model (3.6) has $\mathcal{I}$-valued trajectory, then, equation (3.8) automatically holds. ■

We can also provide an alternative characterisation of invariance, in terms of existence, a suitable $\mathcal{I}$-valued solution of equation (3.6) as follows. Let J be a basis matrix for the invariant subspace $\mathcal{I}$, there exist matrices X_ℓ for all $\ell \in S_N$ such that

$$A_\ell\,J = J\,X_\ell. \tag{3.9}$$

Consider an $\mathcal{I}$-valued boundary condition $\mathbf{x}_{(i_1,i_2,\dots,i_N)}$ of (3.6). Then, for each $(i_1,i_2,\dots,i_N)$ $\in \mathfrak{Q}_0$, we can write $\mathbf{x}_{(i_1,i_2,\dots,i_N)} = J\,\boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)}$ for some $\boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)}$.
As such, if $(i_1,i_2,\dots,i_N)+\ell \in \mathfrak{Q}_0$, for all $\ell \in S_N$, we can write in view of equation (3.9):

$$\begin{aligned}\mathbf{x}_{(i_1+1,i_2+1,\dots,i_N+1)} &= \sum_{\ell\in S_N} A_\ell\,J\,\boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)+\ell}\\ &= \sum_{\ell\in S_N} J\,X_\ell\,\boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)+\ell}.\end{aligned} \tag{3.10}$$

Now, we define the reduced N-D Fornasini-Marchesini model as

$$\boldsymbol{\xi}_{(i_1+1,i_2+1,\dots,i_N+1)} = X_\ell\,\boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)+\ell}.$$

Then, by substitution, one can verify that $\mathbf{x}_{(i_1,i_2,\dots,i_N)} = J\,\boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)}$ is a solution of equation (3.6) and is $\mathcal{I}$-valued.

3.1.2 N-D controlled invariance

In this part, the notion of controlled invariance will be introduced. Generally speaking, controlled invariant subspaces can be considered as a generalisation of the concept of invariants, in the sense that, while invariant subspaces are the subspaces of the state space where the solution generated by equation (3.6) lies, controlled invariant subspaces are the subspaces of solutions generated by

$$\mathbf{x}_{(i_1+1,i_2+1,\dots,i_N+1)} = \sum_{\ell\in S_N} A_\ell\,\mathbf{x}_{(i_1,i_2,\dots,i_N)+\ell} + B\,\mathbf{u}_{(i_1,i_2,\dots,i_N)}. \tag{3.11}$$

Definition 29. A subspace $\mathcal{V}$ of $\mathcal{X}$ is said to be an N-D controlled invariant subspace for model (3.11), if it is at the same time (A_ℓ, B)-controlled invariant, for all $\ell \in S_N$ in the 1-D counterpart, i.e., if

$$A_\ell \mathcal{V} \subseteq \mathcal{V} + \text{im } B, \qquad \forall\, \ell \in S_N. \tag{3.12}$$

The following facts are easy to establish:

(i) Both $\mathcal{X} = \mathbb{R}^n$ and $\{0\}$ are N-D controlled invariant subspaces;

(ii) when $B = 0$, the notion of N-D controlled invariance reduces to the notion of N-D invariance. In addition, any N-D invariant subspace $\mathcal{V}$ is an N-D controlled invariant subspace for any matrix B, but not vice-versa,

(iii) the addition of the N-D controlled invariant subspaces is N-D controlled invariant. Indeed, using equation (3.12), we immediately see that given two N-D controlled invariant subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, we find

$$A_\ell\,(\mathcal{V}_1 + \mathcal{V}_2) \;=\; A_\ell\,\mathcal{V}_1 + A_\ell\,\mathcal{V}_2 \subseteq (\mathcal{V}_1 + \text{im } B) + (\mathcal{V}_2 + \text{im } B) = (\mathcal{V}_1 + \mathcal{V}_2) + \text{im } B,$$

which implies that $\mathcal{V}_1 + \mathcal{V}_2$ is itself N-D controlled invariant. However, since the intersection of two controlled invariant subspaces is not a controlled invariant subspace in a 1-D case (Basile & Marro, 1992), so in the 2-D setting, the same result is true (Ntogramatzidis, 2012). Moreover, the intersection of two N-D controlled invariant subspaces is not in general considered a N-D controlled invariant subspace also. Indeed, using equation (3.12) and the chain of inclusions,

$$A_\ell\,(\mathcal{V}_1 \cap \mathcal{V}_2) \;\subseteq\; A_\ell\,\mathcal{V}_1 \cap A_\ell\,\mathcal{V}_2 \subseteq (\mathcal{V}_1 + \text{im } B) \cap (\mathcal{V}_2 + \text{im } B) \supseteq (\mathcal{V}_1 + \mathcal{V}_2) + \text{im } B,$$

we cannot conclude that $A_\ell\,(\mathcal{V}_1 \cap \mathcal{V}_2) \subseteq (\mathcal{V}_1 + \mathcal{V}_2) + \text{im } B$.

The following theorem is a simple extension of the results that appear in Karamancioglu and Lewis (1992) and Ntogramatzidis (2012), and it presents the basic system-theoretic explanation of the previous definition.

Theorem 3.1. *Let $\mathcal{V}$ be a subspace of $\mathcal{X}$. A $\mathcal{V}$-valued solution of equation (3.11) exists for any $\mathcal{V}$-valued boundary condition, if and only if $\mathcal{V}$ is an N-D controlled invariant subspace.*

Proof: (**Sufficiency**). Suppose equation (3.12) holds. Let $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ be $\mathcal{V}$-valued on the k-th forward boundary $\mathfrak{Q}_k$. Consider equation (3.11), where all the indexes are in $\mathfrak{Q}_k$. Let $\mathbf{z}_k = \mathbf{x}_{(i_1,i_2,\ldots,i_N)+\ell}$ for all $k \in \{0, 1, \ldots, 2^N - 2\}$ and $\ell \in S_N$. By virtue of (3.12), two vectors $\mathbf{x}$ and $\mathbf{w}$ exist from $\mathcal{V}$ and $\mathcal{X}$ respectively, such that

$$A_\ell\,\mathbf{z}_k = \mathbf{x} + B\,\mathbf{w}.$$

Therefore, by taking $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = -\mathbf{w}$, the solution for (3.11) exists and is $\mathcal{V}$-valued on the next boundary $\mathfrak{Q}_{k+1}$. By induction, a control function $\mathbf{u}_{(i_1,i_2,\ldots,i_N)}$ exists for a given $\mathcal{V}$-valued boundary condition, such that $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ exists and it is $\mathcal{V}$-valued for all $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}$.

(**Necessity**). Assume that equation (3.11) does not hold, so that, there exists $\boldsymbol{\xi}_q \in \mathcal{V}$ for all $q \in \{0, 1, \ldots, 2^N - 2\}$, such that there is no $\mathbf{w} \in \mathbb{R}^m$ and $A_\ell\, \boldsymbol{\xi}_q + B\,\mathbf{w} \in \mathcal{V}$ holds. Hence, a control $\mathbf{u}_{\ell_0}$ cannot be found with a boundary condition $\mathbf{x}_\ell = \boldsymbol{\xi}_q$ such that $\mathbf{x}_{(1,1,\ldots,1)} \in \mathcal{V}$. Consequently, a $\mathcal{V}$-valued solution of equation (3.11) does not exist for the $\mathcal{V}$-valued boundary condition. ■

3.1.3 N-D output-nulling subspaces

The controlled invariant subspace contained in $\ker C$ is particularly important because it provides the basis to solve a number of control problems, especially the disturbance decoupling problem (Basile & Marro, 1992). A particular case of N-D controlled invariance subspace is an N-D output-nulling subspace, which is that for any $\mathcal{V}$-valued boundary condition, the system $\sum = (A_\ell, B, C)$ has a $\mathcal{V}$-valued solution to yield a zero output. Stated differently, a zero output can be obtained by solving the system $\sum = (A_\ell, B, C)$, if and only if the local state $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ remains in $\ker C$ for all $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}_i$. Therefore, an N-D output-nulling subspace is simply a N-D controlled invariant subspace contained in the null space of C. The family of all the N-D output-nulling subspaces of $\sum = (A_\ell, B, C)$ is closed under subspace addition. However, it is not closed under subspace intersection. Hence, this family has a maximum given by the addition of all these elements and is denoted by $\mathcal{V}^*$. The famous algorithm to compute $\mathcal{V}^*$ is introduced in Basile and Marro (1969) in the 1-D case and the other is introduced in Ntogramatzidis (2012) in the 2-D setting. Both of them are extended to the N-D systems by the following lemma.

Lemma 3.2. *The subspace $\mathcal{V}^*$ coincides with the last term of the sequence:*

$$\begin{cases} \mathcal{V}_0 = \ker C, \\ \mathcal{V}_q = \bigcap_{\ell \in S_N} A_\ell^{-1}\,(\mathcal{V}_{q-1} + \operatorname{im} B) \cap \ker C, \qquad q \in \{1, \ldots, \kappa\}, \end{cases} \tag{3.13}$$

where the value $\kappa \le n-1$ is determined by the condition $\mathcal{V}_{\kappa+1} = \mathcal{V}_\kappa$, i.e., $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \mathcal{V}_2 \supset \ldots \supset \mathcal{V}_\kappa = \mathcal{V}_{\kappa+1} = \mathcal{V}^*$.

Proof: Firstly, by induction, we show that sequence (3.13) is monotonically nonincreasing. Clearly, $\mathcal{V}_0 \supseteq \mathcal{V}_1$. To prove $\mathcal{V}_h \supseteq \mathcal{V}_{h+1}$, we assume that $\mathcal{V}_{h-1} \supseteq \mathcal{V}_h$. From our assumption we get

$$\begin{aligned}\mathcal{V}_h &= \bigcap_{\ell \in S_N} A_\ell^{-1}\,(\mathcal{V}_{h-1} + \operatorname{im} B) \cap \ker C \\ &\supseteq \bigcap_{\ell \in S_N} A_\ell^{-1}\,(\mathcal{V}_h + \operatorname{im} B) \cap \ker C = \mathcal{V}_{h+1}.\end{aligned}$$

Thus, the sequence (3.13) is monotonically nonincreasing. It is clear that if $\mathcal{V}_{\kappa+1} = \mathcal{V}_\kappa$ holds, then $\mathcal{V}_j = \mathcal{V}_\kappa$ holds also for all $j \geq \kappa$. Since two subspaces that are subsequent in (3.13) are coincident, if and only if they have the same dimension. The dimension of $\mathcal{V}_0$ is at least one, then, $\mathcal{V}_{\kappa+1}$ must have a dimension at least one less than the dimension $\mathcal{V}_\kappa$, before reaching the stationarity. Hence, in at most n steps, the stationarity of the sequence of subspaces (3.13) is reached. Let the index where sequence (3.13) becomes stationary be n. Our aim now is to show that $\mathcal{V}_\kappa$ is an N-D controlled invariant subspace. Begin with

$$\mathcal{V}_\kappa = \bigcap_{\ell \in S_N} A_\ell^{-1}\,(\mathcal{V}_\kappa + \operatorname{im} B) \cap \ker\ \ C. \tag{3.14}$$

Multiplying both sides of equation (3.14) by A_h with $h \in S_N$, we find

$$\begin{aligned}A_h\,\mathcal{V}_\kappa &= A_h(\bigcap_{\ell \in S_N} A_\ell^{-1}\,(\mathcal{V}_\kappa + \operatorname{im} B) \cap \ker\ \ C) \\ &\subseteq \bigcap_{\ell \in S_N} A_h\,A_\ell^{-1}\,(\mathcal{V}_\kappa + \operatorname{im} B) \cap A_h\ \ker\ \ C \\ &\subseteq A_h\,A_h^{-1}\,(\mathcal{V}_\kappa + \operatorname{im} B) \subseteq (\mathcal{V}_\kappa + \operatorname{im} B) \cap \operatorname{im} A_h \subseteq \mathcal{V}_\kappa + \operatorname{im} B.\end{aligned}$$

Obviously, $\mathcal{V}_\kappa$ is an (A_h, B)-controlled invariant subspace with $h \in S_N$ in the the 1-D case. Therefore, it is an N-D controlled invariant subspace. By the same way of constructing equation (3.13), $\mathcal{V}_q \subseteq \ker\ C$, so that $\mathcal{V}_\kappa$ is also N-D output-nulling. To prove $\mathcal{V}_\kappa$ is the largest N-D output-nulling subspace for equation (3.11), which coincides with $\mathcal{V}^*$, we suppose that $\tilde{\mathcal{V}}$ is another N-D output-nulling subspace. Then, $\tilde{\mathcal{V}}$ is an N-D controlled invariant subspace, which is an (A_h, B)-controlled invariant subspace for $h \in S_N$ in the 1-D case contained in ker C. Hence, $\tilde{\mathcal{V}} \subseteq A_h^{-1}(\tilde{\mathcal{V}} + \operatorname{im}\ B)$. Since this is true for each $h \in S_N$, we get

$$\tilde{\mathcal{V}} \subseteq \bigcap_{\ell \in S_N} A_\ell^{-1}\,(\tilde{\mathcal{V}} + \operatorname{im} B) \cap \ker\ \ C. \tag{3.15}$$

Now, we prove that each term of sequence (3.13) contains $\tilde{\mathcal{V}}$, then, specially, $\mathcal{V}_\kappa \supseteq \tilde{\mathcal{V}}$. From sequence (3.13), it is clear that $\mathcal{V}_0 \supseteq \tilde{\mathcal{V}}$, and to this end, we suppose that $\mathcal{V}_q \supseteq \tilde{\mathcal{V}}$.

Hence

$$\begin{aligned}\mathcal{V}_{q+1} &= \bigcap_{\ell \in S_N} A_\ell^{-1}(\mathcal{V}_q + \operatorname{im} B) \cap \ker\ C\\ &\subseteq \bigcap_{\ell \in S_N} A_\ell^{-1}(\tilde{\mathcal{V}} + \operatorname{im} B) \cap \ker\ C,\end{aligned}$$

in view of (3.15), this includes $\tilde{\mathcal{V}}$. Therefore, $\mathcal{V}_\kappa \supseteq \tilde{\mathcal{V}}$. This implies that $\mathcal{V}^* = \mathcal{V}_\kappa$. ∎

3.1.4 N-D controlled invariant subspaces of feedback type

From structure of the model (3.11) we noticed that this model is closed under the feedback input $\mathbf{u}_{(i_1,i_2,\dots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\dots,i_N)}$, which guarantees the existence of a closed-loop equation given by

$$\mathbf{x}_{(i_1+1,i_2+1,\dots,i_N+1)} = (A_{\ell_0} + B\,F)\,\mathbf{x}_{(i_1,i_2,\dots,i_N)} + \sum_{\ell \in S_N \setminus \{\ell_0\}} A_\ell\,\mathbf{x}_{(i_1,i_2,\dots,i_N)+\ell}. \tag{3.16}$$

The closed-loop equation (3.16) shows that the definition of N-D controlled invariance alone does not automatically ensure the existence of a feedback matrix F, which keeps the state evolutions $\mathbf{x}_{(i_1,i_2,\dots,i_N)}$ on an N-D controlled invariant subspace $\mathcal{V}$ for $\mathcal{V}$-valued boundary conditions. For this reason, the concept of N-D controlled invariance of the feedback type is introduced as a simple extension of what was done in Ntogramatzidis (2012).

Definition 30. Subspace $\mathcal{W}$ is an N-D controlled invariant of the feedback type for equation (3.11) if

- $\mathcal{W}$ is a 1-D controlled invariant subspace for (A_{ℓ_0}, B);
- for all $\ell \in S_N \setminus \{\ell_0\}$, $\mathcal{W}$ is A_ℓ-invariant.

Theorem 3.2. *A subspace $\mathcal{W}$ is an N-D controlled invariant subspace of the feedback type for equation (3.11), if and only if a static feedback input $\mathbf{u}_{(i_1,i_2,\dots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\dots,i_N)}$ exist, such that for any $\mathcal{W}$-valued boundary condition, model (3.11) admits a $\mathcal{W}$-valued solution for all $(i_1, i_2, \dots, i_N) \in \mathfrak{Q}$.*

Proof: (**Only if**). From the previous definition, since $\mathcal{W}$ is an N-D controlled invariant subspace of the feedback type for equation (3.11), it is also a 1-D controlled invariant subspace for (A_{ℓ_0}, B), and this implies that there exist two matrices X_{ℓ_0} and Ω, such that $A_{\ell_0} W = W X_{\ell_0} + B\,\Omega$, where W is a basis matrix of $\mathcal{W}$ (Trentelman, Stoorvogel, & Hautus, 2012). Furthermore, since $\mathcal{W}$ is A_ℓ-invariant, the matrices X_ℓ, $\ell \in S_N \setminus \{\ell_0\}$ exist, such that $A_\ell W = W X_\ell$. Since $\operatorname{im} W = \mathcal{W}$, then a linear equation

$\Omega = -F\,W$ can be solved in F. We obtain the equation $(A_{\ell_0} + BF)\,W = W\,X_{\ell_0}$. The states $\mathbf{x}_{(i_1,i_2,\ldots,i_N)+\ell}$ are in $\mathcal{W}$ with the matrix F in the closed-loop system (3.16), so is $\mathbf{x}_{(i_1+1,i_2+1,\ldots,i_N+1)}$, and then for any $\mathcal{W}$-valued boundary condition, the state $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ lies in $\mathcal{W}$ for all $(i_1,i_2,\ldots,i_N) \in \mathfrak{Q}$.

(**If**). Suppose that the inclusion $[\ (A_{\ell_0}+BF) \quad A_{\ell_1} \quad A_{\ell_2} \quad \ldots \quad A_{\ell_{2^N-2}}\ ](\bigoplus_1^{2^N-1}\mathcal{W}) \subseteq \mathcal{W}$ does not hold, then, there exists $\mathbf{x}_\ell \in \mathcal{W}$ such that $\mathbf{x}_{(1,1,\ldots,1)}$ does not lie on $\mathcal{W}$. This implies that $\mathcal{W}$ is not $(A_{\ell_0}+BF)$-controlled invariant and is not A_ℓ-invariant in a 1-D sense, for all $\ell \in S_N \backslash \{\ell_0\}$. Thus, the previous inclusion must hold. ■

3.1.5 N-D output-nulling subspaces of the feedback type

An N-D output-nulling subspace of the feedback type $\mathcal{W}$ is a N-D controlled invariant subspace of the feedback type contained in ker C. Furthermore, $\mathcal{W}$ is N-D output-nulling of the feedback type, if and only if there exists a matrix F, such that for any $\mathcal{W}$-valued boundary condition, model (3.11) has $\mathcal{W}$-valued to yield a zero output $\mathbf{y}_{(i_1,i_2,\ldots,i_N)}$, for all $(i_1,i_2,\ldots,i_N) \in \mathfrak{Q}$. The family of all N-D output-nulling subspaces of the feedback type is denoted by $\mathcal{W}\,(\Sigma)$. This family is closed under subspace addition. Therefore, the addition of all these subspaces is the maximum of $\mathcal{W}\,(\Sigma)$, which is N-D output-nulling subspaces of the feedback type $\mathcal{W}^*$.

Lemma 3.3. *The subspace $\mathcal{W}^*$ coincides with the last term of sequence:*

$$\begin{cases} \mathcal{W}_0 = \ker C, \\ \mathcal{W}_q = \bigcap_{\ell\in S_N\backslash\{\ell_0\}} A_\ell^{-1}\,\mathcal{W}_{q-1} \cap A_{\ell_0}^{-1}\,(\mathcal{W}_{q-1} + \operatorname{im} B) \cap \ker C, \quad q \in \{1,\ldots,\kappa\}, \end{cases}$$

where the value $\kappa \leq n-1$ is determined by the condition $\mathcal{W}_{\kappa+1} = \mathcal{W}_\kappa$, i.e., $\mathcal{W}_0 \supset \mathcal{W}_1 \supset \mathcal{W}_2 \supset \ldots \supset \mathcal{W}_\kappa = \mathcal{W}_{\kappa+1} = \mathcal{W}^*$.

The proof of this lemma comes directly from Lemma 3.2, with obvious modifications.

3.1.6 Disturbance decoupling problem (DDP)

The notion of the N-D output-nulling subspace of the feedback type plays a significant part in solving the problem of disturbance decoupling without the stability requirement. The model to consider is

$$\mathbf{x}_{(i_1+1,i_2+1,\ldots,i_N+1)} = \sum_{\ell\in S_N} A_\ell\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)+\ell} + B\,\mathbf{u}_{(i_1,i_2,\ldots,i_N)} + H\,\mathbf{w}_{(i_1,i_2,\ldots,i_N)}, \tag{3.17}$$

$$\mathbf{y}_{(i_1,i_2,\ldots,i_N)} = C\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}, \tag{3.18}$$

where $\mathbf{w}_{(i_1,i_2,\dots,i_N)}$ represents a non-measurable disturbance. Using a feedback law $\mathbf{u}_{(i_1,i_2,\dots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\dots,i_N)}$, the closed-loop system is given by

$$\begin{aligned}\mathbf{x}_{(i_1+1,i_2+1,\dots,i_N+1)} &= (A_{\ell_0} + B\,F)\,\mathbf{x}_{(i_1,i_2,\dots,i_N)} + \sum_{\ell\in S_N\setminus\{\ell_0\}} A_\ell\,\mathbf{x}_{(i_1,i_2,\dots,i_N)+\ell} \\ &+ H\,\mathbf{w}_{(i_1,i_2,\dots,i_N)}.\end{aligned} \tag{3.19}$$

The problem of disturbance decoupling is solved by finding conditions ensuring that a feedback law $\mathbf{u}_{(i_1,i_2,\dots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\dots,i_N)}$ exists, such that the output of the closed-loop system is not affected by the disturbance $\mathbf{w}_{(i_1,i_2,\dots,i_N)}$. The other decoupling problem is considered when the disturbance $\mathbf{w}_{(i_1,i_2,\dots,i_N)}$ is measurable. In this case, with the measurable disturbance decoupling problem (MDDP), a feedback law takes the form $\mathbf{u}_{(i_1,i_2,\dots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\dots,i_N)} + S\,\mathbf{w}_{(i_1,i_2,\dots,i_N)}$. In an N-D framework, this problem is solved by the following theorem.

Theorem 3.3. *Consider the largest N-D output-nulling subspaces of feedback type $\mathcal{W}^*$ of system (3.16). Then, the (DDP) is solvable, if and only if*

$$\operatorname{im} H \subseteq \mathcal{W}^*. \tag{3.20}$$

Proof: (**If**). Let F be a feedback type output-nulling friend of $\mathcal{W}^*$. Then, equation (3.17) ensures that for each $\mathcal{W}^*$-valued boundary condition, the local state $\mathbf{x}_{(i_1,i_2,\dots,i_N)}$ remains on $\mathcal{W}^*$ for all $(i_1, i_2, \dots, i_N) \in \mathfrak{Q}$. Therefore, it is contained in the null-space of C, which means that the system is decoupled by the disturbance $\mathbf{w}_{(i_1,i_2,\dots,i_N)}$.
(**Only if**). To prove this part, we suppose that (3.17) is decoupled by the disturbance $\mathbf{w}_{(i_1,i_2,\dots,i_N)}$, which means that there exists a matrix F, such that the output $\mathbf{y}_{(i_1,i_2,\dots,i_N)}$ is identically zero. In the situation of zero disturbance equation (3.17) is still disturbance decoupled, i.e., $\mathbf{x}_{(i_1,i_2,\dots,i_N)}$ remains on the largest subspace $\mathcal{W}^*$ for all $\mathbf{x}_{(i_1,i_2,\dots,i_N)}$, which satisfies the inclusion

$$[\ (A_{\ell_0} + BF) \quad A_{\ell_1} \quad A_{\ell_2} \quad \dots \quad A_{\ell_{2^N-2}}\ ](\bigoplus_{1}^{2^N-1} \mathcal{W}^*) \subseteq \ker(C).$$

Since the system must be disturbance decoupling, then, the inclusion im $H \subseteq \mathcal{W}^*$ must be satisfied with matrix H. ■

Theorem 3.4. *The problem of disturbance decoupling, with a measurable disturbance (MDDP), admits solution, if and only if*

$$\operatorname{im} H \subseteq \mathcal{W}^* + \operatorname{im} B. \tag{3.21}$$

Proof: (**If**). Suppose equation (3.21) holds. Then, this inclusion can be written as im $H = \mathcal{W}_1 + \mathcal{W}_2$, where $\mathcal{W}_1 \subseteq \mathcal{W}^*$ and $\mathcal{W}_2 \subseteq$ im B. Through this decomposition, the closed-loop system with the control input $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)} + S\,\mathbf{w}_{(i_1,i_2,\ldots,i_N)}$ becomes

$$\begin{aligned}\mathbf{x}_{(i_1+1,i_2+1,\ldots,i_N+1)} &= (A_{\ell_0} + B\,F)\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)} + \sum_{\ell \in S_N \setminus \{\ell_0\}} A_\ell\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)+\ell} \\ &+ (B\,S + W_2)\mathbf{w}_{(i_1,i_2,\ldots,i_N)} + W_1\,\mathbf{w}_{(i_1,i_2,\ldots,i_N)},\end{aligned}$$

where $\operatorname{im} W_1 = \mathcal{W}_1$ and $\operatorname{im} W_2 = \mathcal{W}_2$. Since $\operatorname{im} W_2 = \mathcal{W}_2 \subseteq \operatorname{im} B$, matrix S can be selected to satisfy $B\,S + W_2 = 0$. Also, by choosing F to be a feedback type output-nulling friend of $\mathcal{W}^*$, the local state lies on $\mathcal{W}^*$, and the output $\mathbf{y}_{(i_1,i_2,\ldots,i_N)}$ is identically zero for a $\mathcal{W}^*$-valued boundary condition.

(**Only if**). It is obvious. ■

Now, we will use the notion of duality to introduce the dual of all the previous subspaces.

3.1.7 N-D conditioned invariance

N-D conditioned invariance is the dual concept of N-D controlled invariance. Moreover, the notion of N-D conditioned invariance leads to the definition of certain subspaces, usually referred to as N-D conditioned invariant subspaces, which are related to the reconstructing of the local state of the system by observers that have no access to the system's control input.

Definition 31. A subspace $\mathcal{S}$ is a N-D conditioned invariant subspace for Σ, if it is at the same time (C, A_ℓ)-conditioned invariant, for all $\ell \in S_N$, which is equivalent to

$$A_\ell(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}, \qquad \forall\, \ell \in S_N. \tag{3.22}$$

The following lemma presents a formula of conditioned invariance in terms of duality. The dual system of the system Σ is identified by $\Sigma^\top = (A_\ell^\top, C^\top, B^\top)$.

Lemma 3.4. *The orthogonal complement of a N-D controlled invariant subspace for Σ is a N-D conditioned invariant subspace for $\Sigma^\top$, and vice-versa.*

Proof: Consider $\mathcal{H}$ is a N-D controlled invariant subspace for the system Σ. From $A_\ell \mathcal{H} \subseteq \mathcal{H} + \operatorname{im} B$, for all $\ell \in S_N$, we obtain $A_\ell^\top (\mathcal{H} + \operatorname{im} B)^\perp \subseteq \mathcal{H}^\perp$, which in turn generates $A_\ell^\top (\mathcal{H}^\perp \cap \operatorname{im} B^\top) \subseteq \mathcal{H}^\perp$. Thus, $\mathcal{H}^\perp$ is a N-D conditioned invariant subspace for the system $\Sigma^\top$. The opposite implication holds with the same previous steps reversed. ■

3.1.8 N-D input-containing subspace

The N-D output-nulling subspace has a dual, named a N-D input-containing subspace. A N-D input-containing subspace $\mathcal{S}$ is a N-D conditioned invariant subspace containing im B. The family of all N-D input-containing subspaces is denoted by $\mathcal{S}(\Sigma)$. The intersection of all these subspaces is the minimum of this family. It is denoted by $\mathcal{S}^*$ and it also dualizes the subspace $\mathcal{V}^*$. Moreover, the orthogonal complement of the minimum N-D input-containing subspace for $\Sigma^\top$ is the maximum N-D output-nulling subspace for Σ and vice versa, by using Lemma 3.4, which is equivalent to $(\min\, \mathcal{S}(\Sigma^\top))^\perp = \max\, \mathcal{V}(\Sigma)$. The dualizing of Lemma 3.2 for computing $\mathcal{V}^*$ represents the computation of $\mathcal{S}^*$, as showing in the following algorithm.

Lemma 3.5. *The subspace $\mathcal{S}^*$ is the last term of the monotonically nondecreasing sequence:*

$$\begin{cases} \mathcal{S}_0 = \text{im}\, B, \\ \mathcal{S}_q = \bigcap_{\ell \in S_N} A_\ell\, (\mathcal{S}_{q-1} \cap \ker\, C) + \text{im}\, B, \qquad q \in \{1, \dots, \kappa\}, \end{cases}$$

where the value $\kappa \leq n-1$ is determined by the condition $\mathcal{S}_{\kappa+1} = \mathcal{S}_\kappa$, i.e., $\mathcal{S}_0 \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supset \ldots \supset \mathcal{S}_\kappa = \mathcal{S}_{\kappa+1} = \mathcal{S}^*$.

3.1.9 N-D conditioned invariant subspace of output-injection type

The N-D conditioned invariance of the output-injection type is considered as a dual concept of N-D controlled invariance of the feedback type. This concept is related to the reconstructing of the state vector of the system by observers that have no access to the system's control input. The developed approach in the 1-D and 2-D cases in Willems (1981); Ntogramatzidis (2012) has been closely followed here.

Definition 32. A subspace $\mathcal{Z}$ is said to be an N-D conditioned invariant subspace of output injection type for Σ, if $\mathcal{Z}$ is $(A_{\ell_0} + G\, C)\, \mathcal{Z} \subseteq \mathcal{Z}$ for an output-injection matrix $G \in \mathbb{R}^{n \times p}$, and it satisfies the inclusion $A_\ell\, \mathcal{Z} \subseteq \mathcal{Z}$, for all $\ell \in S_N \backslash \{\ell_0\}$.

In other words, $\mathcal{Z}$ is an N-D conditioned invariant subspace of the output injection type for Σ, if it is (C, A_{ℓ_0})-conditioned invariant in the 1-D case and it is A_ℓ-invariant for all $\ell \in S_N \backslash \{\ell_0\}$.
More about the duality is that the orthogonal complement of a N-D controlled invariant subspace of the feedback type for $\Sigma = (A_\ell, B, C)$ is a N-D conditioned invariant subspace of the output injection type for $\Sigma^\top = (A_\ell^\top, C^\top, B^\top)$ and vice-versa.

3.1.10 N-D input-containing subspaces of the output-injection type

Definition 33. A subspace $\mathcal{Z}$ is said to be a N-D input-containing subspace of the output-injection type if it is (C, A_{ℓ_0})-conditioned invariant, contains the image of B in the 1-D case, and it is A_ℓ-invariant, for all $\ell \in S_N \backslash \{\ell_0\}$.

The orthogonal complement of a N-D output-nulling subspace of the feedback type for $\Sigma = (A_\ell, B, C)$ is a N-D input-containing subspace of the output-injection type for $\Sigma^\top = (A_\ell^\top, C^\top, B^\top)$ and vice-versa.

Definition 34. Consider a full row-rank matrix Q, such that $\mathcal{Z} = \ker Q$, where $\mathcal{Z}$ is a subspace of $\mathcal{X}$. The condition that characterises the fact that a system (3.11) is a $\mathcal{Z}$-observer can be ruled by

$$\mathbf{p}_{(i_1+1,i_2+1,\ldots,i_N+1)} = \sum_{\ell \in S_N} K_\ell \, \mathbf{p}_{(i_1,i_2,\ldots,i_N)+\ell} + L \, \mathbf{y}_{(i_1,i_2,\ldots,i_N)}, \tag{3.23}$$

such that if $\mathbf{p}_{(i_1,i_2,\ldots,i_N)} = Q \, \mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ for all $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}$, then $\mathbf{p}_{(i_1,i_2,\ldots,i_N)} = Q \, \mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ for all $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}$.

In other words, a $\mathcal{Z}$-observer maintains the external components of the local state $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ to $\mathcal{Z}$, i.e., if $\mathbf{p}_{(i_1,i_2,\ldots,i_N)} = \mathbf{x}_{(i_1,i_2,\ldots,i_N)}/\mathcal{Z}$ on the boundary, then $\mathbf{p}_{(i_1,i_2,\ldots,i_N)} = \mathbf{x}_{(i_1,i_2,\ldots,i_N)}/\mathcal{Z}$ everywhere.

Theorem 3.5. *A subspace $\mathcal{Z}$ of $\mathcal{X}$ is an N-D input-containing subspace of the output-injection type for Σ if and only, if a $\mathcal{Z}$-observer for Σ exists.*

Proof: (**Only if**). Let $\mathcal{Z}$ be an N-D input-containing subspace for the pair (A_{ℓ_0}, C), Also, it is A_ℓ-invariant for all $\ell \in S_N \backslash \{\ell_0\}$. Then, $\mathcal{Z}$ can be written as $\mathcal{Z} = \ker Q$, where Q satisfies $Q \, A_{\ell_0} = \Psi_0 \, Q + \Phi \, C$, $Q \, A_\ell = \Psi_\ell \, Q$, for all $\ell \in S_N \backslash \{\ell_0\}$ and $Q \, B = 0$. Suppose that equation (3.23) is with $K_\ell = \Psi_\ell$ for $\ell \in S_N$, and $L = \Phi$. The error can be defined as $\mathbf{e}_{(i_1,i_2,\ldots,i_N)} = Q \, \mathbf{x}_{(i_1,i_2,\ldots,i_N)} - \mathbf{p}_{(i_1,i_2,\ldots,i_N)}$. Then, since it is assumed that $\mathbf{p}_{(i_1,i_2,\ldots,i_N)} = Q \, \mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ upon the boundary $\mathfrak{B}$, $\mathbf{e}_{(i_1,i_2,\ldots,i_N)} = 0$ for all $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}$. Thus,

$$\begin{aligned}
\mathbf{e}_{(i_1+1,i_2+1,\ldots,i_N+1)} &= Q \, \mathbf{x}_{(i_1+1,i_2+1,\ldots,i_N+1)} - \mathbf{p}_{(i_1+1,i_2+1,\ldots,i_N+1)} \\
&= Q \, A_\ell \mathbf{x}_{(i_1,i_2,\ldots,i_N)+\ell} \\
&- \Psi_\ell \, \mathbf{p}_{(i_1,i_2,\ldots,i_N)+\ell_i} - \Phi \, C \, \mathbf{x}_{(i_1,i_2,\ldots,i_N)} \\
&= \Psi_\ell \, \mathbf{e}_{(i_1,i_2,\ldots,i_N)+\ell}.
\end{aligned} \tag{3.24}$$

Since the error is zero upon $\mathfrak{Q}$, then it is zero everywhere.

(**If**). Suppose that a $\mathcal{Z}$-observer for Σ exists, so that $\mathbf{p}_{(i_1,i_2,\ldots,i_N)} = Q \, \mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ over the boundary $\mathfrak{Q}$ is given, then, we have $\mathbf{p}_{(i_1,i_2,\ldots,i_N)} = Q\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ over $\mathfrak{Q}$. Consider the boundary condition of equation (3.11), such that $\mathbf{x}_{\ell_0} \in \mathcal{Z} \cap \ker C$, $\mathbf{x}_\ell \in \mathcal{Z}$, for all $\ell \in S_N \backslash \{\ell_0\}$. The $\mathcal{Z}$-observer has the boundary condition $\mathbf{p}_\ell = 0$, for all $\ell \in S_N$. This is compatible with the fact that $\mathbf{p}_{(i_1,i_2,\ldots,i_N)} = Q\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ for $(i_1, i_2, \ldots, i_N) \in S_N$, since for such indexes $(i_1, i_2, \ldots, i_N)$, we have $\mathbf{x}_{(i_1,i_2,\ldots,i_N)} \in \mathcal{Z}$, and then $Q\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)} = 0$. Therefore, by virtue of equation (3.23), we obtain $\mathbf{p}_{(1,1,\ldots,1)} = \sum_{\ell \in S_N} K_\ell\,\mathbf{p}_\ell + L\,C\mathbf{x}_{\ell_0}$, which gives zero since, $\mathbf{x}_{\ell_0} \in \ker C$. By contrast, $\mathbf{x}_{(1,1,\ldots,1)} = \sum_{\ell \in S_N} A_\ell \mathbf{x}_\ell$ leads to $Q\,\mathbf{x}_{(1,1,\ldots,1)} = Q \sum_{\ell \in S_N} A_\ell \mathbf{x}_\ell + Q\,B\,\mathbf{u}_{\ell_0} = \mathbf{p}_{(1,1,\ldots,1)}$, which is zero, as shown above. For choosing arbitrary states $\mathbf{x}_\ell$ for all $\ell \in S_N$ and an inputs $\mathbf{u}_{\ell_0}$, we obtain $Q\,A_{\ell_0}(\mathcal{Z} \cap \ker C) + Q \sum_{\ell \in S_N\{\ell_0\}} A_\ell \mathcal{Z} = \{0\}$ and $Q\,B = 0$. These imply that $A_{\ell_0}(\mathcal{Z} \cap \ker C) + \sum_{\ell \in S_N\{\ell_0\}} A_\ell \mathcal{Z} \subseteq \mathcal{Z}$ and $\operatorname{im} B \subseteq \ker Q = \mathcal{Z}$, which is equivalent to $\mathcal{Z}$ being an input-containing subspace of the output-injection type. ■

The family of all N-D input containing subspaces of the output-injection type is denoted by $\mathcal{Z}(\Sigma)$ and it is closed under subspace intersection. Then, $\mathcal{Z}(\Sigma)$ admits a minimum which is $\mathcal{Z}^*$. Its computation by the duality is as follows.

Lemma 3.6. *$\mathcal{Z}^*$ coincides with the last term of the sequence of subspaces:*

$$\begin{cases} \mathcal{Z}_0 = \operatorname{im} B, \\ \mathcal{Z}_q = A_{\ell_0}\,(\mathcal{Z}_{q-1} \cap \ker C) + \sum_{\ell \in S_N \backslash \{\ell_0\}} A_\ell\,\mathcal{Z}_{q-1} + \operatorname{im} B, \qquad q \in \{1, \ldots, \kappa\}, \end{cases}$$

where the value $\kappa \leq n-1$ is determined by the condition $\mathcal{Z}_{\kappa+1} = \mathcal{Z}_\kappa$, i.e., $\mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \mathcal{Z}_2 \supset \ldots \supset \mathcal{Z}_\kappa = \mathcal{Z}_{\kappa+1} = \mathcal{Z}^*$.

3.1.11 Stability of N-D systems

In this part, a sufficient condition of stability is presented for the linear N-D systems that is described by the model,

$$\mathbf{x}_{(i_1+1,i_2+1,\ldots,i_N+1)} = \sum_{\ell \in S_N} A_\ell\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)+\ell} + B\,\mathbf{u}_{(i_1,i_2,\ldots,i_N)}, \tag{3.25}$$

$$\mathbf{y}_{(i_1,i_2,\ldots,i_N)} = C\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}, \tag{3.26}$$

where $i_1, i_2, \ldots, i_N \in \mathbb{N}$.

Definition 35. The system (3.25) is called asymptotically stable if every solution tends to zero for $i_1 \longrightarrow \infty$, $i_2 \longrightarrow \infty, \ldots$ and $i_N \longrightarrow \infty$.

The starting point of this is the set that is defined as follows:

$$G_\ell \overset{\text{def}}{=} \{j \in \mathbb{N} \,|\, \bar{\ell}(j) = 0 \text{ with } \bar{\ell} \text{ is the column vector representation of } \ell \text{ using the canonical basis of } \mathbb{R}^n\}. \tag{3.27}$$

The definition of G_ℓ and model equations (3.25) and (3.26) give us an expression for the transfer function $H(z_1, z_2, \dots, z_N)$.

$$H(z_1, z_2, \dots, z_N) = C\,\Pi_{k=1}^{N} z_k\,(I_n - \sum_{\ell \in S_N} \Pi_{j \in G_\ell} z_j\, A_\ell)^{-1}\, B, \tag{3.28}$$

where I_n is the identity matrix of size $n \times n$. Model (3.25) is asymptotically stable if and only if

$$\det(I_n - \sum_{\ell \in S_N} \Pi_{j \in G_\ell} z_j\, A_\ell) \neq 0, \tag{3.29}$$

for all $(z_1, z_2, \dots, z_N) \in \{(z_1, z_2, \dots, z_N) : |z_i| \leq 1, \forall\, i \in \mathbb{N}\}$.
A sufficient condition for the asymptotically stable of (3.25) is presented by the following, which is an extension of the theorem reported previously by Kar and Singh (2003). For example, for $N = 3$, $S_3 = \{(0,0,0), (0,0,1), (1,0,0), (0,1,0), (0,1,1), (1,0,1), (1,1,0)\}$. Then, $G_{(0,0,0)} = \{1,2,3\}, G_{(0,0,1)} = \{1,2\}, G_{(1,1,0)} = \{3\}, G_{(0,1,0)} = \{1,3\}, G_{(1,0,0)} = \{2,3\}, G_{(0,1,1)} = \{1\}, G_{(1,0,1)} = \{2\}$, we obtain from this

$$\begin{aligned} H(z_1, z_2, \dots, z_N) \;=&\; C\, z_1 z_2 z_3\, (I_n -\; z_1 z_2 z_3\, A_{(0,0,0)} -\; z_1 z_2\, A_{(0,0,1)} -\; z_2 z_3\, A_{(1,0,0)} \\ &-\; z_1 z_3\, A_{(0,1,0)} -\; z_1\, A_{(0,1,1)} -\; z_2\, A_{(1,0,1)} -\; z_3\, A_{(1,1,0)})^{-1}\, B, \end{aligned}$$

which is the transfer function for the model (3.25) and (3.26) by using the definition of G_ℓ in (3.27).

Theorem 3.6. *System (3.25) is asymptotically stable, if there exist $n \times n$ symmetric positive definite matrices P_ℓ for all $\ell \in S_N$, such that*

$$M = \begin{bmatrix} P_{\ell_0} & 0 & \dots & 0 \\ 0 & P_{\ell_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{\ell_{2^N-2}} \end{bmatrix} - A^T\, P\, A \;>0, \tag{3.30}$$

where $A = [\, A_{\ell_0} \;\; A_{\ell_1} \;\; \dots \;\; A_{\ell_{2^N-2}} \,]$ and $P = (P_{\ell_0} + P_{\ell_1} + \dots + P_{\ell_{2^N-2}})$.

Proof: By the properties of the Schur complement, condition (3.30) is equivalent to

$$\left[\begin{array}{cccc|c} P_{\ell_0} & 0 & \dots & 0 & \\ 0 & P_{\ell_1} & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & A^T\, P \\ 0 & 0 & \dots & P_{\ell_{2^N-2}} & \\ \hline & & P\, A & & P \end{array}\right] > 0. \tag{3.31}$$

Let the condition (3.30) be satisfied. Suppose that

$$det(I_n - \sum_{\ell \in S_N} \Pi_{j \in G_\ell} z_j \, A_\ell) = 0, \tag{3.32}$$

a vector $\mathbf{x} \neq 0$, with size $n \times 1$, exists, such that

$$(I_n - \sum_{\ell \in S_N} \Pi_{j \in G_\ell} z_j \, A_\ell) \, \mathbf{x} = 0, \tag{3.33}$$

or, equivalently

$$\mathbf{x} = (\sum_{\ell \in S_N} \Pi_{j \in G_\ell} z_j \, A_\ell) \, \mathbf{x} = \sum_{\ell \in S_N} A_\ell \, [\, \Pi_{j \in G_\ell} z_j \, I_n \,]^T \, \mathbf{x}. \tag{3.34}$$

We obtain from equations (3.30) and (3.34)

$$\begin{aligned} \mathbf{x}^* \, P \, \mathbf{x} &= \mathbf{x}^* \, [\, \Pi_{j \in G_\ell} z_j^* \, I_n \,] \times A^T \, P \, A \, [\, \Pi_{j \in G_\ell} z_j \, I_n \,]^T \, \mathbf{x} \\ &= \mathbf{x}^* \, [\, \Pi_{j \in G_\ell} z_j^* \, I_n \,] \times \left(\begin{bmatrix} P_{\ell_0} & 0 & \dots & 0 \\ 0 & P_{\ell_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{\ell_{2^N-2}} \end{bmatrix} - M \right) [\, \Pi_{j \in G_\ell} z_j \, I_n \,]^T \, \mathbf{x}, \end{aligned}$$

where $\mathbf{x}^*$ denotes the complex conjugate transpose of $\mathbf{x}$. The last equation could be rearranged as

$$\mathbf{x}^* \, \{ \sum_{\ell \in S_N} P_\ell (1 - |\Pi_{j \in G_\ell} z_j|^2) \} \, \mathbf{x} = -\mathbf{x}^* [\, \Pi_{j \in G_\ell} z_j^* \, I_n \,] \, M \, [\, \Pi_{j \in G_\ell} z_j \, I_n \,]^T \, \mathbf{x}. \tag{3.35}$$

Based on the above equation, we obtain a contradiction. Since $M > 0$, the right-hand side is negative. However, the left-hand side of (3.35) is non-negative because P_{ℓ_0}, $P_{\ell_1}, \dots,$ $P_{\ell_{2^N-1}}$ are positive definite matrices, and $|z_1| \leq 1$, $|z_2| \leq 1, \dots,$ $|z_N| \leq 1$. This implies that (3.29) is valid for any $(z_1, z_2, \dots, z_N) \in \{(z_1, z_2, \dots, z_N) : |z_i| \leq 1, i \in \mathbb{N}\}$. Thus, the theorem is proven. ■

Corollary 3.1. *System (3.25-3.26) is asymptotically stable, if an $n \times n$ symmetric positive definite matrix Q and positive numbers $\alpha_0, \alpha_1, \dots, \alpha_{2^N-2}$ exist, such that*

$$R = \begin{bmatrix} \alpha_0 \, Q & 0 & \dots & 0 \\ 0 & \alpha_1 \, Q & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{2^N-2} \, Q \end{bmatrix} (\sum_{i=0}^{2^N-2} \alpha_i) - A^T \, Q \, A > 0. \tag{3.36}$$

Note that Corollary 3.1 is a special case of Theorem 3.6, by replacing $P_{\ell_0} = \alpha_0 \, Q$, $P_{\ell_1} = \alpha_1 \, Q, \dots, P_{\ell_{2^N-1}} = \alpha_{2^N-2} \, Q$.

Corollary 3.2. *System (3.25-3.26) is asymptotically stable if*

$$I_{(2^N-1)N} - (2^N - 1)\, A^T A > 0. \tag{3.37}$$

Moreover, Corollary 3.2 is a special case of Corollary 3.1, by putting $\alpha_0 = \alpha_1 = \ldots = \alpha_{2^N-2} = 1/(2^N - 1)$, and $Q = I_n$.

3.1.12 Friends and stabilisation

By adapting the arguments used in Ntogramatzidis (2012), a fundamental result of the characterisation of the set of controlled invariant friends of N-D controlled invariant subspace of the feedback type is shown by the following theorem.

Theorem 3.7. *Let $\mathcal{W}$, with a basis matrix W, be an N-D controlled invariant subspace of the feedback type. A matrix F is a feedback type controlled invariant friend of $\mathcal{W}$, if $U = -F\,W$, where U is a solution of $A_{\ell_0}\,W = W\,X_{\ell_0} + B\,U$ for some matrix X_{ℓ_0}.*

Proof: Let F be such that $U = -F\,W$, where U is a solution of $A_{\ell_0}\,W = W\,X_{\ell_0} + B\,U$ for a certain X_{ℓ_0}. Thus, $A_{\ell_0}\,W = W\,X_{\ell_0} - B\,F\,W$, which can also be written as $(A_{\ell_0} + B\,F)W = W\,X_{\ell_0}$. Moreover, $A_\ell\,W = W\,X_\ell$, for all $\ell \in S_N \backslash \{\ell_0\}$. Then, F is a controlled invariant friend of $\mathcal{W}$. Conversely, let F be a controlled invariant friend of $\mathcal{W}$. Then, there exists Λ, such that $(A_{\ell_0} + BF)\,W = W\,\Lambda$. Then, $(A_{\ell_0} + BF)\,W = W\,X_{\ell_0} + B\,U$, with $X_{\ell_0} = \Lambda$ and $U = -F\,W$. ■

Similar to the 1-D case, there are two degrees of freedom of the computation of controlled invariant friends F of the subspaces of the N-D invariant of the feedback type $\mathcal{W}$. One of them comes from solving equation $A_{\ell_0}\,W = W\,X_{\ell_0} + B\,U$ for the unknowns X_{ℓ_0} and U. Indeed, the complete set of its solutions is given by

$$\begin{bmatrix} X_{\ell_0} \\ U \end{bmatrix} = \begin{bmatrix} W & B \end{bmatrix}^{\dagger} A_{\ell_0}\,W + \begin{bmatrix} \Omega_0 \\ \Omega_1 \end{bmatrix} K_1, \tag{3.38}$$

where $\operatorname{im} \begin{bmatrix} \Omega_0 \\ \Omega_1 \end{bmatrix} = \ker \begin{bmatrix} W & B \end{bmatrix}$, and K_1 is an arbitrary matrix of suitable size. The other follows from the solution of $U = -F\,W$, which is given by

$$F = -U(W^\top W)^{-1}\,W^\top + K_2\Omega, \tag{3.39}$$

where $\ker \Omega = \mathcal{W}$ and K_2 is another arbitrary matrix of an appropriate size. Therefore, by indicating the symbol $F = F_{K_1,K_2}$ for the choice of F which depends only on the two arbitrary matrices K_1 and K_2.

The new coordinates of a closed-loop matrix $(A_{\ell_0} + B\,F_{K_1,K_2})$ using the change of basis $\begin{bmatrix} T_1 & T_2 \end{bmatrix}$, where T_1 is a basis matrix of $\mathcal{W}$, and T_2 is such that T is invertible, can be written as

$$
\begin{aligned}
T^{-1}\,(A_{\ell_0}+B\,F_{K_1,K_2})\,T &= \begin{bmatrix} L_1(\ell_0)(K_1,K_2) & L_2(\ell_0)(K_1,K_2) \\ 0 & L_3(\ell_0)(K_1,K_2) \end{bmatrix}, \\
T^{-1}\,A_{\ell_1}\,T &= \begin{bmatrix} L_1(\ell_1) & L_2(\ell_1) \\ 0 & L_3(\ell_1) \end{bmatrix}, \quad T^{-1}\,A_{\ell_2}\,T = \begin{bmatrix} L_1(\ell_2) & L_2(\ell_2) \\ 0 & L_3(\ell_2) \end{bmatrix}, \\
&\ldots,\; T^{-1}\,A_{\ell_{2^N-2}}\,T = \begin{bmatrix} L_1(\ell_{2^N-2}) & L_2(\ell_{2^N-2}) \\ 0 & L_3(\ell_{2^N-2}) \end{bmatrix}. \qquad (3.40)
\end{aligned}
$$

Then, as shown in Basile and Marro (1992); Willems (1981); W. Wonham (1985) and Ntogramatzidis et al. (2008), for the 1-D and 2-D cases, the matrix $L_1(\ell_0)(K_1,K_2)$ does not depend on K_2. Likewise, the matrix $L_3(\ell_0)(K_1,K_2)$ does not depend on K_1. Therefore, by defining

$$
\begin{bmatrix} \mathbf{x}'_{(i_1,i_2,\ldots,i_N)} \\ \mathbf{x}''_{(i_1,i_2,\ldots,i_N)} \end{bmatrix} = T^{-1}\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)},
$$

the closed-loop equation, with the new coordinates, can be expressed as

$$
\begin{aligned}
\begin{bmatrix} \mathbf{x}'_{(i_1+1,i_2+1,\ldots,i_N+1)} \\ \mathbf{x}''_{(i_1+1,i_2+1,\ldots,i_N+1)} \end{bmatrix} &= \begin{bmatrix} L_1(\ell_0)(K_1) & L_2(\ell_0)(K_1,K_2) \\ 0 & L_3(\ell_0)(K_2) \end{bmatrix} \begin{bmatrix} \mathbf{x}'_{(i_1,i_2,\ldots,i_N)} \\ \mathbf{x}''_{(i_1,i_2,\ldots,i_N)} \end{bmatrix} \\
&+ \begin{bmatrix} L_1(\ell_1) & L_2(\ell_1) \\ 0 & L_3(\ell_1) \end{bmatrix} \begin{bmatrix} \mathbf{x}'_{(i_1,i_2,\ldots,i_N)+\ell_1} \\ \mathbf{x}''_{(i_1,i_2,\ldots,i_N)+\ell_1} \end{bmatrix} + \ldots \\
&+ \begin{bmatrix} L_1(\ell_{2^N-2}) & L_2(\ell_{2^N-2}) \\ 0 & L_3(\ell_{2^N-2}) \end{bmatrix} \begin{bmatrix} \mathbf{x}'_{(i_1,i_2,\ldots,i_N)+\ell_{2^N-1}} \\ \mathbf{x}''_{(i_1,i_2,\ldots,i_N)+\ell_{2^N-1}} \end{bmatrix},
\end{aligned}
$$

where $\mathbf{x}'$ and $\mathbf{x}''$ are respectively, the components of the local states that are internal and external to the N-D controlled invariant subspace of the feedback type $\mathcal{W}$. Next, we provide a definition of internal and external stabilisability by using the change of coordinates that was described above.

Definition 36. Let $\mathcal{W}$ be a N-D controlled invariant subspace of the feedback type with a feedback type controlled invariant friend F, we say that $\mathcal{W}$ is:

- internally stabilisable, if for any $\mathcal{W}$-valued boundary condition, there holds $\mathbf{x}'_{(i_1,i_2,\ldots,i_N)} \longrightarrow 0$ as $(i_1,i_2,\ldots,i_N)$ is moving away from $\mathfrak{Q}$, i.e., if and only if K_1 in (3.38) exists, such that $(L_1(\ell_0)(K_1), L_1(\ell_1),\ldots,L_1(\ell_{2^N-2}))$ in (3.40) is asymptotically stable,
- externally stabilisable, if for any $\mathcal{W}$-valued boundary condition, there holds $\mathbf{x}''_{(i_1,i_2,\ldots,i_N)} \longrightarrow \mathcal{W}$ as $(i_1,i_2,\ldots,i_N)$ goes away from $\mathfrak{Q}$, i.e., if and only if K_2 in (3.39) exists, such that $(L_3(\ell_0)(K_2), L_3(\ell_1),\ldots,L_3(\ell_{2^N-2}))$ in (3.40) is asymp-

totically stable.

Hence, K_1 is the only parameter that influences the internal stabilisability of $\mathcal{W}$, while K_2 is the only parameter that influences the external stabilisability of $\mathcal{W}$. An alternative way to compute an internal stabilisability of $\mathcal{W}$ is given by the following theorem.

Theorem 3.8. *Let $\mathcal{W}$ be a N-D controlled invariant subspace of the feedback type, and let r and W be respectively, the dimension and basis matrix of $\mathcal{W}$. Let the first r rows of* $\ker \begin{bmatrix} W & B \end{bmatrix}^{\dagger} A_{\ell_0} W$ *be denoted by Q_0. Then, $\mathcal{W}$ is said to be internally stabilisable, if there exists Υ and an $n \times n$ symmetric positive definite matrices $\Upsilon_{\ell_0}, \Upsilon_{\ell_1}, \ldots, \Upsilon_{\ell_{2^N-2}}$, such that the following linear matrix inequality (LMI) holds:*

$$\left[\begin{array}{ccccc} \Upsilon_{\ell_0} & 0 & \cdots & \star & \star \\ 0 & \Upsilon_{\ell_1} & \cdots & \vdots & \star \\ \vdots & \vdots & \ddots & \star & \vdots \\ 0 & 0 & \cdots & \Upsilon_{\ell_{2^N-2}} - (\sum_{\ell \in S_N \backslash \{\ell_{2^N-2}\}} \Upsilon_\ell) & \star \\ \hline Q_0\, \Upsilon_{\ell_{2^N-2}} + \Omega_0\, \Upsilon & W^{\dagger} A_{\ell_1} W \Upsilon_{\ell_{2^N-2}} & \cdots & W^{\dagger} A_{\ell_{2^N-2}} W \Upsilon_{\ell_{2^N-2}} & \Upsilon_{\ell_{2^N-2}} \end{array}\right] > 0, \quad (3.41)$$

where the symbol $\star$ has been used to abbreviate off-diagonal blocks in symmetric matrices. With $(\Upsilon,\ \Upsilon_{\ell_0}, \Upsilon_{\ell_1}, \ldots, \Upsilon_{\ell_{2^N-2}})$ that is defined by (3.41), a matrix K_1 such that the 2^N-1-tuple matrices $(X_{\ell_0}, X_{\ell_1}, \ldots, X_{\ell_{2^N-2}})$ is asymptotically stable, is given by $K_1 = \Upsilon\, \Upsilon_{\ell_{2^N-2}}^{-1}$.

Proof: By using condition (3.30), $(X_{\ell_0}, X_{\ell_1}, \ldots, X_{\ell_{2^N-2}})$ is asymptotically stable, if there exist $n \times n$ symmetric positive definite matrices P_ℓ for all $\ell \in S_N$, such that

$$\left[\begin{array}{cccc|c} P_{\ell_0} & 0 & \ldots & 0 & \\ 0 & P_{\ell_1} & \ldots & 0 & \\ \vdots & \vdots & \ddots & \vdots & X_\ell^T\, P \\ 0 & 0 & \ldots & P_{\ell_{2^N-2}} & \\ \hline & & P\, X_\ell & & P \end{array}\right] > 0, \quad (3.42)$$

where $X_\ell = [\, X_{\ell_0} \;\; X_{\ell_1} \;\; \ldots \;\; X_{\ell_{2^N-2}} \,]$ and $P = P_{\ell_0} + P_{\ell_1} + \ldots + P_{\ell_{2^N-2}}$. Since $\mathcal{W}$ is an output-nulling subspace of the feedback type and $X_\ell = W^{\dagger} A_\ell\, W$, for all $\ell \in S_N \backslash \{\ell_0\}$. Then, from defining Q_0, we can write $X_{\ell_0} = Q_0 + \Omega_0\, K_1$. Therefore, to obtain relation (3.41), we multiply both sides of previous former (3.42) by the block diagonal matrix $\mathrm{diag}\,\underbrace{(P^{-1}, P^{-1}, \ldots, P^{-1})}_{2^N-1\ times}$ and defining $\Upsilon_\ell = P^{-1}\, P_\ell\, P^{-1}$, for all $\ell \in S_N \backslash \{\ell_{2^N-2}\}$ and $\Upsilon_{\ell_{2^N-2}} = P^{-1}$ together with $\Upsilon = K_1\, \Upsilon_{\ell_1}$. ∎

Parallel to the previous discussion, along with the duality of the internal stabilisability of N-D controlled invariant subspaces of the feedback type, we can establish the notion of the external stabilisability of the N-D conditioned invariant subspaces of the output-injection type. An output-injection matrix of the N-D conditioned invariant subspace of output injection type $\mathcal{Z}$, such that $(A_{\ell_0} + G\,C)\mathcal{Z} \subseteq \mathcal{Z}$, and $A_\ell\,\mathcal{Z} \subseteq \mathcal{Z}$, for all $\ell \in S_N \backslash \{\ell_0\}$ coincides with matrix G, such that $\Theta = -Q\,G$, where Θ is a solution of $Q\,A_{\ell_0} = \Phi_{\ell_0}\,Q + \Theta\,C$ for a suitable Φ_{ℓ_0}, and Q is such that ker $Q = \mathcal{Z}$. Indeed, the complete set of solutions of $Q\,A_{\ell_0} = \Phi_{\ell_0}\,Q + \Theta\,C$ is given by

$$\begin{bmatrix} \Phi_{\ell_0} & \Theta \end{bmatrix} = Q\,A_{\ell_0} \begin{bmatrix} Q \\ C \end{bmatrix}^{\dagger} + K_1 \begin{bmatrix} \Omega_0 & \Omega_1 \end{bmatrix}, \tag{3.43}$$

where $\begin{bmatrix} \Omega_0 & \Omega_1 \end{bmatrix}$ is a full row-rank matrix such that

$$\ker \begin{bmatrix} \Omega_0 & \Omega_1 \end{bmatrix} = \operatorname{im} \begin{bmatrix} Q \\ C \end{bmatrix},$$

and K_1 is an arbitrary matrix of suitable size. In the same way, the set of solution of $\Theta = -Q\,G$ is given by

$$G = -Q^\top\,(Q\,Q^\top)^{-1}\,\Theta + \Omega\,K_2, \tag{3.44}$$

where Ω is a full row-rank matrix such that $\operatorname{im}\Omega = \ker\,Q$ and K_2 is an arbitrary matrix of suitable size. The new coordinates with $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ can be written as

$$\begin{aligned} T^{-1}\,(A_{\ell_0} + B\,F)\,T &= \begin{bmatrix} L_1(\ell_0)(K_1,K_2) & L_2(\ell_0)(K_1,K_2) \\ 0 & L_3(\ell_0)(K_1,K_2) \end{bmatrix}, \\ T^{-1}\,A_{\ell_1}\,T &= \begin{bmatrix} L_1(\ell_1) & L_2(\ell_1) \\ 0 & L_3(\ell_1) \end{bmatrix}, \quad T^{-1}\,A_{\ell_2}\,T = \begin{bmatrix} L_1(\ell_2) & L_2(\ell_2) \\ 0 & L_3(\ell_2) \end{bmatrix}, \\ \ldots, T^{-1}\,A_{\ell_{2^N-1}}\,T &= \begin{bmatrix} L_1(\ell_{2^N-1}) & L_2(\ell_{2^N-1}) \\ 0 & L_3(\ell_{2^N-1}) \end{bmatrix}. \end{aligned} \tag{3.45}$$

As in the 1-D case, it is a straightforward fact that $L_1(\ell_0)(K_1,K_2)$ does not depend on K_1 and $L_3(\ell_0)(K_1,K_2)$ does not depend on K_2.

Definition 37. Let $\mathcal{Z}$ be an N-D conditioned invariant subspace of the output injection type, we say that $\mathcal{Z}$ is:

- internally stabilisable, if and only if matrix K_2 in (3.44) exists, such that $(L_1(\ell_0)(K_2), L_1(\ell_1), \ldots, L_1(\ell_{2^N-1}))$ in (3.45) is asymptotically stable,
- externally stabilisable, if and only if matrix K_1 in (3.43) exists, such that $(L_3(\ell_0)(K_1), L_3(\ell_1), \ldots, L_3(\ell_{2^N-1}))$ in (3.45) is asymptotically stable.

On the same pattern of the 2-D case in Ntogramatzidis (2012), along with the duality, the N-D conditioned invariant subspaces of the output-injection type $\mathcal{Z}$ is externally stabilisable, if and only if an output-injection matrix G and a full row-rank matrix Q

exist such that $Q\,(A_{\ell_0} + G\,C) = \Gamma_{\ell_0}\,Q,\ Q\,A_{\ell_1} = \Gamma_{\ell_1}\,Q,\ Q\,A_{\ell_2} = \Gamma_{\ell_2}\,Q, \ldots, Q\,A_{\ell_{2N-2}} = \Gamma_{\ell_{2N-2}}\,Q$, where $(\Gamma_{\ell_0},\, \Gamma_{\ell_1},\, \Gamma_{\ell_2}, \ldots, \Gamma_{\ell_{2N-2}})$ is asymptotically stable.
The following result is the dual of the one in Theorem 3.8, which presents a computationally tractable test for external stabilisability.

Theorem 3.9. *Let $\mathcal{Z}$ of dimension r be an N-D conditioned invariant subspace of the output-injection type, and let Q be such that* $\ker Q = \mathcal{Z}$*. Let the first $n-r$ columns of $Q\,A_{\ell_0}\begin{bmatrix} Q \\ C \end{bmatrix}^{\dagger}$ be denoted by Ψ_0. Then, $\mathcal{Z}$ is said to be externally stabilisable, if there exist Υ and an $n \times n$ symmetric positive definite matrices $\Upsilon_{\ell_0}, \Upsilon_{\ell_1}, \ldots, \Upsilon_{\ell_{2N-2}}$, such that the following linear matrix inequality (LMI) holds:*

$$\left[\begin{array}{ccccc} \Upsilon_{\ell_0} & 0 & \cdots & \star & \star \\ 0 & \Upsilon_{\ell_1} & \cdots & \vdots & \star \\ \vdots & \vdots & \ddots & \star & \vdots \\ 0 & 0 & \cdots & \Upsilon_{\ell_{2N-2}} - (\sum_{\ell \in S_n \setminus \{\ell_{2N-2}\}} \Upsilon_\ell) & \star \\ \hline \Upsilon_{\ell_{2N-2}}\,\Psi_0 + \Upsilon\,\Omega_0 & \Upsilon_{\ell_{2N-2}}\,Q\,A_{\ell_1}\,Q^{\dagger} & \cdots & \Upsilon_{\ell_{2N-2}}\,Q\,A_{\ell_{2N-2}}\,Q^{\dagger} & \Upsilon_{\ell_{2N-2}} \end{array}\right] > 0, \quad (3.46)$$

where the symbol $\star$ has been used to abbreviate off-diagonal blocks in symmetric matrices. With $(\Upsilon,\ \Upsilon_{\ell_0}, \Upsilon_{\ell_1}, \ldots, \Upsilon_{\ell_{2N-2}})$ that is defined by (3.46), matrix K_1, such that the 2^N-1-tuple matrices $(\Gamma_{\ell_0},\, \Gamma_{\ell_1},\, \Gamma_{\ell_2}, \ldots,\, \Gamma_{\ell_{2N-2}})$ is asymptotically stable, is given by $K_1 = \Upsilon_{\ell_{2N-2}}^{-1}\,\Upsilon$.

3.1.13 Reachability and observability for N-D Fornasini-Marchesini systems

We begin by presenting some notions before addressing the reachability for N-D Fornasini-Marchesini model (3.3,3.4). The starting point is a new definition of the $N-$D state transition matrix parallel to those introduced for the 1-D and 2-D systems. The state transition matrix in the 1-D case of the state space discrete systems is $\boldsymbol{\phi}_k = 0$ for $k < 0$ and $\boldsymbol{\phi}_k = A^k$ for $k \geqslant 0$ (Mertzios & Lewis, 1989). Applying the $z-$transform to both sides

$$\mathbf{x}(t+1) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{3.47}$$

is

$$(z\,I_n - A)\,X\,(z) = z\,\mathbf{x}(0) + B\,U\,(z), \tag{3.48}$$

where $X\,(z)$ and $U\,(z)$ are respectively, the $z-$transforms of $\mathbf{x}(t)$ and $\mathbf{u}(t)$. The matrix $(z\,I_n - A)$ is called the resolvent matrix. Moreover, system (3.47) is always solvable (Mertzios & Lewis, 1989). The Laurent expansion about the infinity of the resolvent

matrix of (3.47) is written as $(z\, I_n - A)^{-1} = \sum_{i=0}^{\infty} A^i\, z^{-i-1}$.
A parallel argument for the 2-D Fornasini-Marchesini second order model can be easily established. The 2-D state transition matrix $\boldsymbol{\phi}_{(i,j)}$ of

$$\mathbf{x}_{(i+1,j+1)} = A_0\, \mathbf{x}_{(i,j)} + A_1\, \mathbf{x}_{(i,j+1)} + A_2\, \mathbf{x}_{(i+1,j)} + B\, \mathbf{u}_{(i,j)} \tag{3.49}$$

is defined as follows:

$$\boldsymbol{\phi}_{(i,j)} = \begin{cases} I_n & \text{for } \; i = j = 0 \\ A_0\, \boldsymbol{\phi}_{(i-1,j-1)} + A_1\, \boldsymbol{\phi}_{(i,j-1)} + A_2\, \boldsymbol{\phi}_{(i-1,j)} & \text{for } \; i \neq 0 \quad \text{or} \quad j \neq 0, \end{cases} \tag{3.50}$$

such that $\boldsymbol{\phi}_{(i,j)} = 0$ for $i < 0$ and/or $j < 0$ (Kurek, 1985; Kaczorek, 1992). In F. L. Lewis (1992) and Poularikas (1998), the $z-$transform

$$X\,(z_1, z_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbf{x}_{(i,j)}\, z_1^{-i}\, z_2^{-j}, \tag{3.51}$$

is associated with the 2-D Fornasini-Marchesini second order model in terms of the negative powers of z_i. Then, computing the $z-$transforms of the 2-D Fornasini-Marchesini second order model (3.49) as in Kung et al. (1977); Kaczorek (1991) and F. L. Lewis (1992) is

$$\begin{aligned}(z_1\, z_2\, I_n - z_1\, A_1 - z_2\, A_2 - A_0)\, X\,(z_1, z_2) \;&=\; B\, U\,(z_1, z_2) + z_2\,(z_1\, I_n - A_2)\, \mathbf{x}\,(z_1, 0) \\ &\quad +\; z_1\,(z_2\, I_n - A_1)\, \mathbf{x}\,(0, z_2) - z_1\, z_2\, I_n\, \mathbf{x}_{(0,0)},\end{aligned}$$

where

$$\begin{aligned}\mathbf{x}\,(z_1, 0) &= \sum_{i=0}^{\infty} \mathbf{x}_{(i,0)}\, z_1^{-i}, \\ \mathbf{x}\,(0, z_2) &= \sum_{j=0}^{\infty} \mathbf{x}_{(0,j)}\, z_2^{-j}.\end{aligned}$$

Then, the 2-D polynomial matrix $P(z_1, z_2)$ of the 2-D Fornasini-Marchesini second order model, which is sometimes called the resolvent matrix, is defined as

$$P(z_1, z_2) = (z_1\, z_2\, I_n - z_1\, A_1 - z_2\, A_2 - A_0). \tag{3.52}$$

For the same model, the regularity condition $\Delta\,(z_1, z_2) = \det\, P(z_1, z_2) = \det(z_1\, z_2\, I_n - z_1\, A_1 - z_2\, A_2 - A_0) \neq 0$ is always satisfied. A polynomial in z_1 and z_2 with degrees n may be written in the form (Kurek, 1985; F. L. Lewis, 1992; Kaczorek, 1992),

$$\Delta\,(z_1, z_2) = \det\, P(z_1, z_2) = \sum_{i=0}^{n} \sum_{j=0}^{n} \mathbf{q}_{(i,j)}\, z_1^{i}\, z_2^{j}, \qquad \text{where } \; \mathbf{q}_{(n,n)} = 1. \tag{3.53}$$

Then, the characteristic equation of the 2-D Fornasini-Marchesini second order model is defined as

$$\Delta\left(z_1, z_2\right) = 0, \tag{3.54}$$

while the Laurent expansion about the infinity of the resolvent matrix for the 2-D Fornasini-Marchesini second order model (F. L. Lewis, 1992; Kaczorek, 1992) is

$$P^{-1}(z_1, z_2) = z_1^{-1}\, z_2^{-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \boldsymbol{\phi}_{(i,j)}\, z_1^{-i}\, z_2^{-j}. \tag{3.55}$$

Theorem 3.10 ((Kurek, 1985))**.** *A state-transition matrix $\boldsymbol{\phi}_{(n+i,n+j)}$ for $i > 0$ or $j > 0$ or $i = j = 0$ can be written in the form:*

$$\boldsymbol{\phi}_{(n+i,n+j)} = -\sum_{s=0}^{n} \sum_{t=0}^{n} \mathbf{q}_{(s,t)}\, \boldsymbol{\phi}_{(s+i,t+j)}, \tag{3.56}$$

where $\mathbf{q}_{(s,t)}$ are coefficients of $\Delta\left(z_1, z_2\right)$.

For $i = j = 0$, Theorem 3.10 may be considered a generalisation of the well-known 2-D Cayley-Hamilton Theorem. The state transition matrix satisfies the characteristic equation (3.53), i.e.,

$$\sum_{s=0}^{n} \sum_{t=0}^{n} \mathbf{q}_{(s,t)}\, \boldsymbol{\phi}_{(s,t)} = 0. \tag{3.57}$$

A proof of this result is given in Kaczorek (1992). We extend this characterisation to the N-D case by showing a generalised result. By adapting the same arguments used in the 1-D and 2-D cases, one can easily see that the N-D state transition matrix $\boldsymbol{\phi}_{(i_1,\ldots,i_N)}$ of the N-D Fornasini-Marchesini model (3.11) is defined as follows:

$$\boldsymbol{\phi}_{(i_1,\ldots,i_N)} = \begin{cases} I_n & \text{for } i_1 = \ldots = i_N = 0 \\ A_{\ell_0}\, \boldsymbol{\phi}_{(i_1-1,\ldots,i_N-1)} + \sum_{\ell \in S_N \setminus \{\ell_0\}} A_\ell\, \boldsymbol{\phi}_{(i_1,\ldots,i_N)-\ell} & \text{for } i_1 \neq 0 \text{ or} \ldots \text{ or } i_N \neq 0, \end{cases}$$

such that $\boldsymbol{\phi}_{(i_1,\ldots,i_N)} = 0$ for $i_1 < 0$ and/or $i_2 < 0$ ... and/or $i_N < 0$. We define the N-D z-transform in terms of negative powers of $z_i,\ i \in \mathbb{N}$, as

$$X\left(z_1, z_2, \ldots, z_N\right) = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \cdots \sum_{i_N=0}^{n} \mathbf{x}_{(i_1,i_2,\ldots,i_N)}\, z_1^{-i_1}\, z_2^{-i_2} \ldots z_N^{-i_N}, \tag{3.58}$$

the N-D polynomial matrix $P(z_1, z_2, \ldots, z_N)$ for (3.11) is written as

$$P(z_1, z_2, \ldots, z_N) = (z_1\, z_2 \ldots z_N\, I_n - \sum_{\ell \in S_N} A_\ell\, z_1^{\ell(1)}\, z_2^{\ell(2)} \ldots z_N^{\ell(N)}). \tag{3.59}$$

A polynomial in $z_1, z_2, \ldots$ and z_N, with degrees n may be presented in the form,

$$\Delta(z_1, z_2, \ldots, z_N) = |\,P(z_1, \ldots, z_N)| = \sum_{i_1=0}^{n} \cdots \sum_{i_N=0}^{n} \mathbf{q}_{(i_1,\ldots,i_N)}\, z_1^{i_1} \ldots z_N^{i_N}, \tag{3.60}$$

where $\mathbf{q}_{\underbrace{(n,\ldots,n)}_{N\ times}} = 1$. Then, the characteristic equation of the N-D Fornasini-Marchesini model (3.11) is defined by

$$\Delta(z_1, z_2, \ldots, z_N) = 0. \tag{3.61}$$

Consider the following Laurent expansion about the infinity of the resolvent matrix for the N-D Fornasini-Marchesini model (3.11) as

$$P^{-1}(z_1, z_2, \ldots, z_N) = (z_1^{-1}\, z_2^{-1} \ldots z_N^{-1}) \sum_{i_1=0}^{n} \cdots \sum_{i_N=0}^{n} \boldsymbol{\phi}_{(i_1,\ldots,i_N)}\, z_1^{-i_1}\, z_2^{-i_2} \ldots z_N^{-i_N}. \tag{3.62}$$

The N-D state transition matrix and characteristic function are useful to present the next theorem, similar to the well-known Cayley-Hamilton theorem.

Theorem 3.11. *A state-transition matrix $\boldsymbol{\phi}_{(n+i_1,\ldots,n+i_N)}$ for $i_1 > 0$ or $i_2 > 0 \ldots$ or $i_N > 0$ or $i_1 = i_2 = \ldots = i_N = 0$ satisfies*

$$\boldsymbol{\phi}_{(n+i_1,\ldots,n+i_N)} = -\sum_{t_1=0}^{n} \cdots \sum_{t_N=0}^{n} \mathbf{q}_{(t_1,\ldots,t_N)}\, \boldsymbol{\phi}_{(t_1+i_1,\ldots,t_N+i_N)}, \tag{3.63}$$

where $\mathbf{q}_{(t_1,\ldots,t_N)}$ are the coefficients of $\Delta(z_1, \ldots, z_N)$.

Proof: The rule for computing the inverse of N-D polynomial matrix $P(z_1, z_2, \ldots, z_N)$ is

$$\det[P(z_1, z_2, \ldots, z_N)]\, I_n = \text{adj}\,[P(z_1, z_2, \ldots, z_N)]\, P(z_1, z_2, \ldots, z_N). \tag{3.64}$$

It can be noted that there always exist real matrices $R_{(t_1,\ldots,t_N)}$, such that

$$\text{adj}\,[P(z_1, z_2, \ldots, z_N)] = \sum_{t_1=0}^{n} \cdots \sum_{t_N=0}^{n} R_{(t_1,\ldots,t_N)}\, z_1^{t_1}\, z_2^{t_2} \ldots z_N^{t_N}. \tag{3.65}$$

Then, by using equations (3.59), (3.60) and (3.65), equality (3.64) becomes

$$\begin{aligned} \sum_{t_1=0}^{n} \cdots \sum_{t_N=0}^{n} \mathbf{q}_{(t_1,\ldots,t_N)}\, z_1^{t_1}\, z_2^{t_2} \ldots z_N^{t_N}\, I_n = & \sum_{t_1=0}^{n} \cdots \sum_{t_N=0}^{n} R_{(t_1,\ldots,t_N)}\, (I_n\, z_1^{t_1+1} \ldots z_N^{t_N+1} \\ & - \sum_{\ell \in S_N} A_\ell\, z_1^{t_1+\ell(1)} \ldots z_N^{t_N+\ell(N)}). \end{aligned} \tag{3.66}$$

Two polynomials are equal, if and only if all coefficients are respectively equal. Therefore, if we change $z_1^{t_1}\, z_2^{t_2}\, \ldots z_N^{t_N}$ into $\boldsymbol{\phi}_{(t_1+i_1,\ldots,t_N+i_N)}$, the equality will still be fulfilled

$$\begin{aligned}\sum_{t_1=0}^{n}\cdots\sum_{t_n=0}^{n}\mathbf{q}_{(t_1,\ldots,t_N)}\,\boldsymbol{\phi}_{(t_1+i_1,\ldots,t_N+i_N)} &= \sum_{t_1=0}^{n}\cdots\sum_{t_n=0}^{n} R_{(t_1,\ldots,t_N)}\,\big(\boldsymbol{\phi}_{(t_1+1+i_1,\ldots,t_N+1+i_N)}\\ &- \sum_{\ell\in S_N} A_\ell\,\boldsymbol{\phi}_{(t_1+\ell(1)+i_1,\ldots,t_N+\ell(N)+i_N)}\big).\end{aligned}$$

■

Then, based on the proof, one can note for $i_1 = i_2 = \ldots = i_N = 0$, Theorem 3.11 may be considered a generalisation of the N-D Cayley-Hamilton Theorem. The state transition matrix satisfies the characteristic equation (3.60), i.e.,

$$\sum_{t_1=0}^{n}\cdots\sum_{t_N=0}^{n}\mathbf{q}_{(t_1,\ldots,t_N)}\,\boldsymbol{\phi}_{(t_1,\ldots,t_N)} = 0. \tag{3.67}$$

It is well known that in the 1-D linear time invariant in both cases: continuous or discrete systems, as described by a pair (A, B), the reachable subspace from the origin is defined by the smallest A-invariant subspace containing the image of B. Correspondingly, the reachable subspace from the zero boundary condition of the local state $\mathbf{x}_{(i,j)}$ in a 2-D case, coincides with $\mathfrak{R}_{(i,j)}$, which is defined by the following 2-D sequence of subspaces:

$$\begin{cases}\mathfrak{R}_{(i,j)} = \{0\}, & (i,j)\in\mathfrak{Q}\\ \mathfrak{R}_{(i,j)} = A_0\,\mathfrak{R}_{(i-1,j-1)} + A_1\,\mathfrak{R}_{(i,j-1)} + A_2\,\mathfrak{R}_{(i-1,j)} + \mathrm{im}\ B, & (i,j)\in\mathfrak{Q}^+\setminus\mathfrak{Q}.\end{cases} \tag{3.68}$$

The reachability subspace coincides with the last term of equation (3.68) at $n-1$ and it is defined by $\mathfrak{R} = \mathfrak{R}_{(n-1,n-1)}$. In the case when $\mathfrak{R} = \mathbb{R}^n$, the system is completely reachable from the origin; see, e.g., (Bisiacco, 1985; Fornasini & Marchesini, 1982; Kaczorek, 2000, 2012; Ntogramatzidis, 2012).

The computation of $\mathfrak{R}$, which is the smallest subspace of $\mathbb{R}^n$ that is at the same time A_0-,A_1- and A_2-invariant subspaces containing im B coincides with the last term of the sequence of subspaces:

$$\begin{cases}\mathfrak{R}_0 = \mathrm{im}\ B,\\ \mathfrak{R}_i = \sum_{j=0}^{2} A_j\,\mathfrak{R}_{i-1} + \mathrm{im}\ B, & i\in\{1,2,\ldots,\kappa\},\end{cases} \tag{3.69}$$

where the value $\kappa \le n-1$ is determined by the condition $\mathfrak{R}_{\kappa+1} = \mathfrak{R}_\kappa$, i.e., $\mathfrak{R}_0 \supset \mathfrak{R}_1 \supset \mathfrak{R}_2 \supset \ldots \supset \mathfrak{R}_\kappa = \mathfrak{R}_{\kappa+1} = \mathfrak{R}$ (Ntogramatzidis, 2012). Then, this characterisation can be extended to the N-D case by showing that the same arguments holds. Consider the following N-D sequence of subspaces:

$$\begin{cases} \mathfrak{R}_{(i_1,\ldots,i_N)} = \{0\}, & (i_1,\ldots,i_N) \in \mathfrak{Q} \\ \mathfrak{R}_{(i_1,\ldots,i_N)} = A_{\ell_0}\,\mathfrak{R}_{(i_1-1,\ldots,i_N-1)} + \sum_{i=1}^{2^N-2} A_\ell \mathfrak{R}_{(i_1,\ldots,i_N)-\ell} + \text{im } B, & (i_1,\ldots,i_N) \in \mathfrak{Q}\backslash\mathfrak{Q}. \end{cases}$$

The reachable subspace $\mathfrak{R}_{(i_1,i_2,\ldots,i_N)}$ from the zero boundary condition of the local state $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ can be derived from this sequence of subspaces. Moreover, the subspace of the greatest dimension should be at $i_1 = i_2 = \ldots = i_N = n-1$. Then, the reachability subspace from the zero boundary condition is defined by $\mathfrak{R} = \mathfrak{R}_{\underbrace{(n-1,\ldots,n-1)}_{N\ times}}$. The system is completely reachable from the zero boundary conditions if $\mathfrak{R} = \mathbb{R}^n$. Similar to the 2-D case in Ntogramatzidis (2012), the computation of $\mathfrak{R}$ is provided by the N-D Cayley-Hamilton theorem, which is obvious from the following result.

Theorem 3.12. *$\mathfrak{R}$ is the smallest subspace of $\mathbb{R}^n$ that is at the same time A_ℓ-invariant subspaces for all $\ell \in S_N$ containing* im B. *In addition, its computation coincides with the last term of the sequence of subspaces:*

$$\begin{cases} \mathfrak{R}_0 = \text{im } B \\ \mathfrak{R}_q = \sum_{i=0}^{2^N-2} A_\ell \mathfrak{R}_{q-1} + \text{im } B \qquad q \in \{1,2,\ldots,\kappa\}, \end{cases} \tag{3.70}$$

where the value of $\kappa \leq n-1$ is determined with the condition $\mathfrak{R}_{\kappa+1} = \mathfrak{R}_\kappa$. Therefore, $\mathfrak{R}_0 \supset \mathfrak{R}_1 \supset \mathfrak{R}_2 \supset \ldots \supset \mathfrak{R}_\kappa = \mathfrak{R}_{\kappa+1} = \mathfrak{R}$.

The nonobservability subspace $\mathcal{Q}$ of Σ can be introduced as a dual of previous discussions and it is the limiting subspace which can be obtained from the recursion:

$$\begin{cases} \mathcal{Q}_0 = \ker\, C \\ \mathcal{Q}_q = \bigcap_{i=0}^{2^N-2} A_\ell^{-1} \mathcal{Q}_{q-1} \cap \ker C \qquad q \in \{1,2,\ldots,\kappa\}. \end{cases} \tag{3.71}$$

3.1.14 Self-boundedness and self-hiddenness for N-D systems

The concepts of self-boundedness and self-hiddenness play a pivotal case in the solution of disturbance decoupling problems, due to the fact that both of them allow for the solution of such problems without the need to make the closed-loop of the system maximally unobservable (Conte & Perdon, 1988). Spatially, the self-boundedness controlled invariant subspaces, as defined in Basile and Marro (1982) in the 1-D and 2-D cases, as defined in Ntogramatzidis (2012), are sometimes useful to solve the disturbance decoupling problem because it is useful to deal with the subspaces of smaller dimension and this often leads to a feedforward compensator of a smaller size. Here, these notions are extended to the N-D systems for the Fornasini-Marchesini second order form. Our assumption to guarantee the generalisation is that the matrix B is of full column-rank.

Definition 38. The subspace $\mathcal{V}$, which is an N-D output-nulling subspace of $\Sigma = (A_\ell, B, C)$ is an N-D self-bounded subspace, if for any $\mathcal{V}$-valued boundary condition, the system Σ has a $\mathcal{V}$-valued solution for any control yields zero output, for all $(i_1, i_2, \dots, i_N) \in \mathfrak{Q}$. Similarly, the N-D output-nulling subspace of the feedback type $\mathcal{W}$ is a N-D self-bounded subspace of the feedback type, if for any $\mathcal{W}$-valued boundary condition, the system Σ has a $\mathcal{W}$-valued solution for any control $\mathbf{u}_{(i_1,i_2,\dots,i_N)} = F\mathbf{x}_{(i_1,i_2,\dots,i_N)}$ yields a zero output, for all $(i_1, i_2, \dots, i_N) \in \mathfrak{Q}$.

The normal consequence of this definition is that both $\mathcal{V}^*$ and $\mathcal{W}^*$ are an N-D self-bounded subspace and an N-D self-bounded subspace of the feedback type respectively, because from their definitions for any boundary condition that belongs to them, any existing controls that maintain the output at zero are such that the local state of Σ lies ker C. Now, alternative characterisation of the N-D self-boundedness is provided by the following result, see Basile and Marro (1982) and Ntogramatzidis (2012).

Theorem 3.13. *Let $\mathcal{V}$ be an N-D output-nulling subspace. $\mathcal{V}$ is an N-D self-bounded subspace, if and only if it satisfies the subspace inclusion $\mathcal{V}^* \cap \operatorname{im} B \subseteq \mathcal{V}$.*

Proof: (**If**). Suppose that $\mathcal{V}^* \cap \operatorname{im} B \subseteq \mathcal{V}$ is satisfied. Consider a $\mathcal{V}$-valued boundary condition for the updated equation (3.11), where all the indexes are in $\mathfrak{Q}_k$, i.e., $\mathbf{x}_{(i_1,i_2,\dots,i_N)+\ell} \in \mathcal{V}$, $\ell \in S_N$. Since $\mathcal{V}$ is a N-D output-nulling subspace and then, it is a N-D controlled invariant subspace, the local states in $\mathfrak{Q}_k$ are given by the sum of two vectors: one of them belongs to $\mathcal{V}$ and the other is in im B, so that $\mathbf{x}_{(i_1+1,i_2+1,\dots,i_N+1)} = \boldsymbol{\phi}_{(i_1,i_2,\dots,i_N)} + \boldsymbol{\psi}_{(i_1,i_2,\dots,i_N)} + B\,\mathbf{u}_{(i_1,i_2,\dots,i_N)}$, where $\boldsymbol{\phi}_{(i_1,i_2,\dots,i_N)} \in \mathcal{V}$ and $\boldsymbol{\psi}_{(i_1,i_2,\dots,i_N)} \in \operatorname{im} B$. By virtue of $\mathcal{V}^* \cap \operatorname{im} B \subseteq \mathcal{V}$, $\boldsymbol{\psi}_{(i_1,i_2,\dots,i_N)} + B\,\mathbf{u}_{(i_1,i_2,\dots,i_N)} \in \mathcal{V}^* \cap \operatorname{im} B$, the local state in $\mathfrak{Q}_k$ is in $\mathcal{V}$. By induction, this argument can be repeated to hold on $\mathfrak{Q}$.
(**Onlyif**). Assume that $\mathcal{V}^* \cap \operatorname{im} B \subseteq \mathcal{V}$ does not hold, so that, there exist a vector $\xi_{(i_1,i_2,\dots,i_N)} \in \mathcal{V}^* \cap \operatorname{im} B$ such that $\boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)} \notin \mathcal{V}$. Consider a $\mathcal{V}$-valued boundary condition for Σ, the obtained local state on the k-th forward $\mathfrak{Q}_k$ from the update equation (3.11), can be written using the (if) part as $\mathbf{x}_{(i_1+1,i_2+1,\dots,i_N+1)} = \boldsymbol{\phi}_{(i_1,i_2,\dots,i_N)} + \boldsymbol{\psi}_{(i_1,i_2,\dots,i_N)} + B\,\mathbf{u}_{(i_1,i_2,\dots,i_N)}$, where $\boldsymbol{\phi}_{(i_1,i_2,\dots,i_N)} \in \mathcal{V}$ and $\boldsymbol{\psi}_{(i_1,i_2,\dots,i_N)} \in \operatorname{im} B$. Since both $\boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)}$ and $\boldsymbol{\psi}_{(i_1,i_2,\dots,i_N)}$ are in im B, the control $\mathbf{u}_{(i_1,i_2,\dots,i_N)}$ can be chosen such that $B\,\mathbf{u}_{(i_1,i_2,\dots,i_N)} = \boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)} - \boldsymbol{\psi}_{(i_1,i_2,\dots,i_N)}$, which implies to $\boldsymbol{\xi}_{(i_1,i_2,\dots,i_N)} = \boldsymbol{\psi}_{(i_1,i_2,\dots,i_N)} + B\,\mathbf{u}_{(i_1,i_2,\dots,i_N)}$. From the assumption, this is in $\mathcal{V}^*$ but not in $\mathcal{V}$, which means that $\mathcal{V}$ is not self-bounded. ■

The same argument of providing an alternative characterisation of the N-D self-bounde-dness subspace of the feedback type is expressed by the following theorem, which is considered a simple extension of what is found in Basile and Marro (1982) and Ntogramatzidis (2012), whose proof is an adaptation of Theorem 1 in Ntogramatzidis (2008).

Theorem 3.14. *Let $\mathcal{W}$ be a N-D output-nulling subspace of the feedback type. $\mathcal{W}$ is a N-D self-bounded subspace of the feedback type, if and only if it satisfies the subspace inclusion $\mathcal{W}^* \cap \mathrm{im}\ B \subseteq \mathcal{W}$.*

Proof: (**If**). Suppose that $\mathcal{W}^* \cap \mathrm{im}\ B \subseteq \mathcal{W}$. Let $\mathcal{W}$-valued boundary condition for Σ. Consider the input $\mathbf{u}_{(i_1,i_2,\ldots,i_N)}$ which yields a zero output, i.e., $\mathbf{y}_{(i_1,i_2,\ldots,i_N)} = 0$. Then, $\mathbf{x}_{(i_1,i_2,\ldots,i_N)} \in \mathcal{W}^*$, by expressing the control $\mathbf{u}_{(i_1,i_2,\ldots,i_N)}$ as $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$, such that F is a feedback type output-nulling friend of $\mathcal{W}^*$. From our assumption, it is clear that $\mathcal{W}^* \cap \mathrm{im}\ B \subseteq \mathcal{W} \subseteq \mathcal{W}^*$. From Lemma 2.3, which is also consistent with an N-D self-bounded subspace of feedback type $\mathcal{W}$, it follows that F is a feedback type controlled invariant friend of $\mathcal{W}$. Thus, since this and $\mathcal{W}^* \cap \mathrm{im}\ B \subseteq \mathcal{W}$, it is found that the states $\mathbf{x}_{(i_1,i_2,\ldots,i_N)} \in \mathcal{W}$ for all $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}$.
(**Only if**). Let $\mathcal{W}$ be self-bounded and let $\mathcal{W}$-valued boundary condition for Σ. Any input function that yields a zero output can be parametrised as $\mathbf{u}_{(i_1,i_2,\ldots,i_n)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$, such that F is a feedback type output-nulling friend of $\mathcal{W}^*$ for all $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}$. From the closed-loop equation,

$$\mathbf{x}_{(i_1+1,\ldots,i_N+1)} = (A_{\ell_0} + B\,F)\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)} + \sum_{\ell \in S_N \setminus \{\ell_0\}} A_\ell\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)+\ell},$$

it follows that the local state remains in $\mathcal{W}$ for all $(i_1, i_2, \ldots, i_N) \in \mathfrak{Q}$, which implies that $\mathcal{W}^* \cap \mathrm{im}\ B \subseteq \mathcal{W}$. ■

Corollary 3.3. *Let F be a feedback type output-nulling friend of $\mathcal{W}^*$. Then, F is a feedback type controlled invariant friend of any N-D self-bounded subspace of the feedback type.*

The proof of this corollary is obvious by using both Lemma 2.3 and Theorem 3.14. A set of all N-D self-bounded subspaces of the feedback type of (3.11), if defined by the set:

$$\Theta\,(\Sigma) \stackrel{\text{def}}{=} \{\mathcal{W} \in \mathcal{W}\,(\Sigma)\,|\,\mathcal{W}^* \cap \mathrm{im}\ B \subseteq \mathcal{W}\}. \tag{3.72}$$

This admits both maximum and minimum elements. More explanation about the existence of these elements is given by the following proposition:

Proposition 3.1. *The set of all N-D self-bounded subspaces of the feedback type $\Theta\,(\Sigma)$ is closed under subspaces addition and intersection.*

Proof: Let $\mathcal{W}_1, \mathcal{W}_2$ be the two subspaces of $\Theta\,(\Sigma)$. Since $\mathcal{W}(\Sigma)$ is closed under subspace addition, then, $\Theta\,(\Sigma)$ is also closed under subspace addition, i.e., $\mathcal{W}_1 + \mathcal{W}_2 \in \Theta\,(\Sigma)$. Moreover, their addition is the smallest element of $\Theta\,(\Sigma)$ containing both $\mathcal{W}_1$ and $\mathcal{W}_2$. However, the addition of all elements of $\Theta\,(\Sigma)$ is the maximum of $\Theta\,(\Sigma)$, which is

trivially $\mathcal{W}^*$.
To prove the closeness under the intersection, given a controlled invariant friend F of $\mathcal{W}^*$, so that, by Corollary 3.3, F is a controlled invariant friend of both $\mathcal{W}_1$ and $\mathcal{W}_2$, which means that $(A_{\ell_0} + B\,F)\,(\mathcal{W}_1 \cap \mathcal{W}_2) \subseteq \mathcal{W}_1 \cap \mathcal{W}_2$. Therefore, $\mathcal{W}_1 \cap \mathcal{W}_2$ is a N-D controlled invariant subspace of the feedback type. Their intersection is the largest subspace of $\Theta\,(\Sigma)$ contained in both $\mathcal{W}_1$ and $\mathcal{W}_2$. Since $\mathcal{W}_1$ and $\mathcal{W}_2$ are N-D output-nulling subspaces of the feedback type and they are of $\Theta\,(\Sigma)$, then, both of them are contained in ker C and contains $\mathcal{W}^* \cap \text{im } B$, and their intersection is as well. The minimum element will be characterised in the sequel. ∎

Based on the relevant literature, the minimum subspace of $\Theta\,(\Sigma)$ in the 1-D case is $\mathcal{V}^* \cap \mathcal{S}^*$, where $\mathcal{V}^*$ is the largest output-nulling subspaces of the system and $\mathcal{S}^*$ is the smallest input-containing subspaces of the same system (Morse, 1973). However, the corresponding elements in the 2-D case is $\mathcal{W}^* \cap \mathcal{Z}^*$, which means the intersection of both the largest output-nulling of the feedback type of the system (3.11) $\mathcal{W}^*$ and the smallest input-containing of the output-injection type subspaces of the same system, does not represent the minimum element of $\Theta\,(\Sigma)$, even if it is a 2-D self-bounded subspace of the feedback type (Ntogramatzidis, 2012). As a result of what is proven by Lemmas 7.9 and 7.10 in Ntogramatzidis (2012), the minimum of $\Theta\,(\Sigma)$ coincides with the intersection of the set of subspaces $(A_0 + B\,F)$-, A_1- and A_2-invariant containing $\mathcal{W}^* \cap \text{im } B$, where F is a feedback type output-nulling friend of $\mathcal{W}^*$, which in the N-D case, can be computed by the following algorithm as a simple generalisation of the one in the 2-D case.

Lemma 3.7. *[Ntogramatzidis (2012)] $\mathcal{H}^*$ is the smallest subspace of $\Theta\,(\Sigma)$, which coincides with the last term of the sequence of subspaces:*

$$\begin{cases} \mathcal{H}_0 = \mathcal{W}^* \cap \text{im } B \\ \mathcal{H}_q = (A_{\ell_0} + B\,F)\,\mathcal{H}_{q-1} + \sum_{i=1}^{2^N-2} A_\ell\,\mathcal{H}_{q-1} + (\mathcal{W}^* \cap \text{im } B), \quad q \in \{1, 2, \ldots, \kappa\}, \end{cases} \tag{3.73}$$

where F is any feedback type output-nulling friend of $\mathcal{W}^*$, and the value $\kappa \leq n-1$ is determined by the equation $\mathcal{H}_{\kappa+1} = \mathcal{H}_\kappa$, i.e., $\mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \ldots \supset \mathcal{H}_\kappa = \mathcal{H}_{\kappa+1} = \mathcal{H}^*$.

The proof comes directly from Lemma 7.10, with a slight modification to the N-D system.

3.1.15 Reachability subspaces on N-D controlled invariant subspaces of feedback type

On the controlled invariant subspace in the 1-D case, on the controlled invariant subspace of the feedback type in the 2-D case or in general in the N-D case $\mathcal{W}$, only from the reachable part, we can reach any point of controlled invariant subspace ($\mathcal{W}$)

from the boundary condition ($\mathcal{W}$-valued boundary condition), with a local state that is solving the system (a $\mathcal{W}$-valued solution of (3.11)). This part is called the reachable part on a controlled invariant subspace (the reachable part on $\mathcal{W}$ and it is denoted by $\mathfrak{R}_{\mathcal{W}}$). The characterisation and computation of $\mathfrak{R}_{\mathcal{W}}$ is given in the following theorem.

Theorem 3.15. *Given an N-D controlled invariant subspace of the feedback type $\mathcal{W}$ with a feedback type controlled invariant friend F. $\mathfrak{R}_{\mathcal{W}}(F)$ is a minimum of the set of $(A_{\ell_0} + B\,F)$-, A_ℓ-invariant subspaces containing $\mathcal{W} \cap \mathrm{im}\ B$, for all $\ell \in S_N \backslash \{\ell_0\}$ that does not depend on F.*

Proof: For any $\mathcal{W}$-valued boundary condition, the control function $\mathbf{u}_{(i_1,i_2,\ldots,i_N)}$ can be written as $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)} + \mathbf{v}_{(i_1,i_2,\ldots,i_N)}$ to generate a $\mathcal{W}$-valued solution of Σ, so that we can define

$$\mathbf{v}_{(i_1,i_2,\ldots,i_N)} \stackrel{\mathrm{def}}{=} \mathbf{u}_{(i_1,i_2,\ldots,i_N)} - F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}.$$

Then,

$$\mathbf{x}_{(i_1+1,\ldots,i_N+1)} = (A_{\ell_0} + B\,F)\,\mathbf{x}_{(i_1,\ldots,i_N)} + \sum_{\ell \in S_N \backslash \{\ell_0\}} A_\ell\,\mathbf{x}_{(i_1,\ldots,i_N)} + B\,\mathbf{v}_{(i_1,\ldots,i_N)}. \quad (3.74)$$

The local state $\mathbf{x}_{(i_1+1,\ldots,i_N+1)}$ remains in $\mathcal{W}$, if the element $B\,\mathbf{v}_{(i_1,\ldots,i_N)}$ belongs to $\mathcal{W}$, and therefore belongs to $\mathcal{W} \cap \mathrm{im}\ B$. That means $\mathbf{v}_{(i_1,\ldots,i_N)} \in B^{-1}\,\mathcal{W}$ for all $(i_1,\ldots,i_N) \in \mathfrak{Q}$. The reachable subspace of (3.74), using the input $\mathbf{v}_{(i_1,\ldots,i_N)}$, is defined by the smallest of the set $(A_{\ell_0} + B\,F)$- and A_ℓ-invariant subspaces containing the subspace $\mathcal{W} \cap \mathrm{im}\ B$, for all $\ell \in S_N \backslash \{\ell_0\}$. To prove that the minimum $\mathfrak{R}_{\mathcal{W}}(F)$ does not depend on F, let F^1 and F^2 be two feedback type controlled invariant friends of $\mathcal{W}$, and let the states $\mathbf{x}_{(i_1,\ldots,i_N)+\ell}$ be in $\mathcal{W}$, then,

$$\mathbf{x}^j_{(i_1+1,\ldots,i_N+1)} = (A_{\ell_0} + B\,F^j)\,\mathbf{x}_{(i_1,\ldots,i_N)} + \sum_{\ell \in S_N \backslash \{\ell_0\}} A_\ell\,\mathbf{x}_{(i_1,\ldots,i_N)} + B\,\mathbf{v}^j_{(i_1,\ldots,i_N)},\ j = 1,2,$$

where the control functions $\mathbf{v}^j_{(i_1,\ldots,i_N)}$ belong to the subspace $B^{-1}\,\mathcal{W}$ for all $(i_1,\ldots,i_N)$ belong to $\mathfrak{Q}$. $\mathbf{v}^2_{(i_1,\ldots,i_N)}$ can be chosen to satisfy $\mathbf{x}^1_{(i_1+1,\ldots,i_N+1)} = \mathbf{x}^2_{(i_1+1,\ldots,i_N+1)}$. In particular, we can choose $\mathbf{v}^2_{(i_1,\ldots,i_N)} = \mathbf{v}^1_{(i_1,\ldots,i_N)} + (F^1 - F^2)\,\mathbf{x}_{(i_1,\ldots,i_N)}$. Then, $\mathbf{v}^2_{(i_1,\ldots,i_N)} \in B^{-1}\,\mathcal{W}$ for all $(i_1,\ldots,i_N)$ belong to $\mathfrak{Q}$. Thus, $\mathfrak{R}_{\mathcal{W}}(F^1) = \mathfrak{R}_{\mathcal{W}}(F^2)$. ■

Remark 3.1. From Theorem 3.15, it is obvious that

- any feedback type controlled invariant friend of $\mathcal{W}$ is also a reachability feedback type controlled invariant friend of $\mathfrak{R}_{\mathcal{W}}$;
- $\mathfrak{R}_{\mathcal{W}} \subseteq \mathfrak{R}^*$;
- $\mathfrak{R}_{\mathcal{W}^*} = \min \Phi\,(\Sigma)$.

In view of Theorem 3.15, the computation of subspace $\mathfrak{R}_{\mathcal{W}}$ comes from the computation of any controlled invariant friend F of $\mathcal{W}$. Then, this follows with the recursion which is given by:

$$\begin{cases} \mathfrak{R}_{\mathcal{W}0} &= \mathcal{W} \cap \mathrm{im}\ B, \\ \mathfrak{R}_{\mathcal{W}q} &= (A_{\ell_0} + B\,F)\,\mathfrak{R}_{\mathcal{W}q-1} + \sum_{i=1}^{2^N-2} A_\ell \mathfrak{R}_{\mathcal{W}q-1} + (\mathcal{W} \cap \mathrm{im}\ B) \;\; q \in \{1, 2, \ldots, \kappa\}, \end{cases}$$

where the value $\kappa \leq n-1$ is determined by the equation $\mathfrak{R}_{\mathcal{W}\kappa+1} = \mathfrak{R}_{\mathcal{W}\kappa}$.

CHAPTER 4

Geometric approach for N-D first-order Fornasini-Marchesini state space models

4.1 Structural invariants for N-D model

In this section, we extend the works of Ntogramatzidis et al. (2008) and Conte and Perdon (1988) to the N-D case by showing that an analogous characterisation holds. Consider an N-dimensional system, as described by Alpay and Dubi (2003) and Matsushita et al. (2013),

$$\begin{aligned}
\mathbf{x}_{(i_1,i_2,\ldots,i_{N-1},i_N)} &= A_1\,\mathbf{x}_{(i_1-1,i_2,\ldots,i_{N-1},i_N)} + \ldots + A_N\,\mathbf{x}_{(i_1,i_2\ldots,i_{N-1},i_N-1)} \\
&\quad + B_1\,\mathbf{u}_{(i_1-1,i_2,\ldots,i_{N-1},i_N)} + \ldots + B_N\,\mathbf{u}_{(i_1,i_2,\ldots,i_{N-1},i_N-1)} \qquad (4.1)\\
\mathbf{y}_{(i_1,i_2,\ldots,i_{N-1},i_N)} &= C\,\mathbf{x}_{(i_1,i_2,\ldots,i_{N-1},i_N)}, \qquad (4.2)
\end{aligned}$$

where, for all admissible values of the indices $i_1, i_2, \ldots, i_{N-1}, i_N$ in $\mathbb{Z}$, the vector $\mathbf{x}_{(i_1,i_2,\ldots,i_{N-1},i_N)} \in \mathcal{X} = \mathbb{R}^n$ represents the local state at $(i_1, i_2, \ldots, i_{N-1}, i_N) \in \underbrace{\mathbb{Z}\times\mathbb{Z}\times\ldots\times\mathbb{Z}}_{N\ times}$ and $\mathbf{u}_{(i_1,i_2,\ldots,i_{N-1},i_N)} \in \mathcal{U} = \mathbb{R}^m$ is the control input. Here, $A_1, \ldots, A_N \in \mathbb{R}^{n\times n}$, $B_1, \ldots, B_N \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$. For the sake of brevity, we denote system (4.1-4.2) by $(A_1, \ldots, A_N, B_1, \ldots, B_N, C)$. Therefore, system (4.1-4.2) is identify by $\Sigma = (A_1, \ldots, A_N, B_1, \ldots, B_N, C)$.

Consider also the separation sets defined, for each $k \in \mathbb{Z}$, as

$$\mathfrak{C}_k \stackrel{\text{def}}{=} \{(i_1, i_2, \ldots, i_N) \in \mathbb{Z}^N \,|\, i_1 + i_2 + \ldots + i_N = k\}. \qquad (4.3)$$

According to Fornasini and Marchesini (1984), the elements of the direct product of the local state spaces on $\mathfrak{C}_k$ are the global states on the same separation sets $\mathfrak{C}_k$ which is given by

$$\mathcal{X}_k \stackrel{\text{def}}{=} \{\mathbf{x}_{(i_1,i_2,\ldots,i_N)} \in \mathbb{Z}^N \,|\, (i_1, i_2, \ldots, i_N) \in \mathfrak{C}_k\}. \qquad (4.4)$$

Consider the usual boundary conditions associated with (4.1). In this case, if the values of $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ on $\mathfrak{C}_0$ are given (i.e., $\mathcal{X}_0$ is determined as a boundary condition), therefore, the update equation (4.1) gives $\mathcal{X}_k$ for all $k > 0$ which means $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ for $i_1 + i_2 + \ldots + i_N > 0$. In the sequel, for any subspace $\mathcal{W}$ of $\mathbb{R}^n$, we say that (4.1) has a $\mathcal{W}$-valued boundary condition, if $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ belongs to $\mathcal{W}$ for all $(i_1, i_2, ..., i_N) \in \mathfrak{C}_0$. Equivalently, for each $k > 0$, (4.1) also has a $\mathcal{W}$-valued solution determined by the global state $\mathcal{X}_k$, if $\mathbf{x}_{(i_1,i_2,\ldots,i_N)} \in \mathcal{W}$ for all $(i_1, i_2, ..., i_N) \in \mathfrak{C}_k$.

4.1.1 Invariant subspaces for N-D FM models

The concept of invariant subspaces developed in this section corresponds to the one for 1-D systems obtainable in Basile and Marro (1992) and the other for 2-D systems presented in Ntogramatzidis et al. (2008). We begin with the autonomous case of the N-D FM models (4.1) as

$$\mathbf{x}_{(i_1,i_2,\ldots,i_{N-1},i_N)} = A_1\,\mathbf{x}_{(i_1-1,i_2,\ldots,i_{N-1},i_N)} + \ldots + A_N\,\mathbf{x}_{(i_1,i_2\ldots,i_{N-1},i_N-1)}, \tag{4.5}$$

with the boundary conditions (4.3) and the corresponding case of global state (4.4).

Definition 39. Let the N-D FM model (4.5) be given and let $\mathcal{I}$ be a subspace of $\mathbb{R}^n$. The subspace $\mathcal{I}$ is an $(A_1, A_2, \ldots, A_N)$-invariant if

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} \mathbf{x} \in \underbrace{\mathcal{I} \times \mathcal{I} \times \ldots \times \mathcal{I}}_{N\ times}, \quad \forall \mathbf{x} \in \mathcal{I}.$$

This property of invariance is equivalent to the following inclusion:

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} \mathcal{I} \subseteq \underbrace{\mathcal{I} \times \mathcal{I} \times \ldots \times \mathcal{I}}_{N\ times} \ . \tag{4.6}$$

where $A_1, A_2, \ldots, A_N$ are matrices representation of linear maps in particular bases.

Definition 40. A subspace $\mathcal{I}$ is $(A_1, A_2, \ldots, A_N)$-invariant subspace, if and only if it is at the same time, A_1-, A_2-, …, A_N-invariant subspace in the 1-D counterpart.

Lemma 4.1. *Let $\mathcal{I}$ be a subspace of $\mathbb{R}^n$, of dimension r and a basis matrix $J \in \mathbb{R}^{n\times r}$. Then, $\mathcal{I}$ is $(A_1, A_2, \ldots, A_N)$-invariant if the matrices $X_1, X_2, \ldots, X_N \in \mathbb{R}^{r\times r}$ exist such* *that*

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} J = \begin{bmatrix} J & 0_{n\times r} & \dots & 0_{n\times r} \\ 0_{n\times r} & J & \dots & 0_{n\times r} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n\times r} & 0_{n\times r} & \dots & J \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \tag{4.7}$$

Proof: It is obvious that the relation of subspace inclusion (4.6) can be expressed in matrix form (4.7). ■

Theorem 4.1. *Let $\mathcal{I} \subseteq \mathbb{R}^r$ be a subspace of $\mathbb{R}^n$. The following two statements are equivalent:*

(i) *$\mathcal{I}$ is said to be $(A_1, A_2, \dots, A_N)$-invariant,*

(ii) *a change of coordinate T of $\mathbb{R}^{n\times n}$ exists such that*

$$\hat{A}_i \overset{def}{=} T^{-1} A_i T = \begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ 0_{(n-r)\times r} & \hat{A}_i^{22} \end{bmatrix}, \quad \text{for } i = 1, 2, \dots, N. \tag{4.8}$$

Proof: $(i) \Longrightarrow (ii)$ Let J be a basis matrix of $\mathcal{I}$. Let $T = \begin{bmatrix} J & T_2 \end{bmatrix}$ be an invertible $n \times n$ matrix. $T^{-1} J$ can be written as $\begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix}$ because J has full column-rank. As such, with $\hat{A}_i \overset{\text{def}}{=} T^{-1} A_i T = \begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ \hat{A}_i^{21} & \hat{A}_i^{22} \end{bmatrix}$, for $i = 1, 2, \dots, N$. Then, by Lemma 4.1, there exist N matrices $X_1, X_2, \dots, X_N$ such that (4.7) holds, it follows that

$$\begin{aligned} \begin{bmatrix} \hat{A}_i^{11} \\ \hat{A}_i^{21} \end{bmatrix} &= \begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ \hat{A}_i^{21} & \hat{A}_i^{22} \end{bmatrix} \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} \\ &= \hat{A}_i T^{-1} J = T^{-1} A_i J = T^{-1} J X_i = \begin{bmatrix} X_i \\ 0_{(n-r)\times r} \end{bmatrix}. \end{aligned} \tag{4.9}$$

For $i = 1, 2, \dots, N$, we get $\hat{A}_i^{21} = 0$.
$(ii) \Longrightarrow (i)$. Suppose that the equation (4.8) holds for a non-singular $(n \times n)$ matrix T. Then, clearly

$$\hat{A}_i \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = \begin{bmatrix} X_i \\ 0_{(n-r)\times r} \end{bmatrix}, \quad \text{for } i = 1, 2, \dots, N, \tag{4.10}$$

holds for $X_i = \hat{A}_i^{11}$. By multiplying both sides of (4.10) by T, we obtain

$$A_i T \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = T \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} X_i, \quad \text{for } i = 1, 2, \dots, N,$$

which by Lemma 4.1, guarantees that $T\begin{bmatrix} I_r \\ 0_{(n-r)\times r}\end{bmatrix}$ is an $(A_1, A_2, \dots, A_N)$-invariant subspace of dimension r. ■

Remark 4.1. Consider an $(A_1, A_2, \dots, A_N)$-invariant subspace $\mathcal{I}$ of dimension r and consider the change of coordinate matrix $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$, such that im $T_1 = \mathcal{I}$ and T_2 is such that T is not singular. The components of $\mathbf{x}_{(i_1,i_2,\dots,i_N)}$ with respect to the new coordinate are defined by

$$T^{-1}\,\mathbf{x}_{(i_1,i_2,\dots,i_N)} \overset{\text{def}}{=} \begin{bmatrix} \mathbf{x}'_{(i_1,i_2,\dots,i_N)} \\ \mathbf{x}''_{(i_1,i_2,\dots,i_N)} \end{bmatrix}. \tag{4.11}$$

Model (4.5) can be rewritten as

$$T^{-1}\,\mathbf{x}_{(i_1,i_2,\dots,i_N)} = T^{-1}\,A_1\,\mathbf{x}_{(i_1-1,i_2,\dots,i_N)} + \dots + T^{-1}\,A_N\,\mathbf{x}_{(i_1,\dots,i_{N-1},i_N-1)}$$

i.e.,

$$T^{-1}\,\mathbf{x}_{(i_1,\dots,i_N)} = (T^{-1}\,A_1\,T)\,T^{-1}\,\mathbf{x}_{(i_1-1,\dots,i_N)} + \dots + (T^{-1}\,A_N\,T)\,T^{-1}\,\mathbf{x}_{(i_1,\dots,i_{N-1},i_N-1)},$$

which gives

$$\begin{aligned} \begin{bmatrix} \mathbf{x}'_{(i_1,i_2,\dots,i_N)} \\ \mathbf{x}''_{(i_1,i_2,\dots,i_N)} \end{bmatrix} &= \begin{bmatrix} \hat{A}_1^{11} & \hat{A}_1^{12} \\ 0_{(n-r)\times r} & \hat{A}_1^{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}'_{(i_1-1,i_2,\dots,i_N)} \\ \mathbf{x}''_{(i_1-1,i_2,\dots,i_N)} \end{bmatrix} + \dots \\ &+ \begin{bmatrix} \hat{A}_N^{11} & \hat{A}_N^{12} \\ 0_{(n-r)\times r} & \hat{A}_N^{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}'_{(i_1,\dots,i_{N-1},i_N-1)} \\ \mathbf{x}''_{(i_1,\dots,i_{N-1},i_N-1)} \end{bmatrix}. \end{aligned} \tag{4.12}$$

These lead to

$$\begin{aligned} \mathbf{x}'_{(i_1,\dots,i_N)} &= \hat{A}_1^{11}\,\mathbf{x}'_{(i_1-1,\dots,i_N)} + \hat{A}_1^{12}\,\mathbf{x}''_{(i_1-1,\dots,i_N)} + \dots + \hat{A}_N^{11}\,\mathbf{x}'_{(i_1,\dots,i_{N-1},i_N-1)} \\ &+ \hat{A}_N^{12}\,\mathbf{x}''_{(i_1,\dots,i_{N-1},i_N-1)}, \end{aligned} \tag{4.13}$$

$$\mathbf{x}''_{(i_1,\dots,i_N)} = \hat{A}_1^{22}\,\mathbf{x}''_{(i_1-1,\dots,i_N)} + \dots + \hat{A}_N^{22}\,\mathbf{x}''_{(i_1,\dots,i_{N-1},i_N-1)}, \tag{4.14}$$

for all $(i_1, i_2, \dots, i_N) \in \mathfrak{C}_0$ in equation (4.13), and for all $(i_1, i_2, \dots, i_N) \in \mathfrak{C}_k$ in equation (4.14). Notice that the equation for $\mathbf{x}''_{(i_1,\dots,i_{N-1},i_N-1)}$ admits only the zero solution for any $\mathcal{I}$-valued boundary condition, with respect to model (4.5). This means that the $\mathcal{I}$-valued of model (4.5) is in the form $\begin{bmatrix} \mathbf{x}'_{(i_1,i_2,\dots,i_N)} \\ 0 \end{bmatrix}$ for $(i_1, i_2, \dots, i_N) \in \mathfrak{C}_k$ and $k > 0$.

From the considerations of Remark 4.1, it emerges that $\mathbf{x}'_{(i_1,i_2,\dots,i_N)}$ represents the components of the local states on an $(A_1, A_2, \dots, A_N)$-invariant subspace $\mathcal{I}$ and it is referred to as the internal components of $\mathcal{K}$, while $\mathbf{x}''_{(i_1,i_2,\dots,i_N)}$ represents the components of the local states on the quotient space $\mathbb{R}^n/\mathcal{I}$ and is referred to as the external components of $\mathcal{I}$ as well. Therefore, equation (4.13) controls the internal dynamics on $\mathcal{I}$ and equation (4.14) governs the external dynamics of the same subspace $\mathcal{I}$.

4.1.2 Internal and external stability of N-D invariant subspaces

Definition 41. The system (4.5) is called asymptotically stable if every solution tends to zero for $i_1 + i_2 + \ldots + i_N \longrightarrow \infty$.

According to Fornasini and Marchesini (1978), asymptotic stability of a N-tuple matrices $(A_1, A_2, \ldots, A_N)$ in equation (4.5) is equivalent to

$$\det(I_n - \sum_{i,j=1}^{N} A_i \prod_{j \neq i} z_j) \neq 0, \tag{4.15}$$

for all $(z_1, z_2, \ldots, z_N) \in \{(z_1, z_2, \ldots, z_N) \in \mathbb{C}^N : |z_i| \leq 1, \forall\, i \in \mathbb{N}\}$. If the N-tuple matrices $(A_1, A_2, \ldots, A_N)$ of system (4.5) is asymptotically stable, we also say that system (4.5) with matrices $A_1, A_2, \ldots, A_N$ is asymptotically stable. However, the following lemma expresses a sufficient condition of being asymptotically stable:

Lemma 4.2. *[Kar and Singh (2003)] The system with matrices $(A_1, A_2, \ldots, A_N)$ in (4.5) is asymptotically stable, if $n \times n$ symmetric positive definite matrices $P_1, P_2, \ldots, P_N$ exist such that*

$$\begin{bmatrix} P_1 & 0 & \ldots & 0 \\ 0 & P_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & P_N \end{bmatrix} - A^T (P_1 + P_2 + \ldots + P_N)\, A > 0, \tag{4.16}$$

where $A = [\, A_1 \;\; A_2 \;\; \ldots \;\; A_N \,]$.

Definition 42. Let $\mathcal{I}$ be an $(A_1, A_2, \ldots, A_N)$-invariant subspace. Consider the decomposition (4.8) in Theorem 4.1. Then, we say that $\mathcal{I}$ is

- internally stable, if $(\hat{A}_1^{11}, \hat{A}_2^{11}, \ldots, \hat{A}_N^{11})$ in (4.13) is asymptotically stable,
- externally stable, if $(\hat{A}_1^{22}, \hat{A}_2^{22}, \ldots, \hat{A}_N^{22})$ in (4.14) is asymptotically stable.

Lemma 4.3. *Let $\mathcal{I}$ be an $(A_1, A_2, \ldots, A_N)$-invariant subspace, of dimension r and with a basis matrix J. Let $X_1, X_2, \ldots, X_N \in \mathbb{R}^{r \times r}$ be matrices such that the relation (4.7) holds. The subspace $\mathcal{I}$ is said to be internally stable, if and only if the N-tuple matrices $(X_1, X_2, \ldots, X_N)$ is asymptotically stable.*

From the relation (4.15), the equivalent model (4.12) is asymptotically stable, if and only if the N-tuple matrices $(\hat{A}_1^{11}, \hat{A}_2^{11}, \ldots, \hat{A}_N^{11})$ and $(\hat{A}_1^{22}, \hat{A}_2^{22}, \ldots, \hat{A}_N^{22})$ are asymptotically stable. Therefore, model (4.5) is asymptotically stable, if and only if any $(A_1, A_2, \ldots, A_N)$-invariant subspace $\mathcal{I}$ is both internally and externally stable.

4.1.3 N-D controlled invariance

In this, the concept of $(A_1, A_2, \ldots, A_N, B_1, B_2, \ldots, B_N)$-controlled invariant subspace of the state-space $\mathcal{X}$ of Σ is introduced. This notion of controlled invariant subspace for N-D systems, which is enjoyed by feedback properties, is useful in several control problems.

Definition 43. Let model (4.1-4.2) representation of an N-D system $\Sigma = (A_1, \ldots, A_N, B_1, \ldots, B_N, C)$ be given. A subspace $\mathcal{V}$ of the state space model $\mathcal{X} = \mathbb{R}^n$ is an $(A_1, \ldots, A_N, B_1, \ldots, B_N)$-controlled invariant subspace, if the condition

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} \mathcal{V} \subset \underbrace{\mathcal{V} \times \mathcal{V} \times \ldots \times \mathcal{V}}_{N\ times} + \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}, \tag{4.17}$$

holds.

The same facts of the definition of N-D controlled invariant subspace in the previous chapter are consistent with this definition. For example, $\{0\}$ and $\mathbb{R}^n$ are N-D controlled invariant subspaces and the sum of N-D controlled invariant subspaces is an N-D controlled invariant subspace.

Remark 4.2. N-D controlled invariant subspaces enjoy these following properties:

- Any $(A_1, A_2, \ldots, A_N, B_1, B_2, \ldots, B_N)$-controlled invariant subspace in the N-D case is also an (A_i, B_i)-controlled invariant subspace, with $i \in \{1, 2, \ldots, N\}$ in the 1-D case, but not vice versa.

- Any N-D invariant subspace is an N-D controlled invariant subspace, but not vice versa.

Definition 44. Given an $(A_1, A_2, \ldots, A_N, B_1, B_2, \ldots, B_N)$-controlled invariant subspace $\mathcal{V}$ of $\mathcal{X}$, a feedback matrix $F : \mathcal{X} \longrightarrow \mathcal{U}$ exists such that

$$\left(\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} F \right) \mathcal{V} \subseteq \underbrace{\mathcal{V} \times \mathcal{V} \times \ldots \times \mathcal{V}}_{N\ times}. \tag{4.18}$$

A matrix F is called a controlled invariant friend of $\mathcal{V}$. The inclusion, as seen in (4.18), can be rewritten as

$$\left(\begin{bmatrix} A_1 + B_1\, F \\ A_2 + B_2\, F \\ \vdots \\ A_N + B_N\, F \end{bmatrix}\right) \mathcal{V} \subseteq \underbrace{\mathcal{V} \times \mathcal{V} \times \ldots \times \mathcal{V}}_{N\ times}. \tag{4.19}$$

The set of all the matrices F that satisfies the previous relation is denoted by $\mathfrak{F}(\mathcal{V})$.

By substituting the control $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ in (4.1), we obtain

$$\begin{aligned} \mathbf{x}_{(i_1,i_2,\ldots,i_N)} &= (A_1 + B_1\, F)\,\mathbf{x}_{(i_1-1,i_2,\ldots,i_N)} + \ldots \\ &\quad + (A_N + B_N\, F)\,\mathbf{x}_{(i_1,\ldots,i_{N-1},i_N-1)}. \end{aligned} \tag{4.20}$$

Then, for a given control and any $\mathcal{V}$-valued boundary condition, the global state $\mathcal{X}_k$ is $\mathcal{V}$-valued for $k > 0$, as discussed in the autonomous case, which ensures that $\mathcal{V}$ is an $(A_1 + B_1\, F, \ldots, A_N + B_N\, F)$-invariant subspace for all controlled invariant friends F of $\mathfrak{F}(\mathcal{V})$.

Proposition 4.1. *Let $\mathcal{V}$ be a subspace of the local state $\mathcal{X}$, with dimension r and a basis matrix V, and let the system $\Sigma = (A_1, \ldots, A_N, B_1, \ldots, B_N, C)$ be given. The following statements are equivalent:*

(i) *$\mathcal{V}$ is said to be an $(A_1, \ldots, A_N, B_1, \ldots, B_N)$-controlled invariant subspace;*

(ii) *matrices X and Ω exist such that*

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} V = \begin{bmatrix} V & 0 & \ldots & 0 \\ 0 & V & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & V \end{bmatrix} X + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} \Omega; \tag{4.21}$$

(iii) *$\mathcal{V}$ satisfies the inclusion (4.19);*

(iv) *The matrices X and F that belong to $\mathbb{R}^{nr\times r}$ and $\mathbb{R}^{m\times n}$ respectively, exist such that*

$$\left(\begin{bmatrix} A_1 + B_1\, F \\ A_2 + B_2\, F \\ \vdots \\ A_N + B_N\, F \end{bmatrix}\right) V = \begin{bmatrix} V & 0 & \ldots & 0 \\ 0 & V & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & V \end{bmatrix} X. \tag{4.22}$$

Proof: We prove that (i) implies (ii). It comes directly from the definition of controlled invariant, as per Definition 43.
$(ii) \implies (iii)$. Suppose that $F = -\Omega(V^\top V)^{-1}V^\top$, then we get $\Omega = -FV$. Then, equation (4.21) becomes

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} V = \begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix} X - \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} FV, \tag{4.23}$$

which can be rewritten in the form (4.19).
The fact that (iii) implies (iv) is clear, because equation (4.22) is a matrix representation of the inclusion, as seen in (4.19).
Now, we prove that (iv) implies (i). Equation (4.22) can be rewritten as

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} V = \begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix} X - \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} FV. \tag{4.24}$$

The proof is completed. ■

4.1.4 N-D output-nulling subspaces

Output-nulling subspaces will play an important and crucial role in many control theory problems, such as disturbance decoupling and model matching problems.

Proposition 4.2. *The family of all N-D controlled invariant subspaces contained in a certain subspace $\mathcal{K}$ of $\mathbb{R}^n$ has a maximum, and it is denoted by $\mathcal{V}^*(\mathcal{K})$.*

Proof: The concept of a controlled invariant subspace has some important characteristics. One of them is that a family of controlled invariant subspaces is not empty, since it contains at least $\{0\}$ and the other is that this family is closed under subspace addition. Thus, the addition of these subspaces is the maximum of this family, which is $\mathcal{V}^*(\mathcal{K})$.

Remark 4.3. The following are satisfied:

- If $\mathcal{K}$ is an N-D controlled invariant subspace, then $\mathcal{V}^*(\mathcal{K}) = \mathcal{K}$.
- If instead of $\mathcal{K}$ we consider ker C, the subspace $\mathcal{V}^*(\mathcal{K})$, in this case is called an N-D output nulling subspace and it is simply denoted by $\mathcal{V}^*$, which is considered a special type of N-D controlled invariant subspace.

- The set of all output-nulling subspaces of (4.1) and (4.2) is referred by $\mathfrak{V}(\mathcal{V})$, and it is closed under the addition of subspaces as in the 1-D case (Basile & Marro, 1992). This addition of subspaces is denoted by $\mathcal{V}^*$, which is the largest output-nulling subspace of $\mathfrak{V}(\mathcal{V})$.

An N-D output-nulling subspace $\mathcal{V}^*$ of global states is defined by existing a control input such that the corresponding global state $\mathcal{X}_k$ is $\mathcal{V}^*$-valued for $k > 0$, with the corresponding output to be zero for all $(i_1, i_2, \ldots, i_N)$, such that $i_1 + i_2 + \ldots + i_N \geqslant 0$, if the $\mathcal{V}^*$-valued boundary condition for (4.1) is given. The static state feedback is used as an alternative way to express the control function, i.e., the required control action that maintains $\mathbf{y}_{(i_1,i_2,\ldots,i_N)} = 0$ and the local state $\mathbf{x}_{(i_1,i_2,\ldots,i_N)} \in \mathcal{V}^*$ can always be expressed as $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\, \mathbf{x}_{(i_1,i_2,\ldots,i_N)}$. The following algorithm is a generalisation of Proposition 2.7 in Conte and Perdon (1988), which enables $\mathcal{V}^*$ to be computed in finite terms.

Proposition 4.3. *The subspace $\mathcal{V}^*$ coincides with the last term of the sequence of subspaces:*

$$\begin{cases} \mathcal{V}_0 = \ker C, \\ \mathcal{V}_q = \ker C \cap \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix}^{-1} (\underbrace{\mathcal{V}_{q-1} \times \mathcal{V}_{q-1} \times \ldots \times \mathcal{V}_{q-1}}_{N\ times} + \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}), \end{cases}$$

where q belongs to the set $\{1, \ldots, \kappa\}$, and the value $\kappa \leq n-1$ is determined by the condition $\mathcal{V}_{\kappa+1} = \mathcal{V}_\kappa$, i.e., $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \mathcal{V}_2 \supset \ldots \supset \mathcal{V}_\kappa = \mathcal{V}_{\kappa+1} = \mathcal{V}^*$.

Proof: The same steps of Lemma 3.2 can be followed for the proof. ■

4.1.5 Internal and external stabilisability of an N-D controlled invariant subspace

For all $F \in \mathfrak{F}(\mathcal{V})$, the controlled invariant subspace $\mathcal{V}$ is an $(A_1 + B_1 F, \ldots, A_N + B_N F)$-invariant subspace. Then, the definition of internal and external stabilisability of a controlled invariant subspace can be investigated by using the notions of internal and external stability of $(A_1 + B_1 F, \ldots, A_N + B_N F)$-invariant subspaces that are introduced in Definition 42.

Definition 45. Let $\mathcal{V}$ be an $(A_1, \ldots, A_N, B_1, \ldots, B_N)$-controlled invariant subspace, then,

- $\mathcal{V}$ is internally stabilisable, if a controlled invariant friend F of $\mathcal{V}$ exists, such that $\mathcal{V}$ is an $(A_1 + B_1 F, \ldots, A_N + B_N F)$-invariant internally stable subspace.

- $\mathcal{V}$ is externally stabilisable, if a controlled invariant friend F of $\mathcal{V}$ exists, such that $\mathcal{V}$ is an $(A_1 + B_1\, F, \ldots, A_N + B_N\, F)$-invariant externally stable subspace.

Choosing a controlled invariant friend F of an N-D controlled invariant subspace $\mathcal{V}$ with dimension r and a basis matrix $V \in \mathbb{R}^{n\times r}$ to satisfy the previous definition is carried out by finding the set of solutions of the linear equation $\Omega = -FV$, where $\Omega \in \mathbb{R}^{m\times r}$ is a solution of (4.21), for some $X \in \mathbb{R}^{Nr\times r}$. In particular, this solution is given by the set of all controlled invariant friends F as

$$F = -\Omega(V^\top V)^{-1}V^\top + \Phi, \tag{4.25}$$

such that $\Phi\, V = 0$, (4.25) can be written as

$$F = F_\Omega + \Phi, \tag{4.26}$$

where $F_\Omega = -\Omega(V^\top V)^{-1}V^\top$ and Φ as before. Consider a change of coordinate matrix $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$, such that T_1 is a basis for $\mathcal{V}$. Then, we have

$$T^{-1}\,(A_i + B_i\, F)\, T = \begin{bmatrix} L_{i,11}(\Omega,\Phi) & L_{i,12}(\Omega,\Phi) \\ 0 & L_{i,22}(\Omega,\Phi) \end{bmatrix} \quad \text{for } \ i = 1, 2, \ldots, N. \tag{4.27}$$

A controlled invariant friend of the N-D invariant subspace $\mathcal{V}$ can be built by choosing the matrices Ω and Φ independently, as shown in the following part.

Lemma 4.4. *The submatrices $L_{i,11}(\Omega,\Phi)$ of equation (4.27) do not depend on Φ. The submatrices $L_{i,22}(\Omega,\Phi)$ of equation (4.27) do not depend on Ω.*

Proof: We start proving that $L_{i,11}(\Omega,\Phi)$ do not depend on Φ. Let $F_\Omega = -\Omega(V^\top V)^{-1}V^\top$, where $\Omega \in \mathbb{R}$ is a solution of equation (4.21) for some X. Consider two matrices Φ' and Φ'' of suitable size, such that $F' = F_\Omega + \Phi'$ and $F'' = F_\Omega + \Phi''$, where $\Phi'\, V = 0$ and $\Phi''\, V = 0$. Now,

$$\begin{aligned} &\begin{bmatrix} L_{i,11}(\Omega,\Phi') - L_{i,11}(\Omega,\Phi'') & L_{i,12}(\Omega,\Phi') - L_{i,12}(\Omega,\Phi'') \\ 0 & L_{i,22}(\Omega,\Phi') - L_{i,22}(\Omega,\Phi'') \end{bmatrix} \\ &= T^{-1}\,(A_i + B_i\, F')\, T - T^{-1}\,(A_i + B_i\, F'')\, T \\ &= T^{-1}\,(A_i + B_i\, F_\Omega + B_i\, \Phi' - A_i - B_i\, F_\Omega - B_i\, \Phi'')\, T \\ &= T^{-1}\, B_i(\Phi' - \Phi'') \begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} 0 & T^{-1}\, B_i(\Phi' - \Phi'')T_2 \end{bmatrix}, \end{aligned}$$

since $\Phi'\, T_1 = \Phi''\, T_1 = 0$. This implies that $L_{i,11}(\Omega,\Phi') = L_{i,11}(\Omega,\Phi'')$.
Now we prove that $L_{i,22}(\Omega,\Phi)$ do not depend on Ω. Let Ω_1 and Ω_2 be two matrices of

suitable size. Since equation (4.21) holds for some matrices X' and X'' as

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} V = \begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix} X' + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} \Omega',$$

and

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} V = \begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix} X'' + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} \Omega''.$$

By subtraction, we obtain

$$\begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix} (X' - X'') = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} (\Omega' - \Omega''). \tag{4.28}$$

Let $F' = F_{\Omega'} + \Phi$ and $F'' = F_{\Omega''} + \Phi$, where $\Phi\, V = 0$ and

$$\begin{aligned} F_{\Omega'} &= -\Omega'(V^\top V)^{-1} V^\top \\ F_{\Omega''} &= -\Omega''(V^\top V)^{-1} V^\top. \end{aligned}$$

A simple computation shows that

$$\begin{aligned} &\begin{bmatrix} L_{i,11}(\Omega', \Phi) - L_{i,11}(\Omega'', \Phi) & L_{i,12}(\Omega', \Phi) - L_{i,12}(\Omega'', \Phi) \\ 0 & L_{i,22}(\Omega', \Phi) - L_{i,22}(\Omega'', \Phi) \end{bmatrix} \\ &= T^{-1}\,(A_i + B_i\, F')\, T - T^{-1}\,(A_i + B_i\, F'')\, T \\ &= T^{-1}\,(A_i + B_i\, F_{\Omega'} + B_i\, \Phi - A_i - B_i\, F_{\Omega''} - B_i\, \Phi)\, T \\ &= \begin{bmatrix} T_1 & T_2 \end{bmatrix}^{-1} B_i(\Omega'' - \Omega')\,(V^\top V)^{-1} V^\top\, T, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &B_i(\Omega'' - \Omega')\,(V^\top V)^{-1} V^\top \begin{bmatrix} T_1 & T_2 \end{bmatrix} = \\ &\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} L_{i,11}(\Omega', \Phi) - L_{i,11}(\Omega'', \Phi) & L_{i,12}(\Omega', \Phi) - L_{i,12}(\Omega'', \Phi) \\ 0 & L_{i,22}(\Omega', \Phi) - L_{i,22}(\Omega'', \Phi) \end{bmatrix}, \end{aligned}$$

which yields

$$B_i(\Omega''-\Omega')\,(V^\top V)^{-1}V^\top\,T_2 = T_1\,L_{i,12}(\Omega',\Phi)-L_{i,12}(\Omega'',\Phi) \\ + T_2\,L_{i,22}(\Omega',\Phi)-L_{i,22}(\Omega'',\Phi).$$

Since T_1 is a column basis matrix for $\mathcal{V}$ and V is a column basis matrix for $\mathcal{V}$, we assume that $T_1 = V$, and then,

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} (\Omega''-\Omega')\,(V^\top V)^{-1}V^\top\,T_2$$
$$= \begin{bmatrix} V\,(L_{1,12}(\Omega',\Phi)-L_{1,12}(\Omega'',\Phi))+T_2\,(L_{1,22}(\Omega',\Phi)-L_{1,22}(\Omega'',\Phi)) \\ V\,(L_{2,12}(\Omega',\Phi)-L_{2,12}(\Omega'',\Phi))+T_2(L_{2,22}(\Omega',\Phi)-L_{2,22}(\Omega'',\Phi)) \\ \vdots \\ V\,(L_{N,12}(\Omega',\Phi)-L_{N,12}(\Omega'',\Phi))+T_2(L_{N,22}(\Omega',\Phi)-L_{N,22}(\Omega'',\Phi)) \end{bmatrix}.$$

By using equation (4.28), we obtain

$$\begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix} (X'-X'')\,(V^\top V)^{-1}V^\top\,T_2$$
$$= \begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix} \begin{bmatrix} L_{1,12}(\Omega',\Phi)-L_{1,12}(\Omega'',\Phi) \\ L_{2,12}(\Omega',\Phi)-L_{2,12}(\Omega'',\Phi) \\ \vdots \\ L_{N,12}(\Omega',\Phi)-L_{N,12}(\Omega'',\Phi) \end{bmatrix}$$
$$+ \begin{bmatrix} T_2 & 0 & \dots & 0 \\ 0 & T_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T_2 \end{bmatrix} \begin{bmatrix} L_{1,22}(\Omega',\Phi)-L_{1,22}(\Omega'',\Phi) \\ L_{2,22}(\Omega',\Phi)-L_{2,22}(\Omega'',\Phi) \\ \vdots \\ L_{N,22}(\Omega',\Phi)-L_{N,22}(\Omega'',\Phi) \end{bmatrix}.$$

Therefore, it follows that

$$\begin{bmatrix} T_2 & 0 & \dots & 0 \\ 0 & T_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T_2 \end{bmatrix} \begin{bmatrix} L_{1,22}(\Omega',\Phi)-L_{1,22}(\Omega'',\Phi) \\ L_{2,22}(\Omega',\Phi)-L_{2,22}(\Omega'',\Phi) \\ \vdots \\ L_{N,22}(\Omega',\Phi)-L_{N,22}(\Omega'',\Phi) \end{bmatrix} = 0,$$

from the fact that T_2 has linearly independent columns that implies $L_{i,22}(\Omega', \Phi) - L_{i,22}(\Omega'', \Phi) = 0$ for $i = 1, \ldots, N$. ■

As a result, a matrix F_Ω,which is used to stabilise a controlled invariant subspace $\mathcal{V}$ internally, is equivalent to finding a matrix F_Ω for which equation (4.22) is verified for some $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$, such that the N-tuple matrices $(X_1, X_2, ..., X_N)$ is asymptotically stable. Hence, the degree of freedom here depends only on the choice of Ω, which is given by

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \\ \Omega \end{bmatrix} = \begin{bmatrix} V & 0 & \ldots & 0 & B_1 \\ 0 & V & \ldots & 0 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & V & B_N \end{bmatrix}^\dagger \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} V + \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_N \\ H_{N+1} \end{bmatrix} K, \tag{4.29}$$

where

$$\operatorname{im} \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_N \\ H_{N+1} \end{bmatrix} = \ker \begin{bmatrix} V & 0 & \ldots & 0 & B_1 \\ 0 & V & \ldots & 0 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & V & B_N \end{bmatrix},$$

and K is an arbitrary matrix of suitable size. For the sake of brevity, we suppose that:

$$\begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \\ G_{N+1} \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} V & 0 & \ldots & 0 & B_1 \\ 0 & V & \ldots & 0 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & V & B_N \end{bmatrix}^\dagger \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} V.$$

■

Then, equation (4.29) becomes

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \\ \Omega \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \\ G_{N+1} \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_N \\ H_{N+1} \end{bmatrix} K. \tag{4.30}$$

There are two possibilities for the solution of this equation. First, when

$$\ker \begin{bmatrix} V & 0 & \dots & 0 & B_1 \\ 0 & V & \dots & 0 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & V & B_N \end{bmatrix} = 0,$$

in this case, only one solution exists to achieve either internally stabilisable or not. The other solution is considered when

$$\ker \begin{bmatrix} V & 0 & \dots & 0 & B_1 \\ 0 & V & \dots & 0 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & V & B_N \end{bmatrix},$$

is different from zero. By finding a matrix K to be $(X_1, X_2, \dots, X_N)$ asymptotically stable, Ω can be computed from equation (4.30), which in order implies to solve (4.21) and (4.22) trivially.

Theorem 4.2. *The N-D controlled invariant subspace $\mathcal{V}$ is internally stabilisable, if there exist a matrix Q and symmetric positive definite matrices $M_1, M_2 \dots, M_N$ of suitable size such that*

$$\begin{bmatrix} M_1 & 0 & \dots & 0 & M_N G_1^\top + Q^\top H_1^\top \\ 0 & M_2 - M_1 & \dots & 0 & M_N G_2^\top + Q^\top H_2^\top \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & M_N - M_{N-1} & M_N G_N^\top + Q^\top H_N^\top \\ G_1 M_N + H_1 Q & G_2 M_N + H_2 Q & \dots & G_N M_N + H_N Q & M_N \end{bmatrix} > 0. \tag{4.31}$$

Defining an $(M_1, M_2 \dots, M_N, Q)$ by (4.31), a matrix K such that $(X_1, X_2 \dots, X_N)$ in equation (4.30) is asymptotically stable, is given by $K = Q\, M_N^{-1}$.

Proof: The N-D controlled invariant subspace $\mathcal{V}$ is internally stabilisable, if and only if the N-tuple matrices $(X_1, X_2 \dots, X_N)$ is asymptotically stable, i.e., if there exist matrices $P_1 = P_1^\top > 0, P_2 = P_2^\top > 0, \dots, P_N = P_N^\top > 0$, such that

$$\begin{bmatrix} P_1 & 0 & \dots & 0 \\ 0 & P_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_N \end{bmatrix} - \begin{bmatrix} X_1^\top \\ X_2^\top \\ \vdots \\ X_N^\top \end{bmatrix} P \begin{bmatrix} X_1 & X_2 & \dots & X_N \end{bmatrix} > 0, \tag{4.32}$$

where $P = (P_1 + P_2 + \ldots, +P_N)$, (4.32) is equivalent to

$$\begin{bmatrix} P_1 & 0 & \ldots & 0 \\ 0 & P_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & P_N \end{bmatrix} - \begin{bmatrix} X_1^\top \\ X_2^\top \\ \vdots \\ X_N^\top \end{bmatrix} P\, P^{-1}\, P \begin{bmatrix} X_1 & X_2 & \ldots & X_N \end{bmatrix} > 0. \tag{4.33}$$

From (4.30), $X_i = G_i + H_i\, K$, for $i = 1, 2, \ldots, N$, and by using the Schur complement Theorem in Golub and Van Loan (2012), (4.33) is equivalent to the existence of symmetric positive definite matrices $\Psi_1 = P_1, \Psi_2 = P_1 + P_2, \ldots, \Psi_{N-1} = P_1 + P_2 + \ldots, P_{N-1}$ and $\Psi_N = P_1 + P_2 + \ldots, P_{N-1} + P_N$, such that

$$\begin{bmatrix} \Psi_1 & 0 & \ldots & 0 & (L_1+H_1K)^\top \Psi_N \\ 0 & \Psi_2-\Psi_1 & \ldots & 0 & (L_2+H_2K)^\top \Psi_N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \Psi_N-\Psi_{N-1} & (L_N+H_NK)^\top \Psi_N \\ \Psi_N(L_1+H_1K) & \Psi_N(L_2+H_2K) & \ldots & \Psi_N(L_N+H_NK) & \Psi_N \end{bmatrix} > 0.$$

Multiplying both sides of the previous matrix by $\text{diag}(\underbrace{\Psi_N, \Psi_N, \ldots, \Psi_N}_{N+1\ times})$ and defining $M_i = \Psi_N^{-1}\Psi_i\,\Psi_N^{-1}$ $(i = 1, 2, \ldots, N-1)$, $M_N = \Psi_N^{-1}$, and $Q = K\Psi_N^{-1}$, we obtain (4.31), which concludes the proof. ■

The internal stabilisation of the N-D controlled invariant subspace $\mathcal{V}$ with a basis matrix V is based on choosing F_Ω as a controlled invariant friend of $\mathcal{V}$, such that the $(A_1 + B_1F_\Omega, \ldots, A_N + B_NF_\Omega)$-invariant subspace is internally stable, which happens if and only if $(X_1, X_2 \ldots, X_N)$ is asymptotically stable, while the external stabilisation of the N-D controlled invariant subspace $\mathcal{V}$ depends on choosing a suitable Φ in (4.26). By placing the control function $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ with $F = F_\Omega + \Phi$ in equation (4.1), we obtain

$$\begin{aligned} \mathbf{x}_{(i_1,i_2,\ldots,i_N)} &= (A_1 + B_1F_\Omega + B_1\Phi)\,\mathbf{x}_{(i_1-1,i_2,\ldots,i_N)} + \ldots \\ &+ (A_N + B_NF_\Omega + B_N\Phi)\,\mathbf{x}_{(i_1,\ldots,i_{N-1},i_N-1)}. \end{aligned}$$

Then, F_Ω stabilised $\mathcal{V}$ internally, in different words, $\mathcal{V}$ is an internally stable $(A_1 + B_1F_\Omega, \ldots, A_N + B_NF_\Omega)$-invariant subspace. To stabilise $\mathcal{V}$ externally, Φ must be found such that $((A_1 + B_1F_\Omega + B_1\Phi), \ldots, (A_N + B_NF_\Omega + B_N\Phi))$ is asymptotically stable and $\Phi\, V = 0$.

Theorem 4.3. *The N-D controlled invariant subspace $\mathcal{V}$ is externally stabilisable, if there exist a matrix R and symmetric positive definite matrices $M_1, M_2 \ldots, M_N, M_{N+1}$*

of suitable size such that

$$\begin{bmatrix} M_1 & 0 & \dots & 0 & (L_{1,\Omega}+B_1R^\top Q^\top)^\top \\ 0 & M_2-M_1 & \dots & 0 & (L_{2,\Omega}+B_2R^\top Q^\top)^\top \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & M_N-M_{N-1} & (L_{N,\Omega}+B_NR^\top Q^\top)^\top \\ L_{1,\Omega}+B_1R^\top Q^\top & L_{2,\Omega}+B_2R^\top Q^\top & \dots & L_{N,\Omega}+B_NR^\top Q^\top & M_{N+1} \end{bmatrix} > 0, \tag{4.34}$$

where $L_{i,\Omega} \stackrel{def}{=} A_i + B_iF_\Omega,\ (i = 1,\dots,N)$ *and with*

$$M_N\, M_{N+1} = I \tag{4.35}$$

.

Proof: The condition $\Phi\, V = 0$ is equivalent to im $\Phi^\top \subseteq$ ker $V^\top$, by considering im $Q =$ ker $V^\top$. Then, im $\Phi^\top$, for some matrix R, this implies to $\Phi^\top = Q\, R$, so that $\Phi = R^\top Q^\top$. The N-tuple matrices $((L_{1,\Omega}+B_1\Phi),\dots,(L_{N,\Omega}+B_N\Phi))$ is asymptotically stable by using Lemma 4.2, if the symmetric positive definite matrices $P_1, P_2, \dots, P_N$ exist such that

$$\begin{bmatrix} P_1 & 0 & \dots & 0 \\ 0 & P_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_N \end{bmatrix} - \begin{bmatrix} (L_{1,\Omega}+B_1\Phi)^\top \\ \vdots \\ (L_{N,\Omega}+B_N\Phi)^\top \end{bmatrix} (P_1 + \dots + P_N) \begin{bmatrix} (L_{1,\Omega}+B_1\Phi) & \dots & (L_{N,\Omega}+B_N\Phi) \end{bmatrix} > 0,$$

by using the Schur complement Theorem in Golub and Van Loan (2012).This condition is equivalent to the existence of symmetric positive definite matrices $M_1 = P_1, M_2 = P_1 + P_2, \dots, M_{N-1} = P_1 + P_2 + \dots, P_{N-1}$ and $M_N = P_1 + P_2 + \dots, P_{N-1} + P_N$, such that

$$\begin{bmatrix} M_1 & 0 & \dots & 0 & (L_{1,\Omega}+B_1\Phi)^\top \\ 0 & M_2-M_1 & \dots & 0 & (L_{2,\Omega}+B_2\Phi)^\top \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & M_N-M_{N-1} & (L_{N,\Omega}+B_N\Phi)^\top \\ (L_{1,\Omega}+B_1\Phi) & (L_{2,\Omega}+B_2\Phi) & \dots & (L_{N,\Omega}+B_N\Phi) & M_N^{-1} \end{bmatrix} > 0. \tag{4.36}$$

Finally, by substituting $\Phi = R^\top Q^\top$ with $M_{N+1} = M_N^{-1}$ from (4.35) in (4.36), we obtain (4.34). ■

4.1.6 Disturbance decoupling problems

The problem of disturbance decoupling by state feedback has been studied and solved by using geometric methods. There are two versions of the problem, the basic version which is a solution without stability. It was presented for the 1-D case by Basile and Marro (1969) and for the 2-D case by Conte and Perdon (1988), while the same problem is solved by using the notion of stability as in W. Wonham (1985) for the 1-D case and by Ntogramatzidis et al. (2008) in the 2-D case. These results have been adopted to the N-D case. Regarding the generalisation, consider the N-D system

$$\begin{aligned}
\mathbf{x}_{(i_1,i_2,\ldots,i_N)} &= A_1\,\mathbf{x}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + A_N\,\mathbf{x}_{(i_1,\ldots,i_{N-1},i_N-1)} \\
&+ B_1\,\mathbf{u}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + B_N\,\mathbf{u}_{(i_1,\ldots,i_{N-1},i_N-1)} \\
&+ H_1\,\mathbf{w}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + H_N\,\mathbf{w}_{(i_1,\ldots,i_{N-1},i_N-1)} \\
\mathbf{y}_{(i_1,i_2,\ldots,i_N)} &= C\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)},
\end{aligned} \tag{4.37}$$

where $\mathbf{x}_{(i_1,i_2,\ldots,i_N)}, \mathbf{u}_{(i_1,i_2,\ldots,i_N)}, \mathbf{w}_{(i_1,i_2,\ldots,i_N)}$ and $\mathbf{y}_{(i_1,i_2,\ldots,i_N)}$ are respectively, the local state, the controlled input, the disturbance and the output vector, and they respectively belong to $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^d$ and $\mathbb{R}^p$. A_k, B_k, H_k and C, for $k = 1, 2, \ldots, N$, which are matrices with dimensions $n\times n, n\times m, n\times d$ and $p\times n$, respectively. The disturbance decoupling problem (DDP) is studied and solved for the N-D Fornasini-Marchesini model (4.37) by finding conditions ensuring that a feedback law $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ exists, such that the disturbance $\mathbf{w}_{(i_1,i_2,\ldots,i_N)}$ does not affect on the output function $\mathbf{y}_{(i_1,i_2,\ldots,i_N)}$ of the closed-loop system. The other decoupling problem is considered when the disturbance $\mathbf{w}_{(i_1,i_2,\ldots,i_N)}$ is measurable. In this case, with the measurable disturbance decoupling problem (MDDP), the feedback law takes the form $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)} + S\,\mathbf{w}_{(i_1,i_2,\ldots,i_N)}$. Hence, the two decoupling problems can be solved along the same lines as Basile and Marro (1969) and Conte and Perdon (1988). The following proposition provides a sufficient condition for the resolvability of the DDP. However, the necessary condition is not satisfied in an N-D case, because it is difficult in general to characterise the set of output-nulling global states, as the set of global states whose local components belong to some subspace $\mathcal{V}$ of $\mathbb{R}^n$.

Proposition 4.4. (i) *A sufficient condition of the DDP for system (4.37) to be solvable is*

$$\operatorname{im}\begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_N \end{bmatrix} \subseteq \underbrace{\mathcal{V}^* \times \mathcal{V}^* \times \ldots \times \mathcal{V}^*}_{N\ times}. \tag{4.38}$$

When this condition is satisfied, any output-nulling friend F of $\mathcal{V}^*$ solves this problem.

(ii) *A sufficient condition of the MDDP for system (4.37) to be solvable is*

$$\operatorname{im}\begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_N \end{bmatrix} \subseteq (\underbrace{\mathcal{V}^* \times \mathcal{V}^* \times \ldots \times \mathcal{V}^*}_{N\ times}) + \operatorname{im}\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}. \tag{4.39}$$

Proof: The proof of case (i) is adapted straightforwardly from the proof of Theorem 3.3 with a few modifications. For the proof of case (ii), let us consider $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d\}$ as a basis of $\mathbb{R}^d$, the q number of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q$ from $\mathbb{R}^m$ have been chosen such that

$$\begin{bmatrix} H_1 \\ H_2 \\ \cdots \\ H_N \end{bmatrix} (\mathbf{w}_i) - \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} (\mathbf{u}_i) \in \underbrace{\mathcal{V}^\star \times \mathcal{V}^\star \times \ldots \times \mathcal{V}^\star}_{N\ times}, \tag{4.40}$$

and define the map $S : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ as $S(\mathbf{w}_i) = \mathbf{u}_i$. By taking F to be any controlled invariant friend of $\mathcal{V}$ and $\mathcal{V}^*$ is an output-nulling subspace of global states, the sufficient condition is completed. ■

Now, to solve the DDP by stabilisability of controlled invariant subspace, we will take into account the requirement of stability of the closed-loop matrices $(A_1+B_1 F, \ldots, A_N+B_N F)$, i.e., a matrix F exists, such that the disturbance $\mathbf{w}$ is decoupled by $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)}$ from the output $\mathbf{y}$ and the N-tuple of the closed-loop system matrices $(A_1 + B_1 F, \ldots, A_N + B_N F)$ is asymptotically stable. Similarly, the MDDP is solvable by finding the matrices F and S, such that the disturbance $\mathbf{w}$ is decoupled by $\mathbf{u}_{(i_1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)} + S\mathbf{w}_{(i_1,i_2,\ldots,i_N)}$ from the output $\mathbf{y}$ and such that the N-tuple of the closed-loop system matrices $(A_1 + B_1 F, \ldots, A_N + B_N F)$ is asymptotically stable. The following theory illustrates the proposed solutions to the two problems of decoupling using the concept of stabilisability.

Proposition 4.5. (i) *The DDP for system (4.37) is solvable if*

- $\operatorname{im} \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_N \end{bmatrix} \subseteq \underbrace{\mathcal{V}^* \times \mathcal{V}^* \times \ldots \times \mathcal{V}^*}_{N\ times},$
- $\mathcal{V}^*$ *is internally and externally stabilisable at the same time.*

If these conditions hold, any output-nulling friend of $\mathcal{V}^*$ that stabilizes $\mathcal{V}^*$ internally and externally is a solution of the problem.

(ii) *The MDDP for system (4.37) is solvable if*

- $\operatorname{im} \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_N \end{bmatrix} \subseteq (\underbrace{\mathcal{V}^* \times \mathcal{V}^* \times \ldots \times \mathcal{V}^*}_{N\ times}) + \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix},$
- $\mathcal{V}^*$ *is both internally and externally stabilisable.*

Proof: The proof of (i) follows by compiling two results, which is summarised by choosing a matrix F to be an output-nulling friend of $\mathcal{V}^*$, such that for the first condition, the closed-loop system,

$$\begin{aligned}\mathbf{x}_{(i_1,i_2,\ldots,i_N)} &= (A_1 + B_1F)\,\mathbf{x}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + (A_N + B_NF)\,\mathbf{x}_{(i_1,\ldots,i_{N-1},i_N-1)}\\ &+ H_1\,\mathbf{w}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + H_N\,\mathbf{w}_{(i_1,\ldots,i_{N-1},i_N-1)},\end{aligned}$$

is not affected by the disturbance $\mathbf{w}$ and for the other condition, the N-tuple of the closed-loop matrices $(A_1+B_1\,F,\ldots,A_N+B_N\,F)$ is asymptotically stable. This if found to be similar for part (ii) as well.

4.1.7 Model matching problem

The other important class of control theory in the 1-D and 2-D setting is the so-called model matching problem. The model matching problem is well-known in the area of 1-D systems (Kuvcera, 1981; Malabre & Kucera, 1984). Moreover, in the 2-D case, it has been studied and solved in several aspects. For instance, Conte and Perdon (1988) gave the solution via the geometric approach, while the proportional state feedback and the dynamical state feedback of a special type used by Paraskevopoulos (1979) and Yasuda (1981) respectively. Sebek (1983) used the polynomial method and recently, the stability has been used by Ntogramatzidis et al. (2008). From these papers, it turns out that a number of control problems, in general, have been solved by recasting the control problems as model matching problems. In the situation considered, the model matching problem has been taken into account, in terms of the geometric terms and with stability for the N-D case. Suppose that a system $\Sigma = (A_1,\ldots,A_N,B_1,\ldots,B_N,C)$ that is governed by (4.1) and (4.2) and a system Σ_M is described by

$$\begin{aligned}\mathbf{x}_{(i_1,i_2,\ldots,i_N)M} &= A_{1M}\,\mathbf{x}_{(i_1-1,i_2,\ldots,i_N)M} + \ldots + A_{NM}\,\mathbf{x}_{(i_1,\ldots,i_{N-1},i_N-1)M}\\ &+ B_{1M}\,\mathbf{u}_{(i_1-1,i_2,\ldots,i_N)M} + \ldots + B_{NM}\,\mathbf{u}_{(i_1,\ldots,i_{N-1},i_N-1)M} \qquad (4.41)\\ \mathbf{y}_{(i_1,i_2,\ldots,i_N)M} &= C_M\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)M}, \qquad (4.42)\end{aligned}$$

which are given the same output spaces. The exact model matching depends on designing the input by the dynamic regulate system Σ_C governed by

$$\begin{aligned}\mathbf{z}_{(i_1,i_2,\ldots,i_N)} &= A_{1C}\,\mathbf{z}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + A_{NC}\,\mathbf{z}_{(i_1,\ldots,i_{N-1},i_N-1)}\\ &+ B_{1C}\,\tilde{\mathbf{u}}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + B_{NC}\,\tilde{\mathbf{u}}_{(i_1,\ldots,i_{N-1},i_N-1)}, \qquad (4.43)\\ \mathbf{u}_{(i_1,i_2,\ldots,i_N)} &= R\,\mathbf{z}_{(i_1,i_2,\ldots,i_N)} + H\,\tilde{\mathbf{u}}_{(i_1-1,i_2,\ldots,i_N)}, \qquad (4.44)\end{aligned}$$

such that the model Σ_M is considered as a result of the input and output behaviour between the connection of the original system Σ and the exact model matching system

Σ_M. The extended system Σ_e can be used to turn the model matching problem into a full information decoupling problem Σ_e which is characterised by the local state $\mathbf{x}_e = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_M \end{bmatrix}$ and matrices

$$A_{ke} = \begin{bmatrix} A_k & 0 \\ 0 & A_{kM} \end{bmatrix}, \quad B_{ke} = \begin{bmatrix} B_k \\ 0 \end{bmatrix}, \quad H_{ke} = \begin{bmatrix} 0 \\ B_{kM} \end{bmatrix}, \quad C_e = \begin{bmatrix} C & - & C_M \end{bmatrix},$$

for $K = 1, 2, \ldots, N$. Then, the system Σ_e with the disturbance $\tilde{\mathbf{u}}$ can be written as

$$\begin{aligned} \mathbf{x}_{e(i_1,i_2,\ldots,i_N)} &= A_{1e}\,\mathbf{x}_{e(i_1-1,i_2,\ldots,i_N)} + \ldots + A_{Ne}\,\mathbf{x}_{e(i_1,\ldots,i_{N-1},i_N-1)} \\ &+ B_{1e}\,\mathbf{u}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + B_{Ne}\,\mathbf{u}_{(i_1,\ldots,i_{N-1},i_N-1)} \\ &+ H_{1e}\,\tilde{\mathbf{u}}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + H_{Ne}\,\tilde{\mathbf{u}}_{(i_1,\ldots,i_{N-1},i_N-1)} \\ \mathbf{y}_{(i_1,i_2,\ldots,i_N)} &= C_e\,\mathbf{x}_{e(i_1,i_2,\ldots,i_N)}. \end{aligned} \tag{4.45}$$

In this case, the exact model matching problem (EMMP) is solvable if the MDDP for the system Σ_e is solvable. In other words, with a feedback law $\mathbf{u}_{(i_1-1,i_2,\ldots,i_N)} = F\,\mathbf{x}_{e(i_1-1,i_2,\ldots,i_N)} + S\,\tilde{\mathbf{u}}_{(i_1-1,i_2,\ldots,i_N)}$, the output of the closed loop of the system Σ_e is not affected by the disturbance $\tilde{\mathbf{u}}$. The EMMP in the N-D case can be solved by using a direct consequence of part (ii) of the Proposition 4.4 as follows.

Proposition 4.6. *The exact model matching problem is solvable if the following condition holds:*

$$\operatorname{im}\begin{bmatrix} H_{1e} \\ H_{2e} \\ \vdots \\ H_{Ne} \end{bmatrix} \subseteq (\underbrace{\mathcal{V}_e^* \times \mathcal{V}_e^* \times \ldots \times \mathcal{V}_e^*}_{N\ times}) + \operatorname{im}\begin{bmatrix} B_{1e} \\ B_{2e} \\ \vdots \\ B_{Ne} \end{bmatrix}, \tag{4.46}$$

where $\mathcal{V}_e^*$ is the largest output-nulling of the system Σ_e that is governed by (4.45).

Now, for solving the model matching problem by the notion of stability, the two following conditions must hold:

- $\operatorname{im}\begin{bmatrix} H_{1e} \\ H_{2e} \\ \vdots \\ H_{Ne} \end{bmatrix} \subseteq \underbrace{\mathcal{V}_e^* \times \mathcal{V}_e^* \times \ldots \times \mathcal{V}_e^*}_{N\ times} + \operatorname{im}\begin{bmatrix} B_{1e} \\ B_{2e} \\ \vdots \\ B_{Ne} \end{bmatrix},$
- $\mathcal{V}_e^*$ is internally stabilisable,

where $\mathcal{V}_e^*$ is the largest output-nulling of the system $\Sigma_e = (A_{1e}, \ldots, A_{Ne}, B_{1e}, \ldots, B_{Ne}, C_e)$. The due substitution can be easily noticed from Propositions 4.4 and 4.5, which are mainly derived from system (4.45) that solves the model matching problem by two ways (output-nulling subspaces and the stabilisation of controlled invariant subspaces) respectively.

CHAPTER 5

Geometric conditions for the existence of solutions of singular multidimensional systems

In the singular model, there is no privileged direction of the evolution of the states. In this chapter, we study the problem of characterising the admissible boundary conditions in a 2-D Fornasini-Marchesini singular model, where various aspects of the situations of local states and inputs are explained. This result is then generalised to N-dimensional systems.[†]

5.1 Existence of solutions for 2-D systems

The state model of a 2-D system that we consider is the implicit first order Fornasini-Marchesini 2-D model given by the following equations Kaczorek (1988):

$$E\,\mathbf{x}_{(i+1,j+1)} = A_1\,\mathbf{x}_{(i+1,j)} + A_2\,\mathbf{x}_{(i,j+1)} + B_1\,\mathbf{u}_{(i+1,j)} + B_2\,\mathbf{u}_{(i,j+1)}, \tag{5.1}$$

$$\mathbf{y}_{(i,j)} = C\,\mathbf{x}_{(i,j)}, \tag{5.2}$$

where, for all admissible values of the indices i and j in $\mathbb{Z}$, the vector $\mathbf{x}_{(i,j)} \in \mathcal{X} = \mathbb{R}^n$ represents the local state and $\mathbf{u}_{(i,j)} \in \mathcal{U} = \mathbb{R}^m$ is the control input. Here, E, A_1, $A_2 \in \mathbb{R}^{q\times n}$ and $B_1, B_2 \in \mathbb{R}^{q\times m}$. The outer state-space, corresponding to the number of equations in (5.1), is denoted by $\underline{\mathcal{X}} = \mathbb{R}^q$. In this implicit model, the matrices E, A_1, A_2 are in general not square, and if they are square (i.e., if $q = n$), they may be singular.

[†]This chapter has been published in 2017, 10th International Workshop on Multidimensional Systems (nDS) by the following reference (Mohamed, Padula, & Ntogramatzidis, 2017).

We are interested in the evolution of 'south-west' causal solutions, given suitable boundary conditions $\mathbf{x}_{(i,j)}$ for $(i,j) \in \mathfrak{C}_0$, where

$$\mathfrak{C}_k = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} |\, i+j=k\}, \quad k \in \mathbb{Z}.$$

Within this setting, the latent variable evolves over the region

$$\mathfrak{C} \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \mathfrak{C}_k = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} |\, i+j \geq 0\}.$$

5.1.1 Formulation of problems

Problem 1. Determining under which conditions on the local states of the separation set $\mathfrak{C}_0$, the local states on a given region of the separation set $\mathfrak{C}_M$, where $M \in \mathbb{N} \setminus \{0\}$, are a solution of the Fornasini-Marchesini model (5.1) with arbitrary inputs. We can formalise this problem as follows. Let $h \in \mathbb{Z}$ and $r \in \mathbb{N}\backslash\{0\}$. Let $M > 0$. There exists

$$\tilde{\mathbf{x}}_{(h,M-h)}, \tilde{\mathbf{x}}_{(h-1,M-h+1)}, \ldots, \tilde{\mathbf{x}}_{(h-r+1,M-h+r-1)} \in \mathcal{X},$$

such that for every $\mathbf{u}_{(i,j)} \in \mathcal{U}$ there exists a solution:

$$\{\mathbf{x}_{(i,j)} \mid i \leqslant h,\, j \leqslant M-h+q-1,\, 0 \leqslant i+j \leqslant M\},$$

of (5.1), such that

$$\begin{cases} \mathbf{x}_{(h,M-h)} = \tilde{\mathbf{x}}_{(h,M-h)} \\ \mathbf{x}_{(h-1,M-h+1)} = \tilde{\mathbf{x}}_{(h-1,M-h+1)} \\ \vdots \\ \mathbf{x}_{(h-r+1,M-h+r-1)} = \tilde{\mathbf{x}}_{(h-r+1,M-h+r-1)}, \end{cases}$$

where the vectors $\mathbf{x}_{(h,M-h)}, \mathbf{x}_{(h-1,M-h+1)}, \ldots, \mathbf{x}_{(h-r+1,M-h+r-1)}$ are local states over a segment of $\mathfrak{C}_M$.

Problem 2. Characterising under which conditions on the local states of the separation set $\mathfrak{C}_0$, there exists an input such that the local states on an assigned region of the separation set $\mathfrak{C}_M$ are a solution of the Fornasini-Marchesini model (5.1) for that input. Let $h \in \mathbb{Z}$ and $r \in \mathbb{N}\backslash\{0\}$. Let $M > 0$. There exists

$$\tilde{\mathbf{x}}_{(h,M-h)}, \tilde{\mathbf{x}}_{(h-1,M-h+1)}, \ldots, \tilde{\mathbf{x}}_{(h-r+1,M-h+r-1)} \in \mathcal{X},$$

such that for a suitable control $\mathbf{u}_{(i,j)}$, there exists a solution:

$$\{\mathbf{x}_{(i,j)} \mid i \leqslant h,\, j \leqslant M-h+r-1,\, 0 \leqslant i+j \leqslant M\},$$

of (5.1), such that

$$\begin{cases} \mathbf{x}_{(h,M-h)} = \tilde{\mathbf{x}}_{(h,M-h)} \\ \mathbf{x}_{(h-1,M-h+1)} = \tilde{\mathbf{x}}_{(h-1,M-h+1)} \\ \vdots \\ \mathbf{x}_{(h-r+1,M-h+r-1)} = \tilde{\mathbf{x}}_{(h-r+1,M-h+r-1)}. \end{cases}$$

We begin by considering Problem 1, which is the problem of the existence of solutions for every arbitrarily assigned input function. The following result shows that Problem 1 can be solved geometrically, by means of recursive conditions on subspaces that can be constructed in an iterative fashion.

Theorem 5.1. *Let $h \in \mathbb{Z}$ and $r \in \mathbb{N}\backslash\{0\}$. Let $M > 0$. Problem 1 admits solutions if and only if:*

(i) $\operatorname{im} \hat{B}_i \subseteq \mathcal{M}_i$ *for all* $i \in \{r, \ldots, r+M-1\}$*;*

(ii) *the local states on* $\mathfrak{C}_0$ *satisfy*

$$\hat{\mathbf{x}}_{(0;h,r+M)} \in \hat{A}_{r+M-1}^{-1} \mathcal{M}_{r+M-1},$$

where $\hat{A}_i$, $\hat{B}_i, \hat{\mathbf{x}}_{(\alpha;\beta,\gamma)}$, $\hat{E}_i$ *and* $\mathcal{M}_i$, $i \in \mathbb{N}$ *are, respectively, given by:*

$$\hat{A}_i = \begin{bmatrix} A_1 & A_2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & A_1 & A_2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & A_1 & A_2 & \ldots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \ldots & A_1 & A_2 \end{bmatrix} \hat{B}_i = \begin{bmatrix} B_1 & B_2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & B_1 & B_2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & B_1 & B_2 & \ldots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \ldots & B_1 & B_2 \end{bmatrix}$$

$$\hat{\mathbf{x}}_{(\alpha;\beta,\gamma)} = \begin{bmatrix} \mathbf{x}_{(\beta,\alpha-\beta)} \\ \mathbf{x}_{(\beta-1,\alpha-\beta+1)} \\ \vdots \\ \mathbf{x}_{(\beta-\gamma+1,\alpha-\beta+\gamma-1)} \end{bmatrix}, \qquad \hat{E}_i = \begin{bmatrix} E & 0 & \ldots & 0 \\ 0 & E & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & \ldots & E \end{bmatrix},$$

and

$$\begin{cases} \mathcal{M}_\ell = \operatorname{im} \hat{E}_\ell \\ \mathcal{M}_i = \hat{E}_i\,(\hat{A}_{i-1}^{-1}\,\mathcal{M}_{i-1}) \qquad i > \ell, \end{cases}$$

where $\hat{A}_i$ and $\hat{B}_i$ are $i\,q \times (i+1)\,n$, $\hat{E}_i$ is a block-diagonal matrix, with i identical blocks E and $\hat{\mathbf{x}}_{(\alpha,\beta;\gamma)}$ is a vector whose components are the local states on the separation set $\mathfrak{C}_\alpha$.

Proof: We prove the claim by letting the local state evolve backwards from the separation set $\mathfrak{C}_M$ to $\mathfrak{C}_0$. Every set of local states on the region of $\mathfrak{C}_M$:

$$\{\mathbf{x}_{(h,M-h)}, \mathbf{x}_{(h-1,M-h+1)}, \dots, \mathbf{x}_{(h-r+1,M-h+r-1)}\},$$

in Problem 1 can be written in a compact form as

$$\hat{\mathbf{x}}_{(M;h,r)} = \begin{bmatrix} \mathbf{x}_{(h,M-h)} \\ \mathbf{x}_{(h-1,M-h+1)} \\ \vdots \\ \mathbf{x}_{(h-r+1,M-h+r-1)} \end{bmatrix}.$$

See Figure 5.1 below.

Clearly, a local state $\mathbf{x}_{(h,M-h)}$ on $\mathfrak{C}_M$ is a solution of model (5.1) for arbitrary inputs $\mathbf{u}_{(h,M-h-1)}$ and $\mathbf{u}_{(h-1,M-h)}$, if and only if $\mathbf{x}_{(h,M-h-1)}$ and $\mathbf{x}_{(h-1,M-h)}$ satisfy

$$A_1\,\mathbf{x}_{(h,M-h-1)} + A_2\,\mathbf{x}_{(h-1,M-h)} + B_1\,\mathbf{u}_{(h,M-h-1)} + B_2\,\mathbf{u}_{(h-1,M-h)} \in \operatorname{im} E.$$

Next, the local state $\mathbf{x}_{(h-1,M-h+1)}$ on the same separation set $\mathfrak{C}_M$ is a solution of model (5.1) for arbitrary inputs $\mathbf{u}_{(h-1,M-h)}$ and $\mathbf{u}_{(h-2,M-h+1)}$, if and only if

$$A_1\,\mathbf{x}_{(h-1,M-h)} + A_2\,\mathbf{x}_{(h-2,M-h+1)} + B_1\,\mathbf{u}_{(h-2,M-h)} + B_2\,\mathbf{u}_{(h-2,M-h)} \in \operatorname{im} E.$$

The previous machinery can be applied to all the vector components of $\hat{\mathbf{x}}_{(M;h,r)}$. Lastly, the local state $\mathbf{x}_{(h-r+1,M-h+r-1)}$ on the same separation set, which is a solution of model (5.1), exists for arbitrary inputs $\mathbf{u}_{(h-r+1,M-h+r-2)}$ and $\mathbf{u}_{(h-r,M-h+r-1)}$, if and only if

$$\begin{aligned} A_1\,\mathbf{x}_{(h-r+1,M-h+r-2)} + A_2\,\mathbf{x}_{(h-r,M-h+r-1)} &+ B_1\,\mathbf{u}_{(h-r+1,M-h+r-2)} \\ &+ B_2\,\mathbf{u}_{(h-r,M-h+r-1)} \in \operatorname{im} E. \end{aligned}$$

These equations can be written in matrix form as

$$\begin{bmatrix} A_1 & A_2 & 0 & \dots & 0 & 0 \\ 0 & A_1 & A_2 & \dots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & A_1 & A_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{(h,M-h-1)} \\ \mathbf{x}_{(h-1,M-h)} \\ \mathbf{x}_{(h-2,M-h+1)} \\ \vdots \\ \mathbf{x}_{(h-r+1,M-h+r-2)} \\ \mathbf{x}_{(h-r,M-h+r-1)} \end{bmatrix} + \begin{bmatrix} B_1 & B_2 & 0 & \dots & 0 & 0 \\ 0 & B_1 & B_2 & \dots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & B_1 & B_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{(h,M-h-1)} \\ \mathbf{u}_{(h-1,M-h)} \\ \mathbf{u}_{(h-2,M-h+1)} \\ \vdots \\ \mathbf{u}_{(h-r+1,M-h+r-2)} \\ \mathbf{u}_{(h-r,M-h+r-1)} \end{bmatrix}$$

$$\in \operatorname{im} \begin{bmatrix} E & 0 & \dots & 0 \\ 0 & E & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & E \end{bmatrix},$$

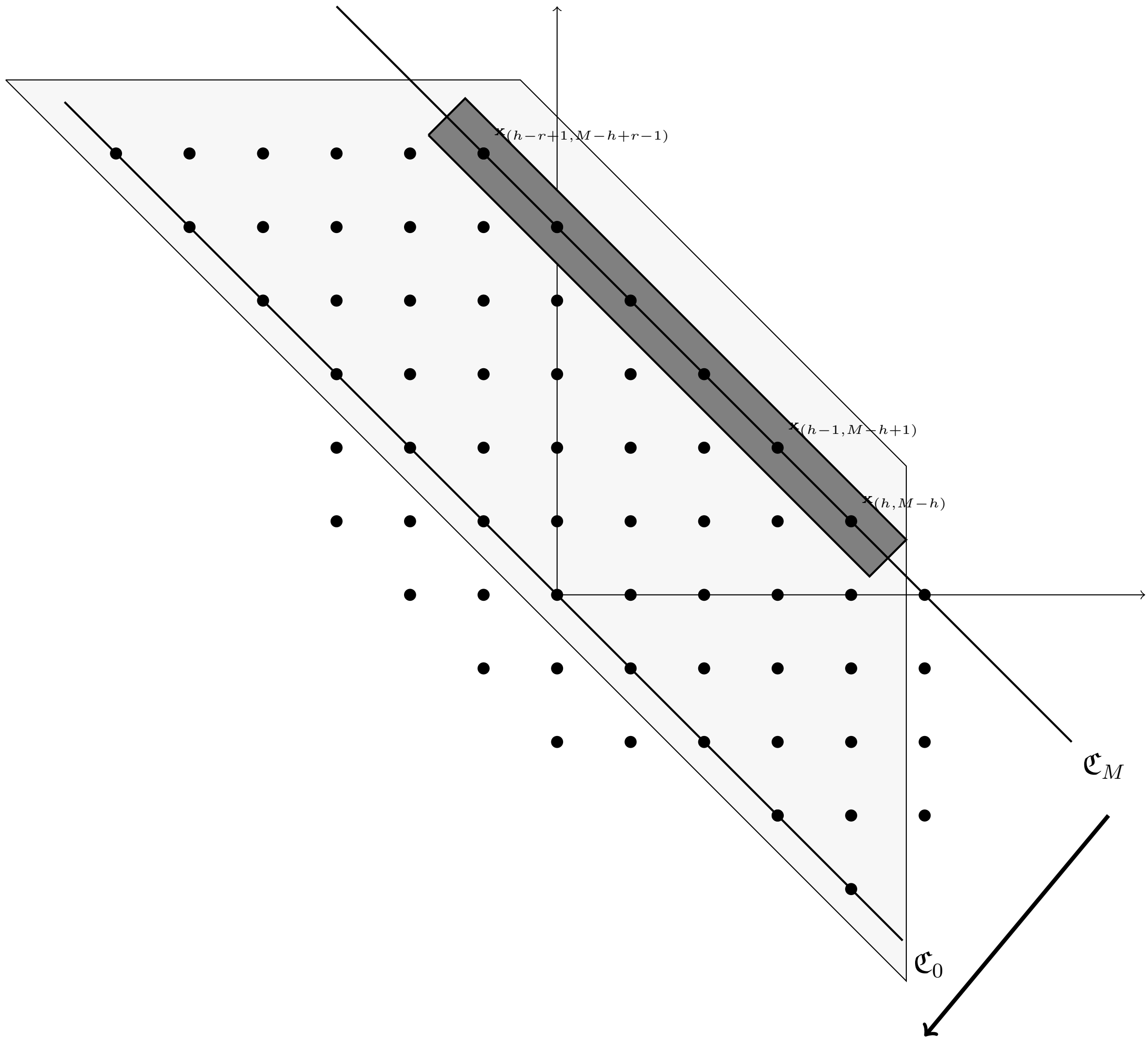

Figure 5.1

where the first and third matrices on the left-hand side have $r \times (r+1)$ blocks, while the one on the right-hand side has $r \times r$ blocks. Using the notations introduced in the statement, we rewrite the latter in compact form as

$$\hat{A}_r \, \hat{\mathbf{x}}_{(M-1;h,r+1)} + \hat{B}_r \, \hat{\mathbf{u}}_{(M-1;h,r+1)} \in \text{im } \hat{E}_r, \tag{5.3}$$

where

$$\hat{\mathbf{u}}_{(M-1;h,r+1)} = \begin{bmatrix} \mathbf{u}_{(h,M-h-1)} \\ \mathbf{u}_{(h-1,M-h)} \\ \vdots \\ \mathbf{u}_{(h-r,M-h+r-1)} \end{bmatrix}.$$

Since $\mathbf{u}_{(i,j)}$ is arbitrary, (5.3) needs to hold in particular when $\mathbf{u}_{(i,j)}$ is identically zero, so that $\hat{\mathbf{x}}_{(M-1;h,r+1)}$ must satisfy:

$$\hat{A}_r \, \hat{\mathbf{x}}_{(M-1;h,r+1)} \in \text{im } \hat{E}_r,$$

which can be written as

$$\hat{\mathbf{x}}_{(M-1;h,r+1)} \in \hat{A}_r^{-1} \text{ im } \hat{E}_r,$$

where $\hat{A}_r^{-1}$ im $\hat{E}_r$ denotes the inverse image of $\hat{E}_r$ through the map $\hat{A}_r$, and is defined also when $\hat{A}_r$ is non-invertible. If the previous condition is satisfied, in order for the left hand-side of (5.3) to be in im $\hat{E}_r$, we must have

$$\text{im } \hat{B}_r \subseteq \text{im } \hat{E}_r.$$

As a result, the condition on the separation set $\mathfrak{C}_{M-1}$ is:

- im $\hat{B}_r \subseteq$ im $\hat{E}_r$;
- $\hat{\mathbf{x}}_{(M-1;h,q+1)} \in \hat{A}_r^{-1}$ im $\hat{E}_r$.

We can write these equations in terms of the following sequence of subspaces:

$$\begin{cases} \mathcal{M}_r = \text{im } \hat{E}_r \\ \mathcal{M}_i = \hat{E}_i \, (\hat{A}_{i-1}^{-1} \, \mathcal{M}_{i-1}) \qquad i > r, \end{cases}$$

as

$$\begin{cases} \text{im } \hat{B}_r \subseteq \mathcal{M}_r \\ \hat{\mathbf{x}}_{(M-1;h,r+1)} \in \hat{A}_r^{-1} \mathcal{M}_r. \end{cases}$$

At the next step, we consider the separation set $\mathfrak{C}_{M-2}$. The recursiveness of the conditions for existence arises because the local states on $\mathfrak{C}_{M-2}$ need to satisfy the equations of model (5.1) for those local states on $\mathfrak{C}_{M-1}$, which guarantees that the conditions at the previous step are met. Using the same notation as in the previous step, we can express these conditions as

- $\operatorname{im} \hat{B}_{r+1} \subseteq \hat{E}_{r+1}(\hat{A}_r^{-1} \operatorname{im} \hat{E}_r)$,
- $\hat{\mathbf{x}}_{(M-2;h,r+2)} \in \hat{A}_{r+1}^{-1}\, \hat{E}_{r+1}\, (\hat{A}_r^{-1} \operatorname{im} \hat{E}_r)$,

which can be written as

$$\begin{cases} \operatorname{im} \hat{B}_{r+1} \subseteq \mathcal{M}_{r+1} \\ \hat{\mathbf{x}}_{(M-2;h,r+2)} \in \hat{A}_{r+1}^{-1} \mathcal{M}_{r+1}. \end{cases}$$

Iterating the previous steps up to $\mathfrak{C}_0$, we obtain the conditions:

$$\begin{cases} \operatorname{im} \hat{B}_{r+M-1} \subseteq \mathcal{M}_{r+M-1} \\ \hat{\mathbf{x}}_{(0;h,r+M)} \in \hat{A}_{r+M-1}^{-1} \mathcal{M}_{r+M-1}. \end{cases}$$

This conclude the proof. ■

The next result addresses Problem 2. Given a subspace $\mathcal{W}$ in $\mathbb{R}^{\nu+\mu}$, we define the projection map as

$$\mathfrak{P}_\nu(\mathcal{W}) \stackrel{\text{def}}{=} \left\{ \xi \in \mathbb{R}^\nu \,\middle|\, \exists\, \omega \in \mathbb{R}^\mu \,:\, \begin{bmatrix} \xi \\ \omega \end{bmatrix} \in \mathcal{W} \right\}.$$

Theorem 5.2. *Let $h \in \mathbb{Z}$ and $r \in \mathbb{N} \backslash \{0\}$. Let $M > 0$. Problem 2 admits solutions, if and only if the local states on $\mathfrak{C}_0$ with suitable controls satisfy:*

$$\hat{\mathbf{x}}_{(0;h,r+M)} \in \mathfrak{P}_{(r+M)\,N}(\hat{F}_{r+M-1}^{-1} \mathcal{M}_{r+M-1}),$$

where $\hat{A}_i$, $\hat{B}_i$, $\hat{\mathbf{x}}_{(\alpha;\beta,\gamma)}$ and $\hat{E}_i$ are the same as in Theorem 5.1, $\mathcal{M}_i$ with $i \in \mathbb{N}$ is a sequence of subspaces given by

$$\begin{cases} \mathcal{M}_\ell \;=\; \operatorname{im} \hat{E}_\ell \\ \mathcal{M}_i \;=\; \hat{E}_i\, \mathfrak{P}_{i\,N}(\hat{F}_{i-1}^{-1}\, \mathcal{M}_{i-1}) \qquad i > \ell, \end{cases} \tag{5.4}$$

where $\hat{F}_i = \begin{bmatrix} \hat{A}_i & \hat{B}_i \end{bmatrix}$ and $\hat{\mathbf{u}}_{(\alpha;\beta,\gamma)}$ is given by

$$\hat{\mathbf{u}}_{(\alpha;\beta,\gamma)} = \begin{bmatrix} \mathbf{u}_{(\beta,\alpha-\beta)} \\ \mathbf{u}_{(\beta-1,\alpha-\beta+1)} \\ \vdots \\ \mathbf{u}_{(\beta-\gamma+1,\alpha-\beta+\gamma-1)} \end{bmatrix}.$$

Proof:

We prove the claim by moving backwards from the separation set $\mathfrak{C}_M$ to $\mathfrak{C}_0$. Consider the set of states on $\mathfrak{C}_M$ with suitable controls. This can be written in compact form as

$$\begin{bmatrix} \hat{\mathbf{x}}_{(M;h,r)} \\ \hat{\mathbf{u}}_{(M;h,r)} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{(h,M-h)} \\ \mathbf{x}_{(h-1,M-h+1)} \\ \vdots \\ \mathbf{x}_{(h-r+1,M-h+r-1)} \\ \mathbf{u}_{(h,M-h)} \\ \mathbf{u}_{(h-1,M-h+1)} \\ \vdots \\ \mathbf{u}_{(-r+1,M-h+r-1)} \end{bmatrix}.$$

There exists a state $\mathbf{x}_{(h,M-h)}$ on $\mathfrak{C}_M$ which is a solution of (5.1) for suitable inputs $\mathbf{u}_{(h,M-h-1)}$ and $\mathbf{u}_{(h-1,M-h)}$, if and only if $\mathbf{x}_{(h,M-h-1)}$ and $\mathbf{x}_{(h-1,M-h)}$ satisfy

$$A_1\,\mathbf{x}_{(h,M-h-1)} + A_2\,\mathbf{x}_{(h-1,M-h)} + \\ B_1\,\mathbf{u}_{(h,M-h-1)} + B_2\,\mathbf{u}_{(h-1,M-h)} \in \operatorname{im} E.$$

The local state $\mathbf{x}_{(h-2,M-h+2)}$ on the same separation set, which is a solution of (5.1) as well, exists for suitable inputs $\mathbf{u}_{(h-2,M-h+1)}$ and $\mathbf{u}_{(h-3,M-h+2)}$, if and only if

$$A_1\,\mathbf{x}_{(h-2,M-h+1)} + A_2\,\mathbf{x}_{(h-3,M-h+2)} + \\ B_1\,\mathbf{u}_{(h-2,M-h+1)} + B_2\,\mathbf{u}_{(h-3,M-h+2)} \in \operatorname{im} E.$$

The previous machinery can be applied to all the vectors components of $\hat{\mathbf{x}}_{(M;h,r)}$ with suitable inputs $\hat{\mathbf{u}}_{(M;h,r)}$. Lastly, the local state $\mathbf{x}_{(h-r+1,N-h+r-1)}$ on the same separation set, which is a solution of (5.1) exists for suitable inputs $\mathbf{u}_{(h-r+1,M-h+r-2)}$ and $\mathbf{u}_{(h-r,M-h+r-1)}$, if and only if

$$A_1\,\mathbf{x}_{(h-q+1,N-h+q-2)} + A_2\,\mathbf{x}_{(h-q,N-h+q-1)} + \\ B_1\,\mathbf{u}_{(h-q+1,N-h+q-2)} + B_2\,\mathbf{u}_{(h-q,N-h+q-1)} \in \operatorname{im} E.$$

These equations can be written as

$$\begin{bmatrix} A_1 & A_2 & 0 & 0\,0 & \dots & 0 & 0 \\ 0 & A_1 & A_2 & 0\,0 & \dots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0\,0 & \dots & A_1 & A_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{(h,M-h-1)} \\ \mathbf{x}_{(h-1,M-h)} \\ \mathbf{x}_{(h-2,M-h+1)} \\ \mathbf{x}_{(h-3,M-h+2)} \\ \vdots \\ \mathbf{x}_{(h-r+1,M-h+r-2)} \\ \mathbf{x}_{(h-r,M-h+r-1)} \end{bmatrix} + \begin{bmatrix} B_1 & B_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & B_1 & B_2 & 0 & \dots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & B_1 & B_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{(h,M-h-1)} \\ \mathbf{u}_{(h-1,M-h)} \\ \mathbf{u}_{(h-2,M-h+1)} \\ \mathbf{u}_{(h-3,M-h+2)} \\ \vdots \\ \mathbf{u}_{(h-r+1,M-h+r-2)} \\ \mathbf{u}_{(h-r,M-h+r-1)} \end{bmatrix}$$

$$\in \operatorname{im} \begin{bmatrix} E & 0 & 0 & \dots & 0 \\ 0 & E & 0 & \dots & 0 \\ 0 & 0 & E & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & E \end{bmatrix}, \tag{5.5}$$

where the first and third matrices on the left-hand side have $r \times (r+1)$ blocks, while the one on the right-hand side has $r \times r$ blocks. Using the notations introduced in the statement, we rewrite (5.5) as

$$\hat{A}_i\,\hat{\mathbf{x}}_{(M-1;h,r+1)} + \hat{B}_i\,\hat{\mathbf{u}}_{(M-1;h,r+1)} \in \operatorname{im}\,\hat{E}_i,$$

which can be written as

$$\begin{bmatrix} \hat{A}_i & \hat{B}_i \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_{(M-1;h,q+1)} \\ \hat{\mathbf{u}}_{(M-1;h,q+1)} \end{bmatrix} \in \operatorname{im}\,\hat{E}_i,$$

and this is equivalent to

$$\begin{bmatrix} \hat{\mathbf{x}}_{(M-1;h,r+1)} \\ \hat{\mathbf{u}}_{(M-1;h,r+1)} \end{bmatrix} \in \begin{bmatrix} \hat{A}_i & \hat{B}_i \end{bmatrix}^{-1} \operatorname{im}\,\hat{E}_i.$$

Using the notation $\hat{F}_i = \begin{bmatrix} \hat{A}_i & \hat{B}_i \end{bmatrix}$, we can rewrite this equation as

$$\begin{bmatrix} \hat{\mathbf{x}}_{(M-1;h,r+1)} \\ \hat{\mathbf{u}}_{(M-1;h,r+1)} \end{bmatrix} \in \hat{F}_i^{-1} \operatorname{im}\,\hat{E}_i.$$

Therefore, the conditions on the separation set $\mathfrak{C}_{M-1}$ can be written as

$$\hat{\mathbf{x}}_{(M-1;h,r+1)} \in \mathfrak{P}_{(r+1)\,N}\,(\hat{F}_r^{-1} \operatorname{im}\hat{E}_r).$$

We can rewrite this condition, by means of the sequence of subspaces in (5.4), where ℓ is chosen to be equal to r:

$$\begin{cases} \mathcal{M}_r &= \operatorname{im}\,\hat{E}_r \\ \mathcal{M}_i &= \hat{E}_i\,\mathfrak{P}_{i,N}(\hat{F}_{i-1}^{-1}\,\mathcal{M}_{i-1}) \qquad i > r, \end{cases}$$

as

$$\hat{\mathbf{x}}_{(M-1;h,r+1)} \in \mathfrak{P}_{(r+1)\,N}\,(\hat{F}_r^{-1}\mathcal{M}_r).$$

At the next step, on the separation set $\mathfrak{C}_{M-2}$, we get

$$\begin{aligned} \hat{\mathbf{x}}_{(M-2;h,r+2)} &\in \mathfrak{P}_{(r+2)\,N}\left(\hat{F}_{r+1}^{-1}\,\hat{E}_{r+1}\,\mathfrak{P}_{(r+1)\,N}\,(\hat{F}_r^{-1} \operatorname{im}\,\hat{E}_r)\right) \\ &= \mathfrak{P}_{(r+2)\,N}\left(\hat{F}_{r+1}^{-1}\mathcal{M}_{r+1}\right). \end{aligned}$$

We have to iterate until we find a condition on the local states of the separation set $\mathfrak{C}_0$:

$$\hat{\mathbf{x}}_{(0;h,r+M)} \in \mathfrak{P}_{(r+M)\,N} \quad \big(\hat{F}_{r+M-1}^{-1}\,\hat{E}_{r+M-1}\,\mathfrak{P}_{(r+M-1)\,N}\hat{F}_{r+M-2}^{-1}\,\hat{E}_{r+M-2}\,\mathfrak{P}_{(r+M-2)\,N}\cdots \hat{F}_{r+1}^{-1}\,\hat{E}_{r+1}\,\mathfrak{P}_{(r+1)\,N}(\hat{F}_r^{-1}\,\mathrm{im}\,\hat{E}_r)\big).$$

Then, by using the sequence of subspaces in (5.4), we get

$$\hat{\mathbf{x}}_{(0;h,r+M)} \in \mathfrak{P}_{(r+M)\,N}\big(\hat{F}_{r+M-1}^{-1}\mathcal{M}_{r+M-1}\big).$$

■

5.2 Existence of solutions of the local states in N-D systems

In the case of N-D systems, we show how the first result considered in the previous section can be generalised in the special case, where a single point in the separation set $\mathfrak{C}_M$ is considered. The second result can be extended along the same lines.

Consider an N-dimensional system:

$$\begin{aligned} E\,\mathbf{x}_{(i_1,i_2,\ldots,i_N)} &= A_1\,\mathbf{x}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + A_N\,\mathbf{x}_{(i_1,\ldots,i_{N-1},i_N-1)} \\ &\quad + B_1\,\mathbf{u}_{(i_1-1,i_2,\ldots,i_N)} + \ldots + B_N\,\mathbf{u}_{(i_1,\ldots,i_{N-1},i_N-1)}. \end{aligned}$$

Consider also the separation sets defined, for each $k \in \mathbb{Z}$, as

$$\mathfrak{C}_k = \{(i_1, i_2, \ldots, i_N) \in \mathbb{Z}^N \mid i_1 + i_2 + \ldots + i_N = k\}.$$

Let $M > 0$. Consider a local state $\mathbf{x}_{(i_1,\ldots,i_N)}$ on the separation set $\mathfrak{C}_M$, i.e. $\sum_{j=1}^N i_j = M$. The local state $\mathbf{x}_{(i_1,\ldots,i_N)}$ depends only a subset of local states on the separation set $\mathfrak{C}_{M-k}$, which is characterised as

$$\begin{aligned} \hat{\mathbf{x}}_{(N,0)} &= x_{i_1,\ldots,i_N} \\ \hat{\mathbf{x}}_{(N,k)} &= \bigsqcup_{j_1=1}^{N}\bigsqcup_{j_2=j_1}^{N}\cdots\bigsqcup_{j_k=j_{k-1}}^{N} \mathbf{x}_{(i_1,\ldots,i_N-\mathbf{1}_{j_1,\ldots,j_k})}, \quad k \geq 1, \end{aligned}$$

where the sequence of operators $\bigsqcup$ appends each new element at the bottom of the vector obtained at the previous iteration and generates the vector at the first iteration, i.e., when $j_1 = j_2 = \cdots = j_k = 1$.

The operator $\mathbf{1}_{j_1,\ldots,j_k}$ subtracts 1 to the j_ℓ-th index i_{j_ℓ}. Note that the sequence $j_1, \ldots, j_k$ has repeated elements and therefore, multiple subtractions to the same index are allowed.

For example, for $N = 4$, $k = 3$, $j_1 = 2$, $j_2 = 2$ and $j_3 = 4$, we have

$$\mathbf{x}_{(i_1,i_2,i_3,i_4-\mathbf{1}_{2,2,4})} = \mathbf{x}_{(i_1,i_2-1-1,i_3,i_4-1)} = \mathbf{x}_{(i_1,i_2-2,i_3,i_4-1)}.$$

The resulting vector on the separation set $\mathfrak{C}_{M-3}$ is

$$\hat{\mathbf{x}}_{(4,3)} = \begin{bmatrix} \mathbf{x}_{(i_1-3,i_2,i_3,i_4)} \\ \mathbf{x}_{(i_1-2,i_2-1,i_3,i_4)} \\ \mathbf{x}_{(i_1-2,i_2,i_3-1,i_4)} \\ \mathbf{x}_{(i_1-2,i_2,i_3,i_4-1)} \\ \mathbf{x}_{(i_1-1,i_2-2,i_3,i_4)} \\ \mathbf{x}_{(i_1-1,i_2-1,i_3-1,i_4)} \\ \mathbf{x}_{(i_1-1,i_2-1,i_3,i_4-1)} \\ \mathbf{x}_{(i_1-1,i_2,i_3-2,i_4)} \\ \mathbf{x}_{(i_1-1,i_2,i_3-1,i_4-1)} \\ \mathbf{x}_{(i_1,i_2-3,i_3,i_4)} \\ \mathbf{x}_{(i_1,i_2-2,i_3-1,i_4)} \\ \mathbf{x}_{(i_1,i_2-2,i_3,i_4-1)} \\ \mathbf{x}_{(i_1,i_2-1,i_3-2,i_4)} \\ \mathbf{x}_{(i_1,i_2-1,i_3-1,i_4-1)} \\ \mathbf{x}_{(i_1,i_2-1,i_3,i_4-2)} \\ \mathbf{x}_{(i_1,i_2,i_3-3,i_4)} \\ \mathbf{x}_{(i_1,i_2,i_3-2,i_4-1)} \\ \mathbf{x}_{(i_1,i_2,i_3-1,i_4-2)} \\ \mathbf{x}_{(i_1,i_2,i_3,i_4-3)} \end{bmatrix}.$$

The input vector $\hat{\mathbf{u}}_{(N,k)}$ on the separation set $\mathfrak{C}_{M-k}$ can be characterised by using a machinery that is identical to the one adopted to define $\hat{\mathbf{x}}_{(N,k)}$.

Let us define the following matrices:

$$T_{N,i}^{(1)} \stackrel{\text{def}}{=} A_i,$$

$$T_{N,i}^{(k)} \stackrel{\text{def}}{=} \left[\begin{array}{c|cccc} T_{N,i}^{(k-1)} & \downarrow T_{N,i+1}^{(k-1)} & \cdots & \downarrow T_{N,N}^{(k-1)} \\ \hline 0 & A_i & & \\ \vdots & & \ddots & \\ 0 & & & A_i \end{array}\right],$$

$$S_{N,i}^{(1)} \stackrel{\text{def}}{=} B_i,$$

$$S_{N,i}^{(k)} \stackrel{\text{def}}{=} \left[\begin{array}{c|ccc} S_{N,i}^{(k-1)} & \downarrow S_{N,i+1}^{(k-1)} & \cdots & \downarrow S_{N,N}^{(k-1)} \\ \hline 0 & B_i & & \\ \vdots & & \ddots & \\ 0 & & & B_i \end{array}\right],$$

where the operator $\downarrow$ aligns the last row of two consecutive matrices of different dimensions and fills the missing entries with zeros, e.g.,

$$M = \left[\begin{array}{cc} I_{3\times 3} & \downarrow I_{2\times 2} \end{array}\right] = \left[\begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \downarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}\right] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ \cline{4-5} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right].$$

Using the introduced notation, we can rewrite the systems'equations in compact form as

$$\left[\begin{array}{cccc} T_{N,1}^{(k)} & \downarrow T_{N,2}^{(k)} & \cdots & \downarrow T_{N,N}^{(k)} \end{array}\right] \hat{\mathbf{x}}_{(N,k)} + \left[\begin{array}{cccc} S_{N,1}^{(k)} & \downarrow S_{N,2}^{(k)} & \cdots & \downarrow S_{N,N}^{(k)} \end{array}\right] \hat{\mathbf{u}}_{(N,k)} = \hat{E}_{N,k}\hat{\mathbf{x}}_{(N,k-1)},$$

with $k \geq 1$, where

$$\hat{E}_{N,k} \stackrel{\text{def}}{=} \begin{bmatrix} E & & \\ & \ddots & \\ & & E \end{bmatrix},$$

is a block-diagonal matrix of suitable size. The matrix $\left[\begin{array}{cccc} T_{N,1}^{(k)} & \downarrow T_{N,2}^{(k)} & \cdots & \downarrow T_{N,N}^{(k)} \end{array}\right]$ has $\sum_{j_1=1}^{N}\sum_{j_2=j_1}^{N}\cdots\sum_{j_k=j_{k-1}}^{N} n$ columns and $\sum_{j_1=1}^{N}\sum_{j_3=j_2}^{N}\cdots\sum_{j_{k-1}=j_{k-2}}^{N} q, \quad k > 1$ rows. Clearly, when $k = 1$, the number of rows is $\dim \mathcal{X} = n$. Note that the number of columns evidently coincides with the number of rows of $\hat{x}_{N,k}$ and, interestingly, also with the number of rows of $T_1^{(k+1)}$, if the system is square.

The number of columns of the matrices $\left[\begin{array}{cccc} S_{N,1}^{(k)} & \downarrow S_{N,2}^{(k)} & \cdots & \downarrow S_{N,N}^{(k)} \end{array}\right]$ can be computed by substituting n with m in the previous equation, while the number of rows is exactly the same. The number of columns evidently coincides with the number of rows of $\hat{\mathbf{u}}_{(N,k)}$, but we no longer have an equality, in general, between the number of columns and the number of rows of $S_1^{(k+1)}$, even when the system is square.*

In view of the previous consideration, it is clear that the matrix $\hat{E}_{N,k}$ has 1 if $k = 1$, or $\sum_{j_1=1}^{N}\sum_{j_2=j_1}^{N}\cdots\sum_{j_{k-1}=j_{k-2}}^{N} 1, \quad k > 1$, diagonal blocks.

System admits a solution for some $\mathbf{x}_{(i_1,\ldots,i_N)}$ and arbitrary input, if the local states

*We have equality when m is equal to the dimension of the outer state space.

on $\mathfrak{C}_{M-1}$ satisfy

$$\operatorname{im}\left[\begin{array}{cccc} S_{N,1}^{(1)} & \downarrow S_{N,2}^{(1)} & \cdots & \downarrow S_{N,N}^{(1)} \end{array}\right] \subseteq \operatorname{im}\hat{E}_{N,1},$$
$$\hat{\mathbf{x}}_{(N,1)} \in \left[\begin{array}{cccc} T_{N,1}^{(1)} & \downarrow T_{N,2}^{(1)} & \cdots & \downarrow T_{N,N}^{(1)} \end{array}\right]^{-1} \operatorname{im}\hat{E}_{N,1},$$

which implies that the local states on $\mathfrak{C}_{M-2}$ must satisfy:

$$\operatorname{im}\left[\,S_{N,1}^{(2)} \downarrow S_{N,2}^{(2)} \cdots \downarrow S_{N,N}^{(2)}\,\right] \subseteq \operatorname{im}\hat{E}_{N,2}\left[\,T_{N,1}^{(1)} \downarrow T_{N,2}^{(1)} \cdots \downarrow T_{N,N}^{(1)}\,\right]^{-1} \operatorname{im}\hat{E}_{N,1}$$
$$\hat{\mathbf{x}}_{(N,2)} \in \left[\,T_{N,1}^{(2)} \downarrow T_{N,2}^{(2)} \cdots \downarrow T_{N,N}^{(2)}\,\right]^{-1} \left(\hat{E}_{N,2}\left[\,T_{N,1}^{(1)} \downarrow T_{N,2}^{(1)} \cdots \downarrow T_{N,N}^{(1)}\,\right]^{-1} \operatorname{im}\hat{E}_{N,1}\right).$$

The following general recursion can be defined as

$$\begin{cases} \mathcal{M}_{N,1} \overset{\text{def}}{=} \operatorname{im}\hat{E}_{N,1} \\ \mathcal{M}_{N,k} \overset{\text{def}}{=} \hat{E}_{N,k}\left(\left[\,T_{N,1}^{(k-1)} \downarrow T_{N,2}^{(k-1)} \cdots \downarrow T_{N,N}^{(k-1)}\,\right]^{-1} \mathcal{M}_{N,k-1}\right), \end{cases}$$

where $k > 1$, from which the solvability condition for some $\mathbf{x}_{(i_1,\ldots,i_N)}$ and arbitrary input can be expressed in terms of the local states on the separation set $\mathfrak{C}_{M-k}$ as

$$\operatorname{im}\left[\begin{array}{cccc} S_{N,1}^{(k)} & \downarrow S_{N,2}^{(k)} & \cdots & \downarrow S_{N,N}^{(k)} \end{array}\right] \subseteq \mathcal{M}_{N,k}, \tag{5.6}$$
$$\hat{\mathbf{x}}(N,k) \in \left[\begin{array}{cccc} T_{N,1}^{(k)} & \downarrow T_{N,2}^{(k)} & \cdots & \downarrow T_{N,N}^{(k)} \end{array}\right]^{-1} \mathcal{M}_{N,k}. \tag{5.7}$$

By using the same machinery also, the solution of Problem 2 can be provided in the form of a recursion for the general N-D case. Let us introduce the matrix:

$$F_N^{(k)} \overset{\text{def}}{=} \left[\begin{array}{cc} \left[\,T_{N,1}^{(k)} \;\cdots\; \downarrow T_{N,N}^{(k)}\,\right] & \left[\,S_{N,1}^{(k)} \;\cdots\; \downarrow S_{N,N}^{(k)}\,\right] \end{array}\right],$$

where $k \geq 1$. By defining the general recursion:

$$\begin{cases} \mathcal{M}_{N,1} \overset{\text{def}}{=} \operatorname{im}\hat{E}_{N,1} \\ \mathcal{M}_{N,k} \overset{\text{def}}{=} \hat{E}_{N,k}\mathfrak{P}_{\aleph_{k-1}}\left((F_N^{(k-1)})^{-1}\mathcal{M}_{N,k-1}\right) \quad k > 1, \end{cases}$$

where $\aleph_k = \sum_{j_1=1}^{N}\sum_{j_2=j_1}^{N}\cdots\sum_{j_k=j_{k-1}}^{N} n$,* the system admits a solution for some $\mathbf{x}_{(i_1,\ldots,i_N)}$ and suitable input, if and only if the local states on $\mathfrak{C}_{M-k}$ satisfy

$$\hat{\mathbf{x}}_{(N,k)} \in \mathfrak{P}_{\aleph_k}\left((F_N^{(k)})^{-1}\mathcal{M}_{N,k}\right). \tag{5.8}$$

*Note that in the 2-D case, when $r = 1$, we have $\aleph_k = (r+k)\,N$.

Conclusion

In this thesis, the generalisation of the N-D geometric theory on the basis of several papers on 2-D systems with the two different types of Fornasini-Marchesini models (first and second order) has been presented. The geometric approach has been developed for N-D Fornasini-Marchesini first and second order models, which have different properties. In particular, the development for the N-D case, for all the definitions of subspaces, together with related properties are found in Conte and Perdon (1988), Ntogramatzidis et al. (2008) and Ntogramatzidis (2012) for 2-D systems. The most powerful notion in the geometric approach is the one of controlled invariant subspace, which provides necessary and sufficient conditions for the solution of the disturbance decoupling problem for the N-D Fornasini-Marchesini second-order model, while it only provides sufficient conditions of the solution of the disturbance decoupling problem and the model matching problem for the N-D Fornasini-Marchesini first order model. These solutions are based on the use of the two different methods: output-nulling subspace, and stabilisation of the controlled invariant subspace. Furthermore, the necessary and sufficient conditions for the existence of solutions of the singular Fornasini-Marchesini models in the geometric framework have been obtained for each one of the two-dimensional cases, and for the N-D case. In the two-dimensional case, two situations for the problem of the existence of solutions are considered in order to guarantee the existence, first for arbitrary input functions, and then, for suitable inputs. Interestingly, the obtained geometric conditions can be checked recursively through a sequence of subspaces, even if the model is not recursive in nature. Then, it has been shown how the found pattern for two-dimensional systems can be generalised to the N-D models, to solve the equivalent problem by using the same machinery. In view of the complexity in the N-D case, attention is restricted on a single local state vector, instead of a family of local states on a separation set. However, a general procedure is outlined for those cases where a set of local states on a certain region of the separation set is assigned. Finally, it is worth mentioning that some of the approaches developed in this thesis find natural application to models, which are even more general than the Fornasini-Marchesini model, such as the Kurek models.

APPENDIX A

Linear Algebra

In the sequel, we will give some geometric background about linear algebra and control theory (definitions and basic properties), which are considered a foundation of the thesis.

A.1 Vector spaces

Let $\mathbb{K}$ be a field. A set of vectors that defines under the operation of vector addition and scalar multiplication over a field $\mathbb{K}$ is called a vector space $\mathcal{X}$, and the following axioms are satisfied with these two operations:

- $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$, (commutative law);
- $(\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{x}_3 = \mathbf{x}_1 + (\mathbf{x}_2 + \mathbf{x}_3)$ for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathcal{X}$, (associative law);
- there exists a unique element denoted by 0, that satisfies: $0 + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{X}$;
- for every element of $\mathcal{X}$, there exists an inverse element $-\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = 0$.
- $k\,(\mathbf{x}_1 + \mathbf{x}_2) = k\,\mathbf{x}_1 + k\,\mathbf{x}_2$ for all $k \in \mathbb{K}$ and for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$;
- $(k_1 + k_2)\,\mathbf{x} = k_1\,\mathbf{x} + k_2\,\mathbf{x}$ for all $k_1, k_2 \in \mathbb{K}$ and for all $\mathbf{x} \in \mathcal{X}$;
- $(k_1\,k_2)\,\mathbf{x} = k_1\,(k_2\,\mathbf{x})$ for all $k_1, k_2 \in \mathbb{K}$ and for all $\mathbf{x} \in \mathcal{X}$;
- $1\,\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{X}$.

In addition, upper- case calligraphic letters such as V,W,X will be used to denote vector spaces, which are considered to be of a finite dimension , The field $\mathbb{K}$ is also supposed to be real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

A.2 Subspaces

A subset $\mathcal{V}$ of the vector space $\mathcal{X}$ over the field $\mathbb{K}$, along with the two operations of vector addition and scalar multiplication, is a subspace, if it is itself a vector space.

A.3 Spanning set

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathcal{X}$, where $\mathcal{X}$ is defined over $\mathbb{K}$. Their span is written span $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ or span$\{\mathbf{x}_i, i \in \mathbb{K}\}$, which is the set of all linear combinations of the $\mathbf{x}_i$ with coefficients in $\mathbb{K}$.

A.4 Linear independence

Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a set of vectors of $\mathcal{X}$. If there exist scalars $\{c_1, \ldots, c_n\}$ of $\mathbb{K}$, such that, the relation $\sum_{i=1}^{n} c_i \, \mathbf{x}_i = 0$ implies $c_i = 0$ for all $i = 1, \ldots, n$, then the set S is said to be linearly independent.

A.5 Basis of a vector space

A basis for the vector space $\mathcal{V}$ over the field $\mathbb{K}$ is a subset of $\mathcal{V}$, which is a spanning set for $\mathcal{V}$ and is linearly independent.
The basis of $\mathcal{V}$ determines the dimension of $\mathcal{V}$, which is the number of vectors of this basis for the vector space.

A.6 Basis matrices of subspaces

Let $\mathcal{V}$ be a subspace of $\mathbb{K}^n$ with a dimension r, $\mathcal{V}$ can be represented by a basis matrix V of $\mathbb{K}^{n \times r}$ such that its columns are linearly independent and span $\mathcal{V}$, i.e., (im $V = \mathcal{V}$ and $\ker V = \{0\}$). The number of columns of a basis matrix for such subspace gives the dimension of a subspace.

A.7 Linear transformation

Let $\mathcal{X}$ and $\mathcal{Y}$ be two vector spaces defined over the same field $\mathbb{K}$. A mapping $\mathcal{A}$ of $\mathcal{X}$ into $\mathcal{Y}$ is said to be a linear transformation if

$$\mathcal{A}(\mathbf{x}_1 + \mathbf{x}_2) = \mathcal{A}(\mathbf{x}_1) + \mathcal{A}(\mathbf{x}_2), \qquad \forall\, \mathbf{x}_1,\, \mathbf{x}_2 \in \mathcal{X}, \tag{A.1}$$

and

$$\mathcal{A}(k\,\mathbf{x}) = k\,\mathcal{A}(\mathbf{x}), \qquad \forall\, k \in \mathbb{K}, \forall\, \mathbf{x} \in \mathcal{X}, \tag{A.2}$$

or (A.1) and (A.2) can be written together as

$$\mathcal{A}(k_1\,\mathbf{x}_1 + k_2\,\mathbf{x}_2) = k_1\,\mathcal{A}(\mathbf{x}_1) + k_2\,\mathcal{A}(\mathbf{x}_2), \qquad \forall\,\mathbf{x}_1,\,\mathbf{x}_2 \in \mathcal{X} \quad k_1, k_2 \in \mathbb{K}. \tag{A.3}$$

A.8 From maps to matrices

Consider a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of $\mathcal{X}$ and a basis $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m\}$ of $\mathcal{Y}$, and let $\mathcal{A}$ be a linear transformation from $\mathcal{X}$ to $\mathcal{Y}$ over $\mathbb{K}$, then

$$\begin{aligned}
\mathcal{A}(\mathbf{x}_1) &= a_{1,1}\,\mathbf{y}_1 + a_{2,1}\,\mathbf{y}_2 + \ldots + a_{m,1}\,\mathbf{y}_m \\
\mathcal{A}(\mathbf{x}_2) &= a_{1,2}\,\mathbf{y}_1 + a_{2,2}\,\mathbf{y}_2 + \ldots + a_{m,2}\,\mathbf{y}_m \\
&\ \vdots \\
\mathcal{A}(\mathbf{x}_n) &= a_{1,n}\,\mathbf{y}_1 + a_{2,n}\,\mathbf{y}_2 + \ldots + a_{m,n}\,\mathbf{y}_m,
\end{aligned}$$

where for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, $a_{i,j} \in \mathbb{K}$. This consideration shows that a linear map $\mathcal{A}$ operates on every vector of $\mathcal{X}$. Indeed, for all $\mathbf{x} \in \mathcal{X}$, there exist coefficients $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{K}$, since $\mathbf{x}$ can be written as a linear combination of vectors of the basis of $\mathcal{X}$ as:

$$\mathbf{x} = \xi_1\,\mathbf{x}_1 + \xi_2\,\mathbf{x}_2 + \ldots + \xi_n\,\mathbf{x}_n.$$

Thus,

$$\begin{aligned}
\mathcal{A}(\mathbf{x}) &= \mathcal{A}\left(\xi_1\,\mathbf{x}_1 + \xi_2\,\mathbf{x}_2 + \ldots + \xi_n\,\mathbf{x}_n\right) \\
&= \xi_1\,\mathcal{A}\left(\mathbf{x}_1\right) + \xi_2\,\mathcal{A}\left(\mathbf{x}_2\right) + \ldots + \xi_n\,\mathcal{A}\left(\mathbf{x}_n\right) \\
&= \xi_1\left(a_{1,1}\,\mathbf{y}_1 + a_{2,1}\,\mathbf{y}_2 + \ldots + a_{m,1}\,\mathbf{y}_m\right) + \xi_2\left(a_{1,2}\,\mathbf{y}_1 + a_{2,2}\,\mathbf{y}_2 + \ldots + a_{m,2}\,\mathbf{y}_m\right) \\
&\quad + \ldots + \xi_n\left(a_{1,n}\,\mathbf{y}_1 + a_{2,n}\,\mathbf{y}_2 + \ldots + a_{m,n}\,\mathbf{y}_m\right) \\
&= \left(a_{1,1}\,\xi_1 + a_{1,2}\,\xi_2 + \ldots + a_{1,n}\,\xi_n\right)\mathbf{y}_1 + \left(a_{2,1}\,\xi_1 + a_{2,2}\,\xi_2 + \ldots + a_{2,n}\,\xi_n\right)\mathbf{y}_2 \\
&\quad + \ldots + \left(a_{m,1}\,\xi_1 + a_{m,2}\,\xi_2 + \ldots + a_{m,n}\,\xi_n\right)\mathbf{y}_m,
\end{aligned}$$

where $\mathcal{A}(\mathbf{x})$ can be written by vectors of components $\begin{bmatrix} a_{1,1}\,\xi_1 + a_{1,2}\,\xi_2 + \ldots + a_{1,n}\,\xi_n \\ \vdots \\ a_{m,1}\,\xi_1 + a_{m,2}\,\xi_2 + \ldots + a_{m,n}\,\xi_n \end{bmatrix}$, with respect to the basis $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of $\mathcal{Y}$, and these components can be computed as

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} a_{1,1}\,\xi_1 + a_{1,2}\,\xi_2 + \ldots + a_{1,n}\,\xi_n \\ a_{2,1}\,\xi_1 + a_{2,2}\,\xi_2 + \ldots + a_{2,n}\,\xi_n \\ \vdots \\ a_{m,1}\,\xi_1 + a_{m,2}\,\xi_2 + \ldots + a_{m,n}\,\xi_n \end{bmatrix}.$$

Definition 46. The matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \in \mathbb{K}^{n\times m},$$

is the matrix of $\mathcal{A}$ with respect to the given basis of $\mathcal{X}$ and $\mathcal{Y}$, where $(a_{1,i}, a_{2,i}, \ldots, a_{m,i})^\top$ is the column of $\mathcal{A}(\mathbf{x}_i)$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$.

A.9 Changes of basis

Let $\mathcal{A} : \mathcal{X} \longrightarrow \mathcal{Y}$ be a linear transformation over the field $\mathbb{K}$, and it is represented by a matrix A with respect to the bases of $\mathcal{X}$ and $\mathcal{Y}$. A change of basis is defined by two matrices P,Q over the field $\mathbb{K}$, where P and Q are nonsingular and their columns are the vectors of the new basis, expressed with respect to the old ones. If $\mathbf{x}$,$\mathbf{y}$ and ξ,η are the old and new coordinates respectively, then $\mathbf{x} = P\xi$, $\mathbf{y} = Q\eta$, and we obtain

$$\eta = Q^{-1}AP\xi = A'\xi,$$

where

$$A' \stackrel{\text{def}}{=} Q^{-1}AP.$$

Remark A.1. As a special case, if $\mathcal{A}$ is a map from $\mathcal{X}$ into itself, then a unique change of basis is represented by the transformation is $T = Q = P$, and it follows that

$$\eta = T^{-1}AT\xi = \acute{A}\xi,$$

where

$$A' \stackrel{\text{def}}{=} T^{-1}AT.$$

A.10 Image and null-space of a linear transformation and matrix

For any linear transformation $\mathcal{A} : \mathcal{X} \longrightarrow \mathcal{Y}$, the image and null-space (or kernel) are the two fundamental concepts that can be defined as

(i) $\text{im } \mathcal{A} \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathcal{Y} \mid \exists\, \mathbf{x} \in \mathcal{X} : \mathbf{y} = \mathcal{A}\,(\mathbf{x})\}$,

(ii) $\ker\ \mathcal{A} \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathcal{X} \mid \mathcal{A}\,(\mathbf{x}) = 0\}$.

Note that im $\mathcal{A}$ is a subspace of $\mathcal{Y}$, while ker $\mathcal{A}$ is a subspace of $\mathcal{X}$. However, the image and null-space of A, where A is the matrix of $\mathcal{A}$, with respect to the given bases of $\mathcal{X}$ and $\mathcal{Y}$, and $A \in \mathbb{K}^{m\times n}$, $n, m \in \mathbb{N} \setminus \{0\}$ are defined as

(i) $\text{im } A \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbb{K}^m \mid \exists\, \mathbf{x} \in \mathbb{K}^n \ :\ \mathbf{y} = A\,(\mathbf{x})\,\}$,

(ii) $\ker\ A \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{K}^n \mid A\,(\mathbf{x}) = 0\}$,

where im A is a subspace of $\mathbb{K}^m$ and ker A is a subspace of $\mathbb{K}^n$.

A.11 Rank and nullity of a matrix

The dimension of im A is referred to as the rank of A, that equals the number of its linearly independent columns, while the nullity of A is the dimension of ker A, i.e.,

$$\begin{aligned} \dim(\text{im } A) &= \text{rank } A, \\ \dim(\ker\ A) &= n - \text{rank } A. \end{aligned}$$

If $\mathcal{A} : \mathcal{X} \longrightarrow \mathcal{Y}$ is a linear transformation, there holds

$$\dim\,(\mathcal{X}) = \dim\,(\ker\ A) + \dim\,(\text{im } A).$$

A.12 Direct sum

The direct sum of two vector spaces $\mathcal{X}$ and $\mathcal{Y}$ can be defined as the vector space:

$$\mathcal{X} \oplus \mathcal{Y} = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \Big| \, \mathbf{x} \in \mathcal{X} \quad \text{and} \quad \mathbf{y} \in \mathcal{Y} \right\}.$$

The direct sum $\mathcal{V} \oplus \mathcal{W}$ of two subspaces $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{X}$and $\mathcal{Y}$ respectively, is a subspace of $\mathcal{X} \oplus \mathcal{Y}$. Moreover, if V is a basis matrix of $\mathcal{V}$ and W is a basis matrix of $\mathcal{W}$, then a basis matrix for $\mathcal{V} \oplus \mathcal{W}$ is given by

$$\text{diag}\{V, W\} = \begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix}.$$

A.13 Orthogonal complement

The orthogonal complement of a subspace $\mathcal{V}$ of $\mathbb{R}^n$ is defined as

$$\mathcal{V}^\perp \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}^\top \mathbf{x} = \mathbf{x}^\top \mathbf{y} = 0, \ \ \forall \mathbf{x} \in \mathcal{V}\}.$$

From the direct sum, we have $\mathcal{V} \oplus \mathcal{V}^\perp = \mathbb{R}^n$.

Lemma A.1. *Let $\mathcal{V}$ and $\mathcal{W}$ be two subspaces of $\mathbb{R}^n$, where $\mathcal{V} \subseteq \mathcal{W}$. Then,*

- $(\mathcal{V}^\perp)^\perp = \mathcal{V}$,
- $\mathcal{V}^\perp \supseteq \mathcal{W}^\perp$.

To represent a subspace in a matrix form, there is an alternative way. A full row-rank matrix is the dual of a full column-rank matrix. Therefore, via both, we can represent the same subspace. A subspace $\mathcal{V}$ of dimension r can be represented by a full row-rank matrix $Q \in \mathbb{R}^{(n-r)\times n}$. In other words, Q satisfies ($\ker Q = \mathcal{V}$ and $\operatorname{im} Q = \mathbb{R}^{n-r}$).

When V is a basis matrix of the subspace $\mathcal{V}$, we notice that the condition $\operatorname{im} V = \mathcal{V}$ is equivalent to the fact that V is of full columns-rank. Dually, the second condition of the last relation $\operatorname{im} Q = \mathbb{R}^{n-r}$ is equivalent to Q having full rows-rank. Stated differently, $Q^\top$ is a basis matrix of $\mathcal{V}^\perp$ such that $\operatorname{im} Q^\top = \mathcal{V}^\perp$ and $\ker Q^\top = \{0\}$. The orthogonal complement of both sides of $\operatorname{im} Q^\top = \mathcal{V}^\perp$ yields $\ker Q = \mathcal{V}$, and $\ker Q^\top = \{0\}$ shows that $Q^\top$ has linearly independent columns. The same is true about Q, that of having linearly independent rows.

A.14 Sum of subspaces

The sum of two subspaces $\mathcal{V}$ and $\mathcal{W}$ of the same vector space $\mathcal{X}$ is defined as

$$\mathcal{V} + \mathcal{W} \stackrel{\text{def}}{=} \{\mathbf{v} + \mathbf{w} \in \mathcal{X} \mid \mathbf{v} \in \mathcal{V} \ \text{ and } \ \mathbf{w} \in \mathcal{W}\}.$$

Moreover, $\mathcal{V}+\mathcal{W}$ is itself a subspace of $\mathcal{X}$, which is the smallest subspace of $\mathcal{X}$ containing the union $\mathcal{V} \cup \mathcal{W}$. For any two basis matrices V and W, respectively of the subspaces $\mathcal{V}$ and $\mathcal{W}$ of the same vector space $\mathcal{X}$, we have

$$\mathcal{V} + \mathcal{W} = \operatorname{im} V + \operatorname{im} W = \operatorname{im} \begin{bmatrix} V & W \end{bmatrix}.$$

A.15 Intersection of subspaces

The intersection of the two subspaces $\mathcal{V}$ and $\mathcal{W}$ of the same vector space $\mathbb{R}^n$ is defined as

$$\mathcal{V} \cap \mathcal{W} = \{\mathbf{z} \in \mathbb{R}^n \,|\, \mathbf{z} \in \mathcal{V} \quad \text{and} \quad \mathbf{z} \in \mathcal{W}\}.$$

The intersection $\mathcal{V} \cap \mathcal{W}$ can never be an empty set. This is because, any subspace of $\mathcal{X}$ contains the origin, then the intersection $\mathcal{V} \cap \mathcal{W}$ contains at least the zero subspace $\{0\}$. The intersection $\mathcal{V} \cap \mathcal{W}$ is the largest subspace of $\mathbb{R}^n$ contained in both $\mathcal{V}$ and $\mathcal{W}$. Let V and W be two basis matrices of the two subspaces $\mathcal{V}$ and $\mathcal{W}$ of the same vector space respectively. We have $(\mathcal{V} \cap \mathcal{W})^{\perp} = \mathcal{V}^{\perp} + \mathcal{W}^{\perp}$, which is equivalent to

$$\mathcal{V} \cap \mathcal{W} = (\mathcal{V}^{\perp} + \mathcal{W}^{\perp})^{\perp}.$$

Lemma A.2. *[Strang (1993)] Let $\mathcal{V}$ and $\mathcal{W}$ be subspaces of $\mathbb{R}^n$ and let $A \in \mathbb{R}^{m\times n}$. Then,*

$$A\,(\mathcal{V} + \mathcal{W}) = A\,\mathcal{V} + A\,\mathcal{W}, \tag{A.4}$$
$$A\,(\mathcal{V} \cap \mathcal{W}) \subseteq A\,\mathcal{V} \cap A\,\mathcal{W}. \tag{A.5}$$

Remark A.2. The subspace $\mathcal{W}$ of $\mathbb{R}^n$ is contained in the subspace $\mathcal{V}$ of $\mathbb{R}^n$, if and only if

$$\mathcal{V} + \mathcal{W} = \mathcal{V}, \tag{A.6}$$
$$\mathcal{V} \cap \mathcal{W} = \mathcal{W}. \tag{A.7}$$

Remark A.3. The union $\mathcal{V} \cup \mathcal{W}$ in general is not a subspace of $\mathcal{X}$, unless one subspace is contained within the other.

A.16 Grassmannian

The Grassmannian $\mathfrak{G}(\mathcal{X})$ of a vector space $\mathcal{X}$ of dimension n consists of all the subspaces of $\mathcal{X}$. Specifically, the set of all subspaces of dimension r of $\mathcal{X}$ is always denoted by $\mathfrak{G}_r(\mathcal{X})$, where $r \in \{0, \ldots, n\}$. There holds,

$$\mathfrak{G}(\mathcal{X}) = \bigcup_{r=0}^{n} \mathfrak{G}_r(\mathcal{X}).$$

The Grassmannian of any vector space cannot be empty because it contains at least the origin and $\mathcal{X}$. Indeed, the addition and intersection of subspaces are two operations that

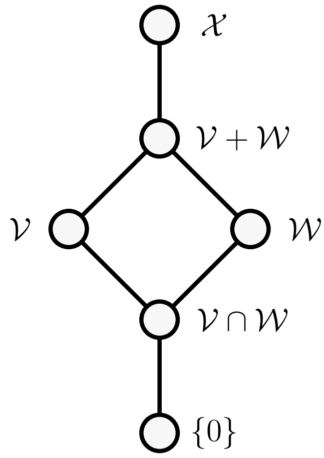

Figure A.1: Lattice $(\mathfrak{G}_{+,\cap;\subseteq}(\mathcal{X}))$.

can be defined in $\mathfrak{G}(\mathcal{X})$. This is because the addition and intersection of two subspaces of $\mathcal{X}$ is a subspace of $\mathcal{X}$. Figure A.1 shows these relations among subspaces, and it is usually called a Hasse diagram or lattice diagram. In particular, each element of $\mathfrak{G}(\mathcal{X})$ in a Hasse diagram is represented as a node and a rising branch from one subspace to an another is where the last subspace contains the previous subspace. There are two nodes representing two subspaces of $\mathfrak{G}(\mathcal{X})$ in the middle of a Hasse diagram that are not connected with the branches (Trentelman et al., 2001).

Remark A.4. Let $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{S}$ be subspaces of $\mathcal{X}$. Then,

- The Grassmann formula for all pairs of subspaces $\mathcal{V}$ and $\mathcal{W}$ is given by

$$\dim(\mathcal{V}+\mathcal{W}) = \dim(\mathcal{V}) + \dim(\mathcal{W}) - \dim(\mathcal{V}\cap\mathcal{W}).$$

  Specifically, if $\mathcal{V}\cap\mathcal{W} = \{0\}$, then $\dim(\mathcal{V}+\mathcal{W}) = \dim(\mathcal{V}) + \dim(\mathcal{W})$.

- The Grassmannian of a vector space in every lattice has a maximum and minimum. The addition of all subspaces of $\mathcal{X}$ gives the maximum, which is $\mathcal{X}$; while the minimum is given by the intersection of all subspaces of $\mathcal{X}$, which is the origin of $\mathcal{X}$:

$$\begin{aligned}\max\mathfrak{G}(\mathcal{X}) &= \sum_{\mathcal{S}\in\mathfrak{G}(\mathcal{X})} \mathcal{S} = \mathcal{X},\\ \min\mathfrak{G}(\mathcal{X}) &= \bigcap_{\mathcal{S}\in\mathfrak{G}(\mathcal{X})} \mathcal{S} = \{0\}.\end{aligned}$$

Theorem A.1. *Given a vector space $\mathcal{X}$ of dimension n, the Grassmannian of $\mathcal{X}$ is a lattice with respect to the binary operations of subspace addition and intersection, and with respect to the inclusion $\subseteq$. In symbols, $(\mathfrak{G}_{+,\cap;\subseteq}(\mathcal{X}))$ is a lattice.*

Lemma A.3. *Consider the subspaces $\mathcal{V}, \mathcal{W}$ and $\mathcal{Z}$ of $\mathbb{R}^n$. There hold*

$$\mathcal{V} \cap (\mathcal{W} + \mathcal{Z}) \supseteq (\mathcal{V} \cap \mathcal{W}) + (\mathcal{V} \cap \mathcal{Z}), \tag{A.8}$$
$$\mathcal{V} + (\mathcal{W} \cap \mathcal{Z}) \subseteq (\mathcal{V} + \mathcal{W}) \cap (\mathcal{V} + \mathcal{Z}). \tag{A.9}$$

However, if any one of these subspaces is contained in any of the others, the previous relations carry with the equality sign, i.e.,

$$\mathcal{V} \cap (\mathcal{W} + \mathcal{Z}) = (\mathcal{V} \cap \mathcal{W}) + (\mathcal{V} \cap \mathcal{Z}), \tag{A.10}$$
$$\mathcal{V} + (\mathcal{W} \cap \mathcal{Z}) = (\mathcal{V} + \mathcal{W}) \cap (\mathcal{V} + \mathcal{Z}). \tag{A.11}$$

A.17 Inverse image of a subspace

Consider $A : \mathcal{X} \longrightarrow \mathcal{Y}$ is a linear map from a vector space $\mathcal{X}$ into a vector space $\mathcal{Y}$. Let $\mathcal{H}$ be a subspace of $\mathcal{Y}$, the inverse image of $\mathcal{H}$ with respect to A is given by

$$A^{-1}\,\mathcal{H} \overset{\text{def}}{=} \{\mathbf{x} \in \mathcal{X} \mid A\,\mathbf{x} \in \mathcal{H}\}. \tag{A.12}$$

It seems clear that, $A^{-1}\,\mathcal{H}$ is a subspace of $\mathcal{X}$, without consideration if A is invertible or not, and we have

$$\dim(A^{-1}\,\mathcal{H}) = \dim(\ker A) + \dim(\mathcal{H} \cap \operatorname{im} A). \tag{A.13}$$

In addition,

$$A^{-1}\,\{0\} = \{\mathbf{x} \in \mathcal{X} \mid A\,\mathbf{x} = \mathbf{0}\} = \ker\ A, \tag{A.14}$$
$$A^{-1}\,\mathcal{Y} = \{\mathbf{x} \in \mathcal{X} \mid A\,\mathbf{x} \in \mathcal{Y}\} = \mathcal{X}. \tag{A.15}$$

There holds

$$A^{-1}\,\mathcal{H} = (A^{\top}\mathcal{H}^{\perp})^{\perp}.$$

Lemma A.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be two subspaces of $\mathbb{R}^m$ and let $A \in \mathbb{R}^{m \times n}$. Then,*

$$(A^{-1}\,\mathcal{H})^{\perp} = A^{\top}\mathcal{H}^{\perp}, \tag{A.16}$$
$$A^{-1}\,(\mathcal{H} \cap \mathcal{K}) = A^{-1}\,\mathcal{H} \cap A^{-1}\,\mathcal{K}, \tag{A.17}$$
$$A^{-1}\,(\mathcal{H} + \mathcal{K}) \supseteq A^{-1}\,\mathcal{H} + A^{-1}\,\mathcal{K}, \tag{A.18}$$
$$A\,(A^{-1}\,\mathcal{H}) = \mathcal{H} \cap \operatorname{im} A, \tag{A.19}$$
$$A^{-1}\,(A\,\mathcal{V}) = \mathcal{V} + \ker A. \tag{A.20}$$

A.18 Change of coordinates

Let $\mathcal{V}$ be a subspace of a vector space $\mathcal{X}$ with dimension r and a basis matrix V. A matrix $T = \left[\begin{array}{c|c} T_1 & T_2 \end{array}\right]$ describes a change of coordinates in $\mathcal{X}$, where the r columns of T_1 form a basis for $\mathcal{V}$, and T_2 is orthogonal to T_1. A basis matrix for $\mathcal{V}$ in the new basis is

$$V' = T^{-1} V = \begin{bmatrix} I_r \\ 0 \end{bmatrix}.$$

Moreover, a vector $\mathbf{x}$ of $\mathcal{V}$ with respect to the transformed basis is given by

$$\mathbf{x}_{\text{new}} = T^{-1} \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ 0 \end{bmatrix},$$

where $\mathbf{x}_1$ is r-dimensional.

Proposition A.1. *Let $A \in \mathbb{R}^{n \times m}$, let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$, and let $\mathcal{W}$ be a subspace of $\mathbb{R}^m$. Let V and W be two basis matrices of $\mathcal{V}$ and $\mathcal{W}$, respectively. The following two statements are equivalent:*

(i) $A\,\mathcal{V} \subseteq \mathcal{W}$,

(ii) there exists a matrix $X \in \mathbb{R}^{\dim \mathcal{W} \times \dim \mathcal{V}}$, such that $A\,V = W\,X$.

Proof: We prove that (i) implies (ii). Let us denote by $\mathbf{v}_i$, the i-th column of V and by $\mathbf{w}_i$, the i-th column of W, so that $V = [\ \mathbf{w}_1 \ \ \mathbf{w}_2 \ \ \ldots \ \ \mathbf{w}_s\ ]$. Then, $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_s\}$ are bases for $\mathcal{V}$ and $\mathcal{W}$ respectively. Since $A\,\mathbf{v}_i \in \mathcal{W}$ for every $i \in \{1, \ldots, s\}$, we can write $A\,\mathbf{v}_i$ as a linear combination of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_s$, using suitable coefficients $\xi_{i,k}$, i.e.,

$$\begin{aligned} A\,\mathbf{v}_1 &= \mathbf{w}_1\,\xi_{1,1} + \mathbf{w}_2\,\xi_{1,2} + \ldots + \mathbf{w}_s\,\xi_{1,s}, \\ &\vdots \\ A\,\mathbf{v}_r &= \mathbf{w}_1\,\xi_{r,1} + \mathbf{w}_2\,\xi_{r,2} + \ldots + \mathbf{w}_s\,\xi_{r,s}. \end{aligned}$$

These equations can be written as $A\,[\ \mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \ldots \ \ \mathbf{v}_r\ ] = [\ \mathbf{w}_1 \ \ \mathbf{w}_2 \ \ \ldots \ \ \mathbf{w}_s\ ]\,X$, where

$$X = \begin{bmatrix} \xi_{1,1} & \xi_{2,1} & \cdots & \xi_{r,1} \\ \xi_{1,2} & \xi_{2,2} & \cdots & \xi_{r,2} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{1,s} & \xi_{2,s} & \cdots & \xi_{r,s} \end{bmatrix}.$$

We now show that (ii) implies (i). Let $\mathbf{v} \in \mathcal{V}$. Then, $\mathbf{v}$ can be written as a linear combination of the columns of V, which are a basis for $\mathcal{V}$, i.e., there exist $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$, such that $\mathbf{v} = \alpha_1 V^1 + \alpha^2 V^2 + \ldots + \alpha^r V^r$. Let $\alpha \stackrel{\text{def}}{=} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}$. Then, using (ii),

$$A\,\mathbf{v} = A\,V\,\alpha = W\,X\,\alpha \in \operatorname{im} W = \mathcal{W}.$$

■

Proposition A.2. *Let $A \in \mathbb{R}^{n\times m}$, let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$, and let $\mathcal{W}$ be a subspace of $\mathbb{R}^m$. Let Q and T be two full row-rank matrices, such that $\ker Q = \mathcal{V}$ and $\ker T = \mathcal{W}$ respectively. The following two statements are equivalent:*

(i) $A\,\mathcal{V} \subseteq \mathcal{W}$,

(ii) there exists a matrix Y of suitable size, such that $T\,A = Y\,Q$.

Proof: We begin proving that (i) implies (ii). Clearly, (i) can be written as $A\,\ker Q \subseteq \ker T$, which implies $T\,A\,\ker Q = \{0\}$. Therefore, $\ker Q \subseteq \ker(T\,A)$, which in turn implies that there exists Y such that $T\,A = Y\,Q$. We prove that (ii) implies (i). Let $\mathbf{v} \in \mathcal{V}$. We have $Q\,\mathbf{v} = \mathbf{0}$, and using (ii) we find $T\,A\,\mathbf{v} = Y\,Q\,\mathbf{v} = \mathbf{0}$. This gives $A\,\mathbf{v} \in \ker T = \mathcal{W}$. ■

A.19 Quotient spaces

Let $\mathcal{V}$ be a subspace of the vector space $\mathcal{X}$. The equivalence relation

$$\mathbf{x} \sim \tilde{\mathbf{x}} \quad \Leftrightarrow \quad \mathbf{x} - \tilde{\mathbf{x}} \in \mathcal{V}, \tag{A.21}$$

on $\mathcal{X}$ satisfies these conditions: it is reflexive (since $\mathbf{x} - \mathbf{x} = \mathbf{0} \in \mathcal{V}$), symmetric (since $\mathbf{x} - \tilde{\mathbf{x}} \in \mathcal{V} \Leftrightarrow \tilde{\mathbf{x}} - \mathbf{x} \in \mathcal{V}$) and transitive (if $\mathbf{x}_1 - \mathbf{x}_2 \in \mathcal{V}$ and $\mathbf{x}_2 - \mathbf{x}_3 \in \mathcal{V}$, by linearity $\mathbf{x}_1 - \mathbf{x}_3 = (\mathbf{x}_1 - \mathbf{x}_2) + (\mathbf{x}_2 - \mathbf{x}_3) \in \mathcal{V}$). Therefore, $\mathcal{X}$ is divided into sets, so that their intersection is empty. Each set contains all the vectors that are equivalent to a given vector. Together, all these sets are called equivalent classes of $\mathcal{X}$ modulo $\mathcal{V}$, and it is denoted by $\mathcal{X}(\text{mod } \mathcal{V})$ or $\mathcal{X}/\mathcal{V}$. The class of the vectors that are equivalent to a vector $\mathbf{x}$ of $\mathcal{X}$ is denoted by $[\mathbf{x}]$.

Definition 47. Let $\mathcal{V}$ be a subspace of a vector space $\mathcal{X}$ over the field $\mathbb{R}$. Quotient space of $\mathcal{X}$ over $\mathcal{V}$ (or modulo $\mathcal{V}$) is defined as the set $\mathcal{X}/\mathcal{V}$ of the equivalence classes of $\mathcal{X}$ modulo $\mathcal{V}$.

In $\mathcal{X}/\mathcal{V}$, both operations of addition and multiplication by scalar can be defined as:

$$(\mathbf{x} + \mathcal{V}) + (\mathbf{y} + \mathcal{V}) \stackrel{\text{def}}{=} (\mathbf{x} + \mathbf{y}) + \mathcal{V} \qquad \forall\, \mathbf{x}, \mathbf{y} \in \mathcal{X}, \tag{A.22}$$

$$\alpha\,(\mathbf{x} + \mathcal{V}) \stackrel{\text{def}}{=} (\alpha\,\mathbf{x}) + \mathcal{V} \qquad \forall\, \alpha \in \mathbb{R}, \forall\, \mathbf{x} \in \mathcal{X}, \tag{A.23}$$

which prove that $\mathcal{X}/\mathcal{Y}$ is a vector space over the field $\mathbb{R}$.

Theorem A.2. *If $\mathcal{X}$ has a finite dimension n and if $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ (with $m < n$) is a basis for $\mathcal{V}$, let $(\mathbf{x}_{m+1}, \mathbf{x}_{m+2}, \ldots, \mathbf{x}_n)$ complete the previous basis to a basis for $\mathcal{X}$. Then, the classes*

$$[\mathbf{x}_{m+1}],\ [\mathbf{x}_{m+2}],\ \ldots\ [\mathbf{x}_n],$$

constitute a basis for $\mathcal{X}/\mathcal{V}$. If $m = n$, $\mathcal{V}$ coincides with $\mathcal{X}$, and $\mathcal{X}/\mathcal{V}$ contains only the zero class.

Corollary A.1. *If $\mathcal{V}$ is a subspace of $\mathcal{X}$, the dimensions of $\mathcal{X}$, $\mathcal{V}$ and $\mathcal{X}/\mathcal{V}$ satisfy*

$$\dim(\mathcal{X}) = \dim(\mathcal{V}) + \dim(\mathcal{X}/\mathcal{V}). \tag{A.24}$$

In particular, if $\mathcal{V}$ has a finite dimension, we have

$$\dim(\mathcal{X}/\mathcal{V}) = \dim(\mathcal{X}) - \dim(\mathcal{V}). \tag{A.25}$$

See more about quotient spaces in Trentelman et al. (2001).

A.20 Eigenvalues

An eigenvalue of a matrix $A \in \mathbb{R}^{n\times n}$ is a real or complex number λ, such that $A\,\mathbf{v} = \lambda\,\mathbf{v}$ for a certain nonzero vector $\mathbf{v} \in \mathcal{X}$. The spectrum of A is the set of all eigenvalues and it is denoted by $\sigma(A)$, i.e., $\lambda \in \sigma(A)$ if and only if $\det(\lambda\, I - A) = 0$. The maximum of the spectrum is called spectral radius $\rho\,(A)$ and is defined as follows:

$$\rho\,(A) = \max\,\{|\lambda|\,|\,\lambda\,\in\,\sigma(A)\}.$$

APPENDIX B

Statement of contribution from author and co-authors for joint paper

To Whom It May Concern,

I, Fatma Mohamed, am first author of the paper "Geometric conditions for the existence of solutions of singular multidimensional systems", presented at the 10th International Workshop on Multidimensional Systems" (NdS 2017), in Zielona-Gora, Poland.

I contributed by 50% in the development of the results presented in this paper and in the preparation of the manuscript. I actively contributed to the development of the original idea for the existence of solution, first for arbitrary input functions, and then, for suitable inputs in the case of implicit first order Fornasini-Marchesini 2-D model.

I, as a co-author, endorse that the level of contribution by the candidate is 50%, and my contribution is estimated at 30%.

Fabrizio Padula

I, as a co-author, endorse that the level of contribution by the candidate is 50%, and my contribution is estimated at 20%.

Lorenzo Ntogramatzidis

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