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Nash Equilibria with Piecewise Quadratic Costs

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Abstract

Inspired by an applied model arising from electric power markets with price caps that is discussed in a previous paper [3], this paper studies the Nash equilibrium problem in which the minimizing players' objective functions are sums of composite separable convex piecewise quadratic functions. Based on a fundamental but previously not proven equivalence between a separable convex piecewise quadratic program and a standard convex quadratic program, we show that the nonsmooth Nash equilibrium problem can be equivalently reformulated as a generalized Nash equilibrium problem with coupled linear constraints. We establish the convergence of a sequential penalized Nash algorithm for solving the reformulated generalized Nash problem under a boundedness condition.

Key words: Piecewise quadratic program, Nash-equilibrium, sequential penalized Nash algorithm
Mathematics Subject Classification: 90C20, 91A06, 91B50

1 Introduction

In a recent paper [3], a Nash-Cournot oligopolistic production model describing an electric power market where the regional prices of electricity sales are subject to caps was formulated and analyzed. This model leads to a Nash equilibrium problem where the players' objective functions are piecewise quadratic functions in the decision variables. Seemingly rather simple, the model raises several computational questions that have not been formally investigated prior to the cited work. Whereas these questions were settled affirmatively therein, due to their fundamental nature, we feel that the questions need to be addressed in a broader context. This constitutes the primary objective of the present paper.

Specifically, this paper deals with a standard Nash equilibrium problem [2, Subsection 1.4.2] where each player's objective function is piecewise quadratic (but of a special type) in all players' decision variables and the players' strategies are compact polyhedra. While the existence

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of an equilibrium solution is trivial, the computation of such an equilibrium solution is a challenge to the state of the art. Indeed, the piecewise quadratic nature of each player's objective function identifies the player's problem as a nonsmooth optimization problem; as such, all existing methods for computing equilibria via the variational inequality (VI) approach [2] are in jeopardy because there is no single-valued formulation of the first-order conditions of an equilibrium solution. As a matter of fact, all methods that depend on such a formulation are invalidated by the nondifferentiability of the players' objectives. Moreover, the multi-valued VI formulation is not applicable either, because simple examples can easily be constructed (see e.g. [3, Example 1]) to show that the defining mapping of such a VI is not monotone. Without such overall monotonicity, no methods exist that can solve the resulting multi-valued VI of the Nash game.

Exploiting the special structure of the applied model, the reference [3] establishes a fundamental complementarity formulation of the nonsmooth Nash-Cournot equilibrium problem with price caps and shows that the well-known Lemke pivoting method [1, 4] can successfully compute an equilibrium solution with a certain restricted multiplier property. In this paper, we analyze the piecewise quadratic Nash equilibrium problem more fully. In order to better motivate our approach, we digress to give a brief historical account of separable convex piecewise quadratic programming, which is a special class of monotropic programs [8].

It is a well-known elementary fact that a separable convex piecewise linear program is equivalent to a standard linear program. While there has been extensive investigation of piecewise quadratic programming [5, 6, 9, 10, 11, 12, 13, 14], especially in the context of "linear-quadratic programming", the extension of the mentioned fact to a separable convex piecewise quadratic program has never been formally established in the literature (although it is natural for one to conjecture its truth). This is not without a reason. Indeed, the conversion appears to be of more conceptual significance than practical appeal. Our contention is that while such an equivalent smooth reformulation of a piecewise quadratic program may not be particularly noteworthy in an optimization context, it in fact offers a viable approach for dealing with the piecewise quadratic Nash equilibrium problem.

Beginning with a formal proof of the aforementioned equivalent reformulation of a separable convex piecewise quadratic program to a standard convex quadratic program, this paper describes a sequential penalized method for computing an equilibrium to a Nash equilibrium problem with separable composite convex piecewise quadratic player objective functions. Our treatment extends that of the special model in [3] and follows a different path. Indeed, via a penalization of the "coupled constraints" in the resulting equivalent smooth formulation of the nonsmooth Nash problem, a technique first proposed in [7] for a generalized Nash game, we no longer need to rely on the "common multiplier" formulation that restricts the class of equilibria to be computed.

2 Piecewise Quadratic Programs

We begin by considering a separable convex piecewise quadratic program of the form:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n} p_i(x_i) \\ \text{subject to} & x \in X, \end{array} \tag{1}$$

where X is a polyhedron in $\Re^n$ and each p_i is a convex piecewise quadratic function of one variable, which we write as

$$p_i(t) \equiv \begin{cases} \frac{1}{2}\, a_{i,0}\, t^2 + b_{i,0}\, t + c_{i,0} & \text{if} \quad t \in (-\infty, \alpha_{i,0}] \\ \frac{1}{2}\, a_{i,j}\, t^2 + b_{i,j}\, t + c_{i,j} & \text{if} \quad t \in [\alpha_{i,j-1}, \alpha_{i,j}], \quad j = 1, \ldots, m \\ \frac{1}{2}\, a_{i,m+1}\, t^2 + b_{i,m+1}\, t + c_{i,m+1} & \text{if} \quad t \in [\alpha_{i,m}, \infty), \end{cases}$$

where

$$-\infty < \alpha_{i,0} < \alpha_{i,1} < \ldots < \alpha_{i,m} < \infty$$

is a partition of the real line into $m + 2$ consecutive closed intervals, within each of which p_i is quadratic; the constants $\{a_{i,0}, a_{i,1}, \ldots, a_{i,m}\}$ and $\{b_{i,0}, b_{i,1}, \ldots, b_{i,m}\}$ satisfy

(continuity) $\frac{1}{2}a_{i,j}\alpha_{i,j}^2 + b_{i,j}\alpha_{i,j} + c_{i,j} = p_i(\alpha_{i,j}) = \frac{1}{2}a_{i,j+1}\alpha_{i,j}^2 + b_{i,j+1}\alpha_{i,j} + c_{i,j+1}$
for all $j = 0, 1, \ldots, m$;

(convexity-1) $a_{i,j}\alpha_{i,j} + b_{i,j} = p_i'(\alpha_{i,j}-) \le p_i'(\alpha_{i,j}+) = a_{i,j+1}\alpha_{i,j} + b_{i,j+1}$
for all $j = 0, 1, \ldots, m$;

(convexity-2) $a_{i,0}$, $a_{i,1}$, $\ldots$, $a_{i,m}$, $a_{i,m+1}$ all nonnegative.

with $p_i'(\alpha_{i,j}-)$ and $p_i'(\alpha_{i,j}+)$ denoting the left- and right-derivative of $p_i(t)$ at the breakpoint $\alpha_{i,j}$, respectively; i.e.,

$$p_i'(\alpha_{i,j}-) \equiv \lim_{\tau \downarrow 0} \frac{p_i(\alpha_{i,j}) - p_i(\alpha_{i,j} - \tau)}{\tau} \quad \text{and} \quad p_i'(\alpha_{i,j}+) \equiv \lim_{\tau \downarrow 0} \frac{p_i(\alpha_{i,j} + \tau) - p_i(\alpha_{i,j})}{\tau}.$$

Our goal is to show that, upon a change of variables, the convex piecewise quadratic program (1) is equivalent to a standard convex quadratic program. For this purpose, we let, for each $i = 1, \ldots, n$ and $j = 0, 1, \ldots, m+1$, $y_{i,j}$ denote the portion of x_i in the interval $[\alpha_{i,j-1}, \alpha_{i,j}]$, where $\alpha_{i,-1} = -\infty$ and $\alpha_{i,m+1} = \infty$. The variables $y_{i,j}$ satisfy the following conditions:

$$0 \le \widehat{\alpha}_{i,j} - y_{i,j} \perp y_{i,j+1} \ge 0, \quad \forall\, j = 0, 1, \ldots, m, \tag{2}$$

where

$$\widehat{\alpha}_{i,j} \equiv \begin{cases} \alpha_{i,0} & \text{if } j = 0 \\ \alpha_{i,j} - \alpha_{i,j-1} & \text{if } j = 1, \ldots, m. \end{cases}$$

In terms of the auxiliary variables $y_{i,j}$, we can write

$$x_i \;=\; \sum_{j=0}^{m+1} y_{i,j}, \quad p_i(x_i) \;=\; p_i(y_{i,0}) + \sum_{j=1}^{m+1} \left[\tfrac{1}{2}\, a_{i,j}\, y_{i,j}^2 + p'_{i,j}(\alpha_{i,j-1})\, y_{i,j} \right], \tag{3}$$

where

$$p_{i,j}(t) \;\equiv\; \tfrac{1}{2}\, a_{i,j}\, t^2 + b_{i,j}\, t + c_{i,j}, \quad j \;=\; 1,\ldots,m+1$$

is the quadratic function that coincides with $p_i(t)$ on the interval $[\alpha_{i,j-1}, \alpha_{i,j}]$. Indeed, if x_i belongs to $[\alpha_{i,k-1}, \alpha_{i,k})$ for some $k = 1,\ldots,m+1$, then $y_{i,j} = \hat{\alpha}_{i,j}$ for all $j = 0,1,\ldots,k-1$, $y_{i,k} = x_i - \alpha_{i,k-1}$, and $y_{i,\ell} = 0$ for all $\ell = k+1,\ldots,m+1$; thus,

$$\begin{aligned} p_i(x_i) \;&=\; \tfrac{1}{2}\, a_{i,k}\, x_i^2 + b_{i,k}\, x_i + c_{i,k} \;=\; p_i(\alpha_{i,k-1}) + (\, a_{i,k}\, \alpha_{i,k-1} + b_{i,k}\,) y_{i,k} + \tfrac{1}{2}\, a_{i,k}\, y_{i,k}^2 \\ &=\; p_i(\alpha_{i,k-1}) + p'_{i,k}(\alpha_{i,k-1})\, y_{i,k} + \tfrac{1}{2}\, a_{i,k}\, y_{i,k}^2; \end{aligned}$$

inductively, it can be shown that

$$\begin{aligned} p_i(\alpha_{i,k-1}) \;&=\; p_i(\alpha_{i,k-2}) + p'_{i,k-1}(\alpha_{i,k-2})\, y_{i,k-1} + \tfrac{1}{2}\, a_{i,k-1}\, y_{i,k-1}^2 \\ &=\; \cdots \;=\; p_i(\alpha_{i,0}) + \sum_{j=1}^{k-1} \left[\tfrac{1}{2}\, a_{i,j}\, y_{i,j}^2 + p'_{i,j}(\alpha_{i,j-1})\, y_{i,j} \right], \end{aligned}$$

from which the representation of $p_i(x_i)$ in (3) follows readily. Consequently, (1) is equivalent to the following quadratic program with linear complementarity constraints:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n} \left\{ p_i(y_{i,0}) + \sum_{j=1}^{m+1} \left[\tfrac{1}{2}\, a_{i,j}\, y_{i,j}^2 + p'_{i,j}(\alpha_{i,j-1})\, y_{i,j} \right] \right\} \\ \text{subject to} & x_i = \displaystyle\sum_{j=0}^{m+1} y_{i,j},\ i = 1,...,n; \quad x \,\in\, X \\ \text{and} & 0 \,\le\, \widehat{\alpha}_{i,j} - y_{i,j} \,\perp\, y_{i,j+1} \,\ge\, 0, \quad \forall\, j \,=\, 0,1,\ldots,m. \end{array} \tag{4}$$

Note that for any given x, there exists a unique set of $\{y_{i,j},\ i = 1,...,n;\ j = 0,...,m+1\}$ satisfying (2) such that $x_i = \sum_{j=0}^{m+1} y_{i,j}$, and thus (3) holds. We call this set of variables $y_{i,j}$ the *piecewise decomposition* of x.

The following result shows that the orthogonality constraints can be dropped from the program (4).

Proposition 1 The two programs (4) and (5), where

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=1}^{n} \left\{ p_i(y_{i,0}) + \sum_{j=1}^{m+1} \left[\tfrac{1}{2}\, a_{i,j}\, y_{i,j}^2 + p'_{i,j}(\alpha_{i,j-1})\, y_{i,j} \right] \right\} & \\ \text{subject to} & x_i = \displaystyle\sum_{j=0}^{m+1} y_{i,j},\ i = 1, ..., n; \quad x \in X & (5) \\ \text{and} & \left. \begin{array}{l} 0 \le \widehat{\alpha}_{i,j} - y_{i,j} \\ 0 \le y_{i,j+1} \end{array} \right\} \quad \forall\, j = 0, 1, \ldots, m. & \end{array}$$

have the same optimal objective value, which is possibly $-\infty$. In particular, if an optimal solution to (5) exists, then there exists an optimal solution $(x, \{y_{i,j}\})$ to (5), in which $\{y_{i,j}\}$ is the piecewise decomposition of x. Obviously, this solution satisfies the complementarity conditions $(\widehat{\alpha}_{i,j} - y_{i,j})y_{i,j+1} = 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$ and is therefore an optimal solution to (4).

Proof. It suffices to show that for any feasible solution (x, y) to (5), there exists $\widehat{y}$ such that $(x, \widehat{y})$ feasible to (4) such that, for all $i = 1, \ldots, n$,

$$p_i(\widehat{y}_{i,0}) + \sum_{j=1}^{m+1} \left[\tfrac{1}{2}\, a_{i,j}\, \widehat{y}_{i,j}^2 + p'_{i,j}(\alpha_{i,j-1})\, \widehat{y}_{i,j} \right] \le p_i(y_{i,0}) + \sum_{j=1}^{m+1} \left[\tfrac{1}{2}\, a_{i,j}\, y_{i,j}^2 + p'_{i,j}(\alpha_{i,j-1})\, y_{i,j} \right]. \quad (6)$$

Once this is established, the proposition follows.

For a given feasible solution $(x, \{y_{i,j}\})$ to (5), let $\{\widehat{y}_{ij}\}$ be the piecewise decomposition of x. In turn, to establish the inequality (6), it suffices to show that if $\delta_k \equiv \min(\widehat{\alpha}_{i,k} - y_{i,k}, y_{i,k+1}) > 0$ for some $k = 0, 1, \ldots, m$, then letting $\widetilde{y}_{i,j} = y_{i,j}$ for all j except for $\widetilde{y}_{i,k} \equiv y_{i,k} + \delta_k$ and $\widetilde{y}_{i,k+1} \equiv y_{i,k+1} - \delta_k$, we have

$$p_i(\widetilde{y}_{i,0}) + \sum_{j=1}^{m+1} \left[\tfrac{1}{2}\, a_{i,j}\, \widetilde{y}_{i,j}^2 + p'_{i,j}(\alpha_{i,j-1})\, \widetilde{y}_{i,j} \right] \le p_i(y_{i,0}) + \sum_{j=1}^{m+1} \left[\tfrac{1}{2}\, a_{i,j}\, y_{i,j}^2 + p'_{i,j}(\alpha_{i,j-1})\, y_{i,j} \right]. \quad (7)$$

Suppose $k = 0$. The above inequality reduces to

$$p_i(\widetilde{y}_{i,0}) + \tfrac{1}{2}\, a_{i,1}\, \widetilde{y}_{i,1}^2 + p'_{i,1}(\alpha_{i,0})\, \widetilde{y}_{i,1} \le p_i(y_{i,0}) + \tfrac{1}{2}\, a_{i,1}\, y_{i,1}^2 + p'_{i,1}(\alpha_{i,0})\, y_{i,1}. \quad (8)$$

Consider the convex quadratic function in δ:

$$c_{i,0} + b_{i,0}\,(\, y_{i,0} + \delta\,) + \tfrac{1}{2}\, a_{i,0}\,(\, y_{i,0} + \delta\,)^2 + \tfrac{1}{2}\, a_{i,1}\,(\, y_{i,1} - \delta\,)^2 + p'_{i,1}(\alpha_{i,0})\,(\, y_{i,1} - \delta\,),$$

which is equal to the right-hand side of (8) when $\delta = 0$ and equal to the left-hand side when $\delta = \delta_0$; the minimum of this function is attained at

$$\delta_{0,\min} \equiv \frac{(\, b_{i,1} + a_{i,1}\alpha_{i,0}\,) - (\, b_{i,0} + a_{i,0}\alpha_{i,0}\,) + a_{i,1}\, y_{i,1} + a_{i,0}\,(\, \alpha_{i,0} - y_{i,0}\,)}{a_{i,0} + a_{i,1}} \ge \delta_0,$$

where $\delta_{0,\min}$ is defined to be ∞ if the denominator is zero. Therefore, (7) holds with $\delta = \delta_0$. Suppose $k > 0$. The inequality (7) reduces to

$$\begin{aligned}&\tfrac{1}{2}\, a_{i,k}\, \widetilde{y}_{i,k}^{\,2} + p'_{i,k}(\alpha_{i,k-1})\, \widetilde{y}_{i,k} + \tfrac{1}{2}\, a_{i,k+1}\, \widetilde{y}_{i,k+1}^{\,2} + p'_{i,k+1}(\alpha_{i,k})\, \widetilde{y}_{i,k+1} \\ &\quad \leq \tfrac{1}{2}\, a_{i,k}\, y_{i,k}^2 + p'_{i,k}(\alpha_{i,k-1})\, y_{i,k} + \tfrac{1}{2}\, a_{i,k+1}\, y_{i,k+1}^2 + p'_{i,k+1}(\alpha_{i,k})\, y_{i,k+1}.\end{aligned}$$

Consider the convex quadratic function in δ:

$$\tfrac{1}{2}\, a_{i,k}\, (\, y_{i,k} + \delta\,)^2 + p'_{i,k}(\alpha_{i,k-1})\, (\, y_{i,k} + \delta\,) + \tfrac{1}{2}\, a_{i,k+1}\, (\, y_{i,k+1} - \delta\,)^2 + p'_{i,k+1}(\alpha_{i,k})\, (\, y_{i,k+1} - \delta\,),$$

whose minimum is attained at

$$\delta_{k,\min} \equiv \frac{(\, b_{i,k+1} + a_{i,k+1}\alpha_{i,k}\,) - (\, b_{i,k} + a_{i,k}\alpha_{i,k}\,) + a_{i,k+1}\, y_{i,k+1} + a_{i,k}\, (\, \widehat{\alpha}_{i,k} - y_{i,k}\,)}{a_{i,k} + a_{i,k+1}} \geq \delta_k,$$

For the same reason as before, (7) holds with $\delta = \delta_k$. □

Proposition 1 is easily applicable to a *composite separable convex piecewise quadratic program* of the form: for some integer $n' > 0$,

$$\begin{array}{ll}\text{minimize} & \displaystyle\sum_{i=1}^{n'} p_i\left((e^i)^T x + f_i\right) \\ \text{subject to} & x \in X,\end{array} \tag{9}$$

where each e^i is an n-dimensional vector and f_i is a scalar. Indeed, (9) is clearly equivalent to

$$\begin{array}{ll}\text{minimize} & \displaystyle\sum_{i=1}^{n'} p_i(t_i) \\ \text{subject to} & x \in X, \\ & t_i = (e^i)^T x + f_i, \quad i = 1, \ldots, n',\end{array} \tag{10}$$

which is in the form of (1) in the variables $(x, t) \in \Re^{n+n'}$. The discussion so far forms the basis for extension to a Nash equilibrium problem in which each player solves a convex *nonsmooth* optimization problem of the form (9); in this case, complication arises because the constraints of the equivalent *smooth* program (10) contain the rival players' strategies. This feature turns the resulting game into one of the generalized type. Another consideration is how the rivals' strategies enter into each individual player's piecewise quadratic objective function. Details are discussed in the next section.

3 The Nash Equilibrium Problem

Consider a Nash equilibrium problem with N players each labeled by $\nu = 1, \ldots, N$. Player ν's optimization problem is to determine a strategy $x^\nu \in \Re^{n_\nu}$ to

$$\begin{array}{ll}\text{minimize} & \theta_\nu(x) \equiv \displaystyle\sum_{i=1}^{n'_\nu} p_i^\nu\left((e^{\nu,i})^T x + f_i^{\,\nu}, x^{-\nu}\right) \\ \text{subject to} & x^\nu \in X^\nu\end{array} \tag{11}$$

for fixed but arbitrary $x^{-\nu} \equiv \left(x^{\nu'} \right)_{\nu' \neq \nu} \in \prod_{\nu' \neq \nu} X^{\nu'}$, where $x \equiv (x^\nu)_{\nu=1}^N$ is the concatenation of all the players' strategies, each X^ν is a polyhedron in $\Re^{n_\nu}$, $e^{\nu,i}$ is a vector in $\Re^n$, with $n \equiv \sum_{\nu=1}^N n_\nu$, f_i^ν is a scalar, and $p_i^\nu(\cdot, x^{-\nu})$ is a convex piecewise quadratic function of one variable given by

$$p_i^\nu(t, x^{-\nu}) \equiv \begin{cases} \frac{1}{2}\, a_{i,0}^\nu\, t^2 + b_{i,0}^\nu(x^{-\nu})\, t + c_{i,0}^\nu & \text{if} \quad t \in (-\infty, \alpha_{i,0}^\nu\,] \\ \frac{1}{2}\, a_{i,j}^\nu\, t^2 + b_{i,j}^\nu(x^{-\nu})\, t + c_{i,j}^\nu & \text{if} \quad t \in [\,\alpha_{i,j-1}^\nu, \alpha_{i,j}^\nu\,],\ j = 1, \ldots, m_i^\nu \\ \frac{1}{2}\, a_{i,m_i^\nu+1}^\nu\, t^2 + b_{i,m_i^\nu+1}^\nu(x^{-\nu})\, t + c_{i,m_i^\nu+1}^\nu & \text{if} \quad t \in [\,\alpha_{i,m_i^\nu}^\nu, \infty\,), \end{cases}$$

where

$$-\infty < \alpha_{i,0}^\nu < \alpha_{i,1}^\nu < \ldots < \alpha_{i,m_i^\nu}^\nu < \infty$$

is a partition of the real line into $m_i^\nu + 2$ consecutive closed intervals, each $b_{i,j}^\nu(x^{-\nu})$ is a scalar affine function of $x^{-\nu}$ such that, for each fixed $x^{-\nu}$, the triple $(a_{i,j}^\nu, b_{i,j}^\nu(x^{-\nu}), c_{i,j}^\nu)$ satisfies the convexity and continuity conditions for $p_i^\nu(\cdot, x^{-\nu})$. Note that under this setting, all the functions $p_i^\nu(\cdot, x^{-\nu})$ have the same number of quadratic pieces, albeit with different breakpoints, all of which are independent of $x^{-\nu}$. Each vector $e^{\nu,i}$ is a concatenation of N subvectors $e^{\nu,\nu',i} \in \Re^{n_{\nu'}}$ for $\nu' = 1, \ldots, N$ so that

$$(\, e^{\nu,i}\,)^T x \equiv \sum_{\nu'=1}^N (\, e^{\nu,\nu',i}\,)^T x^{\nu'}.$$

It is easy to see that a tuple $x^* \equiv (x^{*,\nu})_{\nu=1}^N \in X \equiv \prod_{\nu=1}^N X^\nu$ is a Nash equilibrium if and only if a vector $a^* \equiv (a^{*,\nu})_{\nu=1}^N \in \prod_{\nu=1}^N \partial_{x^\nu} \theta_\nu(x^*)$ exists such that

$$\sum_{\nu=1}^N (\, x^\nu - x^{*,\nu}\,)^T a^{*,\nu} \geq 0, \quad \forall\, (\, x^\nu\,)_{\nu=1}^N \in X;$$

i.e., x^* is a Nash equilibrium if and only if it is a solution to the *multi-valued variational inequality* defined by the pair (X, Θ), where

$$\Theta(x) \equiv \prod_{\nu=1}^N \partial_{x^\nu} \theta_\nu(x),$$

where $\partial_{x^\nu}\theta_\nu(x)$ is the subdifferential of the convex function $\theta(\cdot, x^{-\nu})$ at x^ν. In general, Θ is not monotone (see [3] for a simple example); thus no existing method is applicable to the latter VI. In what follows, we discuss how an equilibrium solution to the above Nash problem can be computed based on the equivalent formulation of each player's nonsmooth optimization as a standard convex quadratic program with auxiliary variables.

Before proceeding further, we should say a few words about each function p_i^ν, which is a "multi-component composite piecewise quadratic function". In the given form, the rivals' variables $x^{-\nu}$ affects this function in 2 ways: one, through the expression $(e^{\nu,i})^T x + f_i^\nu$, which is the first argument in $p_i^\nu\left((e^{\nu,i})^T x + f_i^\nu, x^{-\nu}\right)$, and two, through the linear term in each quadratic piece of $p_i^\nu(\cdot, x^{-\nu})$. Note that both $a_{i,j}^\nu$ and $c_{i,j}^\nu$ are constants: replacing the constant $a_{i,j}^\nu$ by a function of $x^{-\nu}$ destroys the quadratic feature of the overall function $\theta_\nu(x)$; replacing the constant $c_{i,j}^\nu$ by a function of $x^{-\nu}$ has no effect on player ν's optimization; thus the essential term is the linear term of the quadratic pieces. An example of such a function p_i^ν is a firm's revenue function in a Nash-Cournot production game with piecewise linear prices, where each $f_i^\nu = 0$ and $e^{\nu,i}$ is the vector with all components equal to zero except for the (ν, i) component which is equal to one so that $(e^{\nu,i})^T x + f_i^\nu = x_i^\nu$, and where

$$p_i^\nu(t, x^{-\nu}) \equiv -t\,\pi_i(t + S_i^{-\nu}), \tag{12}$$

with $S_i^{-\nu} \equiv \sum_{\nu' \neq \nu} x_i^{\nu'}$ denoting the sum of the rival firms' sales in region i and π_i being the regional price that is a univariate, concave, decreasing, piecewise linear function of sales.

By Proposition 1, (11) is equivalent to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n'_\nu} \left\{ p_i^\nu(y_{i,0}^\nu, x^{-\nu}) + \sum_{j=1}^{m_i^\nu+1} \left[\tfrac{1}{2}\, a_{i,j}^\nu\, (\, y_{i,j}^\nu\,)^2 + (p_{i,j}^\nu)_t'(\alpha_{i,j-1}^\nu, x^{-\nu})\, y_{i,j}^\nu \right] \right\} \\ \text{subject to} & x^\nu \in X^\nu \\ & (\, e^{\nu,i}\,)^T x + f_i^\nu = \displaystyle\sum_{j=0}^{m_i^\nu+1} y_{i,j}^\nu, \quad i = 1, \ldots, n'_\nu \\ \text{and} & \left. \begin{array}{l} 0 \le \widehat{\alpha}_{i,j}^\nu - y_{i,j}^\nu \\ 0 \le y_{i,j+1}^\nu \end{array} \right\} \quad \forall\, j = 0, 1, \ldots, m_i^\nu, \end{array} \tag{13}$$

where $(p_{ij}^\nu)_t'(t, x^{-\nu}) = a_{i,j}^\nu t + b_{i,j}^\nu(x^{-\nu})$ is the partial derivative of the j-th quadratic piece of $p_i^\nu(\cdot, x^{-\nu})$ with respect to t (thus $p_{ij}^\nu(t, x^{-\nu}) \equiv \frac{1}{2} a_{i,j}^\nu t^2 + b_{i,j}^\nu(x^{-\nu}) t + c_{i,j}^\nu$), and

$$\widehat{\alpha}_{i,j}^\nu \equiv \begin{cases} \alpha_{i,0}^\nu & \text{if } j = 0 \\ \alpha_{i,j}^\nu - \alpha_{i,j-1}^\nu & \text{if } j = 1, \ldots, m_i^\nu. \end{cases}$$

Note the presence of the rivals' strategies in the constraints of the above optimization problem. As such, the resulting Nash equilibrium problem is of the generalized type with player ν's variables being (x^ν, y^ν), where $y^\nu = (y_{i,j}^\nu)$. We first state the necessary and sufficient conditions for an equilibrium solution of the game, which are obtained by concatenating the respective Karush-Kuhn-Tucker (KKT) conditions of the individual players' optimization problems (13). For this purpose, we write

$$X^\nu \equiv \{\, x^\nu \in \Re_+^{n_\nu} \,:\, D^\nu x^\nu \le d^\nu \,\},$$

for some matrix D^ν and vector d^ν of appropriate orders. Letting λ^ν and $\mu^\nu_{i,j}$ be the Lagrange multipliers of the constraints $D^\nu x^\nu \leq d^\nu$ and $0 \leq \widehat{\alpha}^\nu_{i,j} - y^\nu_{i,j}$, respectively, we obtain the KKT conditions for the overall Nash game as follows: for all $\nu = 1, \dots, N$,

$$
\begin{array}{l}
0 \leq x^\nu \quad \perp \quad \displaystyle\sum_{i=1}^{n'_\nu} \eta^\nu_i \, e^{\nu,\nu,i} + (\, D^\nu \,)^T \lambda^\nu \\
0 \leq \lambda^\nu \quad \perp \quad d^\nu - D^\nu x^\nu \geq 0 \\
\left.\begin{array}{l}
0 = (\, e^{\nu,i} \,)^T x + f^\nu_i - \displaystyle\sum_{j=0}^{m^\nu_i+1} y^\nu_{i,j} \\
0 = b^\nu_{i,0}(x^{-\nu}) + a^\nu_{i,0} \, y^\nu_{i,0} + \mu^\nu_{i,0} - \eta^\nu_i \\
0 \leq \mu^\nu_{i,0} \perp \widehat{\alpha}^\nu_{i,0} - y^\nu_{i,0} \geq 0 \\
0 \leq y^\nu_{i,m^\nu_i+1} \perp (p^\nu_{i,m^\nu_i+1})'_t(\alpha^\nu_{i,m^\nu_i}, x^{-\nu}) + a^\nu_{i,m^\nu_i+1} y^\nu_{i,m^\nu_i+1} - \eta^\nu_i \geq 0
\end{array}\right\} i = 1, \dots, n'_\nu \\
\left.\begin{array}{l}
0 \leq y^\nu_{i,j} \perp (p^\nu_{i,j})'_t(\alpha^\nu_{i,j-1}, x^{-\nu}) + a^\nu_{i,j} \, y^\nu_{i,j} - \eta^\nu_i + \mu^\nu_{i,j} \geq 0 \\
0 \leq \mu^\nu_{i,j} \perp \widehat{\alpha}^\nu_{i,j} - y^\nu_{i,j} \geq 0
\end{array}\right\} \begin{array}{l} j = 1, \dots, m^\nu_i \\ i = 1, \dots, n'_\nu. \end{array}
\end{array}
\tag{14}
$$

Note the absence of the constants $c^\nu_{i,j}$ in the above conditions. This confirms our previous remark that these constants have no effect on the players' optimization problems. We can convert the above mixed linear complementarity problem into a standard linear complementarity problem (LCP) by eliminating the variables $y^\nu_{i,0}$ and η^ν_i using the two equations:

$$
y^\nu_{i,0} = (\, e^{\nu,i} \,)^T x + f^\nu_i - \sum_{j=1}^{m^\nu_i+1} y^\nu_{i,j}
$$

and

$$
\begin{aligned}
\eta^\nu_i &= b^\nu_{i,0}(x^{-\nu}) + a^\nu_{i,0} \, y^\nu_{i,0} + \mu^\nu_{i,0} \\
&= a^\nu_{i,0} \, f^\nu_i + b^\nu_{i,0}(x^{-\nu}) + a^\nu_{i,0} \left[(\, e^{\nu,i} \,)^T x - \sum_{j=1}^{m^\nu_i+1} y^\nu_{i,j} \right] + \mu^\nu_{i,0}.
\end{aligned}
$$

Substituting these expressions into the rest of the KKT conditions, we obtain, for $\nu = 1, \dots, N$,

$$
\begin{array}{l}
0 \leq x^\nu \quad \perp \quad \displaystyle\sum_{i=1}^{n'_\nu} e^{\nu,\nu,i} \left\{ a^\nu_{i,0} \, f^\nu_i + b^\nu_{i,0}(x^{-\nu}) + a^\nu_{i,0} \left[(\, e^{\nu,i} \,)^T x - \sum_{j=1}^{m^\nu_i+1} y^\nu_{i,j} \right] + \mu^\nu_{i,0} \right\} \\
\qquad\qquad + (\, D^\nu \,)^T \lambda^\nu \geq 0 \\
0 \leq \lambda^\nu \quad \perp \quad d^\nu - D^\nu x^\nu \geq 0
\end{array}
$$

$$
\left.\begin{array}{l}
0 \leq \mu^{\nu}_{i,0} \perp \widehat{\alpha}^{\nu}_{i,0} - (\, e^{\,\nu,i}\,)^T x - f^{\,\nu}_i + \displaystyle\sum_{j=1}^{m^{\nu}_i+1} y^{\nu}_{i,j} \;\geq\; 0 \\
0 \leq y^{\nu}_{i,m+1} \perp (p^{\nu}_{i,m^{\nu}_i+1})'_t(\alpha^{\,\nu}_{i,m^{\nu}_i}, x^{-\nu}) + a^{\,\nu}_{i,m^{\nu}_i+1} y^{\nu}_{i,m^{\nu}_i+1} - a^{\,\nu}_{i,0} f^{\,\nu}_i - b^{\nu}_{i,0}(x^{-\nu}) \\
\qquad - a^{\nu}_{i,0} \left[(\, e^{\,\nu,i}\,)^T x - \displaystyle\sum_{j=1}^{m^{\nu}_i+1} y^{\nu}_{i,j} \right] - \mu^{\nu}_{i,0} \;\geq\; 0
\end{array}\right\} \; i \,=\, 1, \ldots, n'_{\nu}
$$

$$
\left.\begin{array}{l}
0 \;\leq\; y^{\nu}_{i,j} \;\perp\; (p^{\nu}_{i,j})'_t(\alpha^{\,\nu}_{i,j-1}, x^{-\nu}) + a^{\,\nu}_{i,j}\, y^{\nu}_{i,j} - a^{\,\nu}_{i,0}\, f^{\,\nu}_i - b^{\nu}_{i,0}(x^{-\nu}) - \\
\qquad - a^{\nu}_{i,0} \left[(\, e^{\,\nu,i}\,)^T x - \displaystyle\sum_{j=1}^{m^{\nu}_i+1} y^{\nu}_{i,j} \right] - \mu^{\nu}_{i,0} + \mu^{\nu}_{i,j} \;\geq\; 0 \\
0 \;\leq\; \mu^{\nu}_{i,j} \;\perp\; \widehat{\alpha}^{\nu}_{i,j} - y^{\nu}_{i,j} \;\geq\; 0
\end{array}\right\} \; \begin{array}{l} j \,=\, 1, \ldots, m^{\nu}_i \\ i \,=\, 1, \ldots, n'_{\nu}. \end{array}
$$

Handling the case where X is bounded, the *sequential penalized Nash approach* solves a sequence of Nash subproblems obtained by penalizing the constraints $(\, e^{\,\nu,i}\,)^T x + f^{\,\nu}_i - \sum_{j=0}^{m^{\nu}_i+1} y^{\nu}_{i,j} = 0$ that contain the rival players' variable $x^{-\nu}$. Specifically, let $\{\rho_\ell\}$ be an arbitrary sequence of positive scalars such that $\lim_{\ell \to \infty} \rho_\ell = \infty$. For each ℓ, let (x^ℓ, y^ℓ), where $y^\ell \equiv \left(y^{\ell,\nu} \right)_{\nu=1}^{N}$, be an equilibrium solution to a Nash subgame, wherein player ν's optimization problem in the variable (x^ν, y^ν) for fixed $x^{-\nu}$ is:

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{i=1}^{n'_{\nu}} \left\{ p^{\nu}_i(y^{\nu}_{i,0}, x^{-\nu}) + \sum_{j=1}^{m^{\nu}_i+1} \left[\tfrac{1}{2}\, a^{\,\nu}_{i,j}\, (\, y^{\nu}_{i,j}\,)^2 + (p^{\nu}_{i,j})'_t(\alpha^{\nu}_{i,j-1}, x^{-\nu})\, y^{\nu}_{i,j} \right] \right\} \\
& \quad + \dfrac{\rho_\ell}{2} \displaystyle\sum_{i=1}^{n'_{\nu}} \left[(\, e^{\,\nu,i}\,)^T x + f^{\,\nu}_i - \sum_{j=0}^{m^{\nu}_i+1} y^{\nu}_{i,j} \right]^2 \\
\text{subject to} & x^\nu \,\in\, X^\nu \\
\text{and} & \left.\begin{array}{l} 0 \;\leq\; \widehat{\alpha}^{\nu}_{i,j} - y^{\nu}_{i,j} \\ 0 \;\leq\; y^{\nu}_{i,j+1} \end{array}\right\} \quad \forall\, j \,=\, 0, 1, \ldots, m^{\nu}_i,
\end{array}
\tag{15}
$$

where $\rho_\nu > 0$ is a positive penalty parameter. We may assume without loss of generality that y^ℓ satisfies:

$$
(\, \widehat{\alpha}^{\nu}_{i,j} - y^{\ell,\nu}_{i,j}\,)\, y^{\ell,\nu}_{i,j+1} \;=\; 0, \quad \forall\, j \,=\, 0, 1, \ldots, m^{\nu}_i;\; \forall\, i \,=\, 0, 1, \ldots, n'_{\nu}. \tag{16}
$$

There are two issues that need attention: (a) existence of each (x^ℓ, y^ℓ) for given $\rho_\ell > 0$, and (b) convergence of the sequence $\{(x^\ell, y^\ell)\}$ when $\ell \to \infty$. The first issue is complicated by the possible unboundedness of the variables $y^{\nu}_{i,0}$ and y^{ν}_{i,m^{ν}_i+1} (all the other $y^{\nu}_{i,j}$ for $j = 1, \ldots, m^{\nu}_i$

are bounded), a situation that can be eliminated when each X^ν is bounded. The following result formally addresses this issue.

Proposition 2 Suppose that X is bounded and that, for each $x^{-\nu} \in X^{-\nu}$, each $\nu = 1, \ldots, N$, and every $i = 1, \ldots, n'_\nu$,

$$\inf_{y^\nu_{i,0} \le \widehat{\alpha}_{i0}} p^\nu_i(y^\nu_{i,0}, x^{-\nu}) \; > \; -\infty \tag{17}$$

and

$$\inf_{y^\nu_{i,m^\nu_i+1} \ge 0} \left[\tfrac{1}{2}\, a^\nu_{i,m^\nu_i+1}\, (\, y^\nu_{i,m^\nu_i+1}\,)^2 + (p^\nu_{i,m^\nu_i+1})'_t(\alpha^\nu_{i,m^\nu_i}, x^{-\nu})\, y^\nu_{i,m^\nu_i+1} \right] \; > \; -\infty. \tag{18}$$

Then, for every $\rho_\ell > 0$, there exists an equilibrium (x^ℓ, y^ℓ) to the Nash subgame where each player ν's optimization problem is (15).

Proof. Since $p^\nu_i(y^\nu_{i,0}, x^{-\nu})$ and $\frac{1}{2} a^\nu_{i,m^\nu_i+1}(y^\nu_{i,m^\nu_i+1})^2 + (p^\nu_{i,m^\nu_i+1})'_t(\alpha^\nu_{i,m^\nu_i}, x^{-\nu}) y^\nu_{i,m^\nu_i+1}$ are convex quadratic functions of $y^\nu_{i,0}$ and y^ν_{i,m^ν_i+1}, respectively, whose *linear* terms are parameterized by $x^{-\nu}$ belonging to the compact polyhedron $X^{-\nu}$, it follows that

$$\inf_{x^{-\nu} \in X^{-\nu}} \; \inf_{y^\nu_{i,0} \le \widehat{\alpha}_{i0}} p^\nu_i(y^\nu_{i,0}, x^{-\nu}) \; > \; -\infty \tag{19}$$

and

$$\inf_{x^{-\nu} \in X^{-\nu}} \; \inf_{y^\nu_{i,m^\nu_i+1} \ge 0} \left[\tfrac{1}{2}\, a^\nu_{i,m^\nu_i+1}\, (\, y^\nu_{i,m^\nu_i+1}\,)^2 + (p^\nu_{i,m^\nu_i+1})'_t(\alpha^\nu_{i,m^\nu_i}, x^{-\nu})\, y^\nu_{i,m^\nu_i+1} \right] \; > \; -\infty. \tag{20}$$

Fix $\rho_\ell > 0$. For each integer $k > 0$, consider a Nash subgame wherein player ν's problem (15) is augmented by the constraints: $y^\nu_{i,0} \ge -k$ and $y^\nu_{i,m+1} \le k$. Such a game has an equilibrium which we denote $(\widehat{x}^{\,k}, \widehat{y}^{\,k})$. By Proposition 1, it follows that

$$(\, \widehat{\alpha}^\nu_{i,j} - \widehat{y}^{\,k,\nu}_{i,j}\,)\, \widehat{y}^{\,k,\nu}_{i,j+1} \; = \; 0, \quad \forall\, j \; = \; 0, 1, \ldots, m^\nu_i; \; \forall\, i \; = \; 0, 1, \ldots, n'_\nu. \tag{21}$$

We claim that the sequence $\{(\widehat{y}^{\,k}_{i,0}, \widehat{y}^{\,k}_{i,m+1})\}$, and thus $(\widehat{x}^{\,k}, \widehat{y}^{\,k})$, is bounded as $k \to \infty$. Fix any $(\widetilde{x}, \widetilde{y})$ such that for each ν, the pair $(\widetilde{x}^{\,\nu}, \widetilde{y}^{\,\nu})$ is feasible to (15) with the augmented constraints. We then have

$$\begin{aligned}
& \sum_{i=1}^{n'_\nu} \left\{ p^\nu_i(\widehat{y}^{\,k,\nu}_{i,0}, \widehat{x}^{\,k,-\nu}) + \sum_{j=1}^{m^\nu_i+1} \left[\tfrac{1}{2}\, a^\nu_{i,j}\, (\, \widehat{y}^{\,k,\nu}_{i,j}\,)^2 + (p^\nu_{i,j})'_t(\alpha^\nu_{i,j-1}, \widehat{x}^{\,k,-\nu})\, \widehat{y}^{\,k,\nu}_{i,j} \right] + \right. \\
& \qquad \left. \frac{\rho_\ell}{2} \sum_{i=1}^{n'_\nu} \left[(\, e^{\,\nu,i}\,)^T \widehat{x}^{\,k} + f^\nu_i - \sum_{j=0}^{m^\nu_i+1} \widehat{y}^{\,k,\nu}_{i,j} \right]^2 \right\} \\
\le & \sum_{i=1}^{n'_\nu} \left\{ p^\nu_i(\widetilde{y}^\nu_{i,0}, \widehat{x}^{\,k,-\nu}) + \sum_{j=1}^{m^\nu_i+1} \left[\tfrac{1}{2}\, a^\nu_{i,j}\, (\, \widetilde{y}^\nu_{i,j}\,)^2 + (p^\nu_{i,j})'_t(\alpha^\nu_{i,j-1}, \widehat{x}^{\,k,-\nu})\, \widetilde{y}^\nu_{i,j} \right] + \right. \\
& \qquad \left. \frac{\rho_\ell}{2} \sum_{i=1}^{n'_\nu} \left[(\, e^{\,\nu,\nu,i}\,)^T \widetilde{x}^{\,\nu} + \sum_{\nu' \neq \nu} (\, e^{\,\nu,\nu',i}\,)^T \widehat{x}^{\,k,\nu'} + f^\nu_i - \sum_{j=0}^{m^\nu_i+1} \widetilde{y}^\nu_{i,j} \right]^2 \right\}.
\end{aligned}$$

Since the right-hand side is bounded, (19) and (20) imply that

$$\sup_k \left| (e^{\nu,i})^T \widehat{x}^k + f_i^\nu - \sum_{j=0}^{m_i^\nu+1} \widehat{y}_{i,j}^{k,\nu} \right| < \infty,$$

which in turn yields

$$\sup_k \left| \widehat{y}_{i,0}^{k,\nu} + \widehat{y}_{i,m_i^\nu+1}^{k,\nu} \right| < \infty.$$

From this and (21), the boundedness of $\{(\widehat{y}_{i,0}^{k,\nu}, \widehat{y}_{i,m_i^\nu+1}^{k,\nu})\}$ follows readily. Therefore, the sequence $\{(\widehat{x}^k, \widehat{y}^k)\}$ is bounded. It is easy to show that every accumulation point of the latter sequence, at least one of which must exist, is a desired equilibrium asserted by the proposition. □

Clearly, (17) and (18) hold if $a_{i,0}^\nu > 0$ and $a_{i,m_i^\nu}^\nu > 0$ for all pairs (ν, i); in the case where either $a_{i,0}^{nu} = 0$ or $a_{i,m_i^\nu}^\nu = 0$, the (17) and (18) impose some restrictions on the functions $b_{i,0}^\nu(x^{-\nu})$ and $b_{i,m+1}^\nu(x^{-\nu})$ on the set $X^{-\nu}$.

We next address the convergence of the sequence $\{(x^\ell, y^\ell)\}$ when $\ell \to \infty$. To state the optimality conditions for (15), we let $\lambda^{\ell,\nu}$ and $\mu_{i,j}^{\ell,\nu}$ be the Lagrange multipliers of the constraints $D^\nu x^\nu \leq d^\nu$ and $0 \leq \widehat{\alpha}_{i,j}^\nu - y_{i,j}^\nu$, respectively, we obtain the Karush-Kuhn-Tucker (KKT) conditions of (15),

$$0 \leq x^{\ell,\nu} \perp \rho_\ell \sum_{i=1}^{n'_\nu} \left[(e^{\nu,i})^T x^\ell + f_i^\nu - \sum_{k=0}^{m_i^\nu+1} y_{i,k}^{\ell,\nu} \right] e^{\nu,\nu,i} + (D^\nu)^T \lambda^{\ell,\nu} \geq 0$$

$$0 \leq \lambda^{\ell,\nu} \perp d^\nu - D^\nu x^{\ell,\nu} \geq 0$$

$$\left.\begin{array}{l} 0 = b_{i,0}^\nu(x^{-\nu}) + a_{i,0}^\nu y_{i,0}^{\ell,\nu} + \mu_{i,0}^{\ell,\nu} + \rho_\ell \left[\sum_{k=0}^{m+1} y_{i,k}^{\ell,\nu} - (e^{\nu,i})^T x^\ell - f_i^\nu \right] \\ 0 \leq \mu_{i,0}^{\ell,\nu} \perp \widehat{\alpha}_{i,0}^\nu - y_{i,0}^{\ell,\nu} \geq 0 \\ 0 \leq y_{i,m_i^\nu+1}^{\ell,\nu} \perp (p_{i,m_i^\nu+1}^\nu)'_t(\alpha_{i,m_i^\nu}^\nu, x^{-\nu}) + a_{i,m_i^\nu+1}^\nu y_{i,m_i^\nu+1}^{\ell,\nu} + \rho_\ell \left[\sum_{k=0}^{m_i^\nu+1} y_{i,k}^{\ell,\nu} - (e^{\nu,i})^T x^\ell - f_i^\nu \right] \geq 0 \\ \qquad i = 1, \ldots, n'_\nu \end{array}\right\}$$

$$\left.\begin{array}{l} 0 \leq y_{i,j}^{\ell,\nu} \perp (p_{i,j}^\nu)'_t(\alpha_{i,j-1}^\nu, x^{-\nu}) + a_{i,j}^\nu y_{i,j}^{\ell,\nu} + \rho_\ell \left[\sum_{k=0}^{m_i^\nu+1} y_{i,k}^{\ell,\nu} - (e^{\nu,i})^T x^\ell - f_i^\nu \right] + \mu_{i,j}^{\ell,\nu} \geq 0 \\ 0 \leq \mu_{i,j}^{\ell,\nu} \perp \widehat{\alpha}_{i,j}^\nu - y_{i,j}^{\ell,\nu} \geq 0 \\ \qquad j = 1, \ldots, m_i^\nu, \quad i = 1, \ldots, n'_\nu. \end{array}\right\}$$

The next proposition establishes the boundedness of the sequence $\{(x^\ell, y^\ell, \mu^\ell)\}$ under the boundedness of the set X.

Proposition 3 Suppose that X is bounded. Then so is the sequence $\{(x^\ell, y^\ell, \mu^\ell)\}_{\ell=1}^\infty$.

Proof. The boundedness of $\{x^\ell\}$ is trivial. Next we show the boundedness of $\{y^\ell\}$. In turn, it suffices to show that, for all $\nu = 1, \ldots, N$ and $i = 1, \ldots, n'_\nu$,

$$\liminf_{\ell \to \infty} y_{i,0}^{\ell,\nu} > -\infty \quad \text{and} \quad \limsup_{\ell \to \infty} y_{i,m_i^\nu+1}^{\ell,\nu} < \infty.$$

By way of contradiction, assume that $\liminf_{\ell \to \infty} y_{i,0}^{\ell,\nu} = -\infty$ for some pair (ν, i). Without loss of generality, assume that $\lim_{\ell \to \infty} y_{i,0}^{\ell,\nu} = -\infty$. It then follows that $y_{i,0}^{\ell,\nu} < \widehat{\alpha}_{i,0}^\nu$ for all ℓ sufficiently large, which implies, by complementarity, that $\mu_{i,0}^{\ell,\nu} = 0$. Moreover, by (16), we have $y_{i,j}^{\ell,\nu} = 0$ for all $j = 1, \ldots, m_i^\nu + 1$. But this contradicts the equation

$$0 = b_{i,0}^\nu(x^{-\nu}) + a_{i,0}^\nu y_{i,0}^{\ell,\nu} + \mu_{i,0}^{\ell,\nu} + \rho_\ell \left[\sum_{k=0}^{m_i^\nu+1} y_{i,k}^{\ell,\nu} - (e^{\nu,i})^T x^\ell - f_i^\nu \right]. \tag{22}$$

Hence $\{y_{i,0}^{\ell,\nu}\}$ is bounded. Suppose that $\limsup_{\ell \to \infty} y_{i,m_i^\nu+1}^{\ell,\nu} = \infty$. Without loss of generality, we may assume that $\lim_{\ell \to \infty} y_{i,m_i^\nu+1}^{\ell,\nu} = \infty$. Hence, by (16), we have $y_{i,j}^{\ell,\nu} = \widehat{\alpha}_{i,j}^\nu > 0$ for all $j = 0, \ldots, m_i^\nu$, which implies, by complementarity,

$$(p_{i,j}^\nu)_t'(\alpha_{i,j-1}^\nu, x^{-\nu}) + a_{i,j}^\nu y_{i,j}^\nu + \rho_\ell \left[\sum_{k=0}^{m_i^\nu+1} y_{i,k}^{\ell,\nu} - (e^{\nu,i})^T x^\ell - f_i^\nu \right] + \mu_{i,j}^{\ell,\nu} = 0.$$

Finally, we need to show the boundedness of $\{\mu_{i,j}^{\ell,\nu}\}$ for all (ν, i, j). The boundedness of $\{\mu_{i,0}^{\ell,\nu}\}$ follows from (22). That of $\{\mu_{i,j}^{\ell,\nu}\}$ for $j = 1, \ldots, m_i^\nu$ follows from the last two complementarity conditions in the KKT conditions of (15). □.

The next result shows that every accumulation point of the sequence $\{(x^\ell, y^\ell)\}$ is a Nash equilibrium of the original game with piecewise quadratic costs.

Theorem 1 Suppose that the assumptions of Proposition 2 hold. Every accumulation point of the sequence $\{x^\ell\}$ is an equilibrium solution of the Nash game with the players' problems given by (11) for $\nu = 1, \ldots, N$.

proof. The equation (22) implies that

$$\left\{ \rho_\ell \left[\sum_{k=0}^{m_i^\nu+1} y_{i,k}^{\ell,\nu} - (e^{\nu,i})^T x^\ell - f_i^\nu \right] \right\}_{\ell=1}^\infty$$

is bounded. Hence

$$\lim_{\ell \to \infty} \left[\sum_{k=0}^{m_i^\nu+1} y_{i,k}^{\ell,\nu} - (e^{\nu,i})^T x^\ell - f_i^\nu \right] = 0.$$

Define

$$\eta_i^{\ell,\nu} \equiv \rho_\ell \left[(e^{\nu,i})^T x^\ell + f_i^\nu - \sum_{k=0}^{m_i^\nu+1} y_{i,k}^{\ell,\nu} \right]$$

For simplicity, assume that

$$\lim_{\ell\to\infty} (x^\ell, y^\ell, \mu^\ell, \eta^\ell) = (x^*, y^*, \mu^*, \eta^*)$$

exists. We have, for all $i = 1, \ldots, n'_\nu$,

$$\sum_{k=0}^{m_i^\nu+1} y_{i,k}^{*,\nu} = (e^{\nu,i})^T x^* + f_i^\nu,$$

and

$$(\widehat{\alpha}_{i,j}^\nu - y_{i,j}^{*,\nu})\, y_{i,j+1}^{*,\nu} = 0, \quad \forall\, j = 0, 1, \ldots, m_i^\nu.$$

From

$$\begin{aligned} 0 &= \rho_\ell \sum_{i=1}^{n'_\nu} \left[(e^{\nu,i})^T x^\ell + f_i^\nu - \sum_{k=0}^{m_i^\nu+1} y_{i,k}^{\ell,\nu} \right] e^{\nu,\nu,i} + (D^\nu)^T \lambda^{\ell,\nu} \\ 0 &\le \lambda^{\ell,\nu} \perp d^\nu - D^\nu x^{\ell,\nu} \ge 0, \end{aligned}$$

we may assume without loss of generality that, for each ν, the sequence $\{\lambda^{\ell,\nu}\}_{\ell=1}^\infty$ is bounded, and by working with an appropriate subsequence if necessary, that

$$\lim_{\ell\to\infty} \lambda^\ell = \lambda^*$$

exists. Therefore, passing to the limit $\ell \to \infty$ in the KKT conditions of (15), we deduce

$$\begin{aligned}
&0 \le x^{*,\nu} \perp \sum_{i=1}^{n'_\nu} \eta_i^{*,\nu} e^{\nu,\nu,i} + (D^\nu)^T \lambda^{*,\nu} \ge 0 \\
&0 \le \lambda^{*,\nu} \perp d^\nu - D^\nu x^{*,\nu} \ge 0 \\
&\left.\begin{aligned}
&0 = b_{i,0}^\nu(x^{*,-\nu}) + a_{i,0}^\nu y_{i,0}^{*,\nu} + \mu_{i,0}^{*,\nu} - \eta_i^{*,\nu} \\
&0 \le \mu_{i,0}^{*,\nu} \perp \widehat{\alpha}_{i,0}^\nu - y_{i,0}^{*,\nu} \ge 0 \\
&0 \le y_{i,m_i^\nu+1}^{*,\nu} \perp (p_{i,m_i^\nu+1}^\nu)'_t(\alpha_{i,m_i^\nu}^\nu, x^{*,-\nu}) + a_{i,m_i^\nu+1}^\nu y_{i,m_i^\nu+1}^{*,\nu} - \eta_i^{*,\nu} \ge 0
\end{aligned}\right\} i = 1, \ldots, n'_\nu \\
&\left.\begin{aligned}
&0 \le y_{i,j}^{*,\nu} \perp (p_{i,j}^\nu)'_t(\alpha_{i,j-1}^\nu, x^{*,-\nu}) + a_{i,j}^\nu y_{i,j}^{*,\nu} - \eta_i^{*,\nu} + \mu_{i,j}^{*,\nu} \ge 0 \\
&0 \le \mu_{i,j}^{*,\nu} \perp \widehat{\alpha}_{i,j}^\nu - y_{i,j}^{*,\nu} \ge 0
\end{aligned}\right\} \begin{aligned} j &= 1, \ldots, m_i^\nu \\ i &= 1, \ldots, n'_\nu. \end{aligned}
\end{aligned}$$

This shows that $(x^{*,\nu}, y^{*,\nu})$ is an optimal solution of (13) with $x^{-\nu} = x^{*,-\nu}$. Thus x^* is a desired equilibrium of the original game. □

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