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Absolute Stability Of Impulsive Delay System

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Abstract

Absolute stability is a basic and important problem in the design of automatic control systems. This paper initiate the study of absolute stability of impulsive control systems with time delay. Several absolute stability criteria are established by constructing Lyapunov functionals. Some examples are also presented to illustrate the main results.

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1 Introduction

The concept of absolute stability arises in the contexts of both automatic control and the general stability theory. It is a basic and important problem in the design of automatic control systems. Since 1950', there have appeared numerous results about absolute stability of control system described by ordinary differential equations [1]-[4]. On the other hand, in many control problems, such as optimal control models in economics, circuit networks and frequency modulated systems, the underlying systems are under the influence of time delay, and sometimes contain abrupt changes of their states at certain time moments. These phenomena can be successfully described by impulsive delay differential equations and so it is very important to study the problem of absolute stability of impulsive delay control systems. But up to now, there is very few studies about the problem, especially for nonlinear systems. The main reason is that it is very difficult and there is little knowledge available about impulsive functional differential equations. Recently, some papers have studied impulsive functional differential equations theory, impulsive control theory and have derived some good results, see [5]-[9], which give us new motivation to study the absolute stability of impulsive functional control systems.

In this paper, by constructing Lyapunov functional, the absolute stability of some nonlinear impulsive delay control systems is studied. Some sufficient conditions to guarantee the absolute stability for these control systems are established. These criteria are simple and easy to be checked, they are very convenient for engineers and technicians to use. Some examples are given to illustrate the results.

2 Preliminaries

Let R denote the set of real numbers, R^n the space of n-dimensional column vectors $x = (x_1, \cdots, x_n)^T$ with the norm $||x|| = max_{1\leq i\leq n}|x_i|$. Let $||A|| = max_{1\leq i\leq n}\{\sum_{1\leq j\leq m}|a_{ij}|\}$ be the norm of a $n \times m$ matrix $A = (a_{ij})$.

Let

$$\begin{aligned} I &= \{t_k | t_1 < t_2 < \cdots, t_k - t_{k-1} > \alpha > 0, k = 1, 2, \cdots\}; \\ U &= \{h : |h \in C(R, R), \;\; h(0) = 0, \;\; h(\delta)\delta > 0, \delta \neq 0 \;\; \}; \\ \Re &= \{h : |h \in C(R^+, R^+), \;\; h(0) = 0, \;\; h \; is \; strictly \; increasing \; function\}; \end{aligned}$$

$$\begin{aligned} PC([a, b], R^{n\times m}) &= \{\phi : [a, b] \to R^{n\times m} | \phi(t^+) = \phi(t), \forall t \in [a, b); \phi(t^-) \\ & \quad exists \; in \; R^{n\times m}, \forall t \in (a, b] \; and \; \phi(t^-) = \phi(t) \; for \; all \\ & \quad but \; at \; most \; a \; finite \; number \; of \; points \; t \in (a, b]\}; \\ PC([a, \infty), R^{n\times m}) &= \{\phi : [a, \infty) \to R^{n\times m} | \forall b > a, \phi|_{[a,b]} \in PC([a, b], R^{n\times m})\}; \\ C_t &= \{h : |h \in PC([t - \tau, t], R^n)\}. \end{aligned}$$

Let $|||\phi_t||| = sup_{t-\tau\leq\theta\leq t}||\phi(\theta)||$ denote the norm of functions $\phi \in PC([t - \tau, t], R^{n\times m})$, where $\tau > 0$ is constant.

Consider the following control system

$$\begin{cases} x_i' = -a_i x_i + f(x_1, \cdots, x_n, x_1(t-\tau_{i1}), \cdots, x_n(t-\tau_{in})) + H_i(h(\delta)), \\ \delta' = \sum_{i=1}^n p_i x_i - rh(\delta), \quad i = 1, 2, \cdots, n, \quad t \neq t_k, \\ x_i(t_k) = \sum_{j=1}^n d_{ij}(t_k) x_j(t_k^-) + d_{i(n+1)}(t_k)\delta(t_k^-), \\ \delta(t_k) = \sum_{j=1}^n d_{(n+1)j}(t_k) x_j(t_k^-) + d_{(n+1)(n+1)}(t_k)\delta(t_k^-), \end{cases} \tag{2.1}$$

where a_i, d_i, p_i, r and τ_{ij} are constants, $f \in C(R^n \times C_t, R^n)$, $f(0, \cdots, 0) = 0$ and $0 \le \tau_{ij} \le \tau$, $h \in U$ and $H \in C(R, R)$ and $H(0) = 0$.

Without loss of generality, we always assume that $t_0 \le t_1$, where t_0 is the initial time of IVP(initial value problem), we also assume that t_1 is the first instant of I and $t_{k+1} - t_k > \alpha > 0$, α is a constant.

Definition 2.1 *The zero solution of (2.1) is said to be*

(i). stable, if for $\forall \bar{t}_0 \ge t_o$, $\forall \epsilon > 0$, *there exists a* $\sigma = \sigma(\epsilon, \bar{t}_0) > 0$ *such that* $X_{\bar{t}_0} \in PC([\bar{t}_0 - \tau, \bar{t}_0], R^{n+1})$, $|||X_{\bar{t}_0}||| < \sigma$ *implies* $||X(t)|| < \epsilon$ *for all* $t \ge \bar{t}_0$, *where* $X = (x_1, \cdots, x_n, \delta)^T$*;*

(ii). uniformly stable, if the σ *in* (*i*) *is independent of* $\bar{t}_0$*;*

(iii). globally asymptotically stable, if it is stable and all solutions of (2.1) *satisfy* $\lim_{t\to\infty} X(t) = 0$;

(iv). absolutely stable, if for any $h \in U$ *and any* $\tau \ge 0$, *it is globally asymptotically stable.*

3 Main Results

We begin this section by studying the following control system

$$\begin{cases} x_i' &= -a_i x_i + \sum_{j=1}^n a_{ij} x_j + \sum_{j=1}^n b_{ij} x_j(t-\tau_{ij}) + H_i(h(\delta)), \\ \delta' &= \sum_{i=1}^n p_i x_i - r h(\delta), \ \ (i = 1, 2, \cdots, n), \ \ t \neq t_k, \\ x_i(t_k) &= \sum_{j=1}^n d_{ij}(t_k) x_j(t_k^-) + d_{i(n+1)}(t_k)\delta(t_k^-), \\ \delta(t_k) &= \sum_{j=1}^n d_{(n+1)j}(t_k) x_j(t_k^-) + d_{(n+1)(n+1)}(t_k)\delta(t_k^-), \end{cases} \tag{3.2}$$

where $H_i \in C(R, R)$, $h \in U$, $a_i, a_{ij}, b_{ij}, p_i, r, d_{ij}(t_k)$ and τ_{ij} are constants, $0 < \tau_{ij} \leq \tau$ and $k = 1, 2, \cdots$. Denote $X(t) = (x_1(t), \cdots, x_n(t), \delta(t))^T$.

Theorem 3.1 *Assume that there exist constants* $\alpha_i > 0$, $\beta_j > 0$, $i = 1, \cdots, n$, $j = 1, \cdots, n+1$ *such that*

(A_1). $|H_i(s)| \leq \alpha_i |s|$,

(A_2). $-\beta_i a_i + \beta_{n+1}|p_i| + \sum_{j=1}^n \beta_j(|a_{ji}| + |b_{ji}|) < 0$, $i = 1, 2, \cdots, n$,
$\sum_{i=1}^n \beta_i \alpha_i - \beta_{n+1} r < 0$,

(A_3). $\prod_{k=1}^\infty \lambda_k < \infty$,

where $\lambda_k = max_{1 \leq i \leq (n+1)}\{1, \sum_{j=1}^{n+1} \frac{\beta_j |d_{ji}(t_k)|}{\beta_i}\}$, *then the system* (3.2) *is absolutely stable.*

Proof: For any fixed $\tau > 0$ and $h \in U$, define a Lyapunov functional $v(t, X_t)$ as:

$$v(t, X_t) = \sum_{i=1}^n \beta_i [|x_i| + \sum_{j=1}^n |b_{ij}| \int_{t-\tau_{ij}}^t |x_j(s)| ds] + \beta_{n+1}|\delta|, \ \ t \neq t_k,$$

and

$$D^+ v(t, X_t) = limsup_{h \to 0} \frac{v(t+h, X_{t+h}) - v(t, X_t)}{h}, \ \ t \neq t_k.$$

where $X(t)$ is the solution of the system (3.2). Then

$$
\begin{aligned}
v(t_k, X_{t_k}) &= \textstyle\sum_{i=1}^{n} \beta_i[|x_i(t_k)| + \sum_{j=1}^{n} |b_{ij}| \int_{t_k-\tau_{ij}}^{t_k} |x_j(s)|ds] + \beta_{n+1}|\delta(t_k)| \\
&\leq \textstyle\sum_{i=1}^{n} \beta_i[\sum_{j=1}^{n} |d_{ij}(t_k)||x_j(t_k^-)| + |d_{i(n+1)}(t_k)||\delta(t_k^-)|] \\
&\quad \textstyle + \sum_{i=1}^{n} \beta_i \sum_{j=1}^{n} |b_{ij}| \int_{t_k-\tau_{ij}}^{t_k} |x_j(s)|ds \\
&\quad \textstyle +\beta_{n+1}[\sum_{j=1}^{n} |d_{(n+1)j}(t_k)||x_j(t_k^-)| + |d_{(n+1)(n+1)}(t_k)||\delta(t_k^-)|] \\
&= \textstyle\sum_{j=1}^{n}[\sum_{i=1}^{n} \beta_i|d_{ij}(t_k)|]|x_j(t_k^-)| + \sum_{i=1}^{n} \beta_i|d_{i(n+1)}(t_k)||\delta(t_k^-)| \\
&\quad \textstyle + \sum_{i=1}^{n} \beta_i \sum_{j=1}^{n} |b_{ij}| \int_{t_k-\tau_{ij}}^{t_k} |x_j(s)|ds \\
&\quad \textstyle + \sum_{j=1}^{n} \beta_{n+1}|d_{(n+1)j}(t_k)||x_j(t_k^-)| + \beta_{n+1}|d_{(n+1)(n+1)}(t_k)||\delta(t_k^-)| \\
&\leq \textstyle\sum_{j=1}^{n}[\sum_{i=1}^{n+1} \beta_i|d_{ij}(t_k)|]|x_j(t_k^-)| + \sum_{i=1}^{n+1} \beta_i|d_{i(n+1)}(t_k)||\delta(t_k^-)|) \\
&\quad \textstyle + \sum_{i=1}^{n} \beta_i \sum_{j=1}^{n} |b_{ij}| \int_{t_k-\tau_{ij}}^{t_k} |x_j(s)|ds \\
&= \textstyle\sum_{j=1}^{n}[\sum_{i=1}^{n+1} \frac{\beta_i|d_{ij}(t_k)|}{\beta_j}]\beta_j|x_j(t_k^-)| + [\sum_{i=1}^{n+1} \frac{\beta_i|d_{i(n+1)}(t_k)|}{\beta_{n+1}}]\beta_{n+1}|\delta(t_k^-)|) \\
&\quad \textstyle + \sum_{i=1}^{n} \beta_i \sum_{j=1}^{n} |b_{ij}| \int_{t_k-\tau_{ij}}^{t_k} |x_j(s)|ds,
\end{aligned}
$$

From the condition (A_3), we have

$$[\sum_{i=1}^{n+1} \frac{\beta_i|d_{i(n+1)}(t_k)|}{\beta_{n+1}}] \leq \lambda_k, \quad \sum_{i=1}^{n+1} \frac{\beta_i|d_{ij}(t_k)|}{\beta_j} \leq \lambda_k, \quad j = 1, \cdots n$$

and

$$\int_{t_k-\tau_{ij}}^{t_k} |x_j(s)|ds = \int_{t_k^- -\tau_{ij}}^{t_k^-} |x_j(s)|ds, \quad i, j = 1, \cdots n.$$

Thus

$$
\begin{aligned}
v(t_k, X_{t_k}) &\leq \lambda_k[\textstyle\sum_{i=1}^{n} \beta_i|x_i(t_k^-)| + \beta_{n+1}|\delta(t_k^-)| \\
&\quad \textstyle + \sum_{i=1}^{n} \beta_i \sum_{j=1}^{n} |b_{ij}| \int_{t_k^- -\tau_{ij}}^{t_k^-} |x_j(s)|ds] \qquad (3.3) \\
&= \lambda_k v(t_k^-, X_{t_k^-}).
\end{aligned}
$$

Furthermore

$$\begin{aligned} D^+v(t,X_t) &\leq \textstyle\sum_{i=1}^n \beta_i\{-a_i|x_i| + \sum_{j=1}^n |a_{ij}||x_j| \\ &\quad + \textstyle\sum_{j=1}^n |b_{ij}||x_j(t-\tau_{ij})| + |H_i(h(\delta))| \\ &\quad + \textstyle\sum_{j=1}^n |b_{ij}|[|x_j| - |x_j(t-\tau_{ij})|]\} \\ &\quad + \textstyle\sum_{i=1}^n \beta_{n+1}|p_i x_i| - \beta_{n+1} r|h(\delta)| \\ &\leq \textstyle\sum_{i=1}^n \beta_i\{-a_i|x_i| + \sum_{j=1}^n (|a_{ij}| + |b_{ij}|)|x_j| \\ &\quad + |H_i(h(\delta))|\} + \textstyle\sum_{i=1}^n \beta_{n+1}|p_i x_i| - \beta_{n+1} r|h(\delta)| \\ &\leq \textstyle\sum_{i=1}^n \{-\beta_i a_i + \beta_{n+1}|p_i| + \sum_{j=1}^n \beta_j(|a_{ji}| + |b_{ji}|)\}|x_i| \\ &\quad + (\textstyle\sum_{i=1}^n \beta_i\alpha_i - \beta_{n+1} r)|h(\delta)|, \quad t \neq t_k. \end{aligned}$$

From the condition (A_2), it follows that

$$-\beta_i a_i + \beta_{n+1}|p_i| + \sum_{j=1}^n \beta_j(|a_{ji}| + |b_{ji}|) < 0, \quad i = 1, 2, \cdots, n,$$

and

$$\sum_{i=1}^n \beta_i\alpha_i - \beta_{n+1} r < 0.$$

Then, there exists a $\overline{\beta} > 0$ such that

$$-\beta_i a_i + \beta_{n+1}|p_i| + \sum_{j=1}^n \beta_j(|a_{ji}| + |b_{ji}|) < -\overline{\beta}, \quad i = 1, 2, \cdots, n,$$

and

$$\sum_{i=1}^n \beta_i\alpha_i - \beta_{n+1} r < -\overline{\beta}.$$

Therefore, we have

$$D^+v(t,X_t) \leq -\overline{\beta}(\sum_{i=1}^n |x_i| + |h(\delta)|), \quad t \neq t_k. \tag{3.4}$$

Integrating both sides of (3.4) from t_{k-1} to $t \in [t_{k-1}, t_k)$, we obtain

$$v(t, X_t) \le v(t_{k-1}, X_{t_{k-1}}) - \overline{\beta} \int_{t_{k-1}}^{t} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds, \quad t \in [t_{k-1}, t_k), \tag{3.5}$$

and

$$v(t_k^-, X_{t_k^-}) \le v(t_{k-1}, X_{t_{k-1}}) - \overline{\beta} \int_{t_{k-1}}^{t_k} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds. \tag{3.6}$$

By the condition (A_3) we have $\lambda_k \ge 1$. Thus, it is clear from (3.3), (3.5) and (3.6) that, for $t \in [t_k, t_{k+1})$,

$$\begin{aligned} v(t, X_t) &\le v(t_k, X_{t_k}) - \overline{\beta} \textstyle\int_{t_k}^{t} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds \\ &\le \lambda_k v(t_k^-, X_{t_k^-}) - \overline{\beta} \textstyle\int_{t_k}^{t} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds \\ &\le \lambda_k [v(t_{k-1}, X_{t_{k-1}}) - \overline{\beta} \textstyle\int_{t_{k-1}}^{t_k} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds] \\ &\quad - \overline{\beta} \textstyle\int_{t_k}^{t} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds \\ &\le \lambda_k v(t_{k-1}, X_{t_{k-1}}) - \overline{\beta} \textstyle\int_{t_{k-1}}^{t_k} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds \\ &\quad - \overline{\beta} \textstyle\int_{t_k}^{t} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds \\ &= \lambda_k v(t_{k-1}, X_{t_{k-1}}) - \overline{\beta} \textstyle\int_{t_{k-1}}^{t} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds. \end{aligned}$$

Since $\lambda_k \ge 0$, we have

$$\begin{aligned} v(t_{k-1}, X_{t_{k-1}}) &\le \lambda_{k-1} v(t_{k-1}^-, X_{t_{k-1}^-}) \\ &\le \lambda_{k-1} [v(t_{k-2}, X_{t_{k-2}}) - \overline{\beta} \textstyle\int_{t_{k-2}}^{t_{k-1}} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds] \\ &= \lambda_{k-1} v(t_{k-2}, X_{t_{k-2}}) - \lambda_{k-1} \overline{\beta} \textstyle\int_{t_{k-2}}^{t_{k-1}} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds \\ &\le \lambda_{k-1} v(t_{k-2}, X_{t_{k-2}}) - \overline{\beta} \textstyle\int_{t_{k-2}}^{t_{k-1}} (\sum_{i=1}^{n} |x_i(s)| + |h(\delta(s))|) ds. \end{aligned}$$

Thus, for $t \in [t_k, t_{k+1})$,

$$
\begin{aligned}
v(t,X_t) &\le \lambda_k[\lambda_{k-1}v(t_{k-2},X_{t_{k-2}}) - \overline{\beta}\int_{t_{k-2}}^{t_{k-1}}(\textstyle\sum_{i=1}^n |x_i(s)| + |h(\delta(s))|)ds] \\
&\quad - \int_{t_{k-1}}^{t}(\textstyle\sum_{i=1}^n |x_i(s)| + |h(\delta(s))|)ds \\
&= \lambda_k\lambda_{k-1}v(t_{k-2},X_{t_{k-2}}) - \lambda_k\overline{\beta}\int_{t_{k-2}}^{t_{k-1}}(\textstyle\sum_{i=1}^n |x_i(s)| + |h(\delta(s))|)ds \\
&\quad - \int_{t_{k-1}}^{t}(\textstyle\sum_{i=1}^n |x_i(s)| + |h(\delta(s))|)ds \\
&\le \lambda_k\lambda_{k-1}v(t_{k-2},X_{t_{k-2}}) - \overline{\beta}\int_{t_{k-2}}^{t_{k-1}}(\textstyle\sum_{i=1}^n |x_i(s)| + |h(\delta(s))|)ds \\
&\quad - \int_{t_{k-1}}^{t}(\textstyle\sum_{i=1}^n |x_i(s)| + |h(\delta(s))|)ds \\
&\le \lambda_k\lambda_{k-1}v(t_{k-2},X_{t_{k-2}}) - \overline{\beta}\int_{t_{k-2}}^{t}(\textstyle\sum_{i=1}^n |x_i(s)| + |h(\delta(s))|)ds.
\end{aligned}
$$

Repeating this augument gives

$$
\begin{aligned}
v(t,X_t) &\le \textstyle\prod_{i=1}^k \lambda_i v(t_0,X_{t_0}) - \overline{\beta}\int_{t_0}^{t}(\sum_{i=1}^n |x_i(s)| + |h(\delta(s))|)ds \\
&\le \textstyle\prod_{i=1}^k \lambda_i v(t_0,X_{t_0}) \\
&\le \textstyle\prod_{i=1}^k \lambda_i \{\sum_{i=1}^n \beta_i[|x_i(t_0)| + \sum_{j=1}^n |b_{ij}|\int_{t_0-\tau_{ij}}^{t_0}|x_j(s)|ds] + \beta_{n+1}|\delta(t_0)|\} \\
&\le \textstyle\prod_{i=1}^k \lambda_i \{\sum_{i=1}^n \beta_i[|||X_{t_0}||| + \sum_{j=1}^n |b_{ij}||||X_{t_0}|||\tau] + \beta_{n+1}|||X_{t_0}|||\} \\
&= \textstyle\prod_{i=1}^k \lambda_i \{\sum_{i=1}^n \beta_i[1 + \sum_{j=1}^n |b_{ij}|\tau] + \beta_{n+1}\}|||X_{t_0}|||.
\end{aligned} \tag{3.7}
$$

This inequality implies that the system (3.2) is stable.

It remains to show that all solutions of the system (3.2) is such that $\lim_{t\to\infty} X(t) = 0$.

Assume that it is not true. Then, there exists a solution $X(t)$ such that $limsup_{t\to\infty}||X(t)|| > 0$. This, in turn, implies that there exists a sequence $\{T_m\}$ and $\beta > 0$ such that $lim_{m\to\infty}T_m = \infty$ as $m \to \infty$ and $||X(T_m)|| > 2\beta$. Suppose that $T_m \in [t_{k_m}, t_{k_m+1})$. Then either $||X(t)|| \ge \beta$, $t \in [t_{k_m}, t_{k_m+1})$ or there exist two points $s_m, S_m \in [t_{k_m}, t_{k_m+1})$ such that $||X(s_m)|| = \beta$, $||X(S_m)|| = 2\beta$ and $||X(t)|| \ge \beta$, $t \in [s_m, S_m]$. Here, we assume, without loss of generality, that $s_m < S_m$. Consider the second case. We first show that there exists an number $\widetilde{M} > 0$ such that $S_m - s_m \ge \frac{\beta}{\widetilde{M}}$.

Since $||X(S_m)|| = 2\beta$ and $||X(s_m)|| = \beta$,

$$\begin{aligned} \beta &= ||X(S_m)|| - ||X(s_m)|| \leq ||X(S_m) - X(s_m)|| \\ &= || \textstyle\int_{s_m}^{S_m} X'(s)ds|| \leq \int_{s_m}^{S_m} ||f(X_s)||ds, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} f(X_t) = \quad & (-a_1x_1 + \textstyle\sum_{j=1}^{n} a_{1j}x_j + \sum_{j=1}^{n} b_{1j}x_j(t-\tau_{1j}) + H_1(h(\delta)), \cdots, \\ & -a_nx_n + \textstyle\sum_{j=1}^{n} a_{nj}x_j + \sum_{j=1}^{n} b_{nj}x_j(t-\tau_{nj}) + H_n(h(\delta)), \\ & \textstyle\sum_{i=1}^{n} p_ix_i - rh(\delta))^T. \end{aligned} \tag{3.9}$$

From $\prod_{k=1}^{\infty} \lambda_k < \infty$, it follows that there exists an $M > 0$ such that $\prod_{k=1}^{\infty} \lambda_k \leq M$. By (3.7) and the definition of $v(t, X_t)$ and $||X||$, we have

$$\begin{aligned} v(t, X_t) &= \textstyle\sum_{i=1}^{n} \beta_i[|x_i| + \sum_{j=1}^{n} |b_{ij}| \int_{t-\tau_{ij}}^{t} |x_j(s)|ds] + \beta_{n+1}|\delta| \\ &\geq \textstyle\sum_{i=1}^{n} \beta_i|x_i(t)| + \beta_{n+1}|\delta(t)| \\ &\geq min_{1\leq j\leq n}\{\beta_j\}||X(t)||, \quad t \neq t_k, \end{aligned}$$

and

$$v(t, X_t) \leq \prod_{k=1}^{m} \lambda_k v(t_0) \leq Mv(t_0, X_{t_0}), \quad t \in [t_m, t_{m+1}).$$

Thus, for $t \in [t_m, t_{m+1})$, we have

$$min_{1\leq j\leq n}\{\beta_j\}||X(t)|| \leq v(t, X_t) \leq Mv(t_0, X_{t_0}). \tag{3.10}$$

It yields that $||X(t)||$ and $v(t, X_t)$ are bounded since $Mv(t_0, X_{t_0})$ is a constan. Furthermore, from $h \in U$, (A_1) and (3.9), it follows that $||f(X_t)||$ is bounded, which implies that there exists an $\overline{M} > 0$ and an $\widetilde{M} > 0$, such that $||X(t)|| \leq \overline{M}$, $v(t) \leq \widetilde{M}$ and $||f(X_t)|| \leq \widetilde{M}$. In addition, from (3.8)

$$\beta \leq \int_{s_m}^{S_m} ||f(X_s)||ds \leq \widetilde{M}(S_m - s_m),$$

so

$$\beta/\widetilde{M} \leq (S_m - s_m).$$

In summary, there are points $s_m < S_m$, $s_m, S_m \in [t_{k_m}, t_{k_m+1})$ such that $||X(t)|| \geq \beta$, $t \in [s_m, S_m]$ and $\beta/\widetilde{M} \leq S_m - s_m \leq t_{k_m} - t_{k_{m-1}}$. Furthermore, it is easy to see that, if $||X(t)|| \geq \beta$, then there exists $\widetilde{\beta} > 0$ such that $\sum_{i=1}^{n} |x_i| + |h(\delta)| \geq \widetilde{\beta}$ since $h \in U$.

Following we show that $v(t_{k_m}, x_{t_{k_m}}) < 0$ for m large enough.

Since $\lambda_k \geq 1$ and $\prod_{k=1}^{k_j} \lambda_k \leq M$, it follows that $\lim_{j\to\infty} \prod_{k=1}^{j} \lambda_k$ exists. Thus, from the Cauchy criterion, there exists an $N > 0$, such that for any $k_m > k_l \geq N$, $0 \leq \prod_{k=k_1}^{k_m} \lambda_k - \prod_{k=k_1}^{k_l} \lambda_k \leq 1$. Therefore

$$\begin{aligned} 1 &\geq \prod_{k=k_1}^{k_m} \lambda_k - \prod_{k=k_1}^{k_l} \lambda_k \\ &= \prod_{k=k_1}^{k_l} \lambda_k [\prod_{k=k_l+1}^{k_m} \lambda_k - 1] \\ &\geq \prod_{k=k_l+1}^{k_m} \lambda_k - 1. \end{aligned}$$

Thus we have $1 \leq \prod_{k=k_l+1}^{k_m} \lambda_k \leq 2$ for all $k_m > k_l \geq N$. Now let us fix k_l and let $m - l \geq \frac{2\widetilde{M}^2}{\overline{\beta}\widetilde{\beta}\beta}$. Then, from (3.3), (3.5) and (3.6), we can conduct the following estimation

$$
\begin{aligned}
v(t_{k_m}, X_{t_{k_m}}) &\le \lambda_{k_m} v(t_{k_m}^-, X_{t_{k_m}^-}) \\
&\le \lambda_{k_m} v(t_{k_m-1}, X_{t_{k_m-1}}) \le \lambda_{k_m}\lambda_{k_m-1} v(t_{k_m-1}^-, X_{t_{k_m-1}^-}) \\
&\le \cdots \\
&\le \textstyle\prod_{k=k_{(m-1)}+1}^{k_m} \lambda_k v(t_{k_{(m-1)}+1}^-, X_{t_{k_{(m-1)}+1}^-}) \\
&\le \textstyle\prod_{k=k_{(m-1)}+1}^{k_m} \lambda_k [v(t_{k_{(m-1)}}, X_{t_{k_{(m-1)}}}) \\
&\quad -\overline{\beta}\int_{t_{k_{(m-1)}}}^{t_{k_{(m-1)}+1}} (\textstyle\sum_{i=1}^n |x_i(s)| + |h(\delta)|)ds] \\
&\le \textstyle\prod_{k=k_{(m-1)}+1}^{k_m} \lambda_k [v(t_{k_{(m-1)}}, X_{t_{k_{(m-1)}}}) \\
&\quad -\overline{\beta}\int_{s_{k_{(m-1)}}}^{S_{k_{(m-1)}}} (\textstyle\sum_{i=1}^n |x_i(s)| + |h(\delta)|)ds] \\
&\le \textstyle\prod_{k=k_{(m-1)}+1}^{k_m} \lambda_k [v(t_{k_{(m-1)}}, X_{t_{k_{(m-1)}}}) - \overline{\beta}\widetilde{\beta}(S_{k_{(m-1)}} - s_{k_{(m-1)}})] \\
&\le \textstyle\prod_{k=k_{(m-1)}+1}^{k_m} \lambda_k [v(t_{k_{(m-1)}}, X_{t_{k_{(m-1)}}}) - \frac{\overline{\beta}\widetilde{\beta}\beta}{\widetilde{M}}] \\
&\le \textstyle\prod_{k=k_{(m-1)}+1}^{k_m} \lambda_k v(t_{k_{(m-1)}}, X_{t_{k_{(m-1)}}}) - \prod_{k=k_{(m-1)}+1}^{k_m} \lambda_k \frac{\overline{\beta}\widetilde{\beta}\beta}{\widetilde{M}}.
\end{aligned} \tag{3.11}
$$

Similarly, we get

$$
\begin{cases}
v(t_{k_{(m-1)}}, X_{t_{k_{(m-1)}}}) & \le \prod_{k=k_{(m-2)}+1}^{k_{(m-1)}} \lambda_k v(t_{k_{(m-2)}}, X_{t_{k_{(m-2)}}}) - \prod_{k=k_{(m-2)}+1}^{k_{(m-1)}} \lambda_k \frac{\overline{\beta}\widetilde{\beta}\beta}{\widetilde{M}}, \\
& \cdots, \\
v(t_{k_{(l+1)}}, X_{t_{k_{(l+1)}}}) & \le \prod_{k=k_l+1}^{k_{(l+1)}} \lambda_k v(t_{k_l}, X_{t_{k_l}}) - \prod_{k=k_l+1}^{k_{(l+1)}} \lambda_k \frac{\overline{\beta}\widetilde{\beta}\beta}{\widetilde{M}}.
\end{cases} \tag{3.12}
$$

Subsititute (3.12) into (3.11), we get

$$
\begin{aligned}
0 \ &\leq v(t_{k_m}, X_{t_{k_m}}) \\
&\leq v(t_{k_l}, X_{t_{k_l}}) \prod_{k=k_l+1}^{k_m} \lambda_k - \sum_{j=l}^{m-1} \prod_{k=k_j+1}^{k_m} \lambda_k \frac{\overline{\beta}\widetilde{\beta}\beta}{\widetilde{M}} \\
&\leq 2\widetilde{M} - (m-l)\frac{\overline{\beta}\widetilde{\beta}\beta}{\widetilde{M}} < 0.
\end{aligned}
$$

This contradiction implies that $lim_{t\to\infty}||X(t)|| = 0$. So the system (3.2) is globally asymptotically stable for any $\tau > 0$ and $h \in U$ which implies that the system (3.2) is absolutely stable. The proof is complete.

As a special case, when $b_{ij} = 0, i, j = 1, \cdots, n$, the system (3.2) becomes

$$
\left\{
\begin{array}{rcl}
x_i' &=& -a_i x_i + \sum_{j=1}^{n} a_{ij} x_j + H_i(h(\delta)), \\
\delta' &=& \sum_{i=1}^{n} p_i x_i - r h(\delta), \ \ i = 1, 2, \cdots, n, \ \ t \neq t_k, \\
x_i(t_k) &=& \sum_{j=1}^{n} d_{ij}(t_k) x_j(t_k^-) + d_{i(n+1)}(t_k)\delta(t_k^-), \\
\delta(t_k) &=& \sum_{j=1}^{n} d_{(n+1)j}(t_k) x_j(t_k^-) + d_{(n+1)(n+1)}(t_k)\delta(t_k^-),
\end{array}
\right. \tag{3.13}
$$

where $a_i, d_i, p_i, r, d_{ij}(t_k)$ and τ_{ij} are constants, $0 < \tau_{ij} \leq \tau$, $\tau > 0$, $H_i \in C(R, R)$ and $h \in U$. We can get more refined result:

Theorem 3.2 *Assume that there exist constants* $M > 0, \ \ \alpha_i > 0, \ \ \beta_j > 0$, $i = 1, \cdots, n, \ j = 1, \cdots, n+1$ *such that*

(B_1). $|H_i(s)| \leq \alpha_i |s|$,

(B_2). $-\beta_i a_i + \beta_{n+1}|p_i| + \sum_{j=1}^{n} \beta_j |a_{ji}| < 0, \ \ i = 1, 2, \cdots, n,$
$\sum_{i=1}^{n} \beta_i \alpha_i - \beta_{n+1} r < 0,$

(B_3). $\prod_{k=l}^{m} \lambda_k \leq M < \infty, \ \ \forall l \leq m, \ \ m = 1, 2, \cdots,$

where $\lambda_k = max_{1\leq i\leq(n+1)} \sum_{j=1}^{n+1} \frac{\beta_j |d_{ji}(t_k)|}{\beta_i}$. *Then the system* (3.13) *is absolutely stable.*

Proof: For any fixed $\tau > 0$ and $h \in U$, define a Lyapunov function $v(t)$ as:

$$v(t) = v(X(t)) = \sum_{i=1}^{n} \beta_i |x_i| + \beta_{n+1}|\delta|, \ t \neq t_k,$$

where $X(t) = (x_1(t), \cdots, x_n(t), \delta(t))^T$ is the solution of system (3.13). Then the simple computation yields, as in the proof of Theorem 3.1, that

(1).
$$v(t_k) \leq \lambda_k v(t_k^-), \quad k = 1, 2, \cdots, \tag{3.14}$$

(2). there exists a $\overline{\beta} > 0$ such that

$$D^+ v(t) \leq -\overline{\beta}[\sum_{i=1}^{n} |x_i(t)| + |\delta(t)|], \quad t \neq t_k, \tag{3.15}$$

(3).
$$v(t) \leq \prod_{i=1}^{k} \lambda_i v(t_0), \quad t \in [t_k, t_{k+1}). \tag{3.16}$$

(3.16) in turn, implies that the system (3.13) is stable. Next we need to prove that all solutions of the system (3.13) satisfy

$$lim_{t \to \infty} X(t) = 0. \tag{3.17}$$

If (3.17) is not ture, then there exists a solution $X(t)$ such that $limsup_{t\to\infty}||X(t)||$ > 0, using a similar method as that given for the proof of Theorem 3.1, we conclude that

(1). there exist constants $\overline{M} > 0$ and $\widetilde{M} > 0$ such that $||X(t)|| \leq \overline{M}$, $v(t) \leq \widetilde{M}$ and $||f(X(t))|| \leq \widetilde{M}$, where

$$f(X(t)) = \ (-a_1 x_1 + \sum_{j=1}^{n} a_{1j} x_j + H_1(h(\delta)), \cdots,$$
$$-a_n x_n + \sum_{j=1}^{n} a_{nj} x_j + H_n(h(\delta)), \ \sum_{i=1}^{n} p_i x_i - r h(\delta))^T,$$

(2). there exists a constant $\beta > 0$ and sequences $\{s_m\}$ and $\{S_m\}$, such that $s_m < S_m$, $s_m, S_m \in [t_{k_m}, t_{k_m+1})$, and

$$||X(t)|| \geq \beta, \quad t \in [s_m, S_m) \ \ and \ \ S_m - s_m \geq \frac{\beta}{\widetilde{M}},$$

(3). there exists a $\widetilde{\beta} > 0$ such that when $||X(t)|| \geq \beta$,

$$\sum_{i=1}^{n} |x_i| + |h(\delta)| \geq \widetilde{\beta}.$$

For sequence $\{k_m\}$, we claim that

$$limsup_{m\to\infty} \prod_{k=k_1}^{k_m} \lambda_k > 0. \tag{3.18}$$

In fact, if $limsup_{m\to\infty} \prod_{k=k_1}^{k_m} \lambda_k = 0$, then condition (B_3) implies that $lim_{m\to\infty} \prod_{k=1}^{m} \lambda_k = 0$, and (3.16) implies $\lim_{t\to\infty} ||X(t)|| = 0$ which contradicts with $\limsup_{t\to\infty} ||X(t)|| > 0$. So, (3.18) is true.

Without loss of generality, assume that $lim_{m\to\infty} \prod_{k=k_1}^{k_m} \lambda_k = L$ (otherwise, we can choose the subsequence). It follows from the Cauchy criterion that there exists a $N > 0$, such that for any $k_m > k_l \geq N$, $\prod_{k=k_1}^{k_l} \lambda_k \geq L/2$, $0 \leq |\prod_{k=k_1}^{k_m} \lambda_k - \prod_{k=k_1}^{k_l} \lambda_k| \leq L/3$. Therefore

$$\begin{aligned} L/3 \ &\geq |\textstyle\prod_{k=k_1}^{k_m} \lambda_k - \prod_{k=k_1}^{k_l} \lambda_k| \\ &= \textstyle\prod_{k=k_1}^{k_l} \lambda_k |\prod_{k=k_l+1}^{k_m} \lambda_k - 1| \\ &\geq L/2 |\textstyle\prod_{k=k_l+1}^{k_m} \lambda_k - 1|. \end{aligned}$$

Thus, $-2/3 \leq \prod_{k=k_l+1}^{k_m} \lambda_k - 1 \leq 2/3$ and hence $1/3 \leq \prod_{k=k_l+1}^{k_m} \lambda_k \leq 5/3$ for all $k_m > k_l \geq N$. The rest of the proof is similar to that given for Theorem 3.1 and we omit it. The proof is complete.

Remark: One of the differences between Theorem 3.1 and Theorem 3.2 is that in Theorem 3.1, λ_k is not less than 1 no matter how small the value $|d_{ij}(t_k)|$ is. But in Theorem 3.2, λ_k may be less than 1 which yields a larger domain for λ_k since it only requires $\prod_{k=l}^{m} \lambda_k \leq M < \infty, \quad l \leq m = 1, 2, \cdots$. The following examples illustrate the difference.

Example 3.1 *Consider the system*

$$\begin{cases} x' &= -2x + x(t-2) + h(\delta), \\ \delta' &= \frac{1}{2}x - 2h(\delta), \quad t \neq 1, 2, \cdots, \\ x(k) &= \frac{1}{k}x(k^-) + \frac{(-1)^k}{k^2}\delta(k^-), \\ \delta(k) &= (\frac{1}{2} - \frac{1}{k})x(k^-) + (\frac{1+(-1)^k}{2} + \frac{(-1)^k}{k^2})\delta(k^-), \end{cases} \tag{3.19}$$

Using the symbols of Theorem 3.1, we have $a_1 = 2,\ a_{11} = 0,\ b_{11} = 1,\ \alpha_1 = 1,\ p_1 = \frac{1}{2},\ r = 2,\ d_{11}(k) = \frac{1}{k},\ d_{12}(k) = \frac{(-1)^k}{k^2},\ d_{21}(k) = \frac{1}{2} - \frac{1}{k},\ d_{22}(k) = \frac{1+(-1)^k}{2} + \frac{(-1)^k}{k^2}$. *Let us take* $\beta_1 = \beta_2 = 1$. *Then* $-\beta_1 a_1 + \beta_2 p_1 + \beta_1(|a_{11}| + |b_{11}|) = -\frac{1}{2} < 0$, $\beta_1\alpha_1 - \beta_2 r = -1 < 0$ *and*

$$\lambda_1 = 2, \quad \lambda_k = \begin{cases} 1 + \frac{2}{k^2}, & k = 2, 4, \cdots, 2l, \cdots, \\ 1, & k = 3, 5, \cdots, 2l+1, \cdots \end{cases}.$$

Thus, $\prod_{k=1}^{\infty} \lambda_k < \infty$. *It follows from Theorem 3.1 that system* (3.19) *is absolutely stable.*

Example 3.2 *Consider the system*

$$\begin{cases} x' &= -2x + x(t) + h(\delta), \\ \delta' &= \frac{1}{2}x - 2h(\delta), \quad t \neq 1, 2, \cdots, \\ x(k) &= -\frac{1}{2}x(k^-) + \frac{k}{2k+1}\delta(k^-), \\ \delta(k) &= (2^{(-1)^k} - \frac{1}{2})x(k^-) + \frac{1}{k^2}\delta(k^-). \end{cases} \tag{3.20}$$

Using the symbols of Theorem 3.2, we have $a_1 = 2,\ a_{11} = 1,\ b_{11} = 0,\ \alpha_1 = 1,\ p_1 = \frac{1}{2},\ r = 2,\ d_{11}(k) = -\frac{1}{2},\ d_{12}(k) = \frac{k}{2k+1},\ d_{21}(k) = 2^{(-1)^k} - \frac{1}{2},\ d_{22}(k) = \frac{1}{k^2}$.

Let us take $\beta_1 = \beta_2 = 1$. *Then* $-\beta_1 a_1 + \beta_2 p_1 + \beta_1|a_{11}| = -\frac{1}{2} < 0$, $\beta_1\alpha_1 - \beta_2 r = -1 < 0$ *and*

$$\lambda_1 = \frac{4}{3}, \quad \lambda_k = \begin{cases} 2, & k = 2, 4, \cdots, 2l, \cdots, \\ \frac{1}{2}, & k = 5, \cdots, 2l+1, \cdots \end{cases}.$$

Thus, $\prod_{k=l}^{m} \lambda_k \leq 6 < \infty$ *which implies that the system* (3.20) *is absolutely stable.*

Now we turn to study the following nonlinear system :

$$\begin{cases} x_i' &= -a_i x_i + \sum_{j=1}^n f_{ij}(x_j) + \sum_{j=1}^n g_{ij}(x_j(t-\tau_{ij})) + H_i(h(\delta)),\ i=1,2,\cdots,n, \\ \delta' &= \sum_{i=1}^n p_i x_i - rh(\delta),\ \ t \neq t_k, \\ x_i(t_k) &= \sum_{j=1}^n d_{ij}(t_k) x_j(t_k^-) + d_{i(n+1)}(t_k)\delta(t_k^-),\ i=1,2,\cdots,n, \\ \delta(t_k) &= \sum_{j=1}^n d_{(n+1)j}(t_k) x_j(t_k^-) + d_{(n+1)(n+1)}(t_k)\delta(t_k^-), \end{cases} \tag{3.21}$$

where $a_i,\ d_i,\ p_i,\ r,\ d_{ij}(t_k)$ and τ_{ij} are constants, $0 < \tau_{ij} \leq \tau,\ \tau > 0$, $f_{ij}, g_{ij} \in C'(R,R)$, $f_{ij}(0) = g_{ij}(0) = 0$, $H_i \in C(R,R)$ and $h \in U$.

Theorem 3.3 *Assume that there exist constants* $m_{ij},\ M_{ij} \geq 0,\ \alpha_i \geq 0,\ i,j = 1,2,\cdots,n,\ \beta_i > 0,\ i = 1,\cdots,n+1$ *such that*

(1). $|f'_{ij}(s)| \leq m_{ij},\ \ |g'_{ij}(s)| \leq M_{ij},\ \ i,j = i,\cdots,n,$

(2). $-\beta_i a_i + \beta_{n+1} p_i + \sum_{j=1}^n \beta_j (m_{ji} + M_{ji}) < 0, i = 1,2,\cdots,n,$
$\sum_{i=1}^n \beta_i \alpha_i - \beta_{n+1} r < 0,$

(3). $|H_i(s)| \leq \alpha_i |s|,$

(4). $\prod_{k=1}^\infty \lambda_k < \infty,$

where $\lambda_k = max_{1 \leq i \leq (n+1)}\{1, \sum_{j=1}^{n+1} \frac{\beta_j |d_{ji}(t_k)|}{\beta_i}\}$.

Then the system (3.21) is absolutely stable.

Proof: Let

$$v(t) = v(t, X_t) = \sum_{i=1}^n \beta_i [|x_i| + \sum_{j=1}^n M_{ij} \int_{t-\tau_{ij}}^t |x_j(s)| ds] + \beta_{n+1}|\delta|,\ t \neq t_k,$$

where $X(t) = (x_1(t), \cdots, x_n(t), \delta(t))^T$ is the solution of the system (3.21). Then

$$v(t) \geq \sum_{i=1}^n \beta_i |x_i| + \beta_{n+1}|\delta|.$$

Using a similar method as that given in the proof of Theorem 3.1, we can derive that $v(t_k) \leq \lambda_k v(t_k^-)$ for $k = 1, 2, \cdots$ and

$$\begin{aligned} D^+v &\leq \sum_{i=1}^n \beta_i\{-a_i|x_i| + \sum_{j=1}^n |f_{ij}(x_j)| \\ &\quad + \sum_{j=1}^n |g_{ij}(x_j(t-\tau_{ij}))| + |H_i(h(\delta))| \\ &\quad + \sum_{j=1}^n M_{ij}[|x_j| - |x_j(t-\tau_{ij})|]\} \\ &\quad + \sum_{i=1}^n \beta_{n+1}|p_i x_i| - \beta_{n+1} r|h(\delta)| \\ &\leq \sum_{i=1}^n \beta_i\{-a_i|x_i| + \sum_{j=1}^n m_{ij}|x_j| \\ &\quad + \sum_{j=1}^n M_{ij}|x_j(t-\tau_{ij})| + \alpha_i|h(\delta)| \\ &\quad + \sum_{j=1}^n M_{ij}[|x_j| - |x_j(t-\tau_{ij})|]\} \\ &\quad + \sum_{i=1}^n \beta_{n+1}|p_i x_i| - \beta_{n+1} r|h(\delta)| \\ &\leq \sum_{i=1}^n \beta_i\{-a_i|x_i| + \sum_{j=1}^n (m_{ij} + M_{ij})|x_j| \\ &\quad + \alpha_i|h(\delta)|\} + \sum_{i=1}^n \beta_{n+1}|p_i x_i| - \beta_{n+1} r|h(\delta)| \\ &\leq \sum_{i=1}^n \{-\beta_i a_i + \beta_{n+1} p_i + \sum_{j=1}^n \beta_j(m_{ji} + M_{ji})\}|x_i| \\ &\quad + (\sum_{i=1}^n \beta_i \alpha_i - \beta_{n+1} r)|h(\delta)|, \quad t \neq t_k. \end{aligned}$$

Using a similar method as that given in the proof of in Theorem 3.1, we can show that

$$\lim_{t\to\infty} x_i(t) = 0, \quad \lim_{t\to\infty} \delta(t) = 0.$$

Thus, the system (3.21) is absolutely stable. The proof is complete.

Example 3.3 *Consider the system*

$$
\begin{cases}
x' &= -4x + (1 - cosx) + ln(1 + x^2(t-2)) + sin(\sqrt{2}h(\delta)), \\
\delta' &= x - 2h(\delta), \quad t \neq 1, 2, \cdots, \\
x(k) &= \frac{1}{2k}x(k^-) + \frac{1+(-1)^k}{4}\delta(k^-), \\
\delta(k) &= (\frac{1}{3} - \frac{1}{k^2})x(k^-) + (\frac{1}{2} + \frac{1}{k^2})\delta(k^-),
\end{cases}
\tag{3.22}
$$

Using the symbols of Theorem 3.3, we have $f(x) = 1 - cosx$, $g(x(t-2)) = ln(1+x^2(t-2))$, $H(s) = sin\sqrt{2}s$, $a = 4$, $p_1 = 1$, $r = 2$, $d_{11}(k) = \frac{1}{2k}$, $d_{12}(k) = \frac{1+(-1)^k}{4}$, $d_{21}(k) = \frac{1}{3} - \frac{1}{k^2}$, $d_{22}(k) = \frac{1}{2} + \frac{1}{k^2}$, *so* $|f'(x)| \leq 1 = m$, $|g(x)| \leq 1 = M$, $|H(s)| \leq \sqrt{2}|s|$.

Let us take $\beta_1 = \beta_2 = 1$. *Then* $-\beta_1 a + \beta_2 p + \beta_1(m + M) = -1 < 0$, $\beta_1\alpha_1 - \beta_2 r = \sqrt{2} - 2 < 0$ *and*

$$
\lambda_1 = \frac{3}{2}, \quad \lambda_k = \begin{cases} 1 + \frac{1}{k^2}, & k = 2, 4, \cdots, 2l, \cdots, \\ 1, & k = 3, 5, \cdots, 2l+1, \cdots \end{cases} .
$$

So, $\prod_{k=1}^{\infty} \lambda_k < \infty$ *which implies that the system* (3.22) *is absolutely stable.*

Using a similar method, we can establish absolutely stable criteria for the following highly nonlinear system

$$
\begin{cases}
x_i' &= -a_i x_i + g_i(x_1, \cdots, x_n, x_1(t-\tau_{i1}), \cdots, x_n(t-\tau_{in})) + H_i(h(\sigma)), \\
\sigma' &= \sum_{j=1}^n p_j x_j - h(\sigma), \quad i = i, \cdots, n, \quad t \neq t_k, \\
x_i(t_k) &= \sum_{j=1}^n d_{ij}(t_k)x_j(t_k^-) + d_{i(n+1)}(t_k)\sigma(t_k^-), \\
\sigma(t_k) &= \sum_{j=1}^n d_{(n+1)j}(t_k)x_j(t_k^-) + d_{(n+1)(n+1)}(t_k)\sigma(t_k^-),
\end{cases}
\tag{3.23}
$$

where $g_i \in C'$, $g_i(0, \cdots, 0) = 0$, $H_i \in C(R, R)$, $i, j = 1, 2, \cdots, n$ and $h \in U$.

Theorem 3.4 *Assume that there exist constants* $\alpha_i, m_{ij}, M_{ij} \geq 0, i, j = 1, 2, \cdots, n$ *and* $\beta_i > 0, i = 1, 2, \cdots, (n+1)$ *such that*

(1). $|\frac{\partial g_i}{\partial \xi_j}(\xi_1, \cdots, \xi_n, \zeta_1, \cdots, \zeta_n)| \le m_{ij}$,

$|\frac{\partial g_i}{\partial \zeta_j}(\xi_1, \cdots, \xi_n, \zeta_1, \cdots, \zeta_n)| \le M_{ij}$,

$|H_i(s)| \le \alpha_i |s|$,

(2). $-\beta_i a_i + \beta_{n+1} p_i + \sum_{j=1}^n \beta_j (m_{ji} + M_{ji}) < 0, i = 1, 2, \cdots, n$,

$\sum_{i=1}^n \beta_i \alpha_i - \beta_{n+1} < 0$,

(3). $\prod_{k=1}^\infty \lambda_k < \infty$,

where $\lambda_k = max_{1 \le i \le (n+1)} \{1, \sum_{j=1}^{n+1} \frac{\beta_j |d_{ji}(t_k)|}{\beta_i}\}$, $k = 1, 2, \cdots$.

Then system (3.23) is absolutely stable.

Using the idea presented in this paper, we can study absolute stability for the nonlinear system

$$\begin{cases} x_i' &= -a_i x_i + g_i(x_1, \cdots, x_n, x_1(t-\tau_{i1}), \cdots, x_n(t-\tau_{in})) + H_i(h(\sigma)), \\ \sigma' &= G(x_1, \cdots, x_n, x_1(t-\tau_1), \cdots, x_n(t-\tau_n)) - h(\sigma), \quad i = 1, \cdots, n, \quad t \ne t_k, \\ x_i(t_k) &= \sum_{j=1}^n d_{ij}(t_k) x_j(t_k^-) + d_{i(n+1)}(t_k) \sigma(t_k^-), \\ \sigma(t_k) &= \sum_{j=1}^n d_{(n+1)j}(t_k) x_j(t_k^-) + d_{(n+1)(n+1)}(t_k) \sigma(t_k^-), \end{cases} \tag{3.24}$$

where $g_i, G \in C'$, $g_i(0, \cdots, 0) = 0$, $G(0, \cdots, 0) = 0$, $H_i \in C(R, R)$, $0 \le \tau_{ij} \le \tau$, $0 \le \tau_j \le \tau$, $(i, j = 1, 2, \cdots, n)$ and $h \in U$.

Theorem 3.5 *Assume that there exist constants* $\alpha_i, m_{ij}, M_{ij}, c_j, \tilde{c}_j \ge 0, i, j = 1, 2, \cdots, n$ *and* $\beta_i > 0, i = 1, 2, \cdots, (n+1)$ *such that*

(1). $|\frac{\partial g_i}{\partial \xi_j}(\xi_1, \cdots, \xi_n, \zeta_1, \cdots, \zeta_n)| \le m_{ij}, \quad |\frac{\partial g_i}{\partial \zeta_j}(\xi_1, \cdots, \xi_n, \zeta_1, \cdots, \zeta_n)| \le M_{ij}$,

$|\frac{\partial G}{\partial \xi_j}(\xi_1, \cdots, \xi_n, \zeta_1, \cdots, \zeta_n)| \le c_j, \quad |\frac{\partial G}{\partial \zeta_j}(\xi_1, \cdots, \xi_n, \zeta_1, \cdots, \zeta_n)| \le \tilde{c}_{ij}$,

$|H_i(s)| \le \alpha_i |s|, \quad i, j = 1, \cdots, n$,

(2). $-\beta_i a_i + \beta_{n+1}(c_i + \tilde{c}_i) + \sum_{j=1}^n \beta_j (m_{ji} + M_{ji}) < 0, \quad i = 1, 2, \cdots, n$,

$\sum_{i=1}^n \beta_i \alpha_i - \beta_{n+1} < 0$,

(3). $\prod_{k=1}^\infty \lambda_k < \infty$,

where $\lambda_k = max_{1\le i\le(n+1)}\{1, \sum_{j=1}^{n+1}\frac{\beta_j|d_{ji}(t_k)|}{\beta_i}\}$, $k = 1, 2, \cdots$. *Then system* (3.24) *is absolutely stable.*

Proof: Let

$$v(t) = v(t, X_t) = \sum_{i=1}^n \beta_i[|x_i| + \sum_{j=1}^n M_{ij}\int_{t-\tau_{ij}}^t |x_j(s)|ds]$$

$$+\beta_{n+1}\sum_{j=1}^n \tilde{c}_j\int_{t-\tau_j}^t |x_j(s)|ds + \beta_{n+1}|\delta|, \; t \ne t_k,$$

where $X(t) = (x_1(t), \cdots, x_n(t), \delta(t))^T$ is the solution of the system (3.24). Then

$$v(t) \ge \sum_{i=1}^n \beta_i|x_i| + \beta_{n+1}|\delta|.$$

Using a similar method as that given in the proof of Theorem 3.1, we can derive that $v(t_k) \le \lambda_k v(t_k^-)$ for $k = 1, 2, \cdots$ and

$$\begin{aligned}
D^+v \le\; & \sum_{i=1}^n \beta_i\{-a_i|x_i| + |g_i(x_1, \cdots, x_n, x_1(t-\tau_{i1}), \cdots, x_n(t-\tau_{in}))| + |H_i(h(\delta))| \\
& + \sum_{j=1}^n M_{ij}[|x_j| - |x_j(t-\tau_{ij})|]\} + \beta_{n+1}\sum_{j=1}^n \tilde{c}_j[|x_j(t)| - |x_j(t-\tau_j)|] \\
& + \beta_{n+1}|G(x_1, \cdots, x_n, x_1(t-\tau_1), \cdots, x_n(t-\tau_n))| - \beta_{n+1}|h(\delta)| \\
\le\; & \sum_{i=1}^n \beta_i\{-a_i|x_i| + \sum_{j=1}^n m_{ij}|x_j| + \sum_{j=1}^n M_{ij}|x_j(t-\tau_{ij})| + \alpha_i|h(\delta)| \\
& + \sum_{j=1}^n M_{ij}[|x_j| - |x_j(t-\tau_{ij})|]\} + \beta_{n+1}\sum_{j=1}^n \tilde{c}_j[|x_j| - |x_j(t-\tau_{ij})|] \\
& + \beta_{n+1}\sum_{j=1}^n[c_j|x_j| + \tilde{c}_j|x_j(t-\tau_{ij})|] - \beta_{n+1}r|h(\delta)| \\
\le\; & \sum_{i=1}^n \beta_i\{-a_i|x_i| + \sum_{j=1}^n(m_{ij} + M_{ij})|x_j| \\
& + \alpha_i|h(\delta)|\} + \sum_{i=1}^n \beta_{n+1}[c_i + \tilde{c}_i]|x_i| - \beta_{n+1}|h(\delta)| \\
\le\; & \sum_{i=1}^n\{-\beta_i a_i + \beta_{n+1}(c_i + \tilde{c}_i) + \sum_{j=1}^n \beta_j(m_{ji} + M_{ji})\}|x_i| \\
& + (\sum_{i=1}^n \beta_i\alpha_i - \beta_{n+1}r)|h(\delta)|, \quad t \ne t_k.
\end{aligned}$$

Using a similar method as that given in the proof of in Theorem 3.1, we can show that

$$\lim_{t\to\infty} x_i(t) = 0, \quad \lim_{t\to\infty} \delta(t) = 0.$$

Thus, the system (3.24) is absolutely stable. The proof is complete.

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