School of Science
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Connected Domination Critical Graphs

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Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

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October 2015

Dedicated to my father,

Weerapol Kaemawichanurat

Abstract

A dominating set D of a graph G is a vertex subset of $V(G)$ which every vertex of G is either in D or adjacent to a vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D of G is called a connected dominating set if the subgraph of G induced by D is connected. The minimum cardinality of a connected dominating set is called the connected domination number of G and is denoted by $\gamma_c(G)$. A dominating set D of G is called an independent dominating set if D is also an independent set. The minimum cardinality of an independent dominating set is called the independent domination number of G and is denoted by $i(G)$. A vertex subset D is called a total dominating set if every vertex of G is adjacent to a vertex in D. The minimum cardinality of a total dominating set is called the total domination number of G and is denoted by $\gamma_t(G)$.

A graph G is said to be $k-\gamma-$edge critical if the domination number $\gamma(G) = k$ and $\gamma(G+uv) < k$ for any pair of non-adjacent vertices u and v of G. For the connected domination number $\gamma_c(G) = k$, the total domination number $\gamma_t(G) = k$ and the independent domination number $i(G) = k$, a $k-\gamma_c-$edge critical graph, a $k-\gamma_t-$edge critical graph and a $k-i-$edge critical graph are similarly defined. In the context of vertex removal, a graph G is said to be $k-\gamma-$vertex critical if the domination number $\gamma(G) = k$ and $\gamma(G-v) < k$ for any vertex v of G. A $k-\gamma_c-$vertex critical graph, a $k-i-$vertex critical graph, are similarly defined. Moreover, a graph G is said to be $k-\gamma_t-$vertex critical if $\gamma_t(G) = k$ and $\gamma_t(G-v) < k$ for any vertex v of G which is not adjacent to a vertex of degree one.

In this thesis, we investigate the intersection between the classes of critical graphs with respect to different domination numbers. We show that the class of connected $k-\gamma_t-$edge critical graphs is identical to the class of connected $k-\gamma_c-$edge critical graphs if and only if $k = 3$ or 4. In addition, for the vertex critical case, we prove that the class of $2-$connected $k-\gamma_t-$vertex critical graphs is identical to the class of $2-$connected $k-\gamma_c-$vertex critical graphs if and only if $k = 3$ or 4. Moreover, in the class of claw-free graphs, we show that every $k-\gamma-$edge critical graph is also a $k-i-$edge critical graph and vice versa. We also have an analogous result for $k-$

$\gamma-$vertex critical graphs and $k-i-$vertex critical graphs.

For $k-\gamma_c-$vertex critical graphs, we establish the order of $k-\gamma_c-$vertex critical graphs in terms of maximum degree Δ and k. We prove that $\Delta+k \leq n \leq (\Delta-1)(k-1)+3$ and the upper bound is sharp for all integer $k \geq 3$ when Δ is even. It has been proved that every $k-\gamma_c-$vertex critical graph achieving the upper bound is $\Delta-$regular for $k=2$ or 3. For $k=4$, we prove that every $4-\gamma_c-$vertex critical graph achieving the upper bound is $\Delta-$regular. We further show that, for $k=2,3$ or 4, there exists a $k-\gamma_c-$vertex critical graph of order $(\Delta-1)(k-1)+3$ if and only if Δ is even. For $k \geq 5$, we show that if G is a $k-\gamma_c-$vertex critical graph of smallest possible order, namely $\Delta+k$, then G is isomorphic to a cycle of length $k+2$. We also establish the realizability of $k-\gamma_c-$vertex critical graphs of maximum degree Δ whose order is between the bounds when Δ and k are small.

For maximal $k-\gamma_c-$vertex critical graphs (the $k-\gamma_c-$vertex critical graphs whose connected domination number is decreased after any single edge is added), we characterize some classes for $k=3$. More specifically, we prove that every even order maximal $3-\gamma_c-$vertex critical graph is bi-critical. If the order is odd, then every maximal $3-\gamma_c-$vertex critical graph is $3-$factor critical with exactly one exception.

We investigate the hamiltonian properties of $k-\mathscr{D}-$edge critical graphs where $\mathscr{D} \in \{\gamma_c, \gamma_t, i\}$. We prove that if $k=1,2$ or 3, then every $2-$connected $k-\gamma_c-$edge critical graph is hamiltonian. We provide a class of $l-$connected $k-\gamma_c-$edge critical non-hamiltonian graphs for $k \geq 4$ and $2 \leq l \leq \frac{n-3}{k-1}$. Thus, for $n \geq l(k-1)+3$, the class of $l-$connected $k-\gamma_c-$edge critical non-hamiltonian graphs of order n is empty if and only if $k=1,2$ or 3. In addition, for $k-\gamma_t-$edge critical graphs, we show that these graphs are hamiltonian when $k=2$ or 3 and we provide classes of $2-$connected $k-\gamma_t-$edge critical non-hamiltonian graphs for $k=4$ or 5. For $k-i-$edge critical graphs, we give a construction for a class of $2-$connected non-hamiltonian graphs for $k \geq 3$.

Further, we investigate on the hamiltonian properties of $k-\mathscr{D}-$edge critical graphs where $\mathscr{D} \in \{\gamma_c, \gamma_t, \gamma, i\}$ when the graphs are claw-free. We prove that every $2-$connected $4-\gamma_c-$edge critical claw-free graph is hamiltonian and show that the claw-free condition cannot be relaxed. We further prove that the class of $k-\gamma_c-$edge critical claw-free non-hamiltonian graphs of connectivity two is empty if and only if $k=1,2,3$ or 4. We show that every $3-$connected $k-\gamma_c-$edge critical claw-free graph is hamiltonian for $1 \leq k \leq 6$. For $k-\gamma_t-$edge critical graphs, we show that every $3-$connected $k-\gamma_t-$edge critical claw-free graph is hamiltonian for $2 \leq k \leq 5$. We also show that every $3-$connected $4-\mathscr{D}-$edge critical claw-free graph where $\mathscr{D} \in \{\gamma, i\}$ is hamiltonian.

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Contents

CHAPTER 1

Introduction

1.1 Background

Over the past almost three hundreds years or so since Leonhard Euler published a solution to the famous *Königsberg Bridge Problem*, graph theory has become an elegant and important area of mathematics that has been dramatically developed as evidenced in many thousands of publications. In particular, over the recent thirty years or so, mathematicians have advanced and enriched the area of graph theory by developing innovative and important ideas.

The basic structure of a graph which consists of nodes called *vertices* and lines linking pairs of nodes called *edges* makes it easy to apply and becomes an extraordinary tool in solving many mathematical questions in areas such as topology, geometry and number theory. Indeed, graph theory can be applied to any system consisting of a collection of objects some of which are related. A graph naturally models such systems with the vertices representing the objects and the edges representing the relationship between the objects. The versatility of graph theory is the main reason that has attracted the attention of a number of researchers in areas such as engineering, communication networks, transport and logistics and biology. A discussion on the applications of graph theory can be found in Caccetta [39, 40] and Caccetta and Vijayan [52]. In Caccetta [40], the author introduced a fundamental network design problem and pointed out the ideas to construct a network that meet the requirements and optimizes a networks function performance such as cost, delay time and reliability. To find the optimal solutions, we always appropriately focus on different parameters that correspond to the requirements such as connectivity, diameter, domination number and hamiltonicity of graphs.

The idea of domination number of graphs has been applied to many fields such as optimization, linear algebra, complex network and combinatorial designs. The literature on this topic has been excellently surveyed by Haynes, Hedetniemi and Slater [80, 81] and Hedetniemi and Laskar [92]. In fact, the area of domination theory has various types of domination numbers such as independent domination number, total domination number and connected domination number. For other domination numbers see Henning et al. [94, 98, 100, 103, 104] and Kang et al. [114–117, 119, 131].

The idea of independent domination dates back to 1862 and was inspired by a chessboard puzzle. Jaenisch [110] introduced the problem of finding the minimum number of queens that can be placed on a chessboard under the conditions that (i) the queens are not attacking each other and (ii) the queens can attack every square (beyond the queens themself) on the chessboard. The graph which corresponds to this problem can be constructed by letting the vertices represent the squares and two vertices are adjacent based on the legal move of a queen. The representing graph is known as a *queen graph* and the minimum number of queens is the independent domination number. The concept of independent domination became well known because of many useful applications (see Berge [24], Ore [129] and Cockyane and Hedetniemi [58]). For more examples of works on independent domination see [1, 20–22, 73, 136] and [143]. For a good survey see Goddard and Henning [72].

The concept of total domination was also motivated by a problem, introduced by Berge [24], of placing the minimum number of queens on a chessboard to cover all the squares not occupied by queens. In this case, the queens have to cover each other (each queen can go to at least one of the others by one move). This problem can be presented by a queen graph as well and the minimum number of queens can be placed on the chessboard is called the total domination number. The theory of total domination has been studied and remarkably developed through several hundreds of research papers. There is a group of researchers, Haynes, Henning, Mynhardt, Yeo, van der Merwe and Goddard, who have mainly developed the theory and applications of this parameter as detailed, for example, in [82–89, 93, 101, 102, 105–107, 140–143]. For more examples of works on this parameter see [77–79, 96, 97, 99, 112, 126, 127, 133, 135–137, 145, 146] and [148–150]. Outstanding surveys are detailed in Henning [95] and [107].

The concept of connected domination was introduced formally by Sampathkumar and Walikar [134]. The results on this parameter have been continuously developed in both theory and applications. In particular, it is well known and extensively applied in the areas of operations research and wireless networks. There have been a number of papers on the so called *connected dominating set problem*. The problem of determining a smallest connected dominating set of a graph is well known to be NP-hard (Garey and Johnson [70]). Guha and Khuller [75] pointed out that these problems can be solved by constructing a spanning tree (a connected graph contains no cycle) with many leaves (a vertex which is adjacent to exactly one of the others). For more examples see [2, 19, 53, 90, 111–113, 118, 121, 125, 144] and [147].

The well know hamiltonian problem was first encountered in Sir William Rowan Hamilton's game on a dodechahedron in which each of its 20 corners was given a name of a place. The challenge was to find a closed route passing through every corner via the edges of the dodechahedron such that every corner is visited only once except for the origin. This problem can be modeled by a 3−regular graph (a graph which every vertex is adjacent to exactly three other vertices) of twenty vertices such that all the corners correspond to the vertices of the graphs while the edges of the dodechahedron are the edges of the graph. The problem then is to find a tour that pass through each vertex exactly once. Such a tour is called a *hamiltonian cycle*. The problem becomes much more difficult for general graphs. Until now, the problem to find a simple necessary and sufficient condition for a hamiltonian cycle remains unsolve. However, the idea of hamiltonian graphs can be widely applied in many research areas such as Industrial Engineering, Physics and Astronomy, Earth and Planetary Sciences, Immunology and Microbiology, Arts and Humanities. In fact, more than twenty thousand research papers related to this topic have been published. For example, see [22, 29–31, 56, 57, 60, 61, 63–67, 108, 109, 113, 122–124, 136] and [151–153].

Given a set of graph parameters, the question that arises typically in extremal graph theory is finding the relationships between the parameters. The basic method involves fixing some of the parameters and considering how the others change, for example see Caccetta [32–38] and [43]. For a given parameter $\mathscr{P}$ of a graph, it would be interesting to restrict attention to the so called *critical graphs with respect to* $\mathscr{P}$ under a single operation such as vertex or edge deletion or edge addition. That is to say these graphs whose parameter $\mathscr{P}$ changes whenever a single vertex or edge deletion or single edge addition occurs. For examples see Ananchuen and Caccetta [9–12], Ananchuen et al. [3, 5–8, 13–18], Bollobás [25], Bondy and Murty [26], Caccetta [41, 42], Caccetta et al. [44–51], Krishnamoorthy and Nandakumar [120] and Murty [128].

In this thesis, we mainly focus on the critical graphs with respect to connected domination number under the single operation of edge addition as well as critical graphs with respect to connected domination number under the single operation of vertex deletion. However, we also establish some related results on critical graphs with respect to domination number, total domination number and independent domination number. Moreover, we investigate the hamiltonian properties of these critical graphs.

In the following, Section 1.2, we give the notation and terminology for the graphs that we use throughout this thesis. In Section 1.3, we give a summary of all results in each chapter.

1.2 Terminology

Our basic graph theoretic notation and terminology follows for the most part that of Bondy and Murty [27]. Thus G denotes a finite graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. Throughout this thesis all graphs are simple (i.e. no loops or multiple edges). For $v \in V(G)$, the *open neighborhood* $N_G(v)$ of a vertex v in G is the set of vertices $u \in V(G)$ for which $uv \in E(G)$. The *closed neighborhood* $N_G[v]$ of a vertex v in G is the set of vertices $N_G(v) \cup \{v\}$. The *degree* $deg_G(v)$ of a vertex v in G is $|N_G(v)|$. For an integer $r \geq 1$, an $r-$*regular graph* is a graph whose vertices all have degree r. The *maximum degree* of a graph G is denoted by $\Delta(G)$ and the *minimum degree* of a graph G is denoted by $\delta(G)$. For subsets $X, Y \subseteq V(G)$, $N_Y(X)$ denotes the set of vertices $y \in Y$ for which $yx \in E(G)$ for some $x \in X$. It might happen that $N_Y(X) \cap X \neq \emptyset$. For a subgraph H of G, we use $N_Y(H)$ instead of $N_Y(V(H))$. If $X = \{x\}$, we use $N_Y(x)$ instead of $N_Y(\{x\})$. Moreover if $x \in Y$ and $N_Y(x) = \emptyset$, then x is called an *isolated vertex* in Y. We use $E(X,Y)$ to denote the set of all edges having one end vertex in X and the other one in Y. The set $\overline{X}$ denotes $V(G) - X$ and $E(X,G)$ denotes $E(G[X]) \cup E(X, \overline{X})$, that is, the set of edges having at least one end vertex in X. For $y \in Y$, the *private neighbor set* $PN(y,Y)$ of a vertex y respect to Y is a set of vertices $x \notin Y$ such that $N_Y(x) = \{y\}$. We denote $PN(y,Y) \cap X$ by $PN_X(y,Y)$.

A *matching* M is a set of pairwise non-adjacent edges (no two edges in M joining to a common vertex). A matching is *perfect* if it covers every vertex of G. A graph G is $s-$*factor critical* if $G - S$ contains a perfect matching for any vertex subset S of order s of G, moreover, if $s = 2$, we say that G is *bi-critical*. An *independent set* is a set of pairwise non-adjacent vertices. The maximum cardinality an independent set is called the *independence number* and is denoted by $\alpha(G)$.

The join of two disjoint graphs G and H, $G \vee H$, is the graph obtained from the union of G and H by joining every vertex of G to every vertex of H. Moreover, for disjoint graphs $H_1, H_2, ..., H_m$, the join $H_1 \vee H_2 \vee ... \vee H_m$ is the graph obtained from the union of $H_1, H_2, ..., H_m$ by joining every vertex of H_i to every vertex of H_{i+1} for $1 \leq i \leq m-1$. For a graph G of order n, the *Mycielskian* $\mu(G)$ of G is the graph containing a copy of G as an induced subgraph together with another $n+1$ vertices which are vertices u' corresponding to each vertex u of G and a vertex x. The vertex x is joined to every vertex u' and, moreover, a vertex u' is joined to a vertex v of G if $uv \in E(G)$, namely, $\mu(G)$ has the vertex set $V(G) \cup V' \cup \{x\}$ where $V' = \{u' | u \in V(G)\}$

and the edge set $E(G) \cup \{uv' | uv \in E(G)\} \cup \{v'x | v' \in V'\}$. As an example, let G be a graph which is a union of a path u, v, w of length two and an isolated vertex z. The Myscielskian of G is illustrated by Figure 1.1.

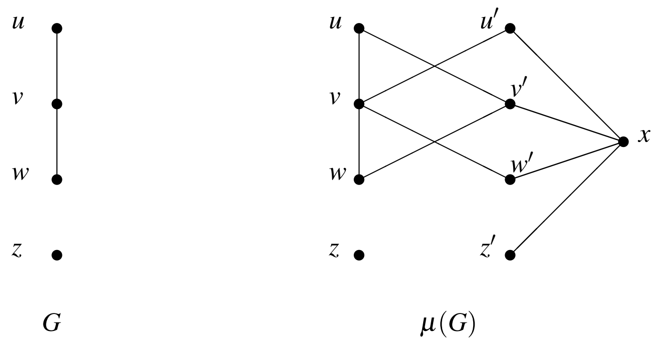

Figure 1.1 : The Myscielskian of G

The number of components of a graph G is denoted by $\omega(G)$, in particular, the number of odd components (a component of odd order) is denoted by $\omega_o(G)$. For a connected graph G, a *cut set* S is a vertex subset which $\omega(G-S) > 1$, moreover if $S = \{c\}$, then c is called a *cut vertex*. The *connectivity* $\kappa(G)$ of a graph G is the minimum cardinality of a cut set. A graph G is *s-connected* if $\kappa(G) \geq s$. When no ambiguity occurs, we abbreviate $N_G(v), N_G[v], \Delta(G), \delta(G), \alpha(G)$ and $\kappa(G)$ to $N(v), N[v], \Delta, \delta, \alpha$ and κ, respectively.

A *tree* is a connected graph that contains no cycle. A *leaf* of a tree, or an *end vertex* of a graph, is a vertex of degree one. A *support vertex* is a vertex adjacent to a leaf. A disjoint union of trees is called a *forest*. A *star* $K_{1,n}$ is a tree of order $n+1$ containing n leaves, in particular if $n = 3$, a star $K_{1,3}$ is called a *claw*. For integers $s_1, s_2, s_3 \geq 1$, let $u_1, u_2, ..., u_{s_1+1}; v_1, v_2, ..., v_{s_2+1}$ and $w_1, w_2, ..., w_{s_3+1}$ be three disjoint paths of length s_1, s_2 and s_3, respectively. The *net* N_{s_1,s_2,s_3} is constructed by adding edges $u_{s_1+1}v_{s_2+1}, v_{s_2+1}w_{s_3+1}$ and $w_{s_3+1}u_{s_1+1}$ (see Figure 1.2).

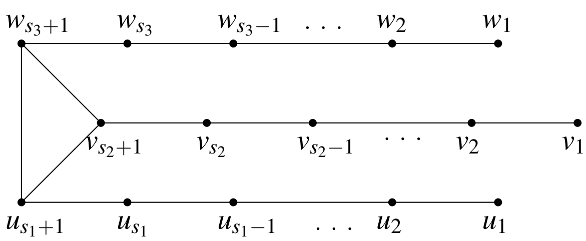

Figure 1.2 : The net N_{s_1,s_2,s_3}

For a family of graphs $\mathscr{F}$, a graph G is said to be *$\mathscr{F}$-free* if there is no induced subgraph of G isomorphic to H for all $H \in \mathscr{F}$. A *spanning subgraph* of a graph G is a subgraph which contains every vertex of G. Moreover, if a tree T is a spanning subgraph of G, then T is a *spanning tree*.

The *distance* between vertices u and v in G is the length of a shortest $(u,v)-$path in G. The *diameter* of G is the maximum distance between two vertices of G. A *hamiltonian path (cycle)* is a path (cycle) that contains every vertex in $V(G)$. A graph G is called *hamiltonian* if it contains a hamiltonian cycle. Moreover, a graph G is *hamiltonian connected* if every pair of vertices of G are joined by a hamiltonian path. For any subgraph F of G and $a,b \in V(G)$, aP_Fb denotes an $(a,b)-$path with all internal vertices in $V(F)$. Note that a and b need not be in $V(F)$. An aP_Fb path is *F-hamiltonian* if it contains all vertices of F. As an example, consider a pair of vertices $a,b \in V(G)$ which are adjacent respectively to $a',b' \in V(F)-\{a,b\}$. If F is a complete subgraph of G, then clearly there exists an $F-$hamiltonian path aP_Fb when $a' \neq b'$ and $|V(F)| > 1$ or when $a' = b'$ and $|V(F)| = 1$.

A *circulant graph* $C_n < a_0, a_1, ..., a_k >$ where $0 < a_0 < a_1 < ... < a_k < \frac{n+1}{2}$ is a graph with vertex set $\{x_0, x_1, ..., x_{n-1}\}$ and edge set $\{x_ix_j | (i-j) \equiv (\pm a_l)(\text{mod } n)$ for some $1 \leq l \leq k\}$. Note that circulant graphs are symmetric.

For subsets $D, X \subseteq V(G)$, D *dominates* X if every vertex in X is either in D or adjacent to a vertex in D. If D dominates X, then we write $D \succ X$, moreover, we also write $a \succ X$ when $D = \{a\}$ and $D \succ x$ when $X = \{x\}$. If $X = V(G)$, then D is a *dominating set* of G and we write $D \succ G$ instead of $D \succ V(G)$. A smallest dominating set is called a *$\gamma-$set*. The order of a $\gamma-$set is called the *domination number* and is denoted by $\gamma(G)$. A vertex subset D is a *connected dominating set* of X if D dominates X and $G[D]$ is connected. We write $D \succ_c X$ if D is a connected dominating set of X. In addition, we also write $a \succ_c X$ when $D = \{a\}$ and $D \succ_c x$ when $X = \{x\}$. Moreover if $X = V(G)$, then D is called a *connected dominating set* of G and we write $D \succ_c G$ instead of $D \succ_c V(G)$. A smallest connected dominating set is called a *γ_c-set*. The order of a γ_c-set is called the *connected domination number* and is denoted by $\gamma_c(G)$. A vertex subset D *totally dominates* X if every vertex in X is adjacent to a vertex of D. We write $D \succ_t X$ if D totally dominates X. If $X = V(G)$, then D is called a *total dominating set* of G and we write $D \succ_t G$ instead of $D \succ_t V(G)$. A smallest total dominating set is called a *γ_t-set*. The order of a γ_t-set is called the *total domination number* and is denoted by $\gamma_t(G)$. Clearly $\gamma_t(G) \geq 2$. If a dominating set D of G is independent, then D is called an *independent dominating set*. We write $D \succ_i G$ if D is an independent dominating set of G. A smallest independent dominating set is called an $i-$*set*. The *independent domination number* $i(G)$ of G is the cardinality of an $i-$set. As an example, consider the net $N_{3,3,3}$ in Figure 1.3. We see that $\{u_2, v_2, w_2, w_4\}$ is a $\gamma-$set and also an $i-$set. Moreover, $\{w_4, w_3, w_2, u_4, u_3, u_2, v_4, v_3, v_2\}$ is a γ_c-set and $\{u_3, u_2, v_3, v_2, w_3, w_2\}$ is a γ_t-set. Hence, $\gamma(N_{3,3,3}) = 4, i(N_{3,3,3}) = 4, \gamma_c(N_{3,3,3}) = 9$ and $\gamma_t(N_{3,3,3}) = 6$.

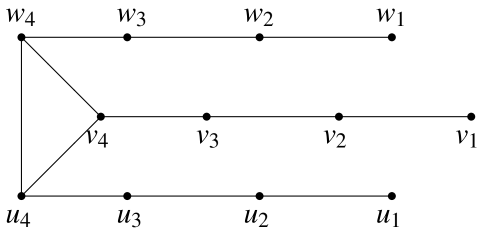

Figure 1.3 : The net $N_{3,3,3}$

A graph G is said to be *k domination edge critical*, or $k-\gamma-$*edge critical*, if $\gamma(G) = k$ and $\gamma(G+uv) < k$ for any pair of non-adjacent vertices u and v of G. A *k connected domination edge critical graph* ($k-\gamma_c-$*edge critical graph*), a *k independent domination edge critical graph* ($k-i-$*edge critical graph*) and a *k total domination edge critical graph* ($k-\gamma_t-$*edge critical graph*) can be similarly defined.

In the context of vertex removal, a graph G is said to be *k domination vertex critical*, or $k-\gamma-$vertex critical, if $\gamma(G) = k$ and $\gamma(G-v) < k$ for any vertex v of G. A *k connected domination vertex critical graph* ($k-\gamma_c-$*vertex critical graph*) and a *k independent domination vertex critical graph* ($k-i-$*vertex critical graph*) can be similarly defined. Observe that a disconnected graph does not have a connected dominating set. Hence, if a $k-\gamma_c-$vertex critical graph G contains a cut vertex c, then we cannot find a connected dominating set of $G-c$. To avoid this, we always focus on $2-$connected graphs when we study $k-\gamma_c-$vertex critical graphs. We observe also that a graph containing an isolated vertex does not have a total dominating set. A graph G is said to be a *k total domination vertex critical graph* ($k-\gamma_t-$*vertex critical graph*) if $\gamma_t(G) = k$ and $\gamma_t(G-v) < k$ for a vertex v of G which is not a support vertex.

If a graph is either $k-\gamma-$edge critical or $k-\gamma-$vertex critical, then we call it a $\gamma-$*critical graph*. An $i-$*critical graph*, a γ_t-*critical graph* and a γ_c-*critical graph* can be defined similarly. A $k-\gamma-$vertex critical graph G is *maximal* if $\gamma(G+uv) < k$ for any pair of non-adjacent vertices u and v of G. A *maximal* $k-\gamma_c-$*vertex critical graph* can be similarly defined.

1.3 Summary

In the previous sections a brief history of graphs, graph parameters was given together with the definition of critical graphs with respect to various types of domination numbers.

Chapter Two reviews some of the voluminous literature on the subject of domination critical graphs and hamiltonian graphs that are pertinent to the major part of this research.

Chapter Three is concerned with the intersection of the classes of critical graphs with respect to different domination numbers. We show that the class of connected $k-\gamma_t-$edge critical graphs is identical to the class of $k-\gamma_c-$edge critical graphs if and only if $k=3$ or 4. Further, we prove that the class of $2-$connected $k-\gamma_t-$vertex critical graphs is identical to the class of $k-\gamma_c-$vertex critical graphs if and only if $k=3$ or 4. Moreover, in the class of claw-free graphs, a $k-\gamma-$edge critical graph is also a $k-i-$edge critical graph and vice versa. We also have analogous results on vertex critical graphs.

Chapter Four is concerned with the order of $k-\gamma_c-$vertex critical graphs. We prove that $\Delta+k \leq n \leq (\Delta-1)(k-1)+3$. We show that our upper bound is sharp for all integer $k \geq 3$ when Δ is even. We prove that $4-\gamma_c-$vertex critical graphs of order 3Δ are $\Delta-$regular. We show that $k-\gamma_c-$vertex critical graphs of order $(\Delta-1)(k-1)+3$ need not be $\Delta-$regular when $k=5$ or 6. We further show that, for $k=2,3$ or 4, there exists a $k-\gamma_c-$vertex critical graph of order $(\Delta-1)(k-1)+3$ if and only if Δ is even. For $k \geq 5$, the $k-\gamma_c-$vertex critical graphs G of smallest possible order, namely $\Delta+k$, are characterized and shown to be precisely a cycle $G=C_{k+2}$ on $k+2$ vertices. We conclude Chapter Four by establishing the realizability of $k-\gamma_c-$vertex critical graphs of order n with maximum degree Δ for $\Delta+k \leq n \leq (\Delta-1)(k-1)+3$ when k and Δ are small.

Chapter Five is concerned with maximal $k-\gamma_c-$vertex critical graphs. We characterize maximal $3-\gamma_c-$vertex critical graphs of connectivity three. We show that any maximal $3-\gamma_c-$vertex critical graph with minimum degree $\delta \geq 4$ contains at least eight vertices. We also characterize all maximal $3-\gamma_c-$vertex critical graphs with $\delta \geq 4$ containing eight vertices. For a positive integer $l \geq 2$, the maximal $3-\gamma_c-$vertex critical graphs G of connectivity l which a minimum cut set of G is an independent set are characterized and shown to be precisely the Mycielskian of a complete graph K_l. For the matching property, we show that every maximal $3-\gamma_c-$vertex critical graph G is bi-critical if the order of G is even. Furthermore, if the order is odd, then G is

3−factor critical with only one exception.

Chapter Six is mainly concerned with the hamiltonian properties of $k-\mathscr{D}-$edge critical graphs where $\mathscr{D} \in \{\gamma_c, \gamma_t, i\}$. We establish that if $k = 1,2$ or 3, then every 2−connected $k-\gamma_c-$edge critical graph is hamiltonian. We give a construction for a class of $l-$connected $k-\gamma_c-$edge critical non-hamiltonian graphs for $k \geq 4$ and $2 \leq l \leq \lfloor \frac{n-3}{k-1} \rfloor$. Hence, for $n \geq l(k-1)+3$, the class of $l-$connected $k-\gamma_c-$edge critical non-hamiltonian graphs of order n is empty if and only if $k = 1,2$ or 3. In addition, we show that all 2−connected $k-\gamma_t-$edge critical graphs are hamiltonian for $k = 2$ or 3 and we provide classes of 2−connected $k-\gamma_t-$edge critical non-hamiltonian graphs for $k = 4$ or 5. In the class of 2−connected $k-i-$edge critical non-hamiltonian graphs, we show that it is non-empty when $k \geq 3$.

Chapter Seven is concerned with the hamiltonian properties of $k-\mathscr{D}-$edge critical claw-free graphs where $\mathscr{D} \in \{\gamma_c, \gamma_t, \gamma, i\}$. We prove that all 2−connected $4-\gamma_c-$edge critical claw-free graphs are hamiltonian and show that the claw-free condition cannot be relaxed. We prove that the class of $k-\gamma_c-$edge critical claw-free non-hamiltonian graphs of connectivity two is empty if and only if $k = 1,2,3$ or 4. We show that every 3−connected $k-\gamma_c-$edge critical claw-free graph is hamiltonian for $1 \leq k \leq 6$. When $\mathscr{D} = \gamma_t$, we show that every 3−connected $k-\gamma_t-$edge critical claw-free graph is hamiltonian for $2 \leq k \leq 5$. When $\mathscr{D} \in \{\gamma, i\}$, we show that every $4-\mathscr{D}-$edge critical claw-free graph is hamiltonian.

We conclude this thesis with Chapter Eight which contains possible future research and open problems.

CHAPTER 2

Literature Review

In this chapter, we state a number of results from the literature that we make use of in our work. We begin with results on domination theory and followed by results on hamiltonian graphs.

2.1 Domination Numbers

We begin with a result of Allan and Laskar [1] which is the relationship between the domination number and the independent domination number of claw-free graphs.

Theorem 2.1.1. [1] If G is a claw-free graph, then $\gamma(G) = i(G)$.

However, in this thesis, we mainly focus on critical graphs with respect to domination numbers. In the following, we give the results which are related in such graphs.

Sumner and Blitch [138] introduced the concept of $k-\gamma-$edge critical graphs and investigated properties and parameters of these such graphs such as the diameter, matching property and hamiltonian property. Moreover, these graphs have been studied by many authors, for example; Ananchuen and Plummer [13, 14, 16] and [18] studied matching property, Wojcicka [151], Flandrin et al. [66], Favaron et al. [65] and Tian et al. [139] studied the hamiltonian property. The following result of Sumner and Blitch [138] was established in the early study on $\gamma-$critical graphs.

Theorem 2.1.2. [138] Let G be a $k-\gamma-$edge critical graph. If $k = 1$, then G is K_n. Moreover, if $k = 2$, then $\overline{G} = \cup_{i=1}^{n} K_{1,n_i}$.

The next basic property of $k-\gamma-$edge critical graphs is observed by Ananchuen et al. [5].

Lemma 2.1.3. [5] Let u and v be a pair of non-adjacent vertices in a $k-\gamma-$edge critical graph G and let D_{uv} be a $\gamma-$set of $G+uv$. Then

(1) $|D_{uv}| = k-1$ and

(2) $|D_{uv} \cap \{u,v\}| = 1$.

In the context of vertex criticality, Ananchuen and Plummer [15] and [17] investigated matching property of $3-\gamma-$vertex critical graphs. We next provide the following observation on $k-\gamma-$vertex critical graphs which follows easily by the definition.

Observation 2.1.4. [17] Let G be a $k-\gamma-$vertex critical graph and, for a vertex $v \in V(G)$, let D_v be a $\gamma-$set of $G-v$. Then $|D_v| = k-1$.

The study on γ_t-edge critical graphs was started by van der Merwe et al. [142] and continued by a number of researchers (for example, Goddard et al. [71], Henning and van der Merwe [101] and van der Merwe and Loizeaux [141]). As a total dominating set does not contain an isolated vertex, we must always have $\gamma_t(G) \geq 2$. By this observation, Henning and van der Merwe [101] pointed out the structure of $k-\gamma_t-$edge critical graphs when $k = 2$.

Theorem 2.1.5. [101] A graph G is $2-\gamma_t-$edge critical if and only if G is a complete graph.

Van der Merwe et al. [141] and [142] established fundamental properties of $4-\gamma_t-$edge critical graphs described in the following propositions. In what follows, for a pair of non-adjacent vertices u and v of G, D^t_{uv} denotes a γ_t-set of $G+uv$.

Proposition 2.1.6. [141] Let G be a $4-\gamma_t-$edge critical graph and let u and v be a pair of non-adjacent vertices of G. Then at least one of the following holds:

(1) $\{u,v\} \succ G$,

(2) for either u or v, without loss of generality, say u, $\{w,u,v\} \succ G$ for some $w \in N_G(u)$ and $w \notin N_G(v)$,

(3) for either u or v, without loss of generality, say u, $\{x,y,u\} \succ G-v$ and $G[\{x,y,u\}]$ is connected.

Proposition 2.1.7. [142] For any graph G with $\gamma_t(G) = 3$ and a γ_t-set D, either $G[D] = P_3$ or $G[D] = K_3$.

Goddard et al. [71] established the following property of $k-\gamma_t-$vertex critical graphs.

Lemma 2.1.8. [71] Let G be a $k-\gamma_t-$vertex critical graph and, for any vertex $v \in V(G)$, let D^t_v be a γ_t-set of $G-v$. Then

(1) $D_v^t \cap N_G[v] = \emptyset$,

(2) $|D_v^t| = k-1$.

Goddard et al. [71], further, pointed out that the order of $k-\gamma_t-$vertex critical graphs satisfies $\Delta+k \leq n \leq \Delta(k-1)+1$. Mojdeh and Rad [127] proved the existence of these graphs achieving the upper bound $\Delta(k-1)+1$ with respect to the parities of k and Δ.

Theorem 2.1.9. [127] If there exists a $k-\gamma_t-$vertex critical $\Delta-$graph G of order $\Delta(k-1)+1$, then k is odd and Δ is even.

Rad and Sharebaf [133] proved that there is only one $3-\gamma_t-$vertex critical graph of order 9 when $\Delta = 4$. Moreover, they established the existence of $3-\gamma_t-$vertex critical graphs achieving the upper bound for even $\Delta \geq 6$.

Theorem 2.1.10. [133] For even $\Delta \geq 4$, there is a $3-\gamma_t-$vertex critical graph of order $2\Delta+1$.

For the lower bound, Mojdeh and Rad [126] proved the existence of $3-\gamma_t-$vertex critical graphs achieving the bound when Δ is small.

Theorem 2.1.11. [126] There is no $3-\gamma_t-$vertex critical graph of order $\Delta+3$ with $\Delta = 3$ or 5.

Sohn et al. [137] further established the existence of $k-\gamma_t-$vertex critical graphs achieving the lower bound of $\Delta+k$ vertices.

Theorem 2.1.12. [137] For any odd $k \geq 3$ and even $\Delta \geq 2\lfloor\frac{k-1}{2}\rfloor$, there exists a $k-\gamma_t-$vertex critical graph of order $\Delta+k$.

We next provide results on connected domination. We begin with a result of Sampathkumar and Walikar [134] which gives a relationship between the connected domination number and the order of a graph.

Proposition 2.1.13. [134] For a connected graph of order $n \geq 3$, $\gamma_c(G) \leq n-2$ and the bound is best possible.

Hedetniemi and Laskar [91], further, generalized Proposition 2.1.13 in term of maximum degree.

Theorem 2.1.14. [91] For any connected graph G, $\gamma_c(G) \leq n-\Delta$.

The $k-\gamma_c-$edge critical graphs were introduced by Chen et al. [55] and continued in Ananchuen [3] and Kaemawichanurat and Ananchuen [111]. Chen et al. [55] completely characterized $2-\gamma_c-$edge critical graphs and gave many properties of $3-\gamma_c-$edge critical graphs. Kaemawichanurat and Ananchuen [111] gave a characterization of $4-\gamma_c-$edge critical graphs with cut vertices and proved that such graphs contain a perfect matching. The following results on $k-\gamma_c-$edge critical graphs were established by Chen et al. [55].

Lemma 2.1.15. [55] Let G be a $k-\gamma_c-$edge critical graph and, for any pair of non-adjacent vertices u and v of G, let D^c_{uv} be a γ_c-set of $G+uv$. Then

(1) $k-2 \leq |D^c_{uv}| \leq k-1$, in particular if $k=3$, then $|D^c_{uv}|=2$.

(2) $D^c_{uv} \cap \{u,v\} \neq \emptyset$.

(3) If $u \in D^c_{uv}$ and $v \notin D^c_{uv}$, then $N_G(v) \cap D^c_{uv} = \emptyset$.

Lemma 2.1.16. [55] Let G be a $3-\gamma_c-$edge critical graph and I an independent set with $|I|=p \geq 3$. Then the vertices in I may be ordered as $w_1,w_2,...,w_p$ in such a way that there exists a path $P=z_1,z_2,...,z_{p-1}$ in $G-I$ with $\{w_i,z_i\} \succ_c G-w_{i+1}$ for $1 \leq i \leq p-1$.

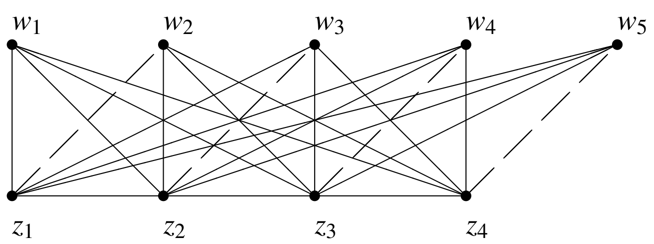

Figure 2.1 : The structure of the independent set I and a path P when $p=5$

Theorem 2.1.17. [55] A graph G is $2-\gamma_c-$edge critical if and only if $\overline{G} = \cup_{i=1}^{n} K_{1,n_i}$ for $n_i \geq 1$ and $n \geq 2$.

Ananchuen [3] provided the following result.

Theorem 2.1.18. [3] Let G be a $3-\gamma_c-$edge critical graph and S a cut set of G with $|S| \geq 2$. Then $\omega(G-S) \leq |S|$.

It is not difficult to see that $3-\gamma_t-$edge critical graphs are $3-\gamma_c-$edge critical graphs. Simmons [136] established the following result on $3-\gamma_t-$edge critical graphs.

Theorem 2.1.19. [136] Let G be a $3-\gamma_t-$edge critical graph. Then $\alpha \leq \delta+2$, moreover if $\alpha = \delta+2$, all vertices of degree δ are contained in every maximum independent set.

We conclude this section with the following lemma, the proof of which uses similar ideas to Favaron et al. [65].

Lemma 2.1.20. Let G be a $3-\gamma_c-$edge critical graph with $\delta \geq 2$ and $\alpha = \delta+2$ and x a vertex of degree δ of G. Then $G[N[x]]$ is a clique (complete graph). Moreover, G has only one vertex of degree δ.

Proof. Let I be a maximum independent set and x a vertex of degree δ. By Theorem 2.1.19, $x \in I$. Let $J = I - \{x\}$ and then $|J| = \delta+1$. Lemma 2.1.16 implies that there exists an ordering $w_1,...,w_{\delta+1}$ of the vertices of J and a path $z_1,...,z_\delta$ such that $\{w_i,z_i\} \succ_c G - w_{i+1}$ for $1 \leq i \leq \delta$. Since I is an independent set and $x \in I$, $xz_i \in E(G)$ for all $1 \leq i \leq \delta$. Thus $N(x) = \{z_1,...,z_\delta\}$. Consider $G + w_iw_j$, where $2 \leq i \neq j \leq \delta+1$. Lemma 2.1.15(2) implies that at least one of w_i or w_j is in $D^c_{w_iw_j}$. Without loss of generality, let $w_i \in D^c_{w_iw_j}$. Lemma 2.1.15(1) yields that $|D^c_{w_iw_j} - \{w_i\}| = 1$. Clearly, $D^c_{w_iw_j} \neq \{w_i,w_j\}$ to dominate J. Suppose $u \in D^c_{w_iw_j} - \{w_i\}$. To dominate x and by the connectedness of $(G+w_iw_j)[D^c_{w_iw_j}]$, $u \in N(x)$. Therefore $u = z_{i'}$ for some $1 \leq i' \leq \delta$. We have $uw_j \notin E(G)$ by Lemma 2.1.15(3). Because z_{j-1} is the only vertex in $N(x)$ that w_j is not adjacent to, $u = z_{j-1}$. Thus $D^c_{w_iw_j} = \{w_i, z_{j-1}\}$. Because $w_iz_{i-1} \notin E(G)$, $z_{i-1}z_{j-1} \in E(G)$. It then follows that $G[N[x]]$ is a clique.

Because I is a maximum independent set, there exists an ordering $q_1,...,q_{\delta+2}$ of the vertices of I and a path $y_1,...,y_{\delta+1}$ satisfying Lemma 2.1.16. Suppose that $deg_G(q_i) = \delta$ for some $1 \leq i \leq \delta+1$. Thus $G[N[q_i]]$ is a clique. We see that $q_1 \succ \{y_1,...,y_{\delta+1}\}$ and so $q_i \neq q_1$. Therefore $N(q_i) = \{y_j | 1 \leq j \leq \delta+1\} - \{y_{i-1}\}$. Since $\{q_i,y_i\} \succ_c G - q_{i+1}$ and $G[N[q_i]]$ is a clique, it follows that $y_i \succ G - q_{i+1}$. Lemma 2.1.16 yields that $y_{i-1}q_{i+1}, y_{i-1}y_i \in E(G)$. Clearly $\{y_i,y_{i-1}\} \succ_c G$, contradicting $\gamma_c(G) = 3$. Therefore $deg_G(q_i) > \delta$ for all $1 \leq i \leq \delta+1$. We have by Theorem 2.1.19 that the only vertex of degree δ of G is $q_{\delta+2}$. This completes the proof. □

For vertex deletion, Ananchuen et al. [7] characterized $k-\gamma_c-$vertex critical graphs when $k \in \{1,2\}$ and, further, established properties of $k-\gamma_c-$vertex critical graphs.

Lemma 2.1.21. [7] Let G be a $k-\gamma_c-$vertex critical graph. If $k = 1$, then the only $1-\gamma_c-$vertex critical graph is a singleton vertex. If $k = 2$, then the $2-\gamma_c-$vertex critical graphs are obtained from K_n for any even number $n \geq 4$ by deleting a perfect matching.

Lemma 2.1.22. [7] Let G be a $k-\gamma_c-$vertex critical graph and u,v two different vertices of G. We, moreover, let D_v^c be a γ_c-set of $G-v$. Then

(1) $D_v^c \cap N_G[v] = \emptyset$,

(2) $|D_v^c| = k-1$ and

(3) $N_G[v] \not\subseteq N_G[u]$.

Actually, when $k \geq 3$, Lemma 2.1.22(3) can be slightly improved regardless of whether or not u and v are adjacent.

Corollary 2.1.23. Let G be a $k-\gamma_c-$vertex critical graph and $u,v \in V(G)$ such that $u \neq v$ and $k \geq 3$. Then $N(v) \not\subseteq N[u]$, in particular, $N(v) \not\subseteq N(u)$.

Proof. Suppose there exist $u,v \in V(G)$ such that $N(v) \subseteq N[u]$. By Lemma 2.1.22(3), it suffices to consider the case $uv \notin E(G)$. Because $\gamma_c(G) \geq 3$, $V(G)-(N[u]\cup\{v\}) \neq \emptyset$. Let $A = V(G)-(N[u]\cup\{v\})$. Consider $G-u$. By Lemma 2.1.22(1), $D_u^c \cap N[u] = \emptyset$. If $v \in D_u^c$, then $D_u^c \cap N(v) = \emptyset$ because $N(v) \subseteq N[u]$. To dominate A, $D_u^c \cap A \neq \emptyset$. It follows that $(G-u)[D_u^c]$ is not connected. Thus $v \notin D_u^c$ and $D_u^c \subseteq A$. Clearly D_u^c does not dominate v, a contradiction and this completes the proof. □

When G is a $k-\gamma_c-$vertex critical graph for $k \geq 3$, it is worth noting that the neighborhood of every vertex of $G-v$ intersects D_v^c.

Lemma 2.1.24. Let G be a $k-\gamma_c-$vertex critical graph for $k \geq 3$ and v a vertex of G. For any vertex $w \in V(G)-\{v\}$, $D_v^c \cap N(w) \neq \emptyset$.

Proof. If $w \in D_v^c$, then $D_v^c \cap N(w) \neq \emptyset$ by the connectedness of $(G-v)[D_v^c]$. But if $w \notin D_v^c$, then $D_v^c \cap N(w) \neq \emptyset$ to dominate w. This completes the proof. □

2.2 Hamiltonian Graphs

Hamiltonian properties of graphs have attracted considerable attention by many researchers. A very simple but useful necessary condition was found by Chvátal and Erdös [57]. They established the following result which is a sufficient condition of a graph to be hamiltonian.

Theorem 2.2.1. [57] Let G be an $l-$connected graph. If $l \geq \alpha$, then G is hamiltonian. Moreover if the inequality is strict, then G is hamiltonian connected.

A well known and very useful property of hamiltonian graphs was established by Chvátal [56].

Proposition 2.2.2. [56] If G is a hamiltonian graph, then $\frac{|S|}{\omega(G-S)} \geq 1$ for every cut set $S \subseteq V(G)$.

A graph G is called *minimal* $2-$*connected non-hamiltonian* if it is a $2-$connected non-hamiltonian graph and H is either hamiltonian or connectivity less than two for any induced subgraph H of G. These graphs have been characterized by Brousek [30] as detailed in the following construction.

For $n_1, n_2, n_3 \geq 3$, let $P = x_1, x_2, ..., x_{n_1}, P' = y_1, y_2, ..., y_{n_2}$ and $P'' = z_1, z_2, ..., , z_{n_3}$ be three disjoint paths of order n_1, n_2 and n_3, respectively. From these, we define the graph P_{n_1,n_2,n_3} by adding the edges

- x_1y_1, y_1z_1, z_1x_1 and
- $x_{n_1}y_{n_2}, y_{n_2}z_{n_3}, z_{n_3}x_{n_1}$.

The graph P_{n_1,n_2,n_3} is illustrated in Figure 2.2(a).

We construct the graph P_{T,n_2,n_3} from the graph P_{n_1,n_2,n_3} by letting $n_1 = 3$ and adding the edge x_1x_3, that is

- $P_{T,n_2,n_3} = P_{3,n_2,n_3} + x_1x_3$.

We can construct the graphs $P_{n_1,T,n_3}, P_{n_1,n_2,T}, P_{T,T,n_3}, P_{T,n_2,T}, P_{n_1,T,T}$ and $P_{T,T,T}$ in the same way. For example, see Figure 2.2(b) for the graph $P_{T,n_2,T}$.

We define the class $\mathscr{P}$ of eight types of graphs as follows.

$$\mathscr{P} = \{P_{p_1,p_2,p_3} : p_i \in \{T, n_i\} \text{ and } n_i \geq 3 \text{ for } 1 \leq i \leq 3\}.$$

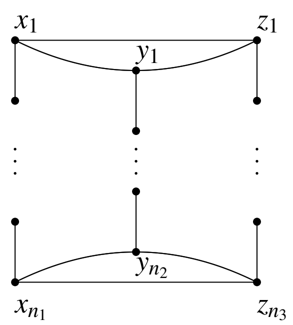

Figure 2.2(a) : The graph P_{n_1,n_2,n_3}

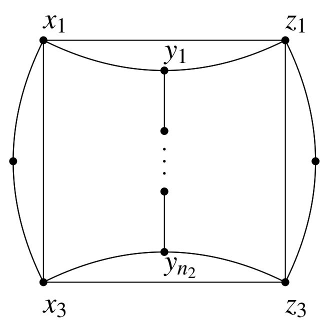

Figure 2.2(b) : The graph $P_{T,n_2,T}$

Theorem 2.2.3. [30] A graph G is a minimal $2-$connected non-hamiltonian claw-free graph if and only if $G \in \mathscr{P}$.

We next determine an operation which was introduced by Ryjáček [132] in order to study hamiltonicities of claw-free graphs. A vertex x is said to be *eligible* if $G[N_G(x)]$ is connected but non complete. For an eligible vertex x let $E_{\overline{G}}(x) = \{vu | u, v \in N_G(x)$ but $uv \notin E(G)\}$ and let G_x be the graph such that $V(G_x) = V(G)$ and $E(G_x) = E(G) \cup E_{\overline{G}}(x)$. The graph G_x is called the *local completion* of G at x. For a claw-free graph G, let $G_0, G_1, ..., G_t$ be a sequence of graphs for which $G = G_0$ and for some eligible vertex x of G_{i-1}, $G_i = (G_{i-1})_x$, at the end, G_t has no eligible vertex. Hence G_t is called the *closure* of G and is denoted by $cl(G)$. Ryjáček [132] showed that $cl(G)$ is well defined. As an example, consider a claw-free graph G in Figure 2.3. We see that $G[N(a)]$ is connected and non complete with $E_{\overline{G}}(a) = \{bd, be\}$. Thus G_a is the graph G adding edges bd and be. We then have $N_{G_a}(d) = \{a, b, c, e, f\}$ and $E_{\overline{G_a}}(d) = \{fc, fa, fe\}$. Therefore $(G_a)_d$ is the graph G_a adding edges fc, fa and fe. We see that the graph $(G_a)_d$ has no eligible vertex. Hence, $cl(G) = (G_a)_d$. In Figure 2.3, we use a *thick line* to denote an edge for the local completion.

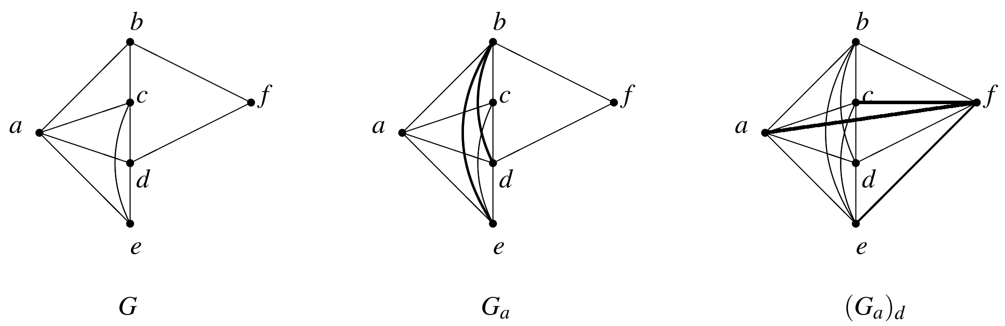

Figure 2.3

We, further, give an example to demonstrate that if we choose a different sequence of vertices for the local completion, then we still obtain the same closure of G. We see that $N(f) = \{b, d\}$ and $E_{\overline{G}}(f) = \{bd\}$. Clearly, $G_f = G + bd$. We then see that c is an eligible vertex of G_f with $N_{G_f}(c) = \{a, b, d, e\}$ and $E_{\overline{G_f}}(c) = \{be\}$. Thus $(G_f)_c = G_f + be$. Clearly $(G_f)_c$ still has b and d as eligible vertices. It is easy to see that both of $((G_f)_c)_d$ and $((G_f)_c)_b$ are the same as $(G_a)_d$.

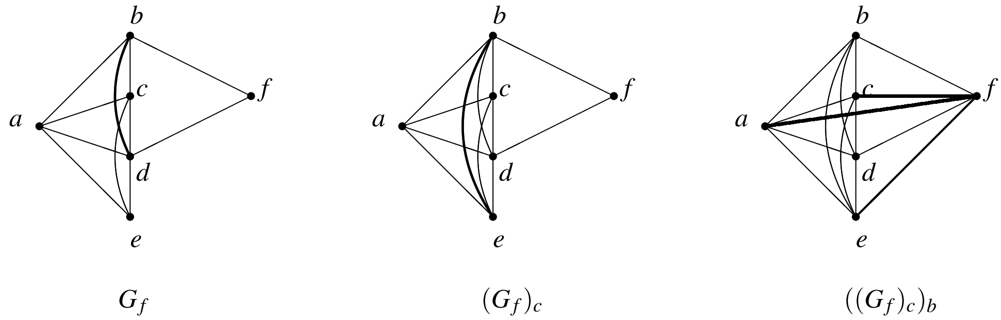

Figure 2.4

Brousek et al. [31] use this operation to establish Theorem 2.2.4. Before we state this theorem let us introduce some classes of graphs.

The Class $\mathscr{F}_1$:

Let Q_1 and Q_2 be two copies of complete graphs of order at least three and Q_3', Q_4' and Q_5' three disjoint non-empty sets of vertices. Let a_3, a_4 and a_5 be three different vertices of Q_1 and b_3, b_4 and b_5 be three different vertices of Q_2. We define a graph G in the class $\mathscr{F}_1$ by adding edges so that the vertices in $\{a_i, b_i\} \cup Q_i'$ form a complete graph for $3 \le i \le 5$.

Observe that $G[\{a_i, b_i\} \cup Q_i']$ is a complete graph of order at least three. We call this subgraph the Q_i. We observe also that $|V(Q_1) \cap V(Q_i)| = 1$ and $|V(Q_2) \cap V(Q_i)| = 1$ for $3 \le i \le 5$. A graph in this class is illustrated by Figure 2.5(a). We use an *oval* to denote a complete subgraph.

The Class $\mathscr{F}_2$:

Let c_1, c_2, c_3, c_1 and c_1', c_2', c_3', c_1' be two disjoint triangles. We, further, let R_1', R_2' and R_3' be three disjoint non-empty sets of vertices and $r \in R_3'$.

Define a graph G in the class $\mathscr{F}_2$ as follows.

- For $1 \le i \le 2$, add edges so that the vertices in $\{c_i', c_i\} \cup R_i'$ form a complete graph,
- add edges so that the vertices in $\{c_3\} \cup R_3'$ form a complete graph and
- add the edge rc_3'.

Observe that, for $1 \le i \le 2$, the subgraph $G[\{c_i, c_i'\} \cup R_i']$ is a complete graph of order at least three. We call this subgraph the R_i. We observe also that $G[\{c_3\} \cup R_3']$ is a complete graph of order a least two and we call this subgraph the R_3. A graph in this class is illustrated by Figure 2.5(b).

The Class $\mathscr{F}_3$:

Let $c_1, c_2, ..., c_5, c_6, c_1$ be a cycle on six vertices and F a copy of a complete graph of order at least three. Let s and s' be two different vertices of F.

Define a graph G in the class $\mathscr{F}_3$ by

- adding the edges sc_1, sc_6 and
- adding the edges $s'c_3, s'c_4$.

A graph in this class is illustrated by Figure 2.5(c).

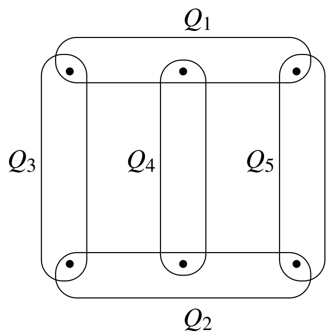

Figure 2.5(a) : The Class $\mathscr{F}_1$

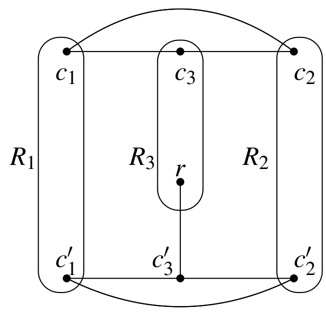

Figure 2.5(b) : The Class $\mathscr{F}_2$

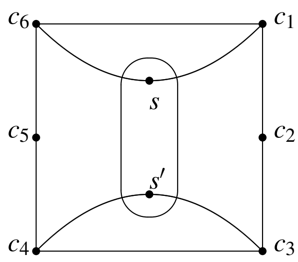

Figure 2.5(c) : The Class $\mathscr{F}_3$

Brousek et al. [31] proved :

Theorem 2.2.4. [31] Let G be a $2-$connected $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}-$free graph. Then either G is hamiltonian, or G is isomorphic to $P_{3,3,3}$ where $P_{3,3,3} \in \mathscr{P}$ or $cl(G) \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$.

Recently, Xiong et al. [152] establish this following lemma.

Theorem 2.2.5. [152] Let G be a $3-$connected $\{K_{1,3}, N_{s_1,s_2,s_3}\}-$free graphs. If $s_1 + s_2 + s_3 \leq 9$ and $s_i \geq 1$, then G is hamiltonian.

CHAPTER 3

Critical Graphs with respect to Domination Numbers

By the concept of connected domination, every connected dominating set of size at least two does not contain an isolated vertex in its induced subgraph. This means that it is also a total dominating set. By the minimality of $\gamma_t(G)$, we always have $\gamma_t(G) \leq \gamma_c(G)$ if $\gamma_c(G) \geq 2$. When the total domination number is 2 or 3, we see that the subgraph induced by a γ_t-set is connected. Therefore, $\gamma_c(G) \leq \gamma_t(G)$. We can conclude by simple structures of γ_t-sets that every graph G with $\gamma_t(G) \in \{2,3\}$ satisfies $\gamma_c(G) = \gamma_t(G)$. However, for $\gamma_t(G) \geq 4$, the subgraph induced by a γ_t-set need not be connected. For example, a cycle $C_8 = c_1, c_2, ..., c_8, c_1$ has $D^t = \{c_1, c_2, c_5, c_6\}$ as one of its γ_t-set and it has $D^c = \{c_1, c_2, ..., c_6\}$ as one of its γ_c-set. The induced subgraph $C_8[D^t]$ consists of two components of paths of length one rather than connected while $C_8[D^c]$ is a path of length five. Therefore, $\gamma_t(G) = \gamma_c(G)$ is not always true for a graph G with $\gamma_t(G) \geq 4$. However, we have noticed that for a $4-\gamma_t-$edge critical graph G, $\gamma_t(G+uv) \leq 3$ for any non-adjacent vertices u, v of G. Thus the subgraph induced by a γ_t-set of $G+uv$ is connected and is therefore a connected dominating set. This gives rise to the following problem : Is the class of connected $4-\gamma_t-$edge critical graphs the same as the class of $4-\gamma_c-$edge critical graphs? When we study edge critical graphs, it spontaneously leads us to consider vertex critical graphs as well. Similarly, for a $2-$connected $4-\gamma_t-$vertex critical graph G, $\gamma_t(G-v) = 3$ for every vertex v of G. Thus a γ_t-set of $G-v$ is a connected dominating set. We may ask the same question whether every $2-$connected $4-\gamma_t-$vertex critical graph is the same as a $4-\gamma_c-$vertex critical graph? In this chapter, we show that the class of connected $k-\gamma_t-$edge critical graphs and the class of $k-\gamma_c-$edge critical graphs are the same if and only if $k = 3$ or 4. Similarly, we show that the class of $2-$connected $k-\gamma_t-$vertex critical graphs and the class of $k-\gamma_c-$vertex critical graphs are the same if and only if $k = 3$ or 4.

For domination number and independent domination number, Allan and Laskar [1] proved that $\gamma(G) = i(G)$ for every claw-free graph G. Although we might obtain a claw as an induced subgraph after adding any single edge in a claw-free graph, we can show that $k-\gamma-$edge critical graphs and $k-i-$edge critical graphs are the same in the class of claw-free graphs.

3.1 γ_c-Critical Graphs and γ_t-Critical Graphs

In this section, we show that the class of connected $k-\gamma_t-$edge critical graphs and the class of $k-\gamma_c-$edge critical graphs are the same if and only if $3 \leq k \leq 4$. Note that $\gamma_t(G) \leq \gamma_c(G)$ when $\gamma_c(G) \geq 2$. We first establish the following theorem.

Theorem 3.1.1. Let G be a connected graph. Then G is a $4-\gamma_t-$edge critical graph if and only if G is a $4-\gamma_c-$edge critical graph.

Proof. Suppose that G is a $4-\gamma_c-$edge critical graph. Thus $\gamma_t(G) \leq \gamma_c(G) = 4$. Suppose that $\gamma_t(G) < 4$. Hence, there exists a γ_t-set D^t of G of size less than 4. Because $|D^t| < 4$, $G[D^t]$ is connected by Proposition 2.1.7. Therefore D^t is a connected dominating set of G of size less than 4, a contradiction. Hence, $\gamma_t(G) = 4$.

Consider $G+uv$ for $uv \notin E(G)$. Because G is $4-\gamma_c-$edge critical, there exists by Lemma 2.1.15(1) a γ_c-set D^c_{uv} of $G+uv$ with $|D^c_{uv}| < 4$. Clearly, D^c_{uv} is a total dominating set of $G+uv$. Therefore $\gamma_t(G+uv) \leq |D^c_{uv}| = \gamma_c(G+uv) < \gamma_c(G) = \gamma_t(G)$. Hence, G is $4-\gamma_t-$edge critical.

Conversely, suppose G is a $4-\gamma_t-$edge critical graph. We first show that $\gamma_c(G) = 4$.

Claim : There exists a connected dominating set of size 4 of G.

Consider $G+uv$ for $uv \notin E(G)$. Let D^t_{uv} be a γ_t-set of $G+uv$. By the criticality, $|D^t_{uv}| < 4$. Clearly $(G+uv)[D^t_{uv}]$ is connected by Proposition 2.1.7. Thus $D^t_{uv} \succ_c G+uv$. We distinguish two cases.

Case 1 : $|D^t_{uv} \cap \{u,v\}| = 1$.

Proposition 2.1.6(3) implies that $|D^t_{uv}| = 3$. We may suppose without loss of generality that $D^t_{uv} \cap \{u,v\} = \{v\}$. Since $D^t_{uv} \succ_c G+uv$ and G is connected, it follows that there exists $w \in V(G) - D^t_{uv}$ such that $wu \in E(G)$ and w must be adjacent to at least one vertex in D^t_{uv}. Because $|D^t_{uv}| = 3$, $D^t_{uv} \cup \{w\}$ is a connected dominating set of size 4 of G.

Case 2 : $|D^t_{uv} \cap \{u,v\}| = 2$.

We then distinguish two subcases according to Proposition 2.1.6(1) and (2).

Subcase 2.1 : $D^t_{uv} = \{u,v\}$.

If there is $w \in N(u) \cap N(v)$, then $\{u,v,w\}$ is a total dominating set of size 3 of G, a contradiction. Hence, $N(u)\cap N(v)=\emptyset$. Because G is connected and $\{u,v\}\succ G$, there exist x,y such that $x\in N(u), y\in N(v)$ and $xy\in E(G)$. Therefore $\{u,v,x,y\}$ is a connected dominating set of size 4 of G.

Subcase 2.2 : $D^t_{uv}=\{u,v,z\}$ for some $z\in V(G)$.

Thus z is adjacent to exactly one of u or v, say v. If there is $y\in N(\{z,v\})\cap N(u)$, then $\{u,v,y,z\}$ is a connected dominating set of size 4 of G. Suppose that $N(\{z,v\})\cap N(u)=\emptyset$. We partition set $V(G)-\{u,v,z\}$ as $A_1=N(u)$ and $A_2=N(\{v,z\})-\{v,z\}$. If $v\succ A_2$, then $\{u,v\}\succ G+uv$. This contradicts the fact that $D^t_{uv}=\{u,v,z\}$ is a smallest total dominating set of $G+uv$. Hence, there is $w\in A_2$ such that $zw\in E(G)$ but $vw\notin E(G)$. Consider $G+vw$. If $|D^t_{vw}\cap\{v,w\}|=1$, then, by similar arguments as in the proof of Case 1, G contains a connected dominating set of size 4. Thus, we suppose $|D^t_{uv}\cap\{v,w\}|=2$. If $D^t_{vw}=\{v,w\}$, then no vertex in D^t_{vw} dominates u because $w\in A_2$ and $A_1\cap A_2=\emptyset$, a contradiction. Therefore $D^t_{vw}=\{a,v,w\}$ for some $a\in V(G)$. In fact $a\in A_1$. Thus a is adjacent to w because $A_1\cap A_2=\emptyset$. Since $vz,wz\in E(G)$, $\{a,v,w,z\}$ is a connected dominating set of size 4 of G and thus, establishing the claim.

We now suppose that $\gamma_c(G)<4$. Thus $\gamma_t(G)\le\gamma_c(G)<4$ contradicting $\gamma_t(G)=4$. Hence, $\gamma_c(G)\ge 4$ and this together with the claim give $\gamma_c(G)=4$.

We finally prove the criticality by considering $G+uv$ for $uv\notin E(G)$. Because G is $4-\gamma_t-$edge critical, there exists a γ_t-set D^t_{uv} of size less than 4 of $G+uv$. Since $|D^t_{uv}|<4$, $(G+uv)[D^t_{uv}]$ is connected by Proposition 2.1.7. Thus $D^t_{uv}\succ_c G+uv$. So $\gamma_c(G+uv)\le|D^t_{uv}|<4=\gamma_c(G)$. This completes the proof of Theorem 3.1.1. □

Let $\mathbb{G}_k=\{G|\gamma_c(G)=\gamma_t(G)=k\}$ and, moreover, let

$\mathbb{T}^e_k$: the class of connected $k-\gamma_t-$edge critical graphs G with $G\in\mathbb{G}_k$ and

$\mathbb{C}^e_k$: the class of connected $k-\gamma_c-$edge critical graphs G with $G\in\mathbb{G}_k$.

In view of Theorem 3.1.1, we have $\mathbb{T}^e_4=\mathbb{C}^e_4$. We next show that $\mathbb{T}^e_k$ and $\mathbb{C}^e_k$ need not be the same for $k\ge 5$.

Theorem 3.1.2. $\mathbb{T}^e_k\neq\mathbb{C}^e_k$ when $k\ge 5$.

Proof. We prove the theorem by providing a graph $G\in\mathbb{T}^e_k-\mathbb{C}^e_k$ when $k\ge 5$. We distinguish the proof by the parity of k.

Case 1 : k is even.

Let $k=2q$ for some positive integer $q\ge 3$. Let G be the graph constructed from q different paths of length two, say $P^i=x^i_1,x^i_2,x^i_3$ for $i=1,2,...,q$ and then adding edges so that the vertices in $\{x^i_1|1\le i\le q\}$ form a clique (see Figure 3.1(a)).

We first show that $\gamma_t(G)=\gamma_c(G)=k=2q$. Note that $\{x^i_1,x^i_2|1\le i\le q\}\succ_c G$. Thus

$\gamma_c(G) \leq 2q$. For $i = 1,2,..,q$, we need at least two vertices to totally dominate each of P^i. So $2q \leq \gamma_t(G)$. Thus $2q \leq \gamma_t(G) \leq \gamma_c(G) \leq 2q$. Hence, $\gamma_t(G) = \gamma_c(G) = 2q$.

We now consider the total domination number of $G+uv$ where $uv \notin E(G)$. If $\{u,v\} = \{x^i_m, x^j_p\}$ where $i \neq j$ and $2 \leq m, p \leq 3$, then $\{x^i_m, x^j_p\} \cup \{x^l_1, x^l_2 | l \neq i, j\} \succ_t G+uv$. So $\gamma_t(G+uv) \leq 2q-2 < \gamma_t(G)$. If $\{u,v\} = \{x^i_1, x^j_p\}$ where $i \neq j$ and $p \in \{2,3\}$, then $\{x^i_1, x^i_2, x^j_p\} \cup \{x^l_1, x^l_2 | l \neq i, j\} \succ_t G+uv$. So $\gamma_t(G+uv) \leq 2q-1 < \gamma_t(G)$. Finally, if $\{u,v\} = \{x^i_1, x^i_3\}$, then $\{x^i_1\} \cup \{x^l_1, x^l_2 | l \neq i\} \succ_t G+uv$ Thus $\gamma_t(G+uv) = 2q-1 < \gamma_t(G)$. So G is $k-\gamma_t-$edge critical and $G \in \mathbb{T}^e_k$.

We now consider the connected domination number of $G+uv$. If $\{u,v\} = \{x^1_3, x^2_3\}$, then by Lemma 2.1.15(2), $D^c_{uv} \cap \{x^1_3, x^2_3\} \neq \emptyset$. Without loss of generality, we may suppose $x^1_3 \in D^c_{uv}$. Since $(G+uv)[D^c_{uv}]$ is connected, we need at least two vertices x^i_1, x^i_2 to dominate P^i for $i \neq 1,2$. If $x^2_3 \in D^c_{uv}$, then $x^2_1, x^2_2 \in D^c_{uv}$ or $x^1_1, x^1_2 \in D^c_{uv}$ by the connectedness of $(G+uv)[D^c_{uv}]$. Therefore $|D^c_{uv}| \geq 2q = k$. Thus G is not critical. Hence, we may assume that $x^2_3 \notin D^c_{uv}$ and so, $x^1_1, x^1_2, x^1_3 \in D^c_{uv}$ by the connectedness of $(G+uv)[D^c_{uv}]$. Further, $x^2_1 \in D^c_{uv}$ for dominating x^2_2. So $|D^c_{uv}| \geq 2q = k$ and G is not a $k-\gamma_c-$edge critical graph. Thus $G \notin \mathbb{C}^e_k$.

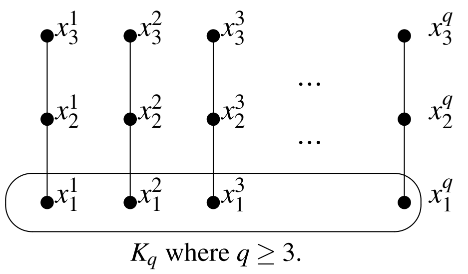

Figure 3.1(a): $G \in \mathbb{T}^e_k - \mathbb{C}^e_k$, k is even

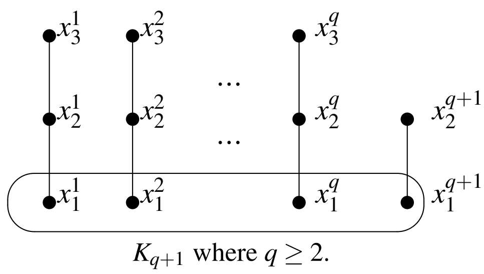

Figure 3.1(b): $G \in \mathbb{T}^e_k - \mathbb{C}^e_k$, k is odd

Case 2 : k is odd.

Let $k = 2q+1$ for some positive integer $q \geq 2$. Let G be the graph constructed from q different paths of length two, say $P^i = x^i_1, x^i_2, x^i_3$ for $i = 1,...,q$ and a path of length one, say $P^{q+1} = x^{q+1}_1, x^{q+1}_2$ and then adding edges so that the vertices in $\{x^i_1 | 1 \leq i \leq q+1\}$ forms a clique (see Figure 3.1(b)).

By similar arguments as in Case 1, we have $\gamma_t(G) = \gamma_c(G) = 2q+1$. To show $\gamma_t(G+uv) < k$ where $uv \notin E(G)$, we can apply similar arguments as in the proof of Case 1 when $\{u,v\} \subseteq \{x^i_l | 1 \leq i \leq q, 1 \leq l \leq 3\}$. We now suppose that $\{u,v\} \cap$

$V(P^{q+1}) \neq \emptyset$. Because $|V(P^{q+1})| = 2$, $|\{u,v\} \cap V(P^{q+1})| = 1$. Without loss of generality, assume that $u \in V(P^{q+1})$ and $v \in V(P^j)$ for some $j \in \{1,...,q\}$. If $u \in \{x_1^{q+1}, x_2^{q+1}\}$ and $v \in \{x_2^j, x_3^j\}$, then $\{u,v\} \cup \{x_1^l, x_2^l | l \neq j, q+1\} \succ_t G+uv$. So $\gamma_t(G+uv) \leq 2q \leq \gamma_t(G)$. Finally if $u = x_2^{q+1}$ and $v = x_1^j$, then $\{x_1^l, x_2^l | l \neq q+1\} \succ_t G+uv$. So $\gamma_t(G+uv) \leq 2q < \gamma_t(G)$ and $G \in \mathbb{T}_k^e$. By considering $G + x_3^1 x_3^2$, we can show that a graph G is not a $k-\gamma_c-$edge critical graph by similar arguments as in Case 1.

Hence, $G \in \mathbb{T}_k^e$ but $G \notin \mathbb{C}_k^e$. Therefore, $\mathbb{T}_k^e \neq \mathbb{C}_k^e$ when $k \geq 5$. This completes the proof of Theorem 3.1.2. □

Chen et al. [55] characterized that a graph G is $2-\gamma_c-$edge critical if and only if $\overline{G} = \cup_{i=1}^n K_{1,n_i}$ for $n_i \geq 1$ and $n \geq 2$. Henning and van der Merwe [101] proved that a graph G is $2-\gamma_t-$edge critical if and only if G is a complete graph (see Theorem 2.1.5). Thus $\mathbb{T}_2^e \neq \mathbb{C}_2^e$. For $k=3$, Ananchuen [3] pointed out that $\mathbb{T}_3^e = \mathbb{C}_3^e$. Thus, Theorems 3.1.1 and 3.1.2 imply the following corollary.

Corollary 3.1.3. $\mathbb{T}_k^e = \mathbb{C}_k^e$ if and only if $3 \leq k \leq 4$.

The next result shows that there exists a graph belonging to $\mathbb{T}_k^e$ and $\mathbb{C}_k^e$.

Theorem 3.1.4. For $k \geq 5$, $\mathbb{T}_k^e \cap \mathbb{C}_k^e \neq \emptyset$.

Proof. Let $G \in \mathbb{C}_k^e$. For all $uv \notin E(G)$ and a γ_c-set D_{uv}^c of $G+uv$, we have D_{uv}^c is also a total dominating set of $G+uv$. Since G is a $k-\gamma_c-$edge critical graph and $\gamma_t(G) = k$, it follows that $\gamma_t(G+uv) \leq |D_{uv}^c| < k = \gamma_t(G)$. Therefore $G \in \mathbb{T}_k^e$, and thus $\mathbb{C}_k^e \subseteq \mathbb{T}_k^e$. To prove the theorem, it suffices to establish a graph G in the class $\mathbb{C}_k^e$. We distinguish two cases according to the parity of k.

Case 1 : k is even.

Let $k = 2m$ for some positive integer $m \geq 3$. For $1 \leq i \leq k$, let K_{n_i} be a copy of a complete graph of order $n_i \geq 1$ and K_k a copy of a complete graph of order k where $V(K_k) = \{x_1, x_2, ...x_k\}$. We obtain a graph G in the class $\mathbb{T}_k^e \cap \mathbb{C}_k^e$ from $K_{n_1}, K_{n_2}, ..., K_{n_k}$ and K_k by adding edges according to the following join operations :

- $K_{n_{2i}} \vee K_{n_{2i-1}}$,
- $x_{2i} \vee (K_{n_{2i-1}} - u_{2i-1})$ and $x_{2i-1} \vee (K_{n_{2i}} - v_{2i})$ for some $u_{2i-1} \in V(K_{n_{2i-1}})$ and $v_{2i} \in V(K_{n_{2i}})$

for $1 \leq i \leq m$. Moreover, for $1 \leq j \leq 2m$,

- $x_j \vee K_{n_j}$.

The following figure illustrates the graph G.

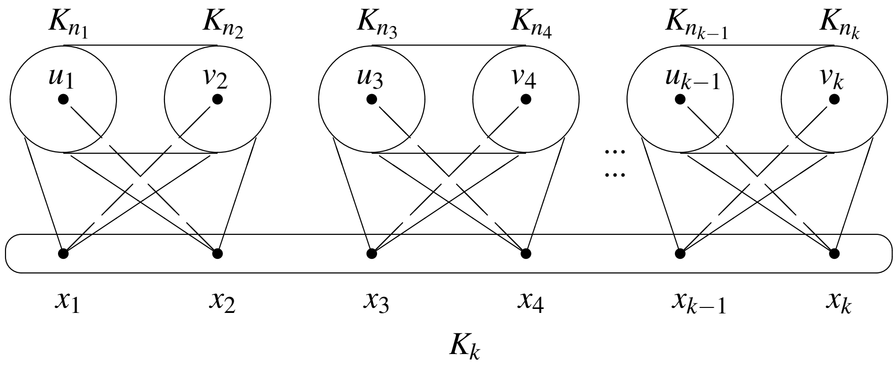

Figure 3.2(a): $G \in \mathbb{T}_k^e \cap \mathbb{C}_k^e$, k is even

We next show that a graph $G \in \mathbb{C}_k^e$. Therefore $\{x_1, x_2, ..., x_k\} \succ_c G$. Thus $\gamma_t(G) \leq \gamma_c(G) \leq k$. By the construction, we need at least two vertices to totally dominate $K_{n_{2i}} \cup K_{n_{2i-1}}$ for $1 \leq i \leq m$. It follows that $\gamma_t(G) \geq k$. Therefore $k \leq \gamma_t(G) \leq \gamma_c(G) \leq k$. Hence, $\gamma_c(G) = \gamma_t(G) = k$.

For establishing the criticality, we consider $G + uv$ where $uv \notin E(G)$. If $\{u, v\} = \{x_{2i}, u_{2i-1}\}$, then $D_{uv}^c = \{x_i | i = 1, 2, ..., k\} - \{x_{2i-1}\}$. Similarly, if $\{u, v\} = \{x_{2i-1}, v_{2i}\}$, then $D_{uv}^c = \{x_i | i = 1, 2, ..., k\} - \{x_{2i}\}$. If $\{u, v\} = \{x_{2i}, q\}$ when q is any vertex in $K_{n_{2j-1}}$ or $K_{n_{2j}}$ for $1 \leq i \neq j \leq m$, then $D_{uv}^c = (\{x_i | i = 1, 2, ..., k\} \cup \{q\}) - \{x_{2j}, x_{2j-1}\}$. We can show that $\gamma_c(G) < k$ when $\{u, v\} = \{x_{2i-1}, q\}$ such that q is a vertex in $K_{n_{2j-1}}$ or $K_{n_{2j}}$ for $1 \leq i \neq j \leq m$ by a similar argument. Further, if $\{u, v\} = \{p, q\}$ when $p \in V(K_{n_{2i}})$ and $q \in V(K_{n_{2j}})$ for $1 \leq i \neq j \leq m$, we have $D_{uv}^c = (\{x_i | i = 1, 2, ..., k\} \cup \{p, q\}) - \{x_{2i-1}, x_{2j}, x_{2j-1}\}$. Moreover, when $p \in V(K_{n_{2i}})$ and $q \in V(K_{n_{2j-1}})$ or $p \in V(K_{n_{2i-1}})$ and $q \in V(K_{n_{2j}})$ or $p \in V(K_{n_{2i-1}})$ and $q \in V(K_{n_{2j-1}})$ for $1 \leq i \neq j \leq m$, we can prove the criticality by similar arguments. Therefore $G \in \mathbb{C}_k^e$.

Case 2 : k is odd.

Let $k = 2m + 1$ for some positive integer $m \geq 2$. For $1 \leq i \leq k - 1$, let K_{n_i} be a copy of a complete graph of order $n_i \geq 1$, $K_{n_k} = K_1$ and K_k a copy of a complete graph of order k such that $V(K_k) = \{x_1, x_2, ...x_k\}$. We obtain a graph G in the class $\mathbb{T}_k^e \cap \mathbb{C}_k^e$ from K_{n_i} for $1 \leq i \leq k$ and K_k by adding edges according to the following join operations :

- $K_{n_{2i}} \vee K_{n_{2i-1}}$,
- $x_{2i} \vee (K_{n_{2i-1}} - u_{2i-1})$ and $x_{2i-1} \vee (K_{n_{2i}} - v_{2i})$ for some $u_{2i-1} \in V(K_{n_{2i-1}})$ and $v_{2i} \in V(K_{n_{2i}})$

for $1 \leq i \leq m$. Moreover, for $1 \leq j \leq 2m + 1$,

- $x_j \vee K_{n_j}$.

The following figure illustrates the graph G. It is worth noting that, in these two constructions of Cases 1 and 2, the graphs $G \in \mathbb{T}_k^e \cap \mathbb{C}_k^e$ when $n_i = 1$ for $1 \leq i \leq k$ were found earlier by Henning and van der Merwe [101].

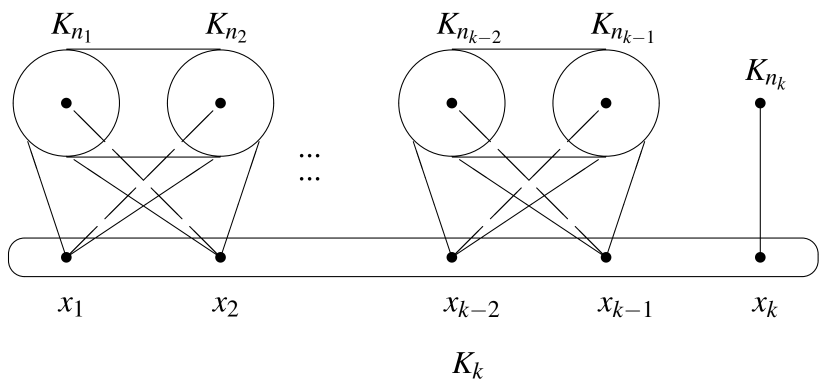

Figure 3.2(b) : $G \in \mathbb{T}_k^e \cap \mathbb{C}_k^e$, k is odd

We can show that $\gamma_c(G) = k$ by similar arguments as in Case 1. We then show the criticality of G. Let $\{a\} = V(K_{n_k})$. Consider $G + uv$ where $uv \notin E(G)$. If $\{u, v\} \subseteq \cup_{i=1}^{k-1}(V(K_{n_i}) \cup \{x_i\})$, we then establish the criticality by similar arguments as k is even. We now consider when $\{u, v\} \cap \{a, x_k\} \neq \emptyset$. If $\{u, v\} = \{x_k, p\}$ for some $p \in V(K_{n_{2i}})$ or $p \in V(K_{n_{2i-1}}), i = 1, 2, ..., m$, then $D_{uv}^c = (\{x_i | i = 1, 2, ..., k\} \cup \{p\}) - \{x_{2i}, x_{2i-1}\}$. If $\{u, v\} = \{a, p\}$ for some $p \in V(K_{n_{2i}})$ or $p \in V(K_{n_{2i-1}}), i = 1, 2, ..., m$, then $D_{uv}^c = (\{x_i | i = 1, 2, ..., k\} \cup \{p\}) - \{x_{2i-1}, x_k\}$ or $D_{uv}^c = (\{x_i | i = 1, 2, ..., k\} \cup \{p\}) - \{x_{2i}, x_k\}$, respectively. Finally, if $\{u, v\} = \{a, x_i\}$ for $1 \leq i \leq k-1$, then $D_{uv}^c = \{x_i | i = 1, 2, ..., k-1\}$. In either case, $\gamma_c(G + uv) < k$. Therefore, $G \in \mathbb{C}_k^e$ and this completes the proof of Theorem 3.1.4. □

In the following, we show that the class of 2−connected $k - \gamma_t-$vertex critical graphs and the class of 2−connected $k - \gamma_c-$vertex critical graphs are the same if and only if $3 \leq k \leq 4$. We first give the following theorem.

Theorem 3.1.5. Let G be a 2−connected graph. Then G is a $4 - \gamma_t-$vertex critical graph if and only if G is a $4 - \gamma_c-$vertex critical graph.

Proof. Note that for any $v \in V(G)$, v is not a support vertex and $G - v$ is connected since G is 2−connected. Let G be a $4 - \gamma_c-$vertex critical graph. Thus $\gamma_t(G) \leq \gamma_c(G) = 4$. If $\gamma_t(G) < 4$, then there exists a γ_t-set D^t of size less than 4 of G. Therefore $G[D^t]$ is connected by Proposition 2.1.7. Thus $D^t \succ_c G$ and we have $\gamma_c(G) \leq 3$, a contradiction. Hence, $\gamma_t(G) = 4$.

We next show that $\gamma_t(G-v) < \gamma_t(G)$ for a vertex v of $V(G)$. Hence $\gamma_t(G-v) \leq \gamma_c(G-v) = 3$ by Lemma 2.1.22(2). Thus $\gamma_t(G-v) < \gamma_t(G)$ as required.

Conversely, suppose G is $4-\gamma_t-$vertex critical. We first show that $\gamma_c(G) = 4$. Let $v \in V(G)$. Consider $G-v$. Lemma 2.1.8(2) yields that $|D^t_v| = 3$. Proposition 2.1.7 gives also that $(G-v)[D^t_v]$ is connected. Therefore $D^t_v \succ_c G-v$. Lemma 2.1.8(1) thus implies there is no vertex of D^t_v adjacent to v. Since G is connected, there exists $w \in V(G) - D^t_v$ such that $vw \in E(G)$ and w is adjacent to at least one vertex of D^t_v. Thus $D^t_v \cup \{w\}$ is a γ_c-set of size 4 of G. We now have $\gamma_c(G) \leq 4$. Suppose there exists D^c which is a γ_c-set of size less than 4. Since $G[D^c]$ is connected, there is no isolated vertex in $G[D^c]$. Thus $D^c \succ_t G$ and so, $\gamma_t(G) \leq |D^c| < 4 = \gamma_t(G)$, a contradiction. Thus $\gamma_c(G) \geq 4$ and these imply that $\gamma_c(G) = 4$. In the proof of criticality, since $|D^t_v| = 3$, $(G-v)[D^t_v]$ is connected. Hence D^t_v becomes a connected dominating set of $G-v$. That is $\gamma_c(G-v) \leq |D^t_v| = 3 < 4 = \gamma_c(G)$ and this completes the proof of our theorem. □

Let

$\mathbb{T}^v_k$: class of 2−connected $k-\gamma_t-$vertex critical graphs G with $G \in \mathbb{G}_k$ and,
$\mathbb{C}^v_k$: class of 2−connected $k-\gamma_c-$vertex critical graphs G with $G \in \mathbb{G}_k$.

As a consequence of Theorem 3.1.5, we have $\mathbb{T}^v_4 = \mathbb{C}^v_4$. However, we next show that $\mathbb{T}^v_k$ and $\mathbb{C}^v_k$ are different when $k \geq 5$.

Theorem 3.1.6. $\mathbb{T}^v_k \neq \mathbb{C}^v_k$ when $k \geq 5$.

Proof. We prove this theorem by giving a construction of a graph G such that $G \in \mathbb{T}^v_k$ but $G \notin \mathbb{C}^v_k$ when $k \geq 5$. We distinguish two cases according to the parity of k.

Case 1 : k is even.

Let $k = 2m+2$ where $m \geq 2$. Let $P^i = a^i_1, a^i_2, a^i_3, a^i_4$ for $1 \leq i \leq m$. Let $V(G) = \cup^m_{i=1} V(P^i) \cup \{x,y\}$ and $E(G) = \{xy\} \cup \{xa^i_1 | 1 \leq i \leq m\} \cup \{ya^i_4 | 1 \leq i \leq m\}$ (see Figure 3.3(a)).

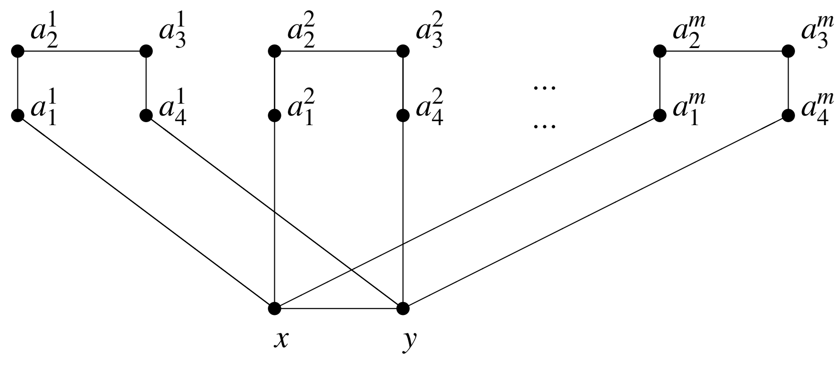

Figure 3.3(a) : $G \in \mathbb{T}^v_k - \mathbb{C}^v_k$, k is even

So $\{x,y\} \cup \{a_1^i, a_4^i | 1 \le i \le m\} \succ_c G$. Thus $\gamma_c(G) \le 2m+2$. Since a γ_c-set of G is also a total dominating set of G, $\gamma_t(G) \le \gamma_c(G) \le 2m+2$. To show that $\gamma_t(G) = \gamma_c(G) = 2m+2$, we need only show that $2m+2 \le \gamma_t(G)$. Let D^t be a γ_t-set of G. We next establish the following claim.

Claim 1 : For $1 \le i \le m$, $|D^t \cap V(P^i)| \ge 2$.

Suppose first that $a_2^i \in D^t$. Thus $a_3^i \in D^t$ or $a_1^i \in D^t$. It follows that $a_3^i, a_2^i \in D^t$ or $a_1^i, a_2^i \in D^t$. We then suppose that $a_2^i \notin D^t$. If $a_3^i \in D^t$, then $a_4^i \in D^t$. Finally, consider when $a_3^i \notin D^t$. Therefore $a_1^i, a_4^i \in D^t$ to dominate a_2^i, a_3^i and we settle Claim 1.

Consider the case $\{x,y\} \subseteq D^t$. By Claim 1, $|D^t| \ge 2m+2$.

We now consider the case when $|\{x,y\} \cap D^t| = 1$. Without loss of generality, assume that $\{x,y\} \cap D^t = \{x\}$. Since $x \in D^t$, x is adjacent to some vertex in D^t. Thus $a_1^i \in D^t$ for some $i \in \{1,...,m\}$. Without loss of generality, let $a_1^1 \in D^t$. Suppose that $a_4^1 \notin D^t$. Since $D^t \succ_t a_4^1$ and $y \notin D^t$, $a_3^1 \in D^t$. Because $a_3^1 \in D^t$ and $a_4^1 \notin D^t$, it follows that $a_2^1 \in D^t$. Hence $\{x, a_1^1, a_2^1, a_3^1\} \subseteq D^t$. Claim 1 gives that $|D^t \cap V(P^i)| \ge 2$ for $2 \le i \le m$. So $|D^t| \ge 2(m-1)+4 = 2m+2$. We then suppose that $a_4^1 \in D^t$. Since $y \notin D^t$, $a_3^1 \in D^t$. Hence, $\{x, a_1^1, a_4^1, a_3^1\} \subseteq D^t$. Similarly, $|D^t| \ge 2(m-1)+4 = 2m+2$.

We finally consider the case when $\{x,y\} \cap D^t = \emptyset$. Since $D^t \succ_t \{x,y\}$, $a_1^i, a_4^j \in D^t$ for some $i,j \in \{1,...,m\}$. Suppose first that $i = j$. Without loss of generality, let $i = j = 1$. Since $x,y \notin D^t$, $a_1^1, a_4^1 \in D^t$ and $a_1^1 a_4^1 \notin E(G)$, it follows that $a_2^1, a_3^1 \in D^t$ and thus $\{a_1^1, a_2^1, a_3^1, a_4^1\} \subseteq D^t$. Claim 1 yields that $|V(P^i) \cap D^t| \ge 2$ for $2 \le i \le m$. So $|D^t| \ge 2(m-1)+4 = 2m+2$. We now consider $j \ne i$. Without loss of generality, let $i = 1, j = 2$. Since $\{x,y\} \cap D^t = \emptyset$ and $a_1^1, a_4^2 \in D^t$, it follows that we need at least three vertices in $D^t \cap V(P^l)$ to totally dominate P^l for $l \in \{1,2\}$. Claim 1 thus implies $|D^t| \ge 2(m-2)+3+3 = 2m+2$. Hence, $2m+2 \le \gamma_t(G) \le \gamma_c(G) \le 2m+2$ and we have that $\gamma_t(G) = \gamma_c(G) = 2m+2$. We show that $\gamma_t(G-v) < \gamma_t(G)$ for a vertex v of G. Consider $G-v$. We have to show that $|D_v^t| = 2m+1$. Suppose first that $v = a_1^i$. Thus $D_v^t = \{a_3^i, a_4^i, y\} \cup \{a_2^j, a_3^j | 1 \le i \ne j \le m\}$ and $|D^t| = 2(m-1)+3 = 2m+1$. We then suppose that $v = a_2^i$. Thus $D_v^t = \{x, y, a_4^i\} \cup \{a_2^j, a_3^j | 1 \le j \ne i \le m\}$ and $|D_v^t| = 2(m-1)+3 = 2m+1$. When $v = x$, we have $D_v^t = \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^i, a_3^i | 2 \le i \le m\}$ and $|D_v^t| = 2(m-1)+3 = 2m+1$. We can prove the criticality when $v = a_4^i, v = a_3^i$ and $v = y$ where $i \in \{1,...,m\}$ by the same arguments as when $v = a_1^i, v = a_2^i$ and $v = x$, respectively. Hence, $G \in \mathbb{T}_k^v$. Consider $G-x$. By Lemma 2.1.22(1), $y \notin D_x^c$. It follows that $(G-x)[D_x^c]$ is not connected. Therefore the graph G is not a $k-\gamma_c-$vertex critical. So $G \notin \mathbb{C}_k^v$.

Case 2 : k is odd.

Let $k = 2m+1$ when $m \ge 2$. Let $P^i = a_1^i, a_2^i, a_3^i, a_4^i$ for $2 \le i \le m$ and $P^1 = a_1^1, a_2^1, a_3^1$.

Let $V(G) = \cup_{i=1}^{m} V(P^i) \cup \{x,y\}$ and $E(G) = \{xy, a_3^1 y\} \cup \{xa_1^i | 1 \leq i \leq m\} \cup \{ya_4^i | 2 \leq i \leq m\}$ (see Figure 3.3(b)).

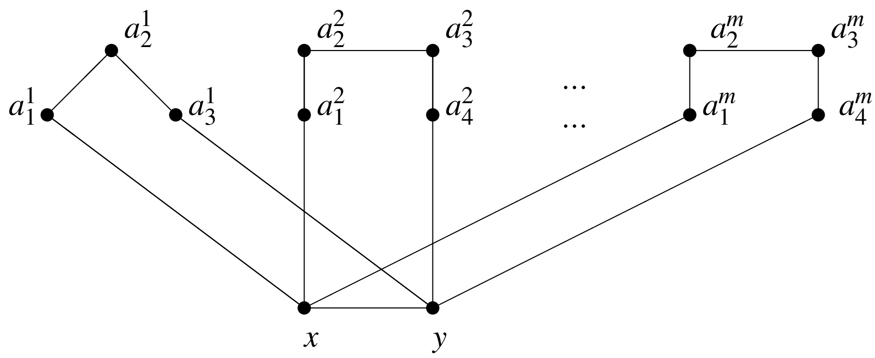

Figure 3.3(b) : $G \in \mathbb{T}_k^v - \mathbb{C}_k^v$, k is odd

We see that $\{x,y,a_1^1\} \cup \{a_1^i, a_4^i | 2 \leq i \leq m\} \succ_c G$. Thus $\gamma_c(G) \leq 2(m-1)+3 = 2m+1$. To show that $\gamma_t(G) = \gamma_c(G) = 2m+1$, we need only show that $\gamma_t(G) \geq 2m+1$. Let D^t be a γ_t-set of G. We establish the following claim.

Claim 2 : For $2 \leq i \leq m$, $|D^t \cap V(P^i)| \geq 2$.

By applying the same arguments as in the proof of Claim 1, $|D^t \cap V(P^i)| \geq 2$ for all i such that $|V(P^i)| = 4$.

We consider the case when $\{x,y\} \subseteq D^t$. To dominate $a_2^1, a_1^1 \in D^t$ or $a_3^1 \in D^t$. Hence, $\{a_1^1, x, y\} \subseteq D^t$ or $\{a_3^1, x, y\} \subseteq D^t$. By Claim 2, $|D^t \cap V(P^i)| \geq 2$ for $2 \leq i \leq m$. So $|D^t| \geq 2(m-1)+3 = 2m+1$.

We consider the case when $|\{x,y\} \cap D^t| = 1$. Without loss of generality, assume that $\{x,y\} \cap D^t = \{x\}$. Since $x \in D^t$ and $y \notin D^t$, it follows that $a_1^i \in D^t$ for some $i \in \{1,...,m\}$. Suppose that $i > 1$, without loss of generality $i = 2$. Thus $a_1^2 \in D^t$. Since $y \notin D^t$ and $D^t \succ_t P^1$, it follows that $|D^t \cap V(P^1)| \geq 2$. Because $D^t \succ_t a_4^2$, $\{x, a_1^2, a_2^2, a_3^2\} \subseteq D^t$ when $a_4^2 \notin D^t$ and $\{x, a_1^2, a_3^2, a_4^2\} \subseteq D^t$ when $a_4^2 \in D^t$. Claim 2 thus implies $\gamma_t(G) = |D^t| \geq 2(m-2)+2+4 = 2m+2 > 2m+1 = \gamma_c(G)$, a contradiction. We then suppose that $i = 1$. Since $y \notin D^t$, $D^t \succ_t a_3^1$ and $a_1^1 a_3^1 \notin E(G)$, it follows that $|D^t \cap V(P^1)| \geq 2$. Claim 2 yields that $|D^t \cap V(P^j)| \geq 2$ for $j \in \{2,...,m\}$. Therefore $|D^t| \geq 2(m-1)+2+1 = 2m+1$.

We consider the case when $\{x,y\} \cap D^t = \emptyset$. To dominate $\{x,y\}$, $\{a_1^i, a_3^1\} \subseteq D^t$ or $\{a_1^i, a_4^j\} \subseteq D^t$ for some $1 \leq i \leq m, 2 \leq j \leq m$.

Suppose that $\{a_1^i, a_4^j\} \subseteq D^t$ for some $1 \leq i \leq m, 2 \leq j \leq m$. Since $x,y \notin D^t$, $|D^t \cap$

$V(P^1)| \geq 2$. We may assume that $i > 1$. If $i \neq j$, then $|D^t \cap V(P^i)| = |D^t \cap V(P^j)| = 3$ to dominate a_4^i and a_1^j because $x, y \notin D^t$. Claim 2 implies that $\gamma_t(G) = |D^t| \geq 2(m-3)+3+3+2 = 2m+2 > 2m+1 = \gamma_c(G)$, a contradiction. Hence, $i = j$. Since $a_1^i, a_4^i \in D^t, x, y \notin D^t$ and $a_1^i a_4^i \notin E(G)$, it follows that $a_2^i, a_3^i \in D^t$. Claim 2 thus implies $\gamma_t(G) = |D^t| \geq 2(m-2)+2+4 = 2m+2 > 2m+1 = \gamma_c(G)$, again a contradiction. We now assume that $i = 1$. Hence $\{a_1^1, a_2^1\} \subseteq D^t$ and $\{a_2^j, a_3^j, a_4^j\} \subseteq D^t$ to totally dominate a_1^j. Thus $|D^t| \geq 2(m-2)+2+3 = 2m+1$.

We now suppose that $\{a_1^i, a_3^1\} \subseteq D^t$ for some $1 \leq i \leq m$. If $i = 1$, then $D^t \cap V(P^1) = \{a_1^1, a_2^1, a_3^1\}$ because $a_1^1 a_3^1 \notin E(G)$. Claim 2 then implies that $|D^t| \geq 2(m-1)+3 = 2m+1$. If $i > 1$, without loss of generality let $i = 2$, then $a_2^1 \in D^t$ because $a_3^1 \in D^t$ and $y \notin D^t$. Since $a_1^2 \in D^t$ and $x, y \notin D^t$, it follows that $|D^t \cap V(P^2)| = 3$ to totally dominate a_4^2. Claim 2 yields that $|D^t| \geq 2(m-2)+2+3 = 2m+1$. Hence, $2m+1 \leq \gamma_t(G) \leq \gamma_c(G) \leq 2m+1$. Therefore, $\gamma_t(G) = \gamma_c(G) = 2m+1$.

We show that $\gamma_t(G-v) < \gamma_t(G)$ for a vertex v of G. Consider $G-v$. We have to show that $|D_v^t| = 2m$. Suppose first that $v = x$, then $D_v^t = \{a_2^i, a_3^i | 2 \leq i \leq m\} \cup \{a_2^1, a_3^1\}$ and $|D_v^t| = 2(m-1)+2 = 2m$. Similarly, $|D_y^t| = 2m$. We then suppose $v = a_1^1$. Thus $D_v^t = \{a_2^i, a_3^i | 2 \leq i \leq m\} \cup \{a_3^1, y\}$ and $|D_v^t| = 2(m-1)+2 = 2m$. We also show that $|D_{a_3^1}^t| = 2m$ by a similar argument as $v = a_1^1$. If $v = a_2^1$, then $D_v^t = \{a_2^i, a_3^i | 2 \leq i \leq m\} \cup \{x, y\}$ and $|D_v^t| = 2(m-1)+2 = 2m$. If $v = a_1^i$ for $2 \leq i \leq m$, then $D_v^t = \{a_2^j, a_3^j | 2 \leq j \neq i \leq m\} \cup \{a_3^i, a_4^i\} \cup \{a_1^1, a_2^1\}$. It follows that $|D_v^t| = 2(m-2)+2+2 = 2m$. Further, if $v = a_4^i$ for $2 \leq i \leq m$, then $D_v^t = \{a_2^j, a_3^j | 2 \leq j \neq i \leq m\} \cup \{a_1^i, a_2^i\} \cup \{a_3^1, a_2^1\}$. It follows that $|D_v^t| = 2(m-2)+2+2 = 2m$. If $v = a_2^i$ for $2 \leq i \leq m$, then $D_v^t = \{a_2^j, a_3^j | 2 \leq j \neq i \leq m\} \cup \{a_1^1, a_4^i, x, y\}$. It follows that $|D_v^t| = 2(m-2)+4 = 2m$. Finally, if $v = a_3^i$ for $2 \leq i \leq m$, then $D_v^t = \{a_2^j, a_3^j | 2 \leq j \neq i \leq m\} \cup \{a_1^1, a_1^i, x, y\}$. It also follows that $|D_v^t| = 2(m-2)+4 = 2m$. Hence, $G \in \mathbb{T}_k^v$.

We can show that G is not a $k-\gamma_c-$vertex critical graph by the same arguments as in Case 1. Hence, $G \notin \mathbb{C}_k^v$ and this completes the proof of our theorem. □

Goddard et al. [71] mentioned that K_2 is a $2-\gamma_t-$vertex critical graph while Ananchuen et al. [7] claimed that a $2-\gamma_c-$vertex critical graph is K_{2n} with a perfect matching deleted where $n \geq 2$. Thus $\mathbb{T}_2^v \neq \mathbb{C}_2^v$. Ananchuen et al. [7] also pointed out that $\mathbb{T}_3^v = \mathbb{C}_3^v$. By Theorems 3.1.5 and 3.1.6, we can conclude the following corollary.

Corollary 3.1.7. $\mathbb{T}_k^v = \mathbb{C}_k^v$ if and only if $3 \leq k \leq 4$.

3.2 $\gamma-$Critical Graphs and $i-$Critical Graphs

We first note that an $i-$set is a dominating set. By the minimality of $\gamma(G)$, we must have $\gamma(G) \leq i(G)$. We note, further, by Theorem 2.1.1 that $\gamma(G) = i(G)$ if a graph G is claw-free. Although we might obtain a claw as an induced subgraph after adding any single edge to a claw-free graph, we can show that in the class of claw-free graphs $\gamma-$edge critical graphs and $i-$edge critical graphs are the same.

Theorem 3.2.1. Let G be a claw-free graph. Then G is a $k-\gamma-$edge critical graph if and only if G is a $k-i-$edge critical graph.

Proof. Let G be in the class of claw-free graphs. Suppose first that G is a $k-\gamma-$edge critical graph. By Theorem 2.1.1, $i(G) = k$. Let u, v be any non-adjacent vertices of G. Consider $G+uv$. Lemma 2.1.3(1) implies that a graph $G+uv$ has a $\gamma-$set of size $k-1$. Choose D_{uv} to be a $\gamma-$set of $G+uv$ such that $(G+uv)[D_{uv}]$ contains the minimum number of edges. Lemma 2.1.3(2) implies that $|D_{uv} \cap \{u,v\}| = 1$. Without loss of generality, let $u \in D_{uv}$. Suppose that $(G+uv)[D_{uv}]$ contains an edge xy. We can choose a vertex $z \in \{x,y\}$ which is not a vertex u. Let $Z = PN(z, D_{uv})$. By the minimality of D_{uv}, $Z \neq \emptyset$. Let $w \in \{x,y\} - \{z\}$. Since $N_Z(w) = \emptyset$, it follows by claw-freeness of G that $(G+uv)[Z]$ is complete. So $z' \succ Z$ and $N_{D_{uv}}(z') = \{z\}$ for all $z' \in Z$. Let $D'_{uv} = (D_{uv} - z) \cup \{z'\}$. Hence $D'_{uv} \succ G+uv$ but $(G+uv)[D'_{uv}]$ contains less edges than $(G+uv)[D_{uv}]$, contradicting the choice of D_{uv}. Thus D_{uv} is an independent set. Therefore $D_{uv} \succ_i G+uv$. This implies that $i(G+uv) \leq |D_{uv}| < k$ and G is a $k-i-$edge critical graph.

Conversely, let G be a $k-i-$edge critical graph. By Theorem 2.1.1, $\gamma(G) = k$. Let u, v be any non-adjacent vertices of G. Consider $G+uv$. By the criticality of G, $i(G+uv) < k$. Because $\gamma(G+uv) \leq i(G+uv)$, G is a $k-\gamma-$edge critical graph as required. □

By using the same idea as Theorem 3.2.1, we also have the analogous result on vertex critical graphs. We conclude this section with the following theorem.

Theorem 3.2.2. Let G be a claw-free graph. Then G is a $k-\gamma-$vertex critical graph if and only if G is a $k-i-$vertex critical graph.

Proof. Let G be in the class of claw-free graph. Suppose first that G is a $k-\gamma-$vertex critical graph. Theorem 2.1.1 implies that $i(G) = k$. Let v be a vertex of G. Consider $G-v$. By Observation 2.1.4, a graph $G-v$ has a $\gamma-$set of size $k-1$. Choose D_v to be a $\gamma-$set of $G-v$ such that $(G-v)[D_{vv}]$ contains the minimum number of edges. Suppose that $(G-v)[D_v]$ contains an edge xy. Let $X = PN(x, D_v)$. By the minimality of D_v, $X \neq \emptyset$. Since $N_X(y) = \emptyset$, it follows by claw-freeness of G that $(G-v)[X]$ is complete. So $x' \succ X$ and $N_{D_v}(x') = \{x\}$ for all $x' \in X$. Let $D'_v = (D_v - x) \cup \{x'\}$. Hence $D'_v \succ G-v$ but $(G-v)[D'_v]$ contains less edges than $(G-v)[D_v]$, contradicting the choice of D_v. Thus D_v is an independent set. Therefore $D_v \succ_i G-v$. This implies that $i(G-v) \leq |D_v| < k$ and G is a $k-i-$vertex critical graph.

Conversely, let G be a $k-i-$vertex critical graph. Theorem 2.1.1 gives that $\gamma(G) = k$. Let v any vertex of G. Consider $G-v$. By the criticality of G, $i(G-v) < k$. Because $\gamma(G-v) \leq i(G-v)$, G is a $k-\gamma-$vertex critical graph as required. □

Van der Merwe et al. [143] established the existence of $3-\gamma-$edge critical graphs for arbitrary $i(G) \geq 3$. Moreover, Ao et al. [20] showed that, for $k \geq 4$, there exists a connected $k-\gamma-$edge critical graph G with $i(G) > k$. Thus the claw-free condition is necessary to prove Theorems 3.2.1 and 3.2.2.

CHAPTER 4

Bounds on the Order of Connected Domination Vertex Critical Graphs

This chapter focuses on the order of $k-\gamma_c-$vertex critical graphs. The study on $k-\gamma_c-$vertex critical graphs was initially started by Ananchuen et al. [7] and [8]. They characterized some $3-\gamma_c-$vertex critical graphs with respect to connectivity and introduced three new infinite families of $3-\gamma_c-$vertex critical graphs. They also studied a matching property of these graphs. For a $3-\gamma_c-$vertex critical graph G of even order, they proved that if G is $K_{1,7}-$free, then G contains a perfect matching (see also Henning and Yeo [105]). Moreover, if G is $K_{1,4}-$free or $K_{1,5}-$free and $5-$connected, then G is bi-critical. For a $3-\gamma_c-$vertex critical graph G of odd order, if G is $K_{1,6}-$free, then G is $1-$factor critical (see also Henning and Yeo [105]). Moreover, if G is $K_{1,3}-$free and $\delta \geq 4$, then G is $3-$factor critical. For surveys of the results in $k-\gamma_c-$vertex critical graphs, see [4] and [130]. The objective of this chapter is to investigate the order of $k-\gamma_c-$vertex critical graphs.

4.1 Introduction

In this section we detail some related results and our main theorems. On $k-\gamma-$vertex critical graphs, Brigham et al. [28] established the upper bound of the order of $k-\gamma-$vertex critical graphs, namely, $n \leq (\Delta+1)(k-1)+1$. They further conjectured that every $k-\gamma-$vertex critical graph on $(\Delta+1)(k-1)+1$ vertices is $\Delta-$regular. This conjectured was proved by Fulman et al. [69].

Theorem 4.1.1. [69] If G is a $k-\gamma-$vertex critical graph of order $(\Delta+1)(k-1)+1$, then it is $\Delta-$regular.

On $k-\gamma_t-$vertex critical graphs, Goddard et al. [71] pointed out that the order of $k-\gamma_t-$vertex critical graphs satisfies $\Delta+k \leq n \leq \Delta(k-1)+1$. They further raised the question of characterizing the $k-\gamma_t-$vertex critical graphs that achieve the upper and lower bounds. Wang et al. [146] and Mojdeh and Rad [127] independently proved the following theorem.

Theorem 4.1.2. ([146] and [127]) If G is a $k-\gamma_t-$vertex critical graph of order $\Delta(k-1)+1$, then it is $\Delta-$regular.

For the lower bound, there have been a number of papers on the so called *existence problem*. These problems investigated the existence of $k-\gamma_t-$vertex critical graphs of order $\Delta+k$ according to the parities of Δ and k. For $k=3$, Mojdeh and Rad [126] claimed that there is a $3-\gamma_t-$vertex critical graph of order $\Delta+3$ for all even Δ but there is no such graph when $\Delta=3$ or 5. Chen and Sohn [54] and Wang et al. [150] independently proved that there is no $3-\gamma_t-$vertex critical graph of order $\Delta+3$ when $\Delta=7$ and $\delta\geq 2$ and, further, provided a class of $3-\gamma_t-$vertex critical graphs of order $\Delta+3$ for odd $\Delta\geq 9$ and $\delta\geq 2$. Therefore, they obtained that :

Theorem 4.1.3. ([54] and [150]) For Δ and $\delta\geq 2$, there is a $3-\gamma_t-$vertex critical graph of order $\Delta+3$ if and only if $\Delta\neq 3,5$ and 7.

For $k=4$, Hassankhani and Rad [78] proved that there is no $4-\gamma_t-$vertex critical graph of order $\Delta+4$ when $\Delta=3$ or 5. Sohn et al. [137], further, proved that such graphs do not exists for $\Delta=7$. They provided classes of $4-\gamma_t-$vertex critical graphs of order $\Delta+4$ for all even Δ and odd $\Delta\geq 9$. That is, they proved that :

Theorem 4.1.4. [137] For Δ and $\delta\geq 2$, there is a $4-\gamma_t-$vertex critical graph of order $\Delta+4$ if and only if $\Delta\neq 3,5$ and 7.

In this chapter, on $k-\gamma_c-$vertex critical graphs, we establish the upper and lower bounds of $k-\gamma_c-$vertex critical graphs in terms of Δ and k. More specifically, we prove that :

Theorem 4.1.5. Let G be a $k-\gamma_c-$vertex critical graph of order n and $k\geq 2$. Then

$$\Delta+k\leq n\leq(\Delta-1)(k-1)+3$$

and the upper bound is sharp for all $k\geq 2$ when Δ is even.

In view of Theorems 4.1.1 and 4.1.2, we naturally come up with the question whether every $k-\gamma_c-$vertex critical graph achieving the upper bound is $\Delta-$regular. Interestingly, it turns out that these graphs are $\Delta-$regular for $k=2,3$ or 4 but do not need to be $\Delta-$regular for $k=5$ or 6 when $\Delta=3$. We see that $(\Delta-1)(k-1)+3<\Delta(k-1)+1$ for $k=4$. By Theorems 3.1.5 and 4.1.5, we obtain the sharp upper bound for $4-\gamma_t-$vertex critical graphs when Δ is even which is 3Δ. We, further, prove that :

Theorem 4.1.6. For $\Delta \geq 2$ and $k = 2,3$ or 4, there is a $k-\gamma_c-$vertex critical graph of order $(\Delta-1)(k-1)+3$ if and only if Δ is even.

Theorem 4.1.6 implies that, for $\Delta \geq 2$ and $k = 3$ or 4, there is a $k-\gamma_t-$vertex critical graph of order $(\Delta-1)(k-1)+3$ if and only if Δ is even.

For the lower bound of $\Delta+k$, we easily obtain the existence of $k-\gamma_c-$vertex critical graphs for $k = 3$ and 4 by Theorems 3.1.5, 4.1.3 and 4.1.4. For $k \geq 5$, we prove that :

Theorem 4.1.7. For $k \geq 5$, G is a $k-\gamma_c-$vertex critical graph of order $\Delta+k$ if and only if G is a cycle, C_{k+2}, on $k+2$ vertices.

We finally study the realizability of $k-\gamma_c-$vertex critical graphs of order between $\Delta+k$ and $(\Delta-1)(k-1)+3$ when Δ and k are small.

4.2 The Upper and Lower Bounds of $k-\gamma_c-$Vertex Critical Graphs

For $k \leq 2$, $k-\gamma_c-$vertex critical graphs are characterized by Lemma 2.1.21. So, in this section, we focus on $k \geq 3$. Recall that, for vertex subsets X and Y of $V(G)$, we use $E(X,Y)$ to denote the set of all edges having one end vertex in X and the other one in Y. The set $\overline{X}$ denotes $V(G)-X$ and $E(X,G)$ denotes $E(G[X]) \cup E(X,\overline{X})$, that is, the set of edges having at least one end vertex in X. We recall also that, for a vertex v of G, D_v^c is a γ_c-set of $G-v$. To establish Theorem 4.1.5, we need to prove these following lemmas.

Lemma 4.2.1. Let G be a $k-\gamma_c-$vertex critical graph of order n and v a vertex of G. We further let H be the subgraph of G such that $V(H) = V(G)-\{v\}$ and $E(H) = E(D_v^c, G)$. Then

$$n \leq |E(H)|+2 = |E(D_v^c, G)|+2.$$

Proof. Because $D_v^c \succ_c G-v$, H is connected. Thus $|V(H)|-1 \leq |E(H)| = |E(D_v^c, G)|$. Since $|V(H)| = n-1$,

$$n \leq |E(H)|+2 = |E(D_v^c, G)|+2$$

and this completes the proof of Lemma 4.2.1. □

Lemma 4.2.2. Let G be a $k-\gamma_c-$vertex critical graph and v a vertex of G. Then

$$|E(D_v^c, G)| \leq (\Delta-1)(k-1)+1$$

and the equality holds if every vertex in D_v^c has degree Δ and the induced subgraph $(G-v)[D_v^c]$ contains exactly $k-2$ edges.

Proof. Consider $G-v$ and let $H' = (G-v)[D_v^c]$. Lemma 2.1.22(2) implies that $|D_v^c| = k-1$. Because H' is connected, it follows that $|E(H')| \geq k-2$ edges. Hence, each vertex u in D_v^c is joined to at most $\Delta - deg_{D_v^c}(u)$ vertices in $\overline{D_v^c}$. Therefore

$$deg_{\overline{D_v^c}}(u) \leq \Delta - deg_{D_v^c}(u).$$

Hence

$$|E(D_v^c, \overline{D_v^c})| \leq \Sigma_{u \in D_v^c}(\Delta - deg_{D_v^c}(u)) = \Delta|D_v^c| - \Sigma_{u \in D_v^c} deg_{D_v^c}(u) = \Delta|D_v^c| - 2|E(H')|$$

and so

$$|E(D_v^c, G)| = |E(D_v^c, \overline{D_v^c})| + |E(H')| \leq \Delta|D_v^c| - |E(H')| = (\Delta-1)(k-1)+1$$

because $|D_v^c| = k-1$ and $|E(H')| \leq k-2$. We see that the equality holds when $deg_G(u) = deg_{D_v^c}(u) + deg_{\overline{D_v^c}}(u) = \Delta$ for all $u \in D_v^c$ and $|E(H')| = k-2$. This completes the proof of Lemma 4.2.2. □

The following corollary establishes the lower bound of the connected domination number of a graph when the maximum degree and the order are given.

Corollary 4.2.3. For integer $k \geq 2$, let G be a graph with maximum degree Δ of order $(\Delta-1)(k-1)+3$. Then $\gamma_c(G) \geq k$.

Proof. Suppose there exists a connected dominating set D of order at most $k-1$. We further let H'' be the subgraph of G such that $V(H'') = V(G)$ and $E(H'') = E(D, G)$. So $|E(G[D])| \geq k-2$ and $deg_{\overline{D}}(u) \leq \Delta - deg_D(u)$ for all $u \in D$. By similar arguments in Lemma 4.2.2, we have

$$|E(D, \overline{D})| \leq \Delta|D| - 2|E(G[D])|.$$

Thus

$$|E(D, G)| \leq |E(D, \overline{D})| + |E(G[D])| \leq \Delta|D| - |E(G[D])| \leq (\Delta-1)(k-1)+1.$$

Therefore $|E(H'')| \leq (\Delta-1)(k-1)+1$. Because $D \succ_c G$ and $V(G) = V(H'')$, it follows that H'' is connected. Thus $|V(H'')|-1 \leq |E(H'')|$. These imply that

$$n = |V(H'')| \leq (\Delta-1)(k-1)+2$$

contradicting $n = (\Delta-1)(k-1)+3$. Therefore $\gamma_c(G) \geq k$. □

The following lemma gives the existence of a connected dominating set of some circulant graph. Recall from Chapter 1 that a circulant graph $C_n < a_0, a_1, ..., a_k >$ where $0 < a_0 < a_1 < ... < a_k < \frac{n+1}{2}$ is a graph with vertex set $\{x_0, x_1, ..., x_{n-1}\}$ and edge set $\{x_i x_j | (i-j) \equiv (\pm a_l)(\text{mod } n)$ for some $1 \leq l \leq k\}$.

Lemma 4.2.4. For all positive integers $k \geq 3$ and even $\Delta \geq 4$, let $n = (\Delta-1)(k-1)+3$. Further, let $C_n < a_0, a_1, ..., a_{\frac{\Delta-2}{2}} >$ be a circulant graph where $a_l = 1 + l(k-1)$ for $l \in \{0, 1, ..., \frac{\Delta-2}{2}\}$. Then $\{x_0, x_1, ..., x_{k-2}\} \succ_c G - x_{\frac{\Delta}{2}(k-1)+1}$.

Proof. Since $a_0 = 1$, it follows that every pair of consecutive vertices are adjacent, namely, $x_i x_{i+1} \in E(G)$ for $i = 0, 1, ..., n-2$ and $x_{n-1} x_0 \in E(G)$. In particular, $G[\{x_0, x_1, ..., x_{k-2}\}]$ is connected.

We partition $V(G) - \{x_{\frac{\Delta}{2}(k-1)+1}\}$ into V_1, V_2 and V_3 where

- $V_1 = \{x_{n-1}, x_0, x_1, ..., x_{k-2}, x_{k-1}\}$,
- $V_2 = \{x_k, x_{k+1}, ..., x_{\frac{\Delta}{2}(k-1)}\}$ and
- $V_3 = \{x_{\frac{\Delta}{2}(k-1)+2}, x_{\frac{\Delta}{2}(k-1)+3}, ..., x_{n-2}\}$.

Let $y \in V(G) - \{x_{\frac{\Delta}{2}(k-1)+1}\}$. We will show that y is either in $\{x_0, x_1, ..., x_{k-2}\}$ or is adjacent to a vertex in this set. Suppose first that $y \in V_1$. If $y \in \{x_0, x_1, ..., x_{k-2}\}$, there is nothing to prove. If $y = x_{n-1}$, then $yx_0 \in E(G)$. Moreover, if $y = x_{k-1}$, then $yx_{k-2} \in E(G)$.

We now consider the case when $y \in V_2$. We show that

$$V_2 = \{x_{j+i(k-1)} : 1 \leq j \leq k-1, 1 \leq i \leq \frac{\Delta-2}{2}\}.$$

We can, further, partition V_2 into $\frac{\Delta-2}{2}$ sets of size $k-1$.

$$\begin{aligned}V_2 = &\{x_k, x_{k+1}, ..., x_{2k-2}\} \cup \{x_{1+2(k-1)}, x_{2+2(k-1)}, ..., x_{(k-1)+2(k-1)}\} \cup ...\\ &\{x_{1+i(k-1)}, x_{2+i(k-1)}, ..., x_{(k-1)+i(k-1)}\} \cup ...\\ &\{x_{1+(\frac{\Delta-2}{2})(k-1)}, x_{2+(\frac{\Delta-2}{2})(k-1)}, ..., x_{(k-1)+(\frac{\Delta-2}{2})(k-1)}\}.\end{aligned}$$

Clearly,

$\{x_k, x_{k+1}, ..., x_{2k-2}\} = \{x_{j+(k-1)} : 1 \leq j \leq k-1\}$,

$\{x_{1+2(k-1)}, x_{2+2(k-1)}, ..., x_{(k-1)+2(k-1)}\} = \{x_{j+2(k-1)} : 1 \leq j \leq k-1\}$,

$$\vdots$$

$\{x_{1+i(k-1)}, x_{2+i(k-1)}, ..., x_{(k-1)+i(k-1)}\} = \{x_{j+i(k-1)} : 1 \leq j \leq k-1\}$,

$$\vdots$$

$\{x_{1+(\frac{\Delta-2}{2})(k-1)}, x_{2+(\frac{\Delta-2}{2})(k-1)}, ..., x_{(k-1)+(\frac{\Delta-2}{2})(k-1)}\} = \{x_{j+(\frac{\Delta-2}{2})(k-1)} : 1 \leq j \leq k-1\}$.

Hence

$$V_2 = \{x_{j+i(k-1)} : 1 \leq j \leq k-1, 1 \leq i \leq \frac{\Delta-2}{2}\}.$$

Therefore $y = x_{j+i(k-1)}$ for some $j \in \{1,2,...,k-1\}$ and $i \in \{1,2,...,\frac{\Delta-2}{2}\}$. We show that $yx_{j-1} \in E(G)$. Clearly

$$(j+i(k-1)) - (j-1) = 1+i(k-1).$$

Recall that $a_i = 1+i(k-1)$. Thus $1+i(k-1)-a_i = 0$. Therefore

$$((j+i(k-1))-(j-1)) \equiv (a_i)(modn).$$

Hence $yx_{j-1} \in E(G)$.

We now consider the case when $y \in V_3$. We show that

$$V_3 = \{x_{(n-2)-i(k-1)-j} : 0 \leq j \leq k-2, 0 \leq i \leq \frac{\Delta-4}{2}\}.$$

Clearly, we can also partition V_3 into $\frac{\Delta-2}{2}$ sets of size $k-1$. Note that $(n-2)-$

$(\frac{\Delta-4}{2})(k-1)-(k-2)=\frac{\Delta}{2}(k-1)+2.$

$$\begin{aligned}V_3 =&\{x_{n-2},x_{(n-2)-1},...,x_{(n-2)-(k-2)}\}\cup\\ &\{x_{(n-2)-(k-1)},x_{(n-2)-(k-1)-1},...,x_{(n-2)-(k-1)-(k-2)}\}\cup\\ &\{x_{(n-2)-2(k-1)},x_{(n-2)-2(k-1)-1},...,x_{(n-2)-2(k-1)-(k-2)}\}\cup ...\\ &\{x_{(n-2)-i(k-1)},x_{(n-2)-i(k-1)-1},...,x_{(n-2)-i(k-1)-(k-2)}\}\cup ...\\ &\{x_{(n-2)-(\frac{\Delta-4}{2})(k-1)},x_{(n-2)-(\frac{\Delta-4}{2})(k-1)-1},...,x_{(n-2)-(\frac{\Delta-4}{2})(k-1)-(k-2)}\}.\end{aligned}$$

Since

$$\{x_{n-2},x_{(n-2)-1},...,x_{(n-2)-(k-2)}\}=\{x_{(n-2)-j}:0\le j\le k-2\},$$

$$\begin{aligned}&\{x_{(n-2)-(k-1)},x_{(n-2)-(k-1)-1},...,x_{(n-2)-(k-1)-(k-2)}\}\\ &\qquad=\{x_{(n-2)-(k-1)-j}:0\le j\le k-2\},\end{aligned}$$

$$\begin{aligned}&\{x_{(n-2)-2(k-1)},x_{(n-2)-2(k-1)-1},...,x_{(n-2)-2(k-1)-(k-2)}\}\\ &\qquad=\{x_{(n-2)-2(k-1)-j}:0\le j\le k-2\},\\ &\qquad\vdots\end{aligned}$$

$$\begin{aligned}&\{x_{(n-2)-i(k-1)},x_{(n-2)-i(k-1)-1},...,x_{(n-2)-i(k-1)-(k-2)}\}\\ &\qquad=\{x_{(n-2)-i(k-1)-j}:0\le j\le k-2\},\\ &\qquad\vdots\end{aligned}$$

$$\begin{aligned}&\{x_{(n-2)-(\frac{\Delta-4}{2})(k-1)},x_{(n-2)-(\frac{\Delta-4}{2})(k-1)-1},...,x_{(n-2)-(\frac{\Delta-4}{2})(k-1)-(k-2)}\}\\ &\qquad=\{x_{(n-2)-(\frac{\Delta-4}{2})(k-1)-j}:0\le j\le k-2\},\end{aligned}$$

it follows that

$$V_3=\{x_{(n-2)-i(k-1)-j}:0\le j\le k-2, 0\le i\le \frac{\Delta-4}{2}\}.$$

Thus $y=x_{(n-2)-i(k-1)-j}$ for some $j\in\{0,1,...,k-2\}$ and $i\in\{0,1,...,\frac{\Delta-4}{2}\}$. We show that $yx_{k-2-j}\in E(G)$. Clearly,

$$(n-2)-i(k-1)-j-(k-2-j)=(\Delta-i-2)(k-1)+2$$

because $n=(\Delta-1)(k-1)+3$. Recall that $a_{i+1}=1+(i+1)(k-1)$. Thus

$$(\Delta-i-2)(k-1)+2-(-a_{i+1})=(\Delta-1)(k-1)+3=n.$$

Therefore

$$(n-2)-i(k-1)-j-(k-2-j)\equiv(-a_{i+1})(modn).$$

It follows that $yx_{k-2-j}\in E(G)$ and this completes the proof of Lemma 4.2.4. □

We are now ready to prove Theorem 4.1.5. For convenience, we restate the theorem.

Theorem 4.1.5. Let G be a $k-\gamma_c-$vertex critical graph of order n and $k\geq 2$. Then

$$\Delta+k\leq n\leq(\Delta-1)(k-1)+3$$

and the upper bound is sharp for all $k\geq 2$ when Δ is even.

Proof. Let G be a $k-\gamma_c-$vertex critical graph and v a vertex of G. Moreover, let H be the subgraph of G defined as in Lemma 4.2.1, namely, $V(H)=V(G)-\{v\}$ and $E(H)=E(D_v^c,G)$. By Lemmas 4.2.1 and 4.2.2, we have that

$$n\leq(\Delta-1)(k-1)+3$$

and this proves the upper bound.

To establish the lower bound, let $a\in V(G)$ be such that $deg_G(a)=\Delta$. Lemma 2.1.22(2) implies that $|D_a^c|=k-1$, moreover, Lemma 2.1.22(1) yields that D_a^c and $N[a]$ are disjoint. Thus

$$\Delta+k=(k-1)+(\Delta+1)=|D_a^c|+|N[a]|=|D_a^c\cup N[a]|\leq|V(G)|=n$$

extablishing the lower bound.

It is easy to see that C_n is $(n-2)-\gamma_c-$vertex critical. Moreover C_n satisfies $\Delta=2$ and $\Delta+k=2+(n-2)=n=(2-1)((n-2)-1)+3=(\Delta-1)(k-1)+3$. Hence, the upper and lower bounds are sharp when $\Delta=2$.

For all positive integers $k\geq 3$ and even $\Delta\geq 4$, let $n=(\Delta-1)(k-1)+3$. Further, let $C_n<a_0,a_1,...,a_{\frac{\Delta-2}{2}}>$ be a circulant graph where $a_l=1+l(k-1)$ for $l\in\{0,1,...,\frac{\Delta-2}{2}\}$. We show that $C_n<a_0,a_1,...,a_{\frac{\Delta-2}{2}}>$ is a $k-\gamma_c-$vertex critical graph with maximum degree Δ of order n. Let $G=C_n<a_0,a_1,...,a_{\frac{\Delta-2}{2}}>$ and $V(G)=\{x_0,x_1,...,x_{n-1}\}$.

For $x_i\in\{x_0,x_1,...,x_{n-1}\}$, we have $x_ix_{j_1},x_ix_{j_2}\in E(G)$ for all

$j_1\in J_1=\{i+1,i+1+(k-1),i+1+2(k-1),...,i+1+(\frac{\Delta-2}{2})(k-1)\}$ and

$$j_2 \in J_2 = \{i-1, i-1-(k-1), i-1-2(k-1), ..., i-1-(\tfrac{\Delta-2}{2})(k-1)\}$$

all indices are taken modulo n. Thus $N(x_i) = \{x_{j_1} : j_1 \in J_1\} \cup \{x_{j_2} : j_2 \in J_2\}$. Since $n = (\Delta-1)(k-1)+3 > \Delta+1$, $J_1 \cap J_2 = \emptyset$. Hence

$$deg_G(x_i) = |N(x_i)| = |\{x_{j_1} : j_1 \in J_1\}| + |\{x_{j_2} : j_2 \in J_2\}| = \Delta.$$

We now ready to prove that $\gamma_c(G) = k$. We see that $((\frac{\Delta}{2}(k-1)+1)-(k-1)) \equiv (a_{\frac{\Delta-2}{2}})(\text{mod } n)$. Thus $x_{k-1}x_{\frac{\Delta}{2}(k-1)+1} \in E(G)$. By Lemma 4.2.4, $\{x_0, x_1, ..., x_{k-1}\} \succ_c G$ and hence $\gamma_c(G) \leq k$. Corollary 4.2.3 implies that $\gamma_c(G) = k$.

We finally establish the criticality of G. For all $y \in V(G)$, consider $G-y$. Because G is symmetric, we suppose without loss of generality that $y = x_{\frac{\Delta}{2}(k-1)+1}$. Lemma 4.2.4 implies that $\{x_0, x_1, ..., x_{k-2}\} \succ_c G-y$. Hence $\gamma_c(G-y) < k$.

Thus, for $k \geq 3$ and even $\Delta \geq 4$, there exists a $k-\gamma_c-$vertex critical graph with maximum degree Δ of order n and this completes the proof of Theorem 4.1.5. □

4.3 The $k-\gamma_c-$Vertex Critical Graphs achieving the Upper Bound

Obviously, for $k=2$, every $2-\gamma_c-$vertex critical graph of order n is $(n-2)-$regular by Lemma 2.1.21. For $k=3$, we have that $(\Delta-1)(k-1)+3 = \Delta(k-1)+1$. Then $3-\gamma_c-$vertex critical graphs of order $(\Delta-1)(k-1)+3$ are $\Delta-$regular by Theorem 4.1.2. For $k=4$, we show that every $4-\gamma_c-$vertex critical graph of order $(\Delta-1)(k-1)+3$ is $\Delta-$regular. For $k=5$ or 6, we provide $k-\gamma_c-$vertex critical graphs with $\Delta=3$ of order $2k+1$ which are not $3-$regular.

Lemma 4.3.1. Let G be a $k-\gamma_c-$vertex critical graph of order $n = (\Delta-1)(k-1)+3$. Then

(1) If $u \in D_v^c$, then $deg_G(u) = \Delta$,

(2) for all $x, y \in D_v^c$, x, y are adjacent to different vertices in $V(G)-(D_v^c \cup \{v\})$ and

(3) $(G-v)[D_v^c]$ is a tree.

Proof. (1) Suppose there exists $u \in D_v^c$ such that $deg_G(u) < \Delta$. By Lemma 4.2.2,

$$|E(D_v^c, G)| < (\Delta-1)(k-1)+1.$$

Lemma 4.2.1 thus implies $n \leq |E(D_v^c, G)|+2 < (\Delta-1)(k-1)+3$, a contradiction. This completes the proof of (1).

Before proving (2) and (3), it is worth establishing the following claim. Let H be defined as in Lemma 4.2.1, namely, $V(H) = V(G) - \{v\}$ and $E(H) = E(D_v^c, G)$. Thus $|E(H)| = |E(D_v^c, G)| \leq (\Delta - 1)(k-1) + 1$ by Lemma 4.2.2.

Claim : H is a tree.

Suppose to the contrary that H is not a tree. Let T be a spanning tree of H. So $|E(T)| < |E(H)|$. Since $V(T) = V(H) = V(G) - \{v\}$, $n = |V(T)| + 1$. Hence, $|V(T)| = |E(T)| + 1$. These imply that

$$n = |V(T)| + 1 = (|E(T)| + 1) + 1 < |E(H)| + 2 \leq (\Delta - 1)(k-1) + 3 = n,$$

a contradiction. Thus establishing the claim.

(2) Suppose $x, y \in D_v^c$ are adjacent to $w \in V(G) - (D_v^c \cup \{v\})$. Since $(G-v)[D_v^c]$ is connected, there is a path from x to y with internal vertices are in D_v^c. Because $xw, yw \in E(H)$, H is not a tree. This contradicts the claim. Thus x and y are not adjacent to any common vertex in $V(G) - (D_v^c \cup \{v\})$. This establishes (2).

(3) Because $(G-v)[D_v^c]$ is a subgraph of H, it follows by the claim that $(G-v)[D_v^c]$ is a tree. This completes the proof of Lemma 4.3.1. □

Theorem 4.3.2. Let G be a $4-\gamma_c-$vertex critical graph of order 3Δ. Then G is $\Delta-$regular.

Proof. By using Lemma 4.3.1(1), we need only show that for each vertex $v \in V(G)$, there exists $w \in V(G)$ such that $v \in D_w^c$. Consider $G - v$. By Lemma 2.1.22(2), let $D_v^c = \{a_i : 1 \leq i \leq 3\}$ and $A_i = N_{\overline{D_v^c}}(a_i)$ for $i = 1, 2, 3$. It follows from Lemma 4.3.1(2) that $A_i \cap A_j = \emptyset$ for $i \neq j$.

Lemma 4.3.1(3) yields $(G-v)[D_v^c]$ is a path P_3 of order 3. Without loss of generality, assume that $(G-v)[D_v^c] = a_1, a_2, a_3$. Consider $G - a_2$. Lemma 2.1.22(1) gives that $(A_2 \cup \{a_1, a_3\}) \cap D_{a_2}^c = \emptyset$. To dominate a_i for all $i \in \{1,3\}$, $|D_{a_2}^c \cap A_i| \geq 1$. Suppose $|D_{a_2}^c \cap A_i| \geq 2$ for some $i \in \{1,3\}$. Thus there are two vertices in $D_{a_2}^c$ with a_i as a common neighbor contradicting Lemma 4.3.1(2). Therefore $|D_{a_2}^c \cap A_i| = 1$ for $i = 1, 3$. It follows by Lemma 2.1.22(2) that $|D_{a_2}^c - (A_1 \cup A_3)| = 1$. Since v is the only one remaining vertex, $v \in D_{a_2}^c$. Thus G is $\Delta-$regular and this completes the proof of Theorem 4.3.2. □

We are now ready to prove Theorem 4.1.6.

Theorem 4.1.6. For $\Delta \geq 2$ and $k = 2, 3$ or 4, there is a $k-\gamma_c-$vertex critical graph of order $(\Delta - 1)(k-1) + 3$ if and only if Δ is even.

Proof. We first consider the case when $k = 2$. As a consequence of Lemma 2.1.21, we have $\Delta = n-2$ and n is even and, clearly, Δ is even. Thus there is a $2-\gamma_c-$vertex critical graph of order $(\Delta-1)(k-1)+3$ if and only if Δ is even. We may assume that k is 3 or 4. By Theorem 4.1.5, there exists a $k-\gamma_c-$vertex critical graphs of order $(\Delta-1)(k-1)+3$ for all $3 \leq k \leq 4$ when Δ is even. Suppose to the contrary that there exists a $k-\gamma_c-$vertex critical graph G of order $(\Delta-1)(k-1)+3$ for some $k \in \{3,4\}$ and odd $\Delta \geq 3$. So $(\Delta-1)(k-1)+3$ is odd. Theorems 4.1.2 and 4.3.2 imply that G is $\Delta-$regular. As Δ is odd, we have that G has an odd number of vertices of odd degree which is a contradiction. Hence, for $3 \leq k \leq 4$, if there exists a $k-\gamma_c-$vertex critical graph of order $(\Delta-1)(k-1)+3$, then Δ is even.

Conversely, let Δ be even. Theorem 4.1.5 yields that there exists a $k-\gamma_c-$vertex critical graph of order $(\Delta-1)(k-1)+3$ for $3 \leq k \leq 4$. This completes the proof. □

For $k \geq 5$, a $k-\gamma_c-$vertex critical graph of order $(\Delta-1)(k-1)+3$ need not be $\Delta-$regular. The graph in Figure 4.1 is G_1 a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 11 and the graph in Figure 4.2 is G_2 a $6-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 13. The graphs G_1 and G_2 are not $3-$regular. Thus the regularity of $k-\gamma_c-$vertex critical graphs of order $(\Delta-1)(k-1)+3$ does not depend on the parity of k.

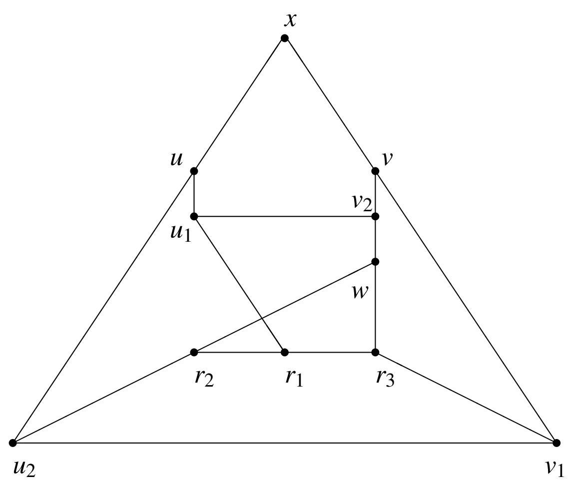

Figure 4.1 : A $5-\gamma_c-$vertex critical graph G_1 of order 11 with $\Delta = 3$

Clearly, G_1 has order 11 and $\Delta = 3$. We now show that it is a $5-\gamma_c-$vertex critical graph. Corollary 4.2.3 gives $\gamma_c(G_1) \geq 5$. We see that $\{u_2, r_2, w, v_2, v\} \succ_c G_1$. Therefore $\gamma_c(G_1) = 5$. We, moreover, see that

$$D^c_x = \{u_2, r_2, w, v_2\}, D^c_u = \{r_2, w, v_2, v\}, D^c_v = \{u, u_2, r_2, w\}, D^c_{u_1} = \{v, v_1, u_2, r_2\},$$

$$D^c_{u_2} = \{u_1, v_2, v, w\}, D^c_{v_1} = \{w, v_2, u_1, u\}, D^c_{v_2} = \{u, u_2, r_2, v_1\}, D^c_w = \{u_1, u, u_2, v_1\},$$

$D^c_{r_1} = \{u_2, v_1, v, v_2\}, D^c_{r_2} = \{v_1, v, v_2, u_1\}$ and $D^c_{r_3} = \{v_2, u_1, u, u_2\}$

Therefore G_1 is a $5-\gamma_c-$vertex critical graph.

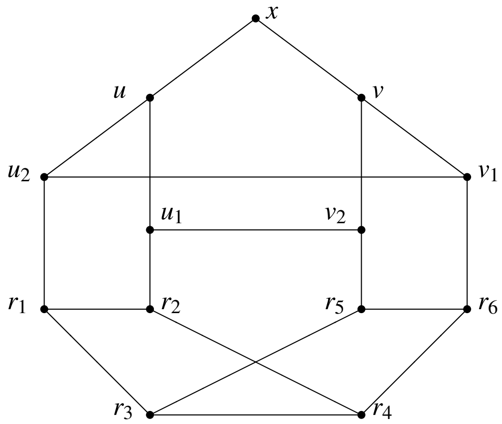

Figure 4.2 : A $6-\gamma_c-$vertex critical graph G_2 of order 13 with $\Delta = 3$

Obviously, G_2 has order 13 and $\Delta = 3$. We show that it is a $6-\gamma_c-$vertex critical graph. Corollary 4.2.3 implies $\gamma_c(G_2) \geq 6$. Clearly $\{u_2, r_1, r_3, r_5, v_2, v\} \succ_c G_2$. Thus $\gamma_c(G_1) = 5$. We, moreover, see that

$D^c_x = \{u_2, r_1, r_3, r_5, v_2\}, D^c_u = \{v, v_2, r_5, r_3, r_1\}, D^c_v = \{u, u_1, r_2, r_4, r_6\},$

$D^c_{u_1} = \{u_2, v_1, v, r_6, r_1\}, D^c_{u_2} = \{u_1, v_2, v, r_5, r_2\}, D^c_{v_1} = \{u, u_1, v_2, r_2, r_5\},$

$D^c_{v_2} = \{u, u_2, r_1, v_1, r_6\}, D^c_{r_1} = \{u, u_1, v_2, r_5, r_6\}, D^c_{r_2} = \{u, u_2, v_1, r_5, r_6\},$

$D^c_{r_3} = \{r_2, u_1, v_2, v, v_1\}, D^c_{r_4} = \{r_5, v_2, u_1, u, u_2\}, D^c_{r_5} = \{v, v_1, u_2, r_1, r_2\}$ and

$D^c_{r_6} = \{v, v_2, u_1, r_2, r_1\}.$

Therefore G_2 is a $6-\gamma_c-$vertex critical graph.

We conclude this section by showing that there are exactly two $5-\gamma_c-$vertex critical graphs of order 11 with $\Delta = 3$ which are G_1 and $G_1 - r_1r_3$.

Lemma 4.3.3. Let G be a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 11 and a vertex x of G of degree 2. If $N(x) = \{u, v\}$, then $|N(u)| = 3$ and $|N(v)| = 3$.

Proof. Suppose to the contrary that $N(v) = \{x, v'\}$. Lemma 2.1.22(1) thus implies $v \notin D^c_{v'}$. To dominate v, $x \in D^c_{v'}$ contradicting Lemma 4.3.1(1). This completes the proof. □

Lemma 4.3.4. Let G be a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 11 and a vertex x of G such that $N(x) = \{u, v\}$. Then $N[u] \cap N[v] = \{x\}$.

Proof. If $uv \in E(G)$, then $N[x] \subseteq N[v]$ contradicting Lemma 2.1.22(3). Thus $uv \notin E(G)$. If there exists $w \in N(v) \cap N(u) - \{x\}$, then $N(x) \subseteq N(w)$ contradicting Corollary 2.1.23. Therefore $N[u] \cap N[v] = \{x\}$ and this completes the proof. □

From now on, let G be a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 11 and a vertex x of G such that $N(x) = \{u, v\}$. By Lemmas 4.3.3 and 4.3.4, there are four different vertices v_1, v_2, u_1 and u_2 in $V(G)$ such that $N(v) = \{x, v_1, v_2\}$ and $N(u) = \{x, u_1, u_2\}$.

Lemma 4.3.5. Let G be a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 11 and a vertex x of G such that $N(x) = \{u, v\}$. Then $N(u_1) \cap N(u_2) = \{u\}$ and $N(v_1) \cap N(v_2) = \{v\}$.

Proof. Suppose there exists $w \in N(u_1) \cap N(u_2) - \{u\}$. Consider $G - w$. Lemma 2.1.22(1) implies that $\{u_1, u_2\} \cap D^c_w = \emptyset$. If $u \in D^c_w$, then $x \in D^c_w$ by the connectedness of $(G-w)[D^c_w]$. But if $u \notin D^c_w$, then $x \in D^c_w$ to dominate u. In both cases, $x \in D^c_w$. This contradicts Lemma 4.3.1(1). Therefore $N(u_1) \cap N(u_2) = \{u\}$ and, similarly, $N(v_1) \cap N(v_2) = \{v\}$. □

Lemma 4.3.6. Let G be a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 11 and a vertex x of degree two in G. Then $(G-x)[D^c_x]$ is a path.

Proof. Suppose that $(G-x)[D^c_x]$ is not a path. By Lemmas 2.1.22(2) and 4.3.1(3), $(G-x)[D^c_x]$ is a claw. Let $D^c_x = \{a, b, c, d\}$ such that a is the center of the claw. For all $y \in \{b, c, d\}$, let $A(y) = N(y) - \{a\}$. Lemma 4.3.1(2) yields that the set of six vertices in $V(G) - \{x, a, b, c, d\}$ can be partitioned into $A(b), A(c)$ and $A(d)$. Consider $G - a$. Lemma 2.1.22(1) gives that $\{b, c, d\} \cap D^c_a = \emptyset$. For all $y \in \{b, c, d\}$, to dominate y, $D^c_a \cap A(y) \neq \emptyset$. Moreover, Lemma 4.3.1(2) gives $|D^c_a \cap A(y)| = 1$ and Lemma 2.1.22(2) gives $|D^c_a - \cup_{y \in \{b,c,d\}} A(y)| = 1$. Since x is the only one remaining vertex, $x \in D^c_a$ contradicting Lemma 4.3.1(1). Therefore $(G-x)[D^c_x]$ is a path and this completes the proof. □

Lemma 4.3.7. Let G be a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 11 and a vertex x of G such that $N(x) = \{u, v\}$. If $N(u) = \{u_1, u_2, x\}$ and $N(v) = \{v_1, v_2, x\}$, then there exists u_i or v_i such that $|N(u) \cap D^c_{v_i}| = 1$ or $|N(v) \cap D^c_{u_i}| = 1$.

Proof. Suppose to the contrary that $|N(u) \cap D^c_{v_i}| > 1$ and $|N(v) \cap D^c_{u_i}| > 1$ for all $i \in \{1, 2\}$. By $\Delta = 3$, $|N(u) \cap D^c_{v_i}| = 2$ and $|N(v) \cap D^c_{u_i}| = 2$. Consider $G - v_1$. Thus

$\{u_1,u_2\} \subseteq D^c_{v_1}$. By Lemmas 2.1.22(1) and 4.3.1(1), v and x are not in $D^c_{v_1}$, respectively. To dominate x, $u \in D^c_{v_1}$, moreover, to dominate v, $v_2 \in D^c_{v_1}$. We have that $D^c_{v_1} = \{v_2,u_1,u,u_2\}$. By the connectedness of $(G-v_1)[D^c_{v_1}]$, v_2u_1 or v_2u_2 is in $E(G)$. Without loss of generality let $v_2u_2 \in E(G)$. Consider $G-v_2$. Similarly, $\{u_1,u_2\} \subseteq D^c_{v_2}$. This contradicts Lemma 2.1.22(1). Thus there exists u_i or v_i such that $|N(u)\cap D^c_{v_i}| = 1$ or $|N(v)\cap D^c_{u_i}| = 1$ and this completes the proof. □

Lemma 4.3.8. Let G be a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 11 and a vertex x of G such that $N(x) = \{u,v\}$, $N(u) = \{u_1,u_2,x\}$ and $N(v) = \{v_1,v_2,x\}$. If $N(u)\cap D^c_{v_i} = \{u_j\}$ for some $i, j \in \{1,2\}$, then $u_jv_{3-i} \in E(G)$.

Proof. Suppose without loss of generality that $N(u)\cap D^c_{v_1} = \{u_1\}$. We will prove that $u_1v_2 \in E(G)$. Suppose to the contrary that $u_1v_2 \notin E(G)$. By Lemmas 2.1.22(1) and 4.3.1(1), v and x are not in $D^c_{v_1}$, respectively. To dominate x and v, u and v_2 are respectively in $D^c_{v_1}$. Lemma 2.1.22(2) together with $(G-v_1)[D^c_{v_1}]$ is connected imply that there exists $w \in V(G)-\{x,u,v,v_1,v_2,u_1,u_2\}$ such that $(G-v_1)[D^c_{v_1}]$ is a path v_2,w,u_1,u. Lemma 4.3.1(2) yields that the three vertices in $V(G)-\{x,u,v,v_1,v_2,u_1,u_2,w\}$ are adjacent to a different vertex in $\{v_2,w,u_1\}$. Let r_1,r_2 and r_3 be vertices such that $r_1u_1, wr_2, v_2r_3 \in E(G)$. By $\Delta = 3$, we have $N(u_1) = \{u,r_1,w\}, N(w) = \{u_1,v_2,r_2\}$ and $N(v_2) = \{v,r_3,w\}$. Since r_1u_1 and v_2r_3 are in $E(G)$, it follows by Lemma 4.3.5 that r_1u_2 and v_1r_3 are not in $E(G)$.

Consider $G-w$. Lemma 4.3.1(1) implies $x \notin D^c_w$. To dominate x, $\{u,v\}\cap D^c_w \neq \emptyset$. Without loss of generality let $u \in D^c_w$. Lemma 2.1.22(1) also implies that $u_1,v_2,r_2 \notin D^c_w$. Since $uu_1, ux \in E(G)$ and $u \in D^c_w$, it follows by Lemma 4.3.1(2) that $r_1, v \notin D^c_w$. We now have that $u_1,v_2,r_2,r_1,x,v,w \notin D^c_w$. Lemma 2.1.22(2) gives that $D^c_w = \{u,u_2,v_1,r_3\}$. Noting that $r_3v_1 \notin E(G)$. The connectedness of $(G-w)[D^c_w]$ implies that $u_2r_3, u_2v_1 \in E(G)$. Therefore $N(u_2) = \{u,r_3,v_1\}$. Since $v_1,r_3 \in D^c_w$, it follows by Lemma 4.3.1(1) that $deg_G(v_1) = 3$ and $deg_G(r_3) = 3$. This implies, by Lemma 4.3.1(2), that $v_1r_2, r_3r_1 \in E(G)$ or $v_1r_1, r_3r_2 \in E(G)$.

Case 1 : $v_1r_2, r_3r_1 \in E(G)$.

Consider $G-r_1$. Lemma 2.1.22(1) thus implies $u_1,r_3 \notin D^c_{r_1}$. If $r_2r_1 \in E(G)$, then $r_2 \notin D^c_{r_1}$ by Lemma 2.1.22(1). Further, if $r_2r_3 \notin E(G)$, then $deg_G(r_2) = 2$. This implies by Lemma 4.3.1(1) that $r_2 \notin D^c_{r_1}$. In both cases, $r_2 \notin D^c_{r_1}$. Lemma 4.3.1(1) gives also that $x \notin D^c_{r_1}$. Lemma 2.1.24 yields $u_2 \in D^c_{r_1}\cap N(u)$ and $v_2 \in D^c_{r_1}\cap N(w)$, moreover, Lemma 2.1.22(2) yields $|D^c_{r_1}-\{u_2,v_2\}| = 2$. As $(G-r_1)[D^c_{r_1}]$ is connected, we must have $D^c_{r_1} = \{u_2,v_2,v_1,v\}$. Thus $D^c_{r_1}$ does not dominate u_1 contradicting $D^c_{r_1} \succ_c G-r_1$. This case cannot occur.

Case 2 : $v_1r_1, r_3r_2 \in E(G)$.

Consider $G-r_2$. Lemma 2.1.22(1) then implies that $w, r_3 \notin D^c_{r_2}$. By using similar arguments to Case 1, r_1 is not in $D^c_{r_2}$ whether it is adjacent to r_2 or not, moreover, $x \notin D^c_{r_2}$. To dominate w, v_2 or u_1 is in $D^c_{r_2}$. Suppose that $v_2 \in D^c_{r_2}$. As $(G-r_2)[D^c_{r_2}]$ is connected, by Lemma 2.1.22(2), we must have $D^c_{r_2} = \{v_2, v, v_1, u_2\}$. But $D^c_{r_2}$ does not dominate u_1 contradicting $D^c_{r_2} \succ_c G-r_2$. Hence $u_1 \in D^c_{r_2}$. Lemma 2.1.24 yields that v_1 or v_2 is in $D^c_{r_2} \cap N(v)$. Since $(G-r_2)[D^c_{r_2}]$ is connected, it follows from Lemma 2.1.22(2) that $D^c_{r_2} = \{u_1, u, u_2, v_1\}$. But $D^c_{r_2}$ does not dominate v_2 contradicting $D^c_{r_2} \succ_c G-r_2$. This case cannot occur.

In both cases we have a contradiction and so $u_1v_2 \in E(G)$ and this completes the proof. □

Theorem 4.3.9. Let G be a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 11. Then G is isomorphic to G_1 or $G_1 - r_1r_3$.

Proof. Because G has odd order, at least one vertex of G has degree two. Let x be a vertex of G such that $N(x) = \{u, v\}$. By Lemmas 4.3.3 and 4.3.4, there are four different vertices u_1, u_2, v_1 and v_2 in G such that $N(u) = \{x, u_1, u_3\}$ and $N(v) = \{x, v_1, v_2\}$.

Lemma 4.3.7 implies that there exists u_i or v_i such that $|N(u) \cap D^c_{v_i}| = 1$ or $|N(v) \cap D^c_{u_i}| = 1$. Without loss of generality let $N(u) \cap D^c_{v_1} = \{u_1\}$. By Lemmas 2.1.22(1) and 4.3.1(1), $v, x \notin D^c_{v_1}$. Thus v_2 and u are in $D^c_{v_1}$ to dominate v and x, respectively. Lemma 4.3.8 gives also that $v_2u_1 \in E(G)$. By Lemma 2.1.22(2), let $\{w\} = D^c_{v_1} = \{v_2, u_1, u\}$. Lemma 4.3.1(3) then implies that w is adjacent to either v_2 or u_1.

We will prove for the case $wv_2 \in E(G)$ and omit the case $wu_1 \in E(G)$ which we can prove by the similar arguments. Suppose that $wv_2 \in E(G)$. Let $\{r_1, r_2, r_3\} = V(G) - \{x, u, v, u_1, u_2, v_1, v_2, w\}$. By $\Delta = 3$ and Lemma 4.3.1(2), w is adjacent to exactly two vertices in $\{r_1, r_2, r_3\}$ and u_1 is adjacent to the vertex in $\{r_1, r_2, r_3\}$ which is not adjacent to w. Without loss of generality let $wr_2, wr_3 \in E(G)$ and $u_1r_1 \in E(G)$.

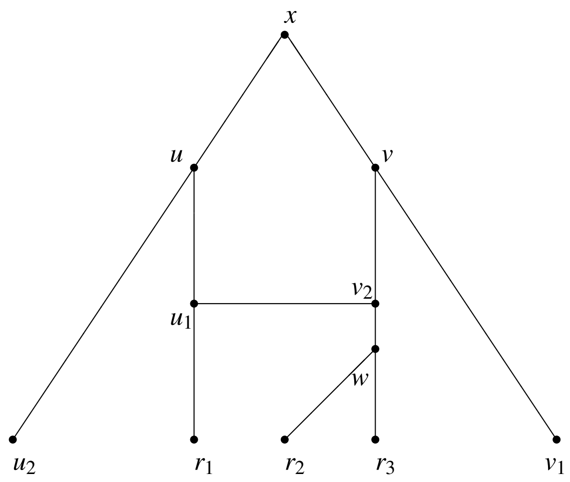

Figure 4.3(a)

We now have that $N(u_1)=\{u,v_2,r_1\}, N(v_2)=\{u_1,w,v\}$ and $N(w)=\{v_2,r_2,r_3\}$. The structure at the moment is displayed by Figure 4.3(a).

We need to characterize $G[\{u_2,v_1,r_1,r_2,r_3\}]$. Consider $G-v_2$. Lemma 2.1.22(1) implies that $\{u_1,w,v\}\cap D^c_{v_2}=\emptyset$, moreover, Lemma 4.3.1(1) gives also that $x\notin D^c_{v_2}$. Thus u and v_1 are in $D^c_{v_2}$ to dominate x and v respectively. Since $u\in D^c_{v_2}$, it follows by the connectedness of $(G-v_2)[D^c_{v_2}]$ that $u_2\in D^c_{v_2}$. To dominate w, $\{r_2,r_3\}\cap D^c_{v_2}\neq\emptyset$. Lemma 2.1.22(2) yields that $|\{r_2,r_3\}\cap D^c_{v_2}|=1$. Without loss of generality let $r_2\in D^c_{v_2}$ Hence $(G-v_2)[\{u_2,r_2,v_1\}]$ is connected and, in fact, it is a path by Lemma 4.3.1(3). We distinguish three cases.

Case 1 : u_2,r_2,v_1 is a path in G.

Thus $N(r_2)=\{w,u_2,v_1\}$. Lemma 4.3.1(2) yields that u_2 and v_1 are adjacent to a different vertex in $\{r_1,r_3\}$. We have two more subcases.

Subcase 1.1 : $u_2r_1\in E(G)$ and $v_1r_3\in E(G)$.

Consider $G-r_3$. Lemma 2.1.22(1) implies that $w,v_1\notin D^c_{r_3}$. If $r_1r_3\in E(G)$, then $r_1\notin D^c_{r_3}$ by Lemma 2.1.22(1). And if $r_1r_3\notin E(G)$, then $deg_G(r_1)=2$ and $r_1\notin D^c_{r_3}$ by Lemma 4.3.1(1). Thus $r_1\notin D^c_{r_3}$ whether r_1 is adjacent to r_3 or not. To dominate v_1, we have r_2 or v is in $D^c_{r_3}$. If $r_2\in D^c_{r_3}$, then $v\notin D^c_{r_3}$ by Lemma 4.3.1(2). Lemma 2.1.24 gives also that $u_1\in D^c_{r_3}\cap N(v_2)$. Moreover, u is in $D^c_{r_3}$ to dominate x. As $(G-r_2)[D^c_{r_3}]$ is connected, by Lemma 2.1.22(2), we must have $D^c_{r_3}=\{r_2,u_2,u,u_1\}$ which does not dominate v. This contradicts $D^c_{r_3}\succ_c G-r_3$. Hence $r_2\notin D^c_{r_3}$ and $v\in D^c_{r_3}$. Lemma 2.1.24 thus implies $u\in D^c_{r_3}\cap N(u_2)$. Since $(G-r_3)[D^c_{r_3}]$ is connected, by Lemma 2.1.22(2), $D^c_{r_3}=\{v,v_2,u_1,u\}$ which does not dominate r_2. This contradicts $D^c_{r_3}\succ_c G-r_3$ and Subcase 1.1 cannot occur.

Subcase 1.2 : $u_2r_3 \in E(G)$ and $v_1r_1 \in E(G)$.

Consider $G-r_3$. By Lemmas 2.1.22(1) and 4.3.1(1), we have that $w,u_2 \notin D^c_{r_3}$ and $x \notin D^c_{r_3}$. By the same arguments as Subcase 1.1, $r_1 \notin D^c_{r_3}$ whether r_1 is adjacent to r_3 or not. To dominate u_2, we have r_2 or u is in $D^c_{r_3}$. If $r_2 \in D^c_{r_3}$, then $u \notin D^c_{r_3}$ by Lemma 4.3.1(2). Thus u_1 is in $D^c_{r_3}$, to dominate u. By the connectedness of $(G-r_2)[D^c_{r_3}]$, $D^c_{r_3} = \{r_2, v_1, v, v_2, u_1\}$ contradicting Lemma 2.1.22(2). Hence $r_2 \notin D^c_{r_3}$ and $u \in D^c_{r_3}$. To dominate r_2, $v_1 \in D^c_{r_3}$. Similarly, $D^c_{r_3} = \{u, u_1, v_2, v, v_1\}$ contradicting Lemma 2.1.22(2) Thus Case 1 cannot occur.

Case 2 : r_2, u_2, v_1 is a path in G.

Thus $N(u_2) = \{u, r_2, v_1\}$. Lemma 4.3.1(2) implies that r_2 and v_1 are adjacent to a different vertex in $\{r_1, r_3\}$.

Subcase 2.1 : $r_2r_3 \in E(G)$ and $v_1r_1 \in E(G)$.

Consider $G-r_1$. Lemma 2.1.22(1) thus implies $u_1, v_1 \notin D^c_{r_1}$. By the same arguments as Subcase 1.1, $r_3 \notin D^c_{r_1}$ whether r_1 is adjacent to r_3 or not. We have that r_2 or w is in $D^c_{r_1}$ to dominate r_3. If $r_2 \in D^c_{r_1}$, then $w \notin D^c_{r_1}$ by Lemma 4.3.1(2). It follows from Lemma 2.1.24 that $v \in D^c_{r_1} \cap N(v_2)$. Lemma 4.3.1(1) gives also that $x \notin D^c_{r_1}$. Thus $(G-r_1)[D^c_{r_1}]$ is not connected, a contradiction. Hence $r_2 \notin D^c_{r_1}$ and $w \in D^c_{r_1}$. Lemma 2.1.24 yields that $u \in D^c_{r_1} \cap N(u_2)$. Since $x \notin D^c_{r_1}$, $(G-r_1)[D^c_{r_1}]$ is not connected, a contradiction. Subcase 2.1 cannot occur.

Subcase 2.2 : $r_2r_1 \in E(G)$ and $v_1r_3 \in E(G)$.

We see that a graph in this subcase is isomorphic to the graph $G_1 - r_1r_3$.

Case 3 : u_2, v_1, r_2 is a path in G.

Thus $N(v_1) = \{v, r_2, u_2\}$. By Lemma 4.3.1(2), r_2 and u_2 are adjacent to a different vertex in $\{r_1, r_3\}$.

Subcase 3.1 : $u_2r_1 \in E(G)$ and $r_2r_3 \in E(G)$.

Thus $r_1 \in N(u_2) \cap N(u_1) - \{u\}$ contradicting Lemma 4.3.5.

Subcase 3.2 : $u_2r_3 \in E(G)$ and $r_2r_1 \in E(G)$.

By adding r_1r_3 to this graph, we see that the graph in this subcase is isomorphic to the graph G_1 by relabeling r_2 and r_3 to r'_3 and r'_2, respectively. and this completes the proof of Case 3.

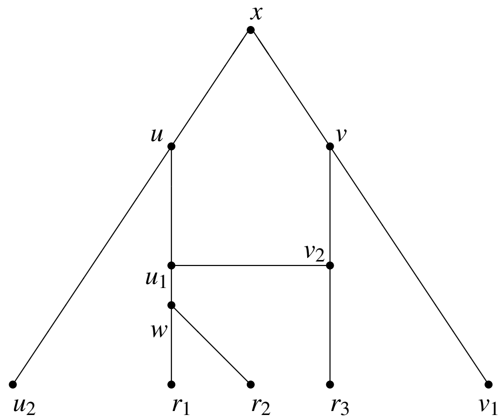

Figure 4.3(b)

When $wu_1 \in E(G)$, similarly, let $\{r_1, r_2, r_3\} = V(G) - \{x, u, v, u_1, u_2, v_1, v_2, w\}$. By $\Delta = 3$ and Lemma 4.3.1(2), w is adjacent to exactly two vertices in $\{r_1, r_2, r_3\}$ while v_2 is adjacent to the vertex in $\{r_1, r_2, r_3\}$ which is not adjacent to w. Without loss of generality let $wr_1, wr_2 \in E(G)$ and $v_2r_3 \in E(G)$. The structure in this case is displayed by Figure 4.3(b) and we can prove by the similar arguments to the case when $wv_2 \in E(G)$. This completes the proof of this theorem. □

4.4 The $k-\gamma_c-$Vertex Critical Graphs achieving the Lower Bound

By Theorem 3.1.5, $k-\gamma_c-$vertex critical graphs of order $\Delta+k$ for $k \leq 4$ have been studied by [54,78,126,137] and [150] in the sense of $k-\gamma_t-$vertex critical graphs. We then turn our attention to $k \geq 5$. The main objective of this section is to prove that, for $k \geq 5$, every $k-\gamma_c-$vertex critical graph of order $\Delta+k$ is isomorphic to C_{k+2}. The following lemma (Lemma 4.4.1) is the main tool to prove Theorem 4.1.7 and we can prove this in two ways. We first state this lemma and prove Theorem 4.1.7 and then we give the two proofs of Lemma 4.4.1 in the following two subsections.

Lemma 4.4.1. Let G be a $k-\gamma_c-$vertex critical graph of order $\Delta+k$ and a be a vertex of degree Δ. We, further, let $H = G[V(G) - N[a]]$. Then H is isomorphic to a path $P_{k-1} = x_1, x_2, ..., x_{k-1}$, moreover, $N(a) \cap N(x_i) \neq \emptyset$ if and only if $i = 1$ or $k-1$.

We are now ready to give a proof of Theorem 4.1.7.

Theorem 4.1.7. For $k \geq 5$, G is a $k-\gamma_c-$vertex critical graph of order $\Delta+k$ if and only if G is a cycle, C_{k+2}, on $k+2$ vertices.

Proof. Clearly, a cycle C_{k+2} is a $k-\gamma_c-$vertex critical graph of order $\Delta+k$.

Conversely, let G be a $k-\gamma_c-$vertex critical graph of order $\Delta+k$. We show that G is a C_{k+2}. Let a be a vertex of maximum degree. Lemma 4.4.1 yields that $(G-a)[D^c_a]$ is a path $x_1,x_2,...,x_{k-1}$ of order $k-2$. Moreover, let $A=N(a)$ and

$$L = D^c_a - \{x_2,x_3\}.$$

So $|L| = k-3$. Because $D^c_a \succ A$, it follows by Lemma 4.4.1 that $\{x_1,x_{k-1}\} \succ_c A$. If there exists $x \in A$ joining both x_1 and x_{k-1}, then $\{x\} \cup L \succ_c G$ contradicting $\gamma_c(G)=k$. Therefore $N_A(x_1)\cap N_A(x_{k-1}) = \emptyset$. Thus A is partitioned into $X_1 = N_A(x_1)$ and $X_{k-1} = N_A(x_{k-1})$.

Claim 1 : $E(X_1,X_{k-1}) = \emptyset$.

Suppose there exists an edge xy such that $x \in X_1$ and $y \in X_{k-1}$. So $\{x,y\}\cup L \succ_c G$ contradicting $\gamma_c(G)=k$. Hence $E(X_1,X_{k-1}) = \emptyset$, thus establishing Claim 1.

Claim 2 : For $i \in \{1,k-1\}$, each $w \in X_i$, there exists a vertex $z \in X_{k-i}$ such that $z \succ X_i - \{w\}$.

We may assume without loss of generality that $w \in X_{k-1}$. Consider $G-w$. Lemma 2.1.22(1) implies that neither a nor x_{k-1} in D^c_w. Lemma 2.1.24 implies also that $D^c_w \cap N(a) \neq \emptyset$ and $D^c_w \cap (\{x_{k-2}\}\cup X_{k-1}) \neq \emptyset$ because $N(x_{k-1}) = \{x_{k-2}\}\cup X_{k-1}$.

We show that $x_{k-2} \in D^c_w$. Suppose to the contrary that $x_{k-2} \notin D^c_w$. Hence $D^c_w \cap X_{k-1} \neq \emptyset$ to dominate x_{k-1}. Thus, there exists $x \in D^c_w \cap X_{k-1}$. Since $N(x_{k-2}) = \{x_{k-1}, x_{k-3}\}$, we have by Lemma 2.1.24 that $x_{k-3} \in N(x_{k-2})\cap D^c_w$. As $(G-w)[D^c_w]$ is connected and $x_{k-1} \notin E(G)$, $(G-w)[D^c_w]$ must have a path $P = x_{k-3},x_{k-4},...,x_2,x_1,y_1,y_2, ...,y_{t-1},x$ where $y_t = x$ and $y_1,y_2,...,y_t$ are in $D^c_w \cap (A\cup\{a\})$. Because $xx_1 \notin E(G)$, $t \geq 2$. Since $a \notin D^c_w$, there exists $i \in \{1,2,...,t-1\}$ such that $y_i \in X_1$ and $y_{i+1} \in X_{k-1}$. This contradicts Claim 1. Thus $x_{k-2} \in D^c_w$.

As $(G-w)[D^c_w]$ is connected and $D^c_w \cap N(a) \neq \emptyset$, we must have $\{x_{k-2},x_{k-3},...,x_2, x_1,z\} \subseteq D^c_w$ for some $z \in X_1$. Lemma 2.1.22(2) then yields that $D^c_w = \{x_{k-2},x_{k-3},...,x_2, x_1,z\}$. Lemma 4.4.1 together with Claim 1 give that $N_{X_{k-1}}(x_i) = \emptyset$ for all $1 \leq i \leq k-2$. Therefore $z \succ X_{k-1} - \{w\}$, thus establishing Claim 2.

It remains to show that $|A| = 2$. Suppose that $|A| > 2$. By the pigeonhole principle, at least one of X_1 and X_{k-1} has more than one vertex. Suppose that $|X_i| > 1$. Let $\{w,w'\} \subseteq X_i$. By Claim 2, there exists $z \in X_{k-i}$ such that $z \succ X_i - \{w\}$. Thus $zw' \in E(G)$ contradicting Claim 1. Therefore $|A| = 2$ and this completes the proof of Theorem 4.1.7. $\square$

By Theorems 4.1.5 and 4.1.7, we can establish the following result.

Corollary 4.4.2. Let G be a $k-\gamma_c-$vertex critical graph with $\Delta \geq 3$ and $k \geq 5$ of order n. Then

$$\Delta + k + 1 \leq n \leq (\Delta - 1)(k-1) + 3.$$

4.4.1 The First Proof of Lemma 4.4.1

The first proof utilizes the idea of Theorem 2.1.14 to show that the maximum degree of the induced subgraph $(G-a)[D_a]$ is at most two. Obviously, a connected graph of maximum degree two is either a cycle or a path. Then it is easy to characterize further the structure between D_a and $N(a)$. Let G be a $k-\gamma_c-$vertex critical graph of order $\Delta + k$ containing a vertex a of degree Δ. In this section, let H be $G[V(G) - N[a]]$. Clearly $|V(H)| = k-1$.

Lemma 4.4.3. $\gamma_c(H) = k-3$ and $\Delta(H) = 2$.

Proof. Lemma 2.1.22(1) gives that $D_a^c \subseteq V(H)$, moreover, Lemma 2.1.22(2) gives also that $|D_a^c| = k-1$. Therefore $V(H) = D_a^c$. By the connectedness of $(G-a)[D_a^c]$, H is connected. Thus $\Delta(H) \geq 2$. By Theorem 2.1.14,

$$\gamma_c(H) \leq |V(H)| - \Delta(H) \leq (k-1) - 2 = k-3.$$

Suppose that $\gamma_c(H) \leq k-4$. Let D be a γ_c-set of H and x a vertex of H which is adjacent to a vertex in $N(a)$. Hence x is adjacent to a vertex in D. Let $y \in N(a) \cap N(x)$. Thus $D \cup \{x,y,a\} \succ_c G$. But $|D \cup \{x,y,a\}| \leq (k-4)+3 \leq k-1$ contradicting $\gamma_c(G) = k$. Therefore $\gamma_c(H) = k-3$. We see that the equality holds if $\Delta(H) = 2$. This completes the proof. □

Lemma 4.4.4. Let D be a γ_c-set of H. Then $N(x) \cap N(a) = \emptyset$ for any vertex x in D.

Proof. Suppose to the contrary that there exists $y \in N(x) \cap N(a)$. Thus $D \cup \{y,a\} \succ_c G$. Lemma 4.4.3 yields that $|D \cup \{y,a\}| = (k-3)+2 = k-1$ contradicting $\gamma_c(G) = k$. This completes the proof. □

We now give the first proof of Lemma 4.4.1.

Proof. We have by Lemma 4.4.3 that $\Delta(H) = 2$. By the connectedness of H, we have that H is either C_{k-1} or P_{k-1}. Suppose to the contrary that H is isomorphic to C_{k-1}. Since G is connected, there exists $x \in V(H)$ such that $N(x) \cap N(a) \neq \emptyset$. Let $H = x_1, x_2, ..., x_{k-1}, x_1$ where $x = x_1$. We see that $\{x_1, x_2, ..., x_{k-3}\}$ is a γ_c-set of H containing a vertex x. This contradicts Lemma 4.4.4. Therefore H is isomorphic to P_{k-1}.

Let $P_{k-1} = x_1, x_2, ..., x_{k-1}$. If $i \in \{1, k-1\}$, then $N(x_i) \cap N(a) \neq \emptyset$ by the 2− connectedness of G.

Conversely, suppose there exists $i \in \{2, 3, ..., k-2\}$ such that $N(x_i) \cap N(a) \neq \emptyset$. We see that $\{x_2, x_3, ..., x_{k-3}\}$ is a γ_c-set of H containing a vertex x_i. This contradicts Lemma 4.4.4. Thus $N(x_i) \cap N(a) \neq \emptyset$ if and only if $i \in \{1, k-1\}$ and this completes the proof of Lemma 4.4.1. □

4.4.2 The Second Proof of Lemma 4.4.1

The second proof is much longer than the first one. The idea of the second proof is based on the the structure between sets at a different distance from a. Let a be a vertex of maximum degree and s the maximum distance from a. For $1 \leq i \leq s$, we have

L_i : the set of vertices at distance i from a.

Clearly $N(a) = L_1$ and $|L_1| = \Delta$. Moreover, for $1 \leq i \leq s-1$, let

V_i : a smallest vertex subset of L_i such that $V_i \succ L_{i+1}$.

Note that $G[V_i]$ needs not be connected. We have the following lemma by the definitions of V_i and L_i.

Lemma 4.4.5. Let D_a^c be a γ_c-set of $G-a$ and $D = \{a\} \cup (\cup_{i=1}^{s-1} V_i)$. Then

(1) $D_a^c = \cup_{i=2}^{s} L_i$,

(2) for $1 \leq i \leq s-1$, $|V_i| \leq |L_{i+1}|$ and

(3) $|D| = k$.

Proof. We see that

$$|\cup_{i=2}^{s} L_i| = |V(G) - N[a]| = n - (\Delta + 1) = \Delta + k - (\Delta + 1) = k - 1.$$

Because $D_a^c \cap (L_1 \cup \{a\}) = \emptyset$, it follows that $D_a^c \subseteq \cup_{i=2}^{s} L_i$. But $|D_a^c| = k-1$, we have that $D_a^c = \cup_{i=2}^{s} L_i$. This establishes (1).

By the minimality of V_i, $PN_{L_{i+1}}(v, V_i) \neq \emptyset$. Hence $|V_i| \leq |L_{i+1}|$ which establishes (2).

We see that $a \succ V_1$. Because $V_i \succ L_{i+1}$, it follows that $V_i \succ V_{i+1}$. Thus $D \succ_c G$. By the minimality of k, $k \leq |D|$. By (1) and (2), we have that

$$|\cup_{i=1}^{s-1} V_i| = |V_1| + |V_2| + ... + |V_{s-1}| \leq |L_2| + |L_3| + ...|L_s| = k - 1.$$

Therefore $|D| = 1 + |\cup_{i=1}^{s-1} V_i| \leq k$. Hence $|D| = k$ which establishes (3). $\square$

Before proving the next lemma, we need to establish a basic property between two finite sequences.

Proposition 4.4.6. Let $\{a_i\}_{i=1}^{s}$ and $\{b_i\}_{i=1}^{s}$ be real finite sequences such that $a_i \leq b_i$ for all i. If $\sum_{i=1}^{s} a_i = \sum_{i=1}^{s} b_i$, then $a_i = b_i$ for $i = 1, ..., s$.

Proof. Since $a_i \leq b_i$ for all $i = 1, ..., n$, there exists a real number $c_i \geq 0$ such that $a_i + c_i = b_i$. Thus

$$\sum_{i=1}^{s} a_i = \sum_{i=1}^{s} b_i = \sum_{i=1}^{s}(a_i + c_i) = \sum_{i=1}^{s} a_i + \sum_{i=1}^{s} c_i.$$

So $\sum_{i=1}^{s} c_i = 0$ and this implies that $c_i = 0$ for all $i = 1, ..., n$. Hence $a_i = b_i$ for all i. $\square$

The following lemma shows the minimum value of s and the relationship between $|V_i|$ and $|L_{i+1}|$.

Lemma 4.4.7. $s \geq 3$ and, for $i = 1, ..., s-1$, $|V_i| = |L_{i+1}|$.

Proof. Clearly $s > 1$. Suppose $s = 2$. Lemma 4.4.5(1) gives that $D_a^c = L_2$. This implies $|L_2| = k-1 \geq 4$. By applying Proposition 2.1.13 to $G[L_2]$, $\gamma_c(G[L_2]) \leq (k-1)-2 = k-3$. Let Y be a γ_c-set of L_2 and $u \in Y$. Since $u \in L_2$, there is $w \in L_1$ adjacent to u. Thus $\{a, w\} \cup Y \succ_c G$. But $\{a, w\} \cup Y$ contains $k-1$ vertices contradicting $\gamma_c(G) = k$. Hence $s \geq 3$.

In view of Lemma 4.4.5 (1), (2) and (3)

$$k = |D| = 1 + |V_1| + ... + |V_{s-1}| \leq 1 + |L_2| + ... + |L_s| = k.$$

Thus $\sum_{i=1}^{s-1} |V_i| = \sum_{i=1}^{s-1} |L_{i+1}|$. By Proposition 4.4.6, $|V_i| = |L_{i+1}|$ for $i = 1, ..., s-1$ and this completes the proof. $\square$

Lemmas 4.4.8-4.4.11 establish a structure of $G[L_i \cup L_{i+1}]$ for $i = 1, ..., s-1$.

Lemma 4.4.8. For any $v \in L_i$, $|N_{L_{i+1}}(v)| \leq 1$ for $i = 1, ..., s-1$.

Proof. Since $PN_{L_{i+1}}(v, V_i) \neq \emptyset$ for all $v \in V_i$, it follows that $|X| \leq |N_{L_{i+1}}(X)|$ for all subset $X \subseteq V_i$. By Hall's Marriage Theorem, there exists a perfect matching M such that every edge in M has an end vertex in V_i and the other one in L_{i+1}.

Claim : $|PN_{L_{i+1}}(v, V_i)| = 1$ for all $v \in V_i$.

Suppose there exists $v' \in V_i$ such that $|PN_{L_{i+1}}(v', V_i)| > 1$. Since $|PN_{L_{i+1}}(v, V_i)| \geq 1$ for all $v \in V_i$, it follows that $|N_{L_{i+1}}(V_i)| \geq |V_i| + 1$. But $|N_{L_{i+1}}(V_i)| \subseteq L_{i+1}$. This implies $|L_{i+1}| \geq |V_i| + 1$ contradicting Lemma 4.4.7, thus establishing the claim.

Suppose there exists $v \in L_i$ such that $|PN_{L_{i+1}}(v,V_i)| > 1$. By the claim $v \in L_i - V_i$. Let $u,w \in N_{L_{i+1}}(v)$. Thus there exist $u',w' \in V_i$ such that $u'u, w'w \in M$. By the claim $PN_{L_{i+1}}(u',V_i) = \{u\}$ and $PN_{L_{i+1}}(w',V_i) = \{w\}$. Thus $(V_i - \{u',w'\}) \cup \{v\} \succ L_{i+1}$ contradicting the minimality of V_i. This completes the proof. □

Lemma 4.4.9. For any $v \in L_i$ and $i = 3,...,s-1$, we have that

(1) If $v \in V_i$ and $s \geq 4$, then $|N_{L_{i-1}}(v)| = 1$.

(2) If $v \in L_i - V_i$ and $s \geq 4$, then $|N_{L_{i-1}}(v)| \leq 2$. Moreover, if $|N_{L_{i-1}}(v)| = 2$, then $|N_{V_{i-1}}(v)| = 1$ and $N_{L_{i+1}}(v) = \emptyset$.

(3) If $v \in L_s$ and $s = 3$, then $|N_{L_2}(v)| \leq 2$. Moreover, if $|N_{L_2}(v)| = 2$, then $|N_{V_2}(v)| = 1$

Proof. We first consider the case when $s \geq 4$. Suppose there exists $v \in L_i$ such that $|N_{L_{i-1}}(v)| \geq 2$. Because $v \in L_i$, there is $v_{i-1} \in V_{i-1}$ adjacent to v. Let $u_{i-1} \in L_{i-1} - \{v_{i-1}\}$ be a vertex which is adjacent to v. If $u_{i-1} \in V_{i-1}$, then by Lemma 4.4.8, $N_{L_i}(u_{i-1}) = \{v\}$. Clearly $G[D - \{u_{i-1}\}]$ is connected. Thus $D - \{u_{i-1}\} \succ_c G$ contradicting the minimality of D. Therefore $u_{i-1} \in L_{i-1} - V_{i-1}$.

Suppose that $v \in V_i$. Because $u_{i-1} \in L_{i-1}$, there is $u_{i-2} \in V_{i-2}$ such that $u_{i-2}u_{i-1} \in E(G)$. Lemma 4.4.8 then yields $N_{L_{i-1}}(u_{i-2}) = \{u_{i-1}\}$. Since $vu_{i-1} \in E(G)$, it follows that $D - \{u_{i-2}\} \succ_c G$ with $k-1$ vertices, a contradiction. Hence if $v \in V_i$, then $|N_{L_{i-1}}(v)| = 1$ and this proves (1).

To prove (2), suppose that $v \in L_i - V_i$ and $v_{i-1}, u_{i-1} \in N_{L_{i-1}}(v)$. Moreover, Lemma 4.4.8 implies that there are different $v_{i-2}, u_{i-2} \in V_{i-2}$ such that $N_{L_{i-1}}(v_{i-2}) = \{v_{i-1}\}$ and $N_{L_{i-1}}(u_{i-2}) = \{u_{i-1}\}$.

Suppose that there exists $w_{i-1} \in N_{L_{i-1}}(v) - \{v_{i-1}, u_{i-1}\}$. Lemma 4.4.8 yields that there is $w_{i-2} \in V_{i-2} - \{v_{i-2}, u_{i-2}\}$ such that $N_{L_{i-1}}(w_{i-2}) = \{w_{i-1}\}$. So $D \cup \{v\} - \{w_{i-2}, u_{i-2}\} \succ_c G$ contradicting the minimality of D. Therefore $|N_{L_{i-1}}(v)| \leq 2$.

By Lemma 4.4.8 and the minimality of V_{i-1}, only one of v_{i-1} or u_{i-1} is in V_{i-1}. If $|N_{L_{i-1}}(v)| = 2$, then $|N_{V_{i-1}}(v)| = 1$.

Suppose that $N_{L_{i+1}}(v) \neq \emptyset$. Lemma 4.4.8 thus implies $|N_{L_{i+1}}(v)| = 1$. Let $\{v_{i+1}\} = N_{L_{i+1}}(v)$. Because $v \notin V_i$, there exists $x_i \in V_i$ such that $x_iv_{i+1} \in E(G)$. By Lemma 4.4.8, $N_{L_{i+1}}(x_i) = \{v_{i+1}\}$. But $\{v_{i+1}, u_{i-1}\} \subseteq N(v)$. Thus $(D \cup \{v\}) - \{x_i, u_{i-2}\} \succ_c G$ contradicting the minimality of D. Hence if $|N_{L_{i-1}}(v)| = 2$, then $N_{L_{i+1}}(v) = \emptyset$ and this proves (2).

Since V_s is not defined in L_s, we can prove that every vertex in L_s has property (3) by the same arguments as when a vertex is in $L_i - V_i$ which is case (2). This completes the proof. □

Lemma 4.4.10. L_i is an independent set for all $i = 2, ..., s-1$.

Proof. Suppose this lemma is false.

Choose $i = min\{j \in \{2, ..., s-1\} : L_j$ is not an independent set $\}$.

Therefore there exist $x_i, y_i \in L_i$ such that $x_iy_i \in E(G)$. Lemma 4.4.8 thus implies there are different $x_{i-1}, y_{i-1} \in V_{i-1}$ such that $N_{L_i}(x_{i-1}) = \{x_i\}$ and $N_{L_i}(y_{i-1}) = \{y_i\}$. If x_i or y_i are in V_i, without loss of generality $x_i \in V_i$, then $D - \{y_{i-1}\} \succ_c G$ contradicting the minimality of D. Hence $x_i, y_i \notin V_i$. We distinguish two cases.

Case 1 : $N_{L_{i+1}}(\{x_i, y_i\}) \neq \emptyset$.

Without loss of generality, $N_{L_{i+1}}(x_i) \neq \emptyset$. By Lemma 4.4.8, let $N_{L_{i+1}}(x_i) = \{x_{i+1}\}$. Since $x_i \notin V_i$, it follows from Lemma 4.4.8 that there exists $z_i \in V_i$ such that $N_{L_{i+1}}(z_i) = \{x_{i+1}\}$. Clearly, $x_i \neq z_i$. We now have that $\{x_{i+1}, y_i\} \subseteq N(x_i)$. Thus $(D \cup \{x_i\}) - \{y_{i-1}, z_i\} \succ_c G$ contradicting the minimality of D. This case cannot occur.

Case 2 : $N_{L_{i+1}}(\{x_i, y_i\}) = \emptyset$.

Let T_i be the component in $(G-a)[L_i]$ containing x_iy_i. By similar arguments, $T_i \cap V_i = \emptyset$ and $N_{L_{i+1}}(T_i) = \emptyset$. Since $V_i \succ L_{i+1}$ and $L_{i+1} \neq \emptyset$, it follows that $L_i - V(T_i) \neq \emptyset$. If $i = 2$, then $(G-a)[D_a^c] = (G-a)[\cup_{j=2}^{s} L_j]$ is not connected with at least two components T_i and $G[\cup_{j=2}^{s} L_j - V(T_i)]$, a contradiction. Thus $i > 2$. By the minimality of i, $L_2, ..., L_{i-1}$ are independent set. Let $T_{j-1} = N_{L_{j-1}}(T_j)$ for $3 \leq j \leq i$. Further, let $H_1 = V(T_i) \cup (\cup_{j=2}^{i-1} T_j)$ and $H_2 = \cup_{j=2}^{s} L_j - H_1$. If there exists an edge xy such that $x \in H_1$ and $y \in H_2$, then $x \in T_j$ and $y \in L_{j-1} \cup L_{j+1}$ for some $j \in \{2, ..., i\}$. Suppose that $y \in L_{j+1}$. Since $x \in T_j$, there exists a vertex $z \in T_{j+1}$ such that $xz \in E(G)$. Lemma 4.4.8 gives that $z = y$. Thus $y \in T_{j+1}$ contradicting $y \in H_2$. Therefore $y \in L_{j-1}$. Because $x \in T_j$, it follows that $y \in N_{L_{j-1}}(T_j)$. Thus $y \in T_{j-1}$ contradicting $y \in H_2$. Hence $G[D_a^c]$ is not connected with at least two components $G[H_1]$ and $G[H_2]$, a contradiction. This completes the proof. □

Lemma 4.4.11. $|L_2| = 2$.

Proof. Suppose to the contrary that $|L_2| = p \geq 3$. We first establish the following claim.

Claim : $G[L_s]$ is connected and $|L_s| \leq 2$.

Lemma 4.4.10 implies that L_j is independent for $j \in \{2, ..., s-1\}$. In view of Lemmas 4.4.8, 4.4.9(1), (2) and (3), $(G-a)[\cup_{j=2}^{s-1} L_j]$ is a forest. By the connectedness of $(G-a)[D_a^c]$, $G[L_s]$ is connected.

Suppose to the contrary that $|L_s| \geq 3$. By Proposition 2.1.13, $\gamma_c((G-a)[L_s]) \leq |L_s| - 2$. Let Y be a γ_c-set of L_s. Since $Y \subseteq L_s$, there exists $w \in V_{s-1}$ which is adjacent to a vertex $y \in Y$. Because $|V_{s-1}| = |L_s|$, $|V_{s-1} - \{w\}| = |L_s| - 1$. Therefore $(D \cup Y) - (V_{s-1} - \{w\}) \succ_c G$. We note by Lemma 4.4.5(3) that $|D| = k$. Thus

$$|(D \cup Y) - (V_{s-1} - \{w\})| \leq k + (|L_s| - 2) - (|L_s| - 1) = k - 1,$$

a contradiction. Hence $|L_s| \leq 2$ and we settle the claim.

Because $|L_2| \geq 3$, there exists $j \in \{3, ..., s\}$ such that $|L_j| < |L_{j-1}|$.

Choose $i = min\{j \in \{3, ..., s\} : |L_j| < |L_{j-1}|\}$.

Thus $|L_2| = |L_3| = ... = |L_{i-1}| > |L_i|$. This implies by Lemmas 4.4.8, 4.4.9(1), (2) and (3) that $(G-a)[\cup_{j=2}^{i-1} L_j]$ is a disjoint union of p paths of length $i-3$ with one end vertex in L_2 and the other end vertex in L_{i-1}. By the connectedness of $(G-a)[D_a^c]$, there exists $v_i \in L_i$ such that $|N_{L_{i-1}}(v_i)| > 1$. Let $v_{i-1} \in N_{V_{i-1}}(v_i)$ and $y_{i-1} \in N_{L_{i-1}}(v_i) - \{v_{i-1}\}$. By Lemma 4.4.9(2) and (3), $N_{L_{i-1}}(v_i) = \{v_{i-1}, y_{i-1}\}$, $N_{V_{i-1}}(v_i) = \{v_{i-1}\}$ and $N_{L_{i+1}}(v) = \emptyset$. These imply that there exist a path P_{i-2} with one end vertex in L_2 and the other end vertex is v_{i-1} and, similarly, a path P'_{i-2} with one end vertex in L_2 and the other end vertex is y_{i-1}.

If $i < s$, then $(G-a)[D_a^c]$ contains a path $P = P_{i-2}, v, P'_{i-2}$ as one of the components. Because $p \geq 3$, it follows that $D_a^c - V(P) \neq \emptyset$. Hence $(G-a)[D_a^c]$ is not connected. Thus $i = s$. This implies $v_i = v_s, v_{i-1} = v_{s-1}$ and $y_{i-1} = y_{s-1}$. Moreover, $|L_2| = |L_3| = ... = |L_{s-1}| \geq 3$ and $(G-a)[\cup_{j=2}^{s-1} L_j]$ is a disjoint union of p paths of length $s-3$.

Case 1 : $|L_s| = 2$.

Let $\{u_s\} = L_s - \{v_s\}$. By the claim, $u_s v_s \in E(G)$. Since $u_s \in L_s$ and $y_{s-1} \in L_{s-1}$, there are vertices $u_{s-1} \in V_{s-1}$, $y_{s-2} \in V_{s-2}$ adjacent to u_s, v_{s-1}, respectively. Lemma 4.4.8 implies that $N_{L_s}(u_{s-1}) = \{u_s\}$ and $N_{L_s}(y_{s-2}) = \{y_{s-1}\}$. Thus $(D \cup \{v_s\}) - \{u_{s-1}, y_{s-2}\} \succ_c G$ contradicting the minimality of D and this case cannot occur.

Case 2 : $|L_s| = 1$.

Hence $\{v_s\} = L_s$. Since $(G-a)[\cup_{l=2}^{s-1} L_l]$ is a disjoint union of p paths, $v_s \succ L_{s-1}$ by the connectedness of $(G-a)[D_a^c]$. Because $v_{s-1} \in L_{s-1}$, there is $v_{s-2} \in V_{s-2}$ adjacent to v_{s-1}. Since $|V_{s-2}| = |L_{s-1}| \geq 3$, $|V_{s-2} - \{v_{s-2}\}| \geq 2$. So $(D \cup \{v_s\}) - (V_{s-2} - \{v_{s-2}\}) \succ_c G$. But $|(D \cup \{v_s\}) - (V_{s-2} - \{v_{s-2}\})| \leq (k+1) - 2 = k - 1$ contradicting the minimality of D. This completes the proof. □

We are now ready to establish the second proof of Lemma 4.4.1.

Proof. Lemma 4.4.7 implies that $|L_2| \geq |L_3| \geq ... \geq |L_s|$. As a graph G is 2$-$connected, it follows from Lemma 4.4.11 that $|L_i| = 2$ for $i = 2, ..., s-1$ and $|L_s| \leq 2$. Lemma 4.4.10 yields that the two vertices in L_i are not adjacent each other. Moreover, by Lemma 4.4.8, each vertex in L_i is adjacent only one vertex in L_{i+1}. By the connectedness of $(G-a)[D_a^c]$, when $|L_s| = 2$, the two vertices in L_s are adjacent and each of them is adjacent to a different vertex in L_{s-1}. Moreover, when $|L_s| = 1$, the two vertices in L_{s-1} are both adjacent to the vertex in L_s. We now have that $G[\cup_{i=2}^{s} L_i]$ is a path of length $k-2$ with the two end vertices are in L_2. This completes the proof. □

4.5 Vertex Critical Graphs of Prescribed Order

In this section we determine the existence of $k-\gamma_c-$vertex critical graphs whose order is between $\Delta+k$ and $(\Delta-1)(k-1)+3$ when Δ and k are given. In view of Lemma 2.1.21, $k-\gamma_c-$vertex critical graphs have been completely characterized for $k=1$ or 2. We then turn our attention to $k \geq 3$. We see that if $\Delta = 2$, then $2+k \leq n \leq (2-1)(k-1)+3$. Thus $n = k+2$. It is not difficult to show that a $k-\gamma_c-$vertex critical graph with $\Delta = 2$ of order $k+2$ is isomorphic to a cycle C_{k+2}. For $\Delta \geq 3$, as the range of n increases, it is difficult to resolve the realizable problem. In the following, we focus on $\Delta = 3$ or 4.

4.5.1 $\Delta = 3$

Our first result in this section is to deny the existence of $3-\gamma_c-$vertex critical graphs with $\Delta = 3$.

Corollary 4.5.1. There is no $3-\gamma_c-$vertex critical graph with $\Delta = 3$.

Proof. By Theorem 4.1.5, n can be 6 or 7. By Theorem 2.1.11, $n \neq 6$. Moreover, Theorem 4.1.6 gives that there is no $3-\gamma_c-$vertex critical graph with $\Delta = 3$ of order 7. Thus there is no $3-\gamma_c-$vertex critical graph with $\Delta = 3$ and this completes the proof. □

For $k \geq 4$, we establish the following theorem.

Theorem 4.5.2. For $k \geq 4$, there is a $k-\gamma_c-$vertex critical graph with $\Delta = 3$ of order $2k$.

Proof. Let $C_k = x_1, x_2, x_3, ..., x_k, x_1$ and $C_k' = y_1, y_2, y_3, ..., y_k, y_1$ be two disjoint cycles of length k. We construct a graph G by adding k edges $x_1y_1, x_2y_2, ..., x_ky_k$ to the two cycles. The following figure shows the graph G. We show that G is a $k-\gamma_c-$vertex critical graph with $\Delta = 3$ of order $2k$.

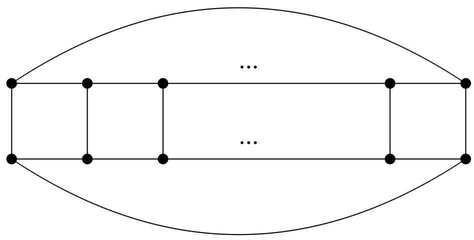

Figure 4.4 : A $k-\gamma_c-$vertex critical graph with $\Delta = 3$ of order $2k$

Clearly, $\{x_1, x_2, ..., x_k\} \succ_c G$. Thus $\gamma_c(G) \leq k$. Suppose that $\gamma_c(G) < k$. Let D be a γ_c-set. Therefore $|D| < k$. Consider k sets $\{x_1, y_1\}, \{x_2, y_2\}, ..., \{x_k, y_k\}$. Since $|D| < k$, at least one of these k sets does not contain any vertex of D. By symmetry, let this set be $\{x_1, y_1\}$. Thus, to dominate x_1, D contains x_2 or x_k. Without loss of generality let $x_2 \in D$. To dominate y_1, D contains y_2 or y_k.

Suppose first that $y_2 \in D$. If $x_k \in D$, then, by the connectedness of $G[D]$, $D \cap \{x_i, y_i\} \neq \emptyset$ for all i where $3 \leq i \leq k$, implying that $|D| \geq |\{x_2, y_2\}| + (k-2) = k$, a contradiction. Therefore $x_k \notin D$. In this case, in order to dominate x_k, we have that x_{k-1} or y_k is in D. In both cases, by the connectedness of $G[D]$, $D \cap \{x_i, y_i\} \neq \emptyset$ for all $3 \leq i \leq k-1$. Thus $\{y_2, x_2, w_3, ..., w_{k-1}\} \subseteq D$ for some $w_i \in \{x_i, y_i\}$. Since $|D| < k$, $D = \{y_2, x_2, w_3, ..., w_{k-1}\}$. Thus D does not dominate z_k where $\{z_k\} = \{x_k, y_k\} - \{w_k\}$. Hence $y_k \in D$. By the connectedness of $G[D]$, $D \cap \{x_i, y_i\} \neq \emptyset$ for all $3 \leq i \leq k-1$, moreover, at least one of $\{x_j, y_j\}$ for some $j \in \{3, 4, ..., k-1\}$ is a subset of D. Therefore $|D| \geq k$, a contradiction. Hence $\gamma_c(G) = k$.

To establish the criticality, consider $G - w_i$ where $w_i \in \{x_i, y_i\}$ and $i \in \{1, 2, 3, ..., k\}$. For $\{z_i\} = \{x_i, y_i\} - \{w_i\}$, we see that $\{z_{i+1}, z_{i+2}, ..., z_k, z_1, z_2, ..., z_{i-1}\} \succ_c G - w_i$. Therefore $\gamma_c(G - w_i) < k$ and G is a $k-\gamma_c-$vertex critical graph. □

Corollary 4.5.3. Let G be a $4-\gamma_c-$vertex critical graph with $\Delta = 3$ of order n. Then $n = 8$.

Proof. Let G be a $4-\gamma_c-$vertex critical graph with $\Delta = 3$. By Theorem 4.1.5, n can be $7, 8$ or 9. Theorem 4.1.4 implies that $n \neq 7$. Theorem 4.1.6 yields also that $n \neq 9$. By Theorem 4.5.2, $n = 8$. □

For $k = 5$ or 6, let us show some $k-\gamma_c-$vertex critical graphs in the following figures. It is a routine exercise to check that they are $k-\gamma_c-$vertex critical graphs.

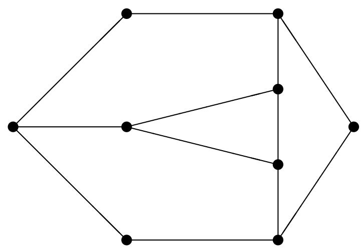

Fig.4.5(a): A $5-\gamma_c-$vertex critical graph of order 9 with $\Delta = 3$

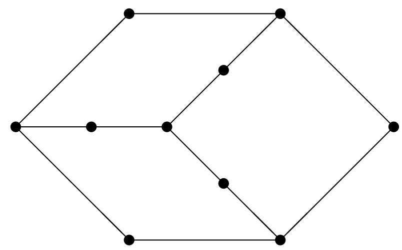

Fig.4.5(b): A $6-\gamma_c-$vertex critical graph of order 10 with $\Delta = 3$

Corollary 4.5.4. Let G be a $5-\gamma_c-$vertex critical graph with $\Delta = 3$ of order n. Then $n = 9, 10$ or 11.

Proof. By Corollary 4.4.2, $n \neq 8$. By Figure 4.5(a), Theorem 4.5.2 and the graph G_1 in Section 4.3, there are $5-\gamma_c-$vertex critical graphs with $\Delta = 3$ of order $9, 10$ or 11. □

For $k = 6$, we have by Corollary 4.4.2 that there is no $6-\gamma_c-$vertex critical graph of order 9. Although we have found $6-\gamma_c-$vertex critical graphs with $\Delta = 3$ of order $10, 12$ or 13 as detailed in Figure 4.5(b), Theorem 4.5.2 and the graph G_2 in Section 4.3, we could not determine the existence of these graphs of order 11. The following table displays all the results that we have for $\Delta = 3$ and $3 \leq k \leq 6$. We use a ✓ to denote *there exists*, a × to denote *there does not exist* and a ? to denote an unresolved case.

Table 1: The existence of $k-\gamma_c-$vertex critical graphs when $\Delta = 3$ for $3 \leq k \leq 6$.

$k\backslash n$	6	7	8	9	10	11	12	13
3	×	×	–	–	–	–	–	–
4	–	×	✓	×	–	–	–	–
5	–	–	×	✓	✓	✓	–	–
6	–	–	–	×	✓	?	✓	✓

4.5.2 $\Delta = 4$

In this subsection we focus on $k-\gamma_c-$vertex critical graphs with $\Delta = 4$ for $k = 3$ or 4. The following two figures show some $k-\gamma_c-$vertex critical graphs for $k = 3$ or 4.

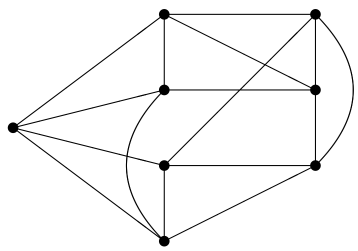

Fig.4.6(a): A $3-\gamma_c-$vertex critical graph of order 8 with $\Delta = 4$

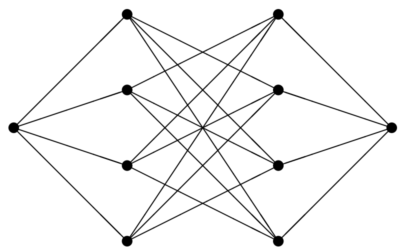

Fig.4.6(b): A $4-\gamma_c-$vertex critical graph of order 10 with $\Delta = 4$

Our first result, is when $k = 3$.

Corollary 4.5.5. There is a $3-\gamma_c-$vertex critical graph with $\Delta = 4$ of order n for all $n = 7, 8$ or 9.

Proof. Let G be a $3-\gamma_c-$vertex critical graph with $\Delta = 4$ of order n. By Theorem 4.1.5, n can be $7, 8$ or 9. Theorem 4.1.6 gives that there exists a $3-\gamma_c-$vertex critical graphs of order 9. By Theorem 2.1.12 and Figure 4.6(a), there are $3-\gamma_c-$vertex critical graphs with $\Delta = 4$ of order $n = 7$ or 8. □

For $k = 4$, by Theorem 4.1.4, Figure 4.6(b) and Theorem 4.1.5, there are $4-\gamma_c-$vertex critical graphs with $\Delta = 4$ of order $n = 8, 10$ or 12. We could not determine the existence of such graphs of order 11. However, we can show, for the smaller case, that there is no $4-\gamma_c-$vertex critical graph with $\Delta = 4$ of order 9 by giving a very long proof as detailed in Theorem 4.5.6. We conclude this work with the following table which displays all the known results when $\Delta = 4$.

Theorem 4.5.6. There is no $4-\gamma_c-$vertex critical graph with $\Delta = 4$ of order 9.

Table 2: The existence of $k-\gamma_c-$vertex critical graphs when $\Delta=4$ for $3\leq k\leq 4$.

$k\backslash n$	7	8	9	10	11	12
3	✓	✓	✓	–	–	–
4	–	✓	×	✓	?	✓

Hereafter, to prove Theorem 4.5.6, let G be a $4-\gamma_c-$vertex critical graph of order 9 with a vertex a of degree $\Delta=4$ and let s be the maximum distance of a vertex from the vertex a in the graph G. For $1\leq i\leq s$, let

L_i : the set of vertices at distance i from a.

Thus $L_1=N(a)$ and $|\cup_{i=2}^{s}L_i|=4$.

Lemma 4.5.7. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta=4$. Then $s\leq 3$ and $|L_3|\leq 1$.

Proof. Suppose that $s\geq 4$. Since $|\cup_{i=2}^{s}L_i|=4$, at least one of L_2 or L_3 contains exactly one vertex. Thus G has a cut vertex contradicting G is $2-$connected. Therefore $s\leq 3$. Suppose to the contrary that $|L_3|\geq 2$. Thus $|L_2|=|L_3|=2$. Let $L_2=\{x_1,x_2\}$ and $L_3=\{y_1,y_2\}$. Since $y_1\in L_3$, it is adjacent to a vertex x_i for some $i\in\{1,2\}$. Similarly, y_2 is adjacent to a vertex x_j for some $j\in\{1,2\}$. By the $2-$connectedness of G, we can find the different i and j. Without loss of generality let $i=1$ and $j=2$. If $x_1x_2\in E(G)$, then $N(y_2)\subseteq N[x_1]$ contradicting Corollary 2.1.23. Thus $x_1x_2\notin E(G)$. Suppose that $x_iy_j\in E(G)$ where $\{i,j\}=\{1,2\}$. Clearly $N(y_i)\subseteq N[y_j]$ contradicting Corollary 2.1.23. Therefore $x_1y_2,x_2y_1\notin E(G)$. Consider $G-a$. Lemma 2.1.22(1) yields that $D_a^c\subseteq\cup_{i=2}^{3}L_i$. As $(G-a)[D_a^c]$ is connected, by Lemma 2.1.22(2), we must have $|D_a^c\cap L_2|=1$. Let $x_i\in D_a^c$. Thus $x_i\succ L_1$. This implies that $N(a)\subseteq N[x_i]$ contradicting Corollary 2.1.23. Thus $|L_3|\leq 1$ and this completes the proof. □

Lemma 4.5.8. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta=4$. Then $N_{L_2}(a')\neq\emptyset$ for all $a'\in L_1$.

Proof. Suppose to the contrary that $N_{L_2}(a')=\emptyset$. Thus $N[a']\subseteq N[a]$ contradicting Lemma 2.1.22(3). This completes the proof. □

We set up the condition for Lemmas 4.5.9 and 4.5.10. Let $a_i,a_j,a_l\in L_1$ and $b'\in L_2$ such that $\{a,b'\}\subseteq N(a_i)\cap N(a_j)\cap N(a_l)$. Moreover, let $Z=\{z_1,z_2,z_t\}$ where $t=2$ or

3 be a set of all possible vertices in $(L_1 \cup L_2) - \{a_i, a_j, a_l\}$ which a_i, a_j and a_l can be adjacent to.

Lemma 4.5.9. If $t = 2$, then at least one of these following situations must occur

(i) $a_jz_1, a_lz_2, a_iz_1, a_iz_2$ and a_ja_l are in $E(G)$ or

(ii) $a_ia_j, a_ia_l \in E(G)$ and a_j and a_l are adjacent to a different vertex in Z.

Proof. Suppose first that there is no edge in $E(G[\{a_i, a_j, a_l\}])$. If a_i is not adjacent to any vertex in Z, then $N(a_i) \subseteq N[a_j]$ contradicting Corollary 2.1.23. Thus a_i is adjacent to a vertex in Z. Similarly, a_j and a_l are adjacent to a vertex in Z. Since $t = 2$, it follows by the pigeonhole principle that at least two of a_i, a_j and a_l are adjacent to the same vertex in Z, a_i and a_j says. Thus $N(a_i) \subseteq N[a_j]$ or $N(a_i) \subseteq N[a_j]$ contradicting Corollary 2.1.23. Thus there is at least one edge in $E(G[\{a_i, a_j, a_l\}])$.

We then suppose that there is exactly one edge $a_ja_l \in E(G[\{a_i, a_j, a_l\}])$. It follows from Lemma 2.1.22(3) that a_j and a_l are adjacent to a different vertex in Z. Moreover $a_i \succ Z$ as otherwise $N(a_i) \subseteq N[a_l]$ or $N(a_i) \subseteq N[a_j]$ contradicting Corollary 2.1.23. This proves *(i)*.

Suppose that there exists a vertex a_i such that $a_i \succ \{a_j, a_l\}$. Thus $N(a_i) = \{a_j, a_l, a, b'\}$ and $\{a_i, a, b'\} \subseteq N(a_j) \cap N(a_l)$. If $a_ja_l \in E(G)$, then $N[a_j] = \{a_i, a_j, a_l, a, b'\} = N[a_l]$ contradicting Lemma 2.1.22(3). Therefore $a_ja_l \notin E(G)$. If a_j is not adjacent to any vertex in Z, then $N(a_j) \subseteq N[a_l]$ contradicting Corollary 2.1.23. Thus a_j is adjacent to a vertex in Z, similarly, so is a_l. But if they are adjacent to the same vertex in Z, then, by $\Delta = 4$, $N(a_j) = N(a_l)$ contradicting Corollary 2.1.23. Thus a_j and a_l are adjacent to a different vertex in Z. This proves *(ii)*. □

Lemma 4.5.10. If $t = 3$, then at least one of these following situations must occur

(i) there is a perfect matching between the sets $\{a_i, a_j, a_l\}$ and Z or

(ii) for some some $\{z, z'\} \subseteq Z$, a_jz, a_lz', a_iz, a_iz' and a_ja_l are in $E(G)$ or

(iii) $a_ia_j, a_ia_l \in E(G)$ and a_j and a_l are adjacent to a different vertex in $\{z, z'\}$.

Proof. We distinguish two cases.

Case 1 : $|N_Z\{a_i, a_j, a_l\}| = 3$.

Suppose first that $N_Z(a_i) = \emptyset$. If a_i is not adjacent to any vertex in $\{a_j, a_l\}$, then $N(a_i) \subseteq N[a_l]$ contradicting Corollary 2.1.23. If a_i is adjacent to exactly one vertex in $\{a_j, a_l\}$, a_j says, then $N[a_i] \subseteq N[a_j]$ contradicting Lemma 2.1.22(3). Thus $a_ia_j, a_ia_l \in E(G)$. By $\Delta = 4$, each of a_j and a_l can be adjacent to at most one vertex in Z. This implies $|N_Z\{a_i, a_j, a_l\}| \leq 2$, a contradiction. Therefore $N_Z(a_i) \neq \emptyset$, similarly, $N_Z(a_j) \neq \emptyset$ and $N_Z(a_l) \neq \emptyset$.

As $\{a,b'\} \subseteq N(a_i) \cap N(a_j) \cap N(a_l)$, by $\Delta = 4$, each of a_i, a_j and a_l is adjacent to at most two vertices in Z. Since $|N_Z\{a_i,a_j,a_l\}| = 3$, there exist two vertices in $\{a_i,a_j,a_l\}$ which are adjacent to a different vertex in Z, without loss of generality let $a_iz_1, a_jz_2 \in E(G)$. If $a_lz_3 \in E(G)$, then $\{a_iz_1, a_jz_2, a_lz_3\}$ is a perfect matching. Suppose that $a_lz_3 \notin E(G)$. Because $N_Z(a_l) \neq \emptyset$, a_l is adjacent to z_1 or z_2. Without loss of generality let $a_lz_1 \in E(G)$. If $a_iz_3 \in E(G)$, then $\{a_iz_3, a_lz_1, a_jz_2\}$ is a perfect matching. We then suppose that $a_iz_3 \notin E(G)$. As $|N_Z\{a_i,a_j,a_l\}| = 3$, we must have $a_jz_3 \in E(G)$. Thus $N(a_j) = \{a,b',z_2,z_3\}$, in particular, $a_ja_i, a_ja_l \notin E(G)$. We show that at least one of a_i and a_l is adjacent to z_2. Suppose to the contrary that a_i and a_l are not adjacent to z_2. If $a_ia_l \in E(G)$, then $N[a_i] = N[a_l]$ contradicting Lemma 2.1.22(3). Thus $a_ia_l \notin E(G)$. Therefore $N(a_i) = \{a,b',z_1\} = N(a_l)$ contradicting Corollary 2.1.23. Hence a_i or a_l is adjacent to z_2.

Clearly if $a_iz_2 \in E(G)$, then $\{a_iz_2, a_jz_3, a_lz_1\}$ is a perfect matching. If $a_lz_2 \in E(G)$, then $\{a_lz_2, a_iz_1, a_jz_3\}$ is a perfect matching. This proves *(i)* and we finish Case 1.

Case 2 : $|N_Z\{a_i,a_j,a_l\}| \leq 2$.

If $|N_Z\{a_i,a_j,a_l\}| < 2$, then it is not difficult to see that there exist two vertices in $\{a_i,a_j,a_l\}$, a_i and a_j say, such that $N(a_i) \subseteq N[a_j]$. This contradicts Corollary 2.1.23. Therefore $|N_Z\{a_i,a_j,a_l\}| = 2$. By Lemma 4.5.9 *(i)* and *(ii)*, we establish *(ii)* and *(iii)*, respectively. This completes the proof. □

Lemma 4.5.11. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. If $L_3 = \{b\}$ and $deg_G(b) = 3$, then $|N_{L_2}(b')| \leq 2$ for all $b' \in L_2$.

Proof. Suppose there exists $b_1 \in L_2$ such that b_1 is adjacent to a_1, a_2 and a_3 in L_1. Since $\Delta = 4$, $N(b_1) = \{a_1,a_2,a_3,b\}$. Let $\{a_4\} = L_1 - \{a_1,a_2,a_3\}$. If $a_4a_i \in E(G)$ for some $i \in \{1,2,3\}$, then $\{a_i,b_1,b\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Thus $a_4a_i \notin E(G)$ for all $i \in \{1,2,3\}$

Claim : $|\{a_1a_2, a_2a_3, a_3a_1\} \cap E(G)| \leq 1$.

Suppose there exists a_i is adjacent to a_j and a_l for $\{i,j,l\} = \{1,2,3\}$. Thus $N(a_i) = \{a_j,a_l,a,b_1\}$. Consider $G - a_i$. By Lemma 2.1.22(1), $\{a_j,a_l,a,b_1\} \cap D^c_{a_i} = \emptyset$. Lemma 2.1.24 gives also that $a_4 \in D^c_{a_i} \cap N(a)$ and $b \in D^c_{a_i} \cap N(b_1)$. As $(G - a_i)[D^c_{a_i}]$ is connected, by Lemma 2.1.22(2), there exists b_2 such that $\{b_2\} \in (L_2 - \{b_1\}) \cap D^c_{a_i}$. Thus $\{a_4,b_2\} \succ \{a_j,a_l\}$. Since $a_4a_j, a_4a_l \notin E(G)$, it follows that $b_2a_j, b_2a_l \in E(G)$. Thus $N(a_j) = \{a,a_i,b_1,b_2\} = N(a_l)$ contradicting Corollary 2.1.23. Hence $|\{a_1a_2,a_2a_3,a_3a_1\} \cap E(G)| \leq 1$ and we settle the claim.

Let $\{b_2,b_3\} = L_2 - \{b_1\}$. We see that $\{b_1,a\} \subseteq N(a_1) \cap N(a_2) \cap N(a_3)$ and $\{b_2,b_3\}$ is the set of all vertices which a_1,a_2 and a_3 can be adjacent to. By Lemma 4.5.9, *(i)* or *(ii)* must be occur for which $\{z_1, z_2\} = \{b_2, b_3\}$.

By the claim, *(ii)* in Lemma 4.5.9 cannot occur. Hence *(i)* occurs. This implies that $N(b) = \{b_1, b_2, b_3\} \subseteq N[a_i]$ contradicting Corollary 2.1.23. Therefore $|N_{L_2}(b')| \leq 2$ for all $b' \in L_2$ and this completes the proof. □

Lemma 4.5.12. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. If $L_3 = \{b\}$ and $deg_G(b) = 3$, then there is at most one vertex $a' \in L_1$ such that $|N_{L_2}(a')| \geq 2$.

Proof. Suppose to the contrary that there exists a_1 and a_2 in L_1 such that $|N_{L_2}(a_1)| \geq 2$ and $|N_{L_2}(a_2)| \geq 2$. Since $N(b) = L_2$, it follows by Corollary 2.1.23 that $|N_{L_2}(a_1)| = 2$ and $|N_{L_2}(a_2)| = 2$. We distinguish two cases.

Case 1 : $N_{L_2}(a_1) = N_{L_2}(a_2)$.

Let $N_{L_2}(a_1) = \{b_1, b_2\}$ and $\{b_3\} = L_2 - \{b_1, b_2\}$. By Lemma 4.5.11, $N_{L_1}(b_i) = \{a_1, a_2\}$ for all $i = 1, 2$. Moreover at least one vertex of b_1 or b_2 is not adjacent to b_3, as otherwise $N[b] \subseteq N[b_3]$ contradicting Lemma 2.1.22(3). Without loss of generality let $b_1 b_3 \notin E(G)$. Clearly $N(b_1) \subseteq N[b_2]$ contradicting Corollary 2.1.23. Case 1 cannot occur.

Case 2 : $N_{L_2}(a_1) \neq N_{L_2}(a_2)$.

Since $|L_2| = 3$, $|N_{L_2}(a_1) \cap N_{L_2}(a_2)| = 1$. Let $N_{L_2}(a_1) = \{b_1, b_2\}$ and $N_{L_2}(a_2) = \{b_2, b_3\}$. Lemma 4.5.11 implies that $N_{L_1}(b_2) = \{a_1, a_2\}$. Let $\{a_3, a_4\} = L_1 - \{a_1, a_2\}$. By Lemmas 4.5.8 and 4.5.11, a_3 and a_4 are adjacent to a different vertex in $\{b_1, b_3\}$. Without loss of generality let $a_3 b_1, a_4 b_3 \in E(G)$. We have that $b_1 b_2, b_1 b_3$ and $b_2 b_3$ are not in $E(G)$ as otherwise $\{b_2, a_2, a\}, \{a_1, b_1, b_3\}$ and $\{b_2, a_1, a\}$ dominate G, respectively, contradicting $\gamma_c(G) = 4$. Thus $N(b_1) = \{a_1, a_3, b\}, N(b_2) = \{a_1, a_2, b\}$ and $N(b_3) = \{a_2, a_4, b\}$. We moreover have that neither $\{a_2 a_3, a_1 a_4\}$ nor $\{a_2 a_4, a_1 a_3\}$ are subset of $E(G)$ as otherwise $\{a_1, b_2, a_2\} \succ_c G$, a contradiction.

Consider $G - b_2$. Lemma 2.1.22(1) thus implies $\{a_1, a_2, b\} \cap D^c_{b_2} = \emptyset$. By Lemma 2.1.24, b_1 or b_2 is in $D^c_{b_2} \cap N(b)$. We may assume without loss of generality that $b_1 \in D^c_{b_2}$. It follows by the connectedness of $(G - b_2)[D^c_{b_2}]$ that $a_3 \in D^c_{b_2}$. Lemma 2.1.22(2) gives also that $|D^c_{b_2} - \{b_1, a_3\}| = 1$. By Lemma 2.1.24, $a_4 \in D^c_{b_2} \cap N(b_3)$. Thus $D^c_{b_2} = \{b_1, a_3, a_4\}$ and $a_3 a_4 \in E(G)$. To dominate a_2, we have $a_2 a_3$ or $a_2 a_4$ is in $E(G)$.

Subcase 2.1 : $a_2 a_3 \in E(G)$.

Thus $a_1 a_4 \notin E(G)$ and $N(a_3) = \{a, a_4, a_2, b_1\}$. Consider $G - a_3$. Lemma 2.1.22(1) yields that $\{a, a_4, a_2, b_1\} \cap D^c_{a_3} = \emptyset$. Since $N(b_3) = \{a_2, a_4, b\}$, it follows from Lemma 2.1.24 that $b \in D^c_{a_3} \cap N(b_3)$. Moreover, $a_1 \in D^c_{a_3}$ to dominate a. By the connectedness of $(G-a_3)[D^c_{a_3}]$, $b_2 \in D^c_{a_3}$. By Lemma 2.1.22(2), $D^c_{a_3} = \{b,a_1,b_2\}$. But $D^c_{a_3}$ does not dominate a_4 contradicting $D^c_{a_3} \succ_c G-a_3$.

Subcase 2.2 : $a_2a_3 \notin E(G)$.

Therefore $a_2a_4 \in E(G)$ and $N(a_4) = \{a,a_2,a_3,b_3\}$. As $\{a_2a_4,a_1a_3\} \nsubseteq E(G)$, we have $a_1a_3 \notin E(G)$. Consider $G-a_4$. Lemma 2.1.22(1) implies that $\{a,a_2,a_3,b_3\} \cap D^c_{a_4} = \emptyset$. Since $N(b_3) = \{a_4,a_2,b\}$, it follows from Lemma 2.1.24 that $b \in D^c_{a_4} \cap N(b_3)$. To dominate a, $a_1 \in D^c_{a_4}$. By the connectedness of $(G-a_4)[D^c_{a_4}]$, b_1 or b_2 is in $D^c_{a_4}$. Lemma 2.1.22(2) yields that $D^c_{a_4}$ is either $\{b,a_1,b_1\}$ or $\{b,a_1,b_2\}$. But $\{b,a_1,b_1\}$ does not dominate a_2 and $\{b,a_1,b_2\}$ does not dominate a_3. These contradict $D^c_{a_4} \succ_c G-a_4$, thus case 2 cannot occur.

Hence, there is at most one vertex $a' \in L_1$ such that $|N_{L_2}(a')| \geq 2$ and this completes the proof. □

Lemma 4.5.13. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. If $L_3 = \{b\}$, then $deg_G(b) = 2$.

Proof. Suppose to the contrary that $deg_G(b) = 3$. Since $|\cup_{i=2}^{3} L_i| = 4$, $|L_2| = 3$. Thus $b \succ L_2$.

Suppose that there exists $a_1 \in L_1$ such that $|N_{L_2}(a_1)| \geq 2$. Hence $|N_{L_2}(a_1)| = 2$ as otherwise $N(b) \subseteq N[a_1]$ contradicting Corollary 2.1.23. Let $N_{L_2}(a_1) = \{b_1,b_2\}$. Moreover, let $\{a_2,a_3,a_4\} = L_1 - \{a_1\}$ and $\{b_3\} = L_2 - \{b_1,b_2\}$. If $b_ib_3 \in E(G)$ for some $i \in \{1,2\}$, then $\{a,a_1,b_i\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Thus $b_ib_3 \notin E(G)$. We see that $\{b,a_1\} \subseteq N(b_1) \cap N(b_2)$. Lemma 2.1.22(3) and Corollary 2.1.23 then imply that b_1 and b_2 are adjacent to a different vertex in $\{a_2,a_3,a_4\}$. Without loss of generality let $b_1a_2, b_2a_3 \in E(G)$. Lemma 4.5.11 yields that $N_{L_1}(b_1) = \{a_1,a_2\}$ and $N_{L_2}(b_2) = \{a_1,a_3\}$. By Lemma 4.5.8, $a_4b_3 \in E(G)$. Consider $G-a$. Lemma 2.1.22(1) implies that $D^c_a \subseteq \{b,b_1,b_2,b_3\}$. As $(G-a)[D^c_a]$ is connected, $D^c_a \neq \{b_1,b_2,b_3\}$. By Lemma 2.1.24, $b_3 \in D^c_a \cap N(a_4)$. Thus either b_1 or b_2 is in D^c_a. If $b_1 \in D^c_a$, then $b_3a_3 \in E(G)$ to dominate a_3. So $N_{L_2}(a_1) = \{b_1,b_2\}$ and $N_{L_2}(a_3) = \{b_2,b_3\}$ contradicting Lemma 4.5.12. Therefore $b_1 \notin D^c_a$ and $b_2 \in D^c_a$. To dominate a_2, $b_3a_2 \in E(G)$. Thus $N_{L_2}(a_1) = \{b_1,b_2\}$ and $N_{L_2}(a_2) = \{b_1,b_3\}$ contradicting Lemma 4.5.12. So $|N_{L_2}(a')| \leq 1$ for all $a' \in L_1$.

We finally consider $G-a$. Lemma 2.1.22(1) yields that $D^c_a \subseteq L_2 \cup L_3$. Suppose that $D^c_a \subseteq L_2$ By the connectedness of $(G-a)[D^c_a]$ and $|L_2| = 3$, there exists $b_i \in L_2$ such that $b_i \succ L_2$. Therefore $N[b] \subseteq N[b_i]$ contradicting Lemma 2.1.22(3). Thus $|D^c_a \cap L_2| \leq 2$. As $D^c_a \succ L_1$, by Lemma 4.5.11, we must have $|D^c_a \cap L_2| = 2$. Let $b_i,b_j \in D^c_a \cap L_2$. Lemma 4.5.11 gives also that $|N_{L_1}(b_i)| = |N_{L_1}(b_j)| = 2$ and $N_{L_1}(b_i) \cap N_{L_1}(b_j) = \emptyset$. Since $|N_{L_2}(a')| \leq 1$ for all $a' \in L_1$, no vertex in L_1 is adjacent to b_l where $\{b_l\} =$

$L_2 - \{b_i, b_j\}$. This contradicts $b_l \in L_2$. Hence $deg_G(b) \leq 2$. As G is $2-$connected, we must have $deg_G(b) = 2$ and this completes the proof. □

Lemma 4.5.14. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. Moreover, let $L_3 = \{b\}$ and $L_2 = \{b_1, b_2, b_3\}$. If $N(b) = \{b_1, b_2\}$, then $G[\{b, b_1, b_2, b_3\}]$ is a path of length three with end vertices b_i and b_3 for some $i \in \{1, 2\}$.

Proof. Clearly, $bb_1, bb_2 \in E(G)$ and $bb_3 \notin E(G)$. By Lemma 2.1.22(3), $b_1b_2 \notin E(G)$ as otherwise $N[b] \subseteq N[b_1]$. If b_3 is not adjacent to b_1 and b_2, then $N(b_3) \subseteq N[a]$ contradicting Corollary 2.1.23. Thus b_3 is adjacent to at least one vertex of b_1 or b_2. If b_3 is adjacent to both b_1 and b_2, then $N(b) \subseteq N[b_3]$ contradicting Corollary 2.1.23. Thus $G[\{b, b_1, b_2, b_3\}]$ is a path with end vertices b_i and b_3 for some $i \in \{1, 2\}$. This completes the proof. □

Lemma 4.5.15. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. If $deg_G(b) = 2$, then $|N_{L_1}(b')| \leq 2$ for all $b' \in L_2$.

Proof. By Lemma 4.5.14, let $G[L_2 \cup L_3]$ be a path b_1, b, b_2, b_3. Because $\{b, b_3\} \subseteq N(b_2)$ and $\Delta = 4$, it follows that $|N_{L_1}(b_2)| \leq 2$.

We first suppose to the contrary that $|N_{L_1}(b_1)| \geq 3$. By $\Delta = 4$, $|N_{L_1}(b_1)| = 3$. Let $N_{L_1}(b_1) = \{a_1, a_2, a_3\}$ and $\{a_4\} = L_1 - \{a_1, a_2, a_3\}$. If $a_ib_2 \in E(G)$ for some $i \in \{1, 2, 3\}$, then $N(b) \subseteq N[a_i]$ contradicting Corollary 2.1.23. Thus $a_ib_2 \notin E(G)$ for all $i \in \{1, 2, 3\}$.

Claim 1 : $|\{a_1a_2, a_2a_3, a_3a_1\}| \leq 1$.

Suppose without loss of generality that $\{a_1a_2, a_2a_3\} \subseteq E(G)$. Thus $N(a_2) = \{a, a_1, a_3, b_1\}$ and $\{a_2, a, b_1\} \subseteq N(a_1) \cap N(a_3)$. Lemma 2.1.22(3) and Corollary 2.1.23 imply that a_1 and a_3 are adjacent to a different vertex in $\{a_4, b_3\}$. Without loss of generality let $a_1b_3, a_3a_4 \in E(G)$. Consider $G - a_2$. By Lemma 2.1.22(1), $\{a, a_1, a_3, b_1\} \cap D^c_{a_2} = \emptyset$. To dominate b_1 and a, $D^c_{a_2}$ contains b and a_4, respectively. As $(G - a_2)[D^c_{a_2}]$ is connected, $b_2 \in D^c_{a_2}$. Lemma 2.1.22(2) yields $D^c_{a_2} = \{b, a_4, b_2\}$. But $D^c_{a_2}$ does not dominate a_1 contradicting $D^c_{a_2} \succ_c G - a_2$. Therefore $|\{a_1a_2, a_2a_3, a_3a_1\}| \leq 1$, thus establishing Claim 1.

We see that $\{a, b_1\} \subseteq N(a_1) \cap N(a_2) \cap N(a_3)$ and $\{a_4, b_3\}$ is the set of all possible vertices which a_1, a_2 and a_3 can be adjacent to. We have by Lemma 4.5.9 that *(i)* or *(ii)* must occur which $\{z_1, z_2\} = \{a_4, b_3\}$.

By Claim 1, *(ii)* in Lemma 4.5.9 cannot occur. Hence *(i)* occurs. Thus $a_ia_4, a_ib_3 \in E(G)$. This implies that $\{a_i, b_1, b\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Therefore $|N_{L_1}(b_1)| \leq 2$.

We now suppose to the contrary that $|N_{L_1}(b_3)| \geq 3$. Similarly, $|N_{L_1}(b_3)| = 3$. Let $N_{L_1}(b_3) = \{a_2, a_3, a_4\}$ and $\{a_1\} = L_1 - \{a_2, a_3, a_4\}$.

Claim 2 : $|\{a_2a_3, a_3a_4, a_4a_2\}| \leq 1$.

Suppose without loss of generality that $\{a_2a_3, a_3a_4\} \subseteq E(G)$. Thus $N(a_3) = \{a, a_2, a_4, b_3\}$. Consider $G - a_3$. By Lemma 2.1.22(1), $\{a, a_2, a_4, b_3\} \cap D^c_{a_3} = \emptyset$. To dominate a and b_3, a_1 and b_2 are respectively in $D^c_{a_3}$. Lemma 2.1.22(2) gives $|D^c_{a_3} - \{a_1, b_2\}| = 1$. Since $b_1b_2 \notin E(G)$, by the connectedness of $(G - a_3)[D^c_{a_3}]$, $a_1b_2 \in E(G)$. If $b_1 \in D^c_{a_3}$, then $a_1b_1 \in E(G)$. Thus $N(b) \subseteq N[a_1]$ contradicting Corollary 2.1.23. Therefore $b_1 \notin D^c_{a_3}$, moreover, $b \in D^c_{a_3}$ to dominate b_1. That is $D^c_{a_3} = \{a_1, b_2, b\}$. Hence $\{a_1, b_2\} \succ \{a_2, a_4\}$. Since $\{a_3, a, b_3\} \subseteq N(a_2) \cap N(a_4)$ and $\Delta = 4$, it follows that a_2 and a_4 are not adjacent to b_1. Therefore b_1 is not adjacent to any vertex in L_1 contradicting $b_1 \in L_2$. Thus $|\{a_2a_3, a_3a_4, a_4a_2\}| \leq 1$ and we settle Claim 2.

We see that $\{a, b_3\} \subseteq N(a_2) \cap N(a_3) \cap N(a_4)$ and $\{a_1, b_1, b_2\}$ is the set of all possible vertices which a_2, a_3 and a_4 can be adjacent to. In view of Lemma 4.5.10, *(i)*, *(ii)* or *(iii)* must occur.

By Claim 2, *(iii)* in Lemma 4.5.10 cannot occur. Suppose that *(ii)* occurs. Thus $\{z, z'\} \neq \{b_1, b_2\}$ as otherwise $N(b) \subseteq N(a_i)$ contradicting Corollary 2.1.23. We have two more cases that $\{z, z'\} = \{a_1, b_1\}$ or $\{z, z'\} = \{a_1, b_2\}$. If $\{z, z'\} = \{a_1, b_1\}$, then $a_ia_1, a_ib_1 \in E(G)$. This implies that $\{a_i, b_1, b_3\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Hence $\{z, z'\} = \{a_1, b_2\}$. Thus $a_1a_j, b_2a_l, a_1a_i, b_2a_i \in E(G)$. We see that $N_{L_1}(b_2) = \{a_l, a_i\}$ and $N_{L_1}(b_3) = \{a_j, a_l, a_i\}$. By Lemma 4.5.8, $a_1b_1 \in E(G)$. Thus $\{a_i, a_1, b_2\} \succ_c G$, a contradiction. That is *(ii)* cannot occur.

Hence *(i)* occurs. Without loss of generality let $a_2a_1, a_3b_1, a_4b_2 \in E(G)$. Moreover, by Corollary 2.1.23, $a_3b_2, a_4b_1 \notin E(G)$ as otherwise $N(b) \subseteq N[a_3]$ and $N(b) \subseteq N[a_4]$ respectively. Consider $G - b_3$. Lemma 2.1.22(1) gives that $\{a_2, a_3, a_4, b_2\} \cap D^c_{b_3} = \emptyset$. By the connectedness of $(G - b_3)[D^c_{b_3}]$, $D^c_{b_3}$ is either $\{a, a_1, b_1\}$ or $\{a_1, b_1, b\}$. If $D^c_{b_3} = \{a, a_1, b_1\}$, then $a_1b_1, a_1b_2 \in E(G)$. Thus $N(b) \subseteq N[a_1]$ contradicting Corollary 2.1.23. Hence $D^c_{b_3} = \{a_1, b_1, b\}$ and $b_1a_1 \in E(G)$. Thus $\{b_1, a_3, b_3\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Therefore $|N_{L_1}(b_3)| \leq 2$ and this completes the proof. □

Lemma 4.5.16. Let G be a $4 - \gamma_c -$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. Then $L_3 = \emptyset$.

Proof. Suppose to the contrary that $L_3 \neq \emptyset$. Lemma 4.5.7 thus implies $|L_3| = 1$. Let $L_3 = \{b\}$. By Lemmas 4.5.13 and 4.5.14, there exist b_1, b_2 and b_3 such that $L_2 = \{b_1, b_2, b_3\}$ and $G[L_2 \cup L_3] = b_1, b, b_2, b_3$ is a path of length three. Consider $G - a$. Lemma 2.1.22(1) gives that $D^c_a \subseteq L_2 \cup L_3$. As $(G - a)[D^c_a]$ is connected, D^c_a must be

$\{b,b_2,b_3\}$ or $\{b_1,b,b_2\}$.

Suppose first that $D_a^c = \{b,b_2,b_3\}$. Thus $\{b_2,b_3\} \succ L_1$. Since $|L_1| = 4$, it follows from Lemma 4.5.15 that $|N_{L_1}(b_2)| = 2, |N_{L_1}(b_3)| = 2$ and $N_{L_1}(b_2) \cap N_{L_1}(b_3) = \emptyset$. Without loss of generality let $N_{L_1}(b_2) = \{a_1,a_2\}$ and $N_{L_1}(b_3) = \{a_3,a_4\}$. Since $a_1b_2, a_2b_2 \in E(G)$ and $N(b) = \{b_1,b_2\}$, it follows by Corollary 2.1.23 that $a_1b_1, a_2b_1 \notin E(G)$. Consider $G - b_3$. Lemma 2.1.22(1) then implies that $\{a_4,a_3,b_2\} \cap D_{b_3}^c = \emptyset$. Lemma 2.1.24 gives that a_1 or a_2 is in $D_{b_3}^c \cap N(a)$, similarly, $b_1 \in D_{b_3}^c \cap N(b)$. Thus $(G - b_3)[D_{b_3}^c]$ is not connected, a contradiction. Therefore $D_a^c \neq \{b,b_2,b_3\}$.

Hence $D_a^c = \{b_1,b,b_2\}$. Similarly, let $N_{L_1}(b_1) = \{a_1,a_2\}$ and $N_{L_1}(b_2) = \{a_3,a_4\}$. Since $N(b) = \{b_1,b_2\}$, it follows by Corollary 2.1.23 that a_1b_2, a_2b_2, a_3b_1 and a_4b_1 are not in $E(G)$. Suppose that $|N_{L_1}(b_3)| = 1$. If $b_3a_i \in E(G)$ for some $i \in \{3,4\}$, then $N[b_3] \subseteq N[b_2]$ contradicting Lemma 2.1.22(3). If $i \in \{1,2\}$, then $N_{L_2}(a_3) = \{b_2\} = N_{L_2}(a_4)$. Lemma 2.1.22(3) and Corollary 2.1.23 give that a_3 and a_4 are adjacent to a different vertex in $\{a_1,a_2\}$. Let $a_ja_i \in E(G)$ for some $j \in \{3,4\}$. Hence $N(b_3) \subseteq N(a_j)$ contradicting Corollary 2.1.23. Therefore $|N_{L_1}(b_3)| \geq 2$. By Lemma 4.5.15, $|N_{L_1}(b_3)| = 2$. If $N_{L_1}(b_3) = \{a_3,a_4\}$, then $N[b_3] \subseteq N[b_2]$ contradicting Lemma 2.1.22(3). If $N_{L_1}(b_3) = \{a_1,a_2\}$, then $\{a_1,b_3,b_2\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Thus b_3 is adjacent to one vertex in $\{a_1,a_2\}$ and one vertex in $\{a_3,a_4\}$. Without loss of generality let $b_3a_2, b_3a_4 \in E(G)$.

Consider $G - a_4$. Lemma 2.1.22(1) yields $\{b_2,b_3,a\} \cap D_{a_4}^c = \emptyset$. To dominate b_3, $a_2 \in D_{a_4}^c$. Lemma 2.1.24 gives also that $b_1 \in D_{a_4}^c \cap N(b)$. To dominate b_2, a_3 or b is in $D_{a_4}^c$. Thus, by Lemma 2.1.22(2), $D_{a_4}^c$ is $\{a_2,b_1,a_3\}$ or $\{a_2,b_1,b\}$. If $D_{a_4}^c = \{a_2,b_1,a_3\}$, then by the connectedness of $(G - a_4)[D_{a_4}^c]$, $a_2a_3 \in E(G)$. If $D_{a_4}^c = \{a_2,b_1,b\}$, then $a_2a_3 \in E(G)$ because $D_{a_4}^c \succ_c G - a_4$. In both cases, $a_2a_3 \in E(G)$. Thus $\{b_1,a_2,b_3\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Thus $L_3 = \emptyset$ and this completes the proof. □

Lemma 4.5.17. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. Moreover, let $x \in L_2$ and $y \in L_1$. If $xy \in E(G)$, then $\{x,y\}$ does not dominate L_2.

Proof. By Lemma 4.5.16, $L_3 = \emptyset$. If $\{x,y\} \succ L_2$, then $\{x,y,a\} \succ_c G$ contradicting $\gamma_c(G) = 4$. This completes the proof. □

Lemma 4.5.18. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. Then $G[L_2]$ is isomorphic to either P_4 or C_4.

Proof. Lemma 4.5.16 implies that $|L_2| = 4$. Consider $G - a$. By Lemma 2.1.22(1), $D_a^c \subseteq L_2$. Since $(G - a)[D_a^c]$ is connected, it contains a path b_1, b_2, b_3 as a spanning subgraph where $\{b_1,b_2,b_3\} \subseteq L_2$. Let $\{b_4\} = L_2 - \{b_1,b_2,b_3\}$. Let $x \in L_1$ such that

$xb_2 \in E(G)$. Thus b_4 is adjacent to a vertex in D_a^c. If $b_2b_4 \in E(G)$, then $\{x, b_2\} \succ L_2$ contradicting Lemma 4.5.17. Thus $b_1b_4 \in E(G)$ or $b_3b_4 \in E(G)$. Moreover, by Lemma 4.5.17, every vertex in L_2 is adjacent to at most two vertices in L_2. This implies that $G[L_2]$ is isomorphic to either P_4 or C_4 and this completes the proof. □

Lemma 4.5.19. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. Then $|N_{L_2}(x)| \leq 2$ for all $x \in L_1$.

Proof. Suppose that x is adjacent to three vertices b_1, b_2 and b_3 in L_2. Lemma 4.5.18 gives that the vertex in $L_2 - \{b_1, b_2, b_3\}$ is adjacent to b_i for some $i \in \{1,2,3\}$. Thus $\{b_i, x\} \succ_c L_2$ contradicting Lemma 4.5.17. Thus $|N_{L_2}(x)| \leq 2$ and this completes the proof. □

Lemma 4.5.20. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. If $(G-a)[D_a^c]$ is a path b_1, b_2, b_3 and $G[L_2]$ is the cycle b_1, b_2, b_3, b_4, b_1, then $|N_{L_1}(b_2)| \leq 1$.

Proof. Suppose to the contrary that $|N_{L_1}(b_2)| \geq 2$. By $\Delta = 4$, $|N_{L_1}(b_2)| = 2$. Let $N_{L_1}(b_2) = \{a_2, a_3\}$. Let $\{a_1, a_4\} = L_1 - \{a_2, a_3\}$. By Lemma 4.5.17, a_2b_4 and a_3b_4 are not in $E(G)$.

Since $\{b_1, b_2, b_3\} \succ_c G-a$, $\{b_1, b_3\} \succ \{a_1, a_4\}$. If $b_ia_1, b_ia_4 \in E(G)$ for some $i \in \{1,3\}$, then $\{b_2, b_i, a_2\} \succ_c G$ contradicting $\gamma_c(G = 4$. Thus b_1 and b_3 are adjacent to a different vertex in $\{a_1, a_4\}$. Without loss of generality let $b_1a_1, b_3a_4 \in E(G)$. If a_ia_j for some $i \in \{2,3\}$ and for some $j \in \{1,4\}$, then, for $y \in \{b_1, b_3\}$ such that $ya_{5-j} \in E(G)$, we have that $\{a_i, b_2, y\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Thus $a_ia_j \notin E(G)$ for all $i \in \{2,3\}$ and $j \in \{1,4\}$. We see that $\{a, b_2\} \subseteq N(a_2) \cap N(a_3)$. By Corollary 2.1.23, a_2 and a_3 are adjacent to a different vertex in $\{b_1, b_3\}$. Without loss of generality let $a_2b_1, a_3b_3 \in E(G)$.

Because $b_4 \in L_2$ and $b_4a_2, b_4a_3 \notin E(G)$, it follows that b_4a_1 or b_4a_4 is in $E(G)$. Therefore $\{a_1, a, a_3\} \succ_c G$ or $\{a_2, a, a_4\} \succ_c G$ respectively. These contradict $\gamma_c(G) = 4$. Thus $|N_{L_1}(b_2)| \leq 1$ and this completes the proof. □

Lemma 4.5.21. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. If $(G-a)[D_a^c]$ is a path b_1, b_2, b_3 and $G[L_2]$ is the cycle b_1, b_2, b_3, b_4, b_1, then $|N_{L_1}(b_3)| \leq 1$.

Proof. Suppose to the contrary that $|N_{L_1}(b_3)| \geq 2$. By $\Delta = 4$, $N_{L_1}(b_3) = 2$. Let $N_{L_1}(b_3) = \{a_3, a_4\}$ and $\{a_1, a_2\} = L_1 - \{a_3, a_4\}$. Since $\{b_1, b_2, b_3\} \succ_c G-a$, $\{b_1, b_2\} \succ \{a_1, a_2\}$. By Lemma 4.5.20, b_2 is adjacent to at most one vertex in $\{a_1, a_2\}$. Thus b_1 is adjacent to at least one vertex in $\{a_1,a_2\}$. Without loss of generality let $b_2a_1 \notin E(G)$ and $b_1a_1 \in E(G)$. We distinguish two cases.

Case 1 : $b_2a_2 \in E(G)$.

By Lemma 4.5.20, $N_{L_1}(b_2) = \{a_2\}$. Since $b_3 \succ \{b_2,b_4,a_3,a_4\}$, it follows from Lemma 4.5.17 that $a_3b_1, a_4b_1 \notin E(G)$. Moreover if $a_1a_i \in E(G)$ for some $i \in \{3,4\}$, then $\{a_i,b_3,b_2\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Thus $a_1a_i \notin E(G)$ for all $i \in \{3,4\}$.

Since $b_2 \succ \{b_1,b_3,a_2\}$, it follows from Lemma 4.5.17 that $a_2b_4 \notin E(G)$. Moreover if $b_4a_1 \in E(G)$, then let $b_1' = b_2, b_2' = b_3, b_3' = b_4$ and $b_4' = b_1$. Thus $\{b_1',b_2',b_3'\} \succ_c G - a$ and b_1',b_2',b_3' is a path but $|N_{L_2}(b_2')| \geq 2$ contradicting Lemma 4.5.20. Hence $b_4a_1 \notin E(G)$. Because $b_4 \in L_2$, it is adjacent to at least one vertex in $\{a_3,a_4\}$. Without loss of generality let $b_4a_3 \in E(G)$.

We now have that $\{a,b_3\} \subseteq N(a_3) \cap N(a_4)$. By Corollary 2.1.23, $a_4a_2 \in E(G)$ and $a_2a_3 \notin E(G)$. Consider $G - a_4$. Lemma 2.1.22(1) implies that $\{a,b_3,a_2\} \cap D^c_{a_4} = \emptyset$. Lemma 2.1.24 yields that $b_1 \in D^c_{a_4} \cap N(b_2)$ and $b_4 \in D^c_{a_4} \cap N(a_3)$. Lemma 2.1.22(2) gives also that $|D^c_{a_4} - \{b_4,b_1\}| = 1$. Moreover, by Lemma 2.1.24, a_1 or a_3 is in $D^c_{a_4} \cap N(a)$. If $a_3 \in D^c_{a_4}$, then $b_1a_2 \in E(G)$ because $D^c_{a_4} \succ a_2$. Thus $\{a_2,a,a_3\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Therefore $a_1 \in D^c_{a_4}$ and, similarly, $b_1a_2 \notin E(G)$. Thus $a_1a_2 \in E(G)$. We then have that $\{a_2,b_2,b_3\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Hence, Case 1 cannot occur.

Case 2 : $b_2a_2 \notin E(G)$.

Because $\{b_1,b_2,b_3\} \succ_c G - a$, $b_1 \succ \{a_1,a_2\}$. Since $b_2 \in L_2$, it is adjacent to at least one vertex in $\{a_3,a_4\}$. Without loss of generality let $b_2a_3 \in E(G)$. Let $b_1' = b_3, b_2' = b_2, b_3' = b_1$ and $b_4' = b_4$. Thus $\{b_1',b_2',b_3'\} \succ_c G - a$ and $G[L_2]$ is the cycle b_1',b_2',b_3',b_4',b_1'. Moreover, $|N_{L_1}(b_3')| = 2$. We can show that Case 2 cannot occur by the same arguments as Case 1.

Therefore $|N_{L_1}(b_3)| \leq 1$ and this completes the proof. □

Lemma 4.5.22. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. Then $G[L_2]$ is a path.

Proof. Suppose to the contrary that $G[L_2]$ is not a path. By Lemma 4.5.18, $G[L_2]$ is a cycle. Let $G[L_2] = b_1,b_2,b_3,b_4,b_1$. Consider $G - a$. Lemma 2.1.22(1) yields that $D^c_a \subseteq L_2$. Since $(G-a)[D^c_a]$ is connected, it is a path. Without loss of generality let $(G-a)[D^c_a]$ be b_1,b_2,b_3. By Lemmas 4.5.20 and 4.5.21, $|N_{L_1}(b_2)| \leq 1$ and $|N_{L_1}(b_3)| \leq 1$. Since $\{b_1,b_2,b_3\} \succ_c L_1$, b_1 is adjacent to at least two vertices in L_1 and b_2 and b_3 are adjacent to a different vertex in $L_1 - N(b_1)$. Let $b_1' = b_3, b_2' = b_2, b_3' = b_1$ and $b_4' = b_4$. We have that $\{b_1',b_2',b_3'\} \succ_c G - a$ and b_1',b_2',b_3',b_4',b_1' is a cycle. But $|N_{L_1}(b_3')| \geq 2$

contradicting Lemma 4.5.21. Thus $G[L_2]$ is a path and this completes the proof. □

Lemma 4.5.23. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. If $D_a^c = \{b_1, b_2, b_3\}$ and $G[L_2]$ is a path b_1, b_2, b_3, b_4, then $|N_{L_1}(b_3)| = 1$.

Proof. Suppose to the contrary that $|N_{L_1}(b_3)| \geq 2$. By $\Delta = 4$, $|N_{L_1}(b_3)| = 2$. Let $N_{L_1}(b_3) = \{a_3, a_4\}$. Since $D_a^c = \{b_1, b_2, b_3\}$, $\{b_1, b_2\} \succ L_1$. If b_1 does not have a private neighbor respect to D_a^c in L_1, then $\{b_2, b_3\} \succ_c G - a$ contradicting Lemma 2.1.22(2). Thus b_1 has a private neighbor a_1 in $L_1 - \{a_3, a_4\}$. That is $a_1b_2, a_1b_3 \notin E(G)$. Let $\{a_2\} = L_1 - \{a_1, a_2, a_3\}$. We distinguish two cases.

Case 1 : $b_2a_2 \notin E(G)$.

Thus $b_1a_2 \in E(G)$. Because $b_2 \in L_2$, it is adjacent to a vertex in L_1. But $b_2a_1, b_2a_2 \notin E(G)$. It is adjacent to a vertex in $\{a_3, a_4\}$. Without loss of generality let $b_2a_3 \in E(G)$.

We have that $b_1a_i \in E(G)$ for all $i \in \{1, 2\}$ and $b_2a_3, b_3a_3 \in E(G)$. Hence $b_4a_i \notin E(G)$ for all $i \in \{1, 2\}$ as otherwise $\{a_i, a, a_3\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Thus $N[b_4] \subseteq N[b_3]$ contradicting Lemma 2.1.22(3). Case 1 cannot occur.

Case 2 : $b_2a_2 \in E(G)$.

Recall that $a_1b_2, a_1b_3 \notin E(G)$. Since $b_2 \succ \{b_1, b_3, a_2\}$, it follows from Lemma 4.5.17 that $a_2b_4 \notin E(G)$.

If $a_ib_1 \in E(G)$ for some $i \in \{3, 4\}$, then $\{b_3, a_i, a\} \succ_c G$, contradicting $\gamma_c(G) = 3$. Moreover if $a_ia_1 \in E(G)$ for some $i \in \{3, 4\}$, then $\{a_i, b_3, b_2\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Therefore $a_ib_1, a_ia_1 \notin E(G)$ for all $i \in \{3, 4\}$.

We see that $\{a, b_3\} \subseteq N(a_3) \cap N(a_4)$. Corollary 2.1.23 implies that there exist different vertices x and y in $\{a_2, b_2, b_4\}$ such that $xa_3, ya_4 \in E(G)$. We distinguish three more subcases.

Subcase 2.1 : $\{x, y\} = \{b_2, b_4\}$.

Without loss of generality let $a_3b_2, a_4b_4 \in E(G)$. This implies that $b_4a_3 \notin E(G)$ otherwise $\{b_2, a_3, a\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Moreover if $b_4a_1 \in E(G)$, then $\{a_1, a, a_3\} \succ_c G$, again a contradiction. Thus $N[b_4] = \{b_4, b_3, a_4\} \subseteq N[a_4]$ contradicting Lemma 2.1.22(3). This subcase cannot occur.

Subcase 2.2 : $\{x, y\} = \{a_2, b_2\}$.

Without loss of generality let $a_3a_2, a_4b_2 \in E(G)$. Recall that $a_2b_4 \notin E(G)$. We see that $\{b_3, b_4\} \subseteq N[b_3] \cap N[b_4]$ and $N[b_3] = \{b_4, b_3, b_2, a_3, a_4\}$. By Lemma 2.1.22(3), b_4 is adjacent to a_1. Thus $\{a_1, a, a_4\} \succ_c G$, a contradiction. This subcase cannot occur.

Subcase 2.3 : $\{x, y\} = \{a_2, b_4\}$.

Without loss of generality let $a_3a_2, a_4b_4 \in E(G)$. Lemma 4.5.17 yields that $b_2a_4 \notin$

$E(G)$. Recall that $b_4a_2 \notin E(G)$. Thus $b_4a_1 \in E(G)$ as otherwise $N[b_4] \subseteq N[b_3]$ contradicting 2.1.22(3). Consider $G-a_3$. Lemma 2.1.22(1) implies that $\{a,a_2,b_3\} \cap D^c_a = \emptyset$. Lemma 2.1.24 gives also that $b_1 \in D^c_{a_3} \cap N(b_2)$ and $b_4 \in D^c_{a_3} \cap N(a_4)$. As $(G-a_3)[D^c_{a_3}]$ is connected, by Lemma 2.1.22(2), we must have $D^c_{a_3} = \{b_1,a_1,b_4\}$. Thus a_1a_2 or b_1a_2 is in $E(G)$. If $a_1a_2 \in E(G)$, then $\{a_1,b_4,b_3\} \succ_c G$ contradicting $\gamma_c(G)=4$. Therefore $b_1a_2 \in E(G)$. Thus $\{a_2,a,a_4\} \succ_c G$ again a contradiction. This subcase cannot occur.

Hence $|N_{L_1}(b_3)| = 1$ and this completes the proof. □

Lemma 4.5.24. Let G be a $4-\gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. If $D^c_a = \{b_1,b_2,b_3\}$ and $G[L_2]$ is a path b_1,b_2,b_3,b_4, then $|N_{L_1}(b_2)| = 1$.

Proof. Suppose to the contrary that $|N_{L_1}(b_2)| \geq 2$. By $\Delta = 4$, $|N_{L_1}(b_2)| = 2$. Let $N_{L_1}(b_2) = \{a_2,a_3\}$. Since $b_2 \succ \{b_1,b_3,a_2,a_3\}$, it follows from Lemma 4.5.17 that $b_4a_2, b_4a_3 \notin E(G)$. Because $D^c_a = \{b_1,b_2,b_3\}$, b_1 has at least one private neighbor with respect to D^c_a in L_1, a_1 say. Thus $b_1a_1 \in E(G)$ and $a_1b_2, a_1b_3 \notin E(G)$. Moreover, let $\{a_4\} = L_1 - \{a_1,a_2,a_3\}$. Since $b_3 \in L_2$, $N_{L_1}(b_3) \neq \emptyset$. By Lemma 4.5.23, $|N_{L_1}(b_3)| = 1$. We distinguish two cases.

Case 1 : $N_{L_1}(b_3) \neq \{a_4\}$.

Since $a_1b_3 \notin E(G)$, b_3 is adjacent to a vertex in $\{b_2,b_3\}$. Without loss of generality let $b_3a_3 \in E(G)$. Because $D^c_a = \{b_1,b_2,b_3\}$, $b_1a_4 \in E(G)$. Thus $a_1b_4 \notin E(G)$ as otherwise $\{a_1,b_1,b_2\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Consider $G-b_2$. By Lemma 2.1.22(1), $\{b_1,a_2,a_3,b_3\} \cap D^c_{b_2} = \emptyset$. To dominate b_3, $b_4 \in D^c_{b_2}$. By the connectedness of $(G-b_2)[D^c_{b_2}]$, $a_4 \in D^c_{b_2}$ and $b_4a_4 \in E(G)$. Thus $\{a_3,a,a_4\} \succ_c G$ contradicting $\gamma_c(G) = 4$. This case cannot occur.

Case 2 : $N_{L_1}(b_3) = \{a_4\}$.

Since $b_3 \succ \{b_2,b_4,a_4\}$, it follows from Lemma 4.5.17 that $a_4b_1 \notin E(G)$. Reminding that $a_2b_4, a_3b_4 \notin E(G)$. Thus $b_4a_1 \in E(G)$ as otherwise $N[b_4] \subseteq N[b_3]$ contradicting Lemma 2.1.22(3). We see that $\{a,b_2\} \subseteq N(a_2) \cap N(a_3)$. By Corollary 2.1.23, there exist different vertices x and y in $\{b_1,a_1,a_4\}$ such that $xa_2, ya_3 \in E(G)$. We distinguish three more subcases.

Subcase 2.1 : $\{x,y\} = \{a_1,a_4\}$.

Without loss of generality let $a_2a_1, a_3a_4 \in E(G)$. Clearly $a_1a_4 \notin E(G)$ as otherwise $\{a_1,a_4,a_3\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Since $\{a_1,b_2\} \subseteq N(b_1) \cap N(a_2)$, it follows from Corollary 2.1.23 that $b_1a_3 \in E(G)$. Consider $G-a_2$. By Lemma 2.1.22(1), $\{a_1,a,b_2\} \cap D^c_{a_2} = \emptyset$. To dominate a_1, we have b_1 or b_4 is in $D^c_{a_2}$. We show that $a_4b_4 \in E(G)$ in both cases. Suppose that $b_1 \in D^c_{a_2}$. Lemma 2.1.24 implies that $b_3 \in D^c_{a_2} \cap N(b_4)$ or, possibly, $a_4 \in D^c_{a_2} \cap N(b_4)$. As $(G-a_2)[D^c_{a_2}]$ is connected, by Lemma

2.1.22(2), we obtain $D^c_{a_2} = \{b_1, a_3, a_4\}$. Thus $a_4b_4 \in E(G)$. We then suppose that $b_4 \in D^c_{a_2}$. By Lemma 2.1.24, $a_3 \in D^c_{a_2} \cap N(b_1)$. As $(G - a_2)[D^c_{a_2}]$ is connected, by Lemma 2.1.22(2), we must have $D^c_{a_2} = \{b_4, a_4, a_3\}$. Thus $a_4b_4 \in E(G)$. This implies that $\{a_3, a, a_4\} \succ_c G$ contradicting $\gamma_c(G = 4$. This subcase cannot occur.

Subcase 2.2 : $\{x, y\} = \{b_1, a_4\}$.

Without loss of generality let $a_2b_1, a_3a_4 \in E(G)$. We see that neither a_3a_1 nor a_4a_1 are in $E(G)$ as otherwise $\{a_3, b_2, b_3\}$ and $\{a_4, b_3, b_2\}$ dominate G respectively. Moreover, $a_4b_4 \notin E(G)$ as otherwise $\{a_2, a, a_4\} \succ_c G$.

We show that $b_1a_3 \notin E(G)$. Suppose to the contrary that $b_1a_3 \in E(G)$. Thus $\{a, b_1, b_2\} \subseteq N(a_2) \cap N(a_3)$. Therefore $a_2a_1 \in E(G)$ as otherwise $N(a_2) \subseteq N(a_3)$ contradicting Corollary 2.1.23. By $\Delta = 4$, $N(a_2) = \{a, b_1, b_2, a_1\}$. Consider $G - a_2$. Lemma 2.1.22(1) implies that $\{a, b_1, b_2, a_1\} \cap D^c_{a_2} = \emptyset$. Thus $b_4 \in D^c_{a_2}$ to dominate a_1 and $a_3 \in D^c_{a_2}$ to dominate b_1. Since $(G - a_2)[D^c_{a_2}]$ is connected and $a_4b_4, a_3b_3 \notin E(G)$, it follows that $D^c_{a_2} = \{a_3, a_4, b_3, b_4\}$ contradicting Lemma 2.1.22(2). Hence $b_1a_3 \notin E(G)$.

Consider $G - a_2$. By Lemma 2.1.22(1), $\{b_1, b_2, a\} \cap D^c_{a_2} = \emptyset$. To dominate b_1, we have that a_1 is in $D^c_{a_2}$. Lemma 2.1.24 thus implies $a_4 \in D^c_{a_2} \cap N(a_3)$. By the connectedness of $(G - a_2)[D^c_{a_2}]$, $D^c_{a_2} = \{a_4, b_3, b_4, a_1\}$ contradicting Lemma 2.1.22(2). This subcase cannot occur.

Subcase 2.3 : $\{x, y\} = \{b_1, a_1\}$.

Without loss of generality let $a_2b_1, a_3a_1 \in E(G)$. Thus $a_4b_4 \notin E(G)$ as otherwise $\{a_2, a, a_4\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Consider $G - a_3$. By Lemma 2.1.22(1), $\{a_1, a,\ b_2\} \cap D^c_{a_3} = \emptyset$. To dominate a_1, we have b_1 or b_4 is in $D^c_{a_3}$. Suppose that $b_1 \in D^c_{b_3}$. As $(G - a_3)[D^c_{a_3}]$ is connected, by Lemma 2.1.22(2), we must have $D^c_{a_3} = \{b_1, a_2, a_4\}$ with $a_2a_4 \in E(G)$. But $D^c_{a_3}$ does not dominate b_4 contradicting $D^c_{a_3} \succ_c G - a_3$. Thus $b_1 \notin D^c_{b_3}$ and $b_4 \in D^c_{a_3}$. To dominate b_1, $a_2 \in D^c_{a_3}$. By the connectedness of $(G - a_3)[D^c_{a_3}]$, $D^c_{a_3} = \{b_4, b_3, a_4, a_2\}$ with $a_2a_4 \in E(G)$. This contradicts Lemma 2.1.22(2) and Case 2 cannot occur.

Hence $|N_{L_1}(b_2)| = 1$ and this completes the proof. □

Lemma 4.5.25. Let G be a $4 - \gamma_c-$vertex critical graph of order 9 and a be a vertex of degree $\Delta = 4$. If $D^c_a = \{b_1, b_2, b_3\}$ and $G[L_2]$ is a path b_1, b_2, b_3, b_4, then $N_{L_1}(b_2) \neq N_{L_1}(b_3)$.

Proof. Lemmas 4.5.23 and 4.5.24 imply that $|N_{L_1}(b_2)| = |N_{L_1}(b_3)| = 1$. Suppose to the contrary that $N_{L_1}(b_2) = N_{L_1}(b_3) = \{a_4\}$. Since $D^c_a = \{b_1, b_2, b_3\}$, $b_1 \succ L_1 - \{a_4\}$. Let $N_{L_1}(b_1) = \{a_1, a_2, a_3\}$. If $b_4a_i \in E(G)$ for some $i \in \{1, 2, 3\}$, then $\{a_i, a, a_4\} \succ_c$

G contradicting $\gamma_c(G) = 4$. Thus $b_4a_i \notin E(G)$ for all i. This implies that $N[b_4] \subseteq N[b_3]$ contradicting Lemma 2.1.22(3). Hence $N_{L_1}(b_2) \neq N_{L_1}(b_3)$ and this completes the proof. □

We are now ready to prove Theorem 4.5.6.

Proof. Suppose there exists a $4-\gamma_c-$vertex critical graph G of order 9 and $\Delta = 4$. Let a be a vertex of degree 4 and, for all $1 \leq i \leq s$, L_i the set of vertices at distance i from a. Lemma 4.5.16 implies that $s = 2$ and Lemma 4.5.22 implies that $G[L_2]$ is a path of length three. Consider $G-a$. By Lemma 2.1.22(1), $D^c_a \subseteq L_2$. By the connectedness of $(G-a)[D^c_a]$, let $(G-a)[D^c_a] = b_1,b_2,b_3$. Since $n=9$, $|L_2 - D^c_a| = 1$. Let $\{b_4\} = L_2 - D^c_a$. Since $D^c_a \succ_c G-a$, b_4 is adjacent to at least one vertex in D^c_a. Since $G[L_2]$ is a path, either b_1b_4 or b_3b_4 is in $E(G)$. Without loss of generality let $b_3b_4 \in E(G)$.

By Lemmas 4.5.23 and 4.5.24, $|N_{L_1}(b_2)| = 1$ and $|N_{L_1}(b_3)| = 1$. Lemma 4.5.25 yields, further, that $N_{L_1}(b_2) \neq N_{L_1}(b_3)$. Let $N_{L_1}(b_2) = \{a_3\}$ and $N_{L_1}(b_3) = \{a_4\}$, moreover, $\{a_1,a_2\} = L_1 - \{a_2,a_4\}$. Thus $b_1a_1, b_1a_2 \in E(G)$. We have that $a_4b_1 \notin E(G)$ and $a_3b_4 \notin E(G)$ as otherwise $\{b_3,a_4,a\}$ and $\{b_2,a_3,a\}$ dominate G, a contradiction.

Since $N[b_3] = \{b_2,b_3,b_4,a_4\}$, it follows from Lemma 2.1.22(3) that b_4 is adjacent to a vertex in $\{a_1,a_2\}$. Without loss of generality let $b_4a_1 \in E(G)$. We see that $\{a,b_1\} \subseteq N(a_1) \cap N(a_2)$. Corollary 2.1.23 then implies that a_2 is adjacent to at least one vertex in $\{a_3,a_4\}$. We distinguish two cases.

Case 1 : $a_2a_4 \in E(G)$.

We show that $a_1a_3 \in E(G)$. Consider $G-a_2$. By Lemma 2.1.22(1), $\{a,b_1,a_4\} \cap D^c_{a_2} = \emptyset$. To dominate a, we have a_1 or a_3 is in $D^c_{a_2}$. We have three more subcases.

Subcase 1.1 : $a_1 \in D^c_{a_2}$ and $a_3 \notin D^c_{a_2}$.

Lemma 2.1.24 implies that $b_3 \in D^c_{a_2} \cap N(b_2)$. As $(G-a_2)[D^c_{a_2}]$ is connected, $b_4 \in D^c_{a_2}$. Lemma 2.1.22(2) gives that $D^c_{a_2} = \{a_1,b_3,b_4\}$. Since $b_3a_3, b_4a_3 \notin E(G)$, $a_1a_3 \in E(G)$. We finish this subcase.

Subcase 1.2 : $a_1 \notin D^c_{a_2}$ and $a_3 \in D^c_{a_2}$.

Lemma 2.1.24 yields that $b_3 \in D^c_{a_2} \cap N(b_4)$. Lemma 2.1.22(2) gives also that $|D^c_{a_2} - \{a_3,b_3\}| = 1$. By the connectedness of $(G-a_2)[D^c_{a_2}]$, $D^c_{a_2} = \{a_3,b_3,b_2\}$. Since $b_2a_1, b_3a_1 \notin E(G)$, $a_1a_3 \in E(G)$. We finish this subcase.

Subcase 1.3 : $a_1 \in D^c_{a_2}$ and $a_3 \in D^c_{a_2}$.

By Lemma 2.1.22(2), $|D^c_{a_2} - \{a_1,a_3\}| = 1$. To dominate b_3, either b_2 or b_4 is in $D^c_{a_2}$. Since neither b_2a_1 nor b_4a_3 are in $E(G)$, it follows that $a_1a_3 \in E(G)$. We finish this subcase and so $a_1a_3 \in E(G)$.

We now have that $N(a_1) = \{a, b_1, b_4, a_3\}$. Thus $a_2b_4 \notin E(G)$ and $a_4b_4 \notin E(G)$ as otherwise $\{a_1, b_4, b_3\}$ and $\{a_1, b_1, b_4\}$ dominate G, respectively, contradicting $\gamma_c(G) = 4$. Consider $G - a_3$. By Lemma 2.1.22(1), $\{a, a_1, b_2\} \cap D^c_{a_3} = \emptyset$. This implies, by Lemma 2.1.24, that $b_3 \in D^c_{a_3} \cap N(b_4)$ and $a_2 \in D^c_{a_3} \cap N(b_1)$. As $(G-a_3)[D^c_{a_3}]$ is connected, by Lemma 2.1.22(2), $D^c_{a_3} = \{b_3, a_2, a_4\}$. But $D^c_{a_3}$ does not dominate a_1 contradicting $D^c_{a_3} \succ_c G - a_3$. Case 1 cannot occur.

Case 2 : $a_2a_4 \notin E(G)$.

Thus $a_2a_3 \in E(G)$. We have that $a_1a_4 \notin E(G)$ as otherwise $\{a_1, b_1, b_2\} \succ_c G$ contradicting $\gamma_c(G) = 4$. Moreover, $a_1a_3 \notin E(G)$ as otherwise $N(a_2) \subseteq N[a_1]$ contradicting Corollary 2.1.23.

Consider $G - a_2$. By Lemma 2.1.22(1), $\{a, b_1, a_3\} \cap D^c_{a_2} = \emptyset$. Thus a_1 or a_4 is in $D^c_{a_2}$ to dominate a. Suppose that $a_1 \in D^c_{a_2}$. As $(G - a_2)[D^c_{a_2}]$ is connected, we obtain $b_4 \in D^c_{a_2}$. Lemma 2.1.22(2) gives that $|D^c_{a_2} - \{a_1, b_4\}| = 1$. Lemma 2.1.24 gives also that $b_3 \in D^c_{a_2} \cap N(b_2)$. Therefore $D^c_{a_2} = \{a_1, b_4, b_3\}$ does not dominate a_3, a contradiction. Thus $a_1 \notin D^c_{a_2}$ and $a_4 \in D^c_{a_2}$. To dominate a_1, $b_4 \in D^c_{a_2}$. Lemma 2.1.24 yields that b_3 is in $D^c_{a_2} \cap N(b_2)$. Lemma 2.1.22(2) then implies that $D^c_{a_2} = \{a_4, b_3, b_4\}$. But $D^c_{a_2}$ does not dominate b_1 contradicting $D^c_{a_2} \succ_c G - a_2$. Thus Case 2 cannot occur and this completes the proof of Theorem 4.5.6. $\square$

CHAPTER 5

Maximal Connected Domination Vertex Critical Graphs

The study of maximal $3-\gamma-$vertex critical graphs was started by Ananchuen et al. [6]. They established the characterization of these graphs of connectivity two. They also proved that maximal $3-\gamma-$vertex critical graphs are bi-critical with some exceptions. The objective of this chapter is to investigate properties of maximal $k-\gamma_c-$vertex critical graphs. In particular, we characterize the structures of some classes and study matching properties of maximal $3-\gamma_c-$vertex critical graphs.

Let $\mathscr{M}(k,l,r)$: the class of maximal $k-\gamma_c-$vertex critical graphs of connectivity l with an induced subgraph of a minimum cut set containing at most r edges.

The chapter is organized as the following. In Section 5.2, we characterize all graphs in $\mathscr{M}(3,3,r)$. We see that a graph G in $\mathscr{M}(3,l,r)$ for any positive integer l is complicated to characterize. Then we focus on all graphs in $\mathscr{M}(3,l,0)$. We establish that a graph G in $\mathscr{M}(3,l,0)$ is isomorphic to $\mu(K_l)$ (the Mycielskian of K_l). We show that a maximal $3-\gamma_c-$vertex critical graph with $\delta \geq 4$ contains at least eight vertices. We conclude this section by characterizing all maximal $3-\gamma_c-$vertex critical graphs with $\delta \geq 4$ containing eight vertices. In Section 5.3, we prove that every maximal $3-\gamma_c-$vertex critical graph of even order is bi-critical. We, moreover, prove that every maximal $3-\gamma_c-$vertex critical graph of odd order is either $3-$factor critical or in the class $\mathscr{M}(3,l,0)$.

5.1 Preliminaries

In this section, we state a number of results from the literature that we make use of in our work. We begin with a result of Ananchuen et al. [7] which gives a characterization of $3-\gamma_c-$vertex critical graphs of connectivity two as detailed in the following theorem.

Theorem 5.1.1. [7] If G is a $3-\gamma_c-$vertex critical graph, then G is either isomorphic to a cycle C_5 of length five or $3-$connected.

They showed that a minimum cut set of $3-\gamma_c-$vertex critical graphs of connectivity three contains at most one edge. They, further, established the two following theorems which are the characterizations of $3-\gamma_c-$vertex critical graphs of connectivity three according to the number of edge of a minimum cut set.

Theorem 5.1.2. [7] Suppose G is a $3-\gamma_c-$vertex critical graph and S is a minimum cut set of G with $|S|=3$. If S is an independent set, then G is isomorphic to one of G_0 or G'_0.

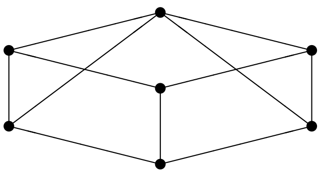

Figure 5.1(a) : G_0

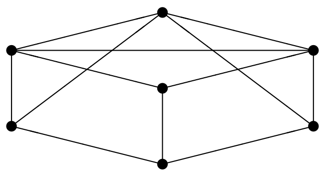

Figure 5.1(b) : G'_0

Theorem 5.1.3. [7] Suppose G is a $3-\gamma_c-$vertex critical graph and S is a minimum cut set of G with $|S|=3$. If S contains an edge, then G has G_1 or G_2 as a spanning subgraph where G_1 and G_2 are defined as follows. Let $i \in \{1,2\}$.

(1) $V(G_i)=\{c_1,c_2,c_3,x,y\}\cup A\cup B$ where A and B are independent.

(2) G_i has $S=\{c_1,c_2,c_2\}$ as a minimum cut set of size 3.

(3) G_i-S contains exactly two components H_1 and H_2 such that $V(H_1)=\{x\}$ and $V(H_2)=\{y\}\cup A\cup B$.

(4) $\{c_1c_2,xc_1,xc_2,xc_3\}\subseteq E(G_i)$, moreover, $yc_3\notin E(G_i)$.

(5) $c_1\succ A, c_2\succ B$ and $c_3\succ A\cup B$.

(6) y is adjacent at least one vertex in A and at least one vertex in $\{c_1,c_2\}$.

(7) Join each vertex in A to every vertex in B with a perfect matching deleted.

(8) For G_1, $y\succ B$. For G_2, $yc_1,yc_2,yb\in E(G)$ for some $b\in V(B)$.

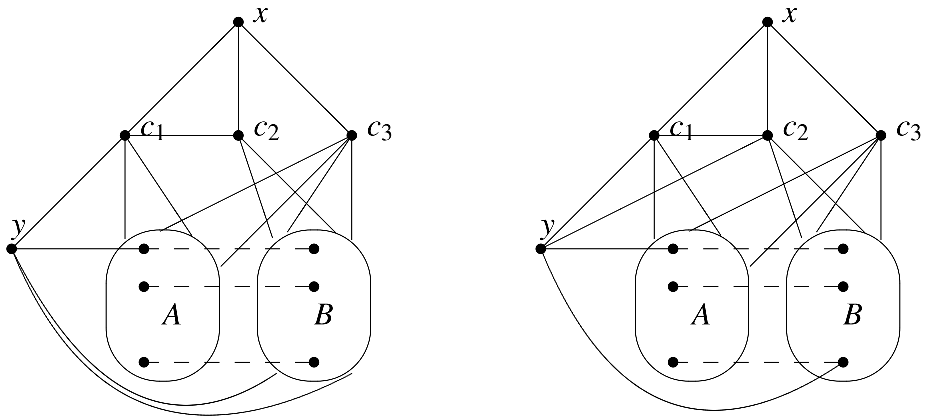

Figure 5.2(a) : G_1 **Figure 5.2(b) :** G_2

We conclude this section by proving the following two lemmas. Let G be a maximal $3-\gamma_c-$vertex critical graph with a vertex cut set S and $C_1, C_2, ..., C_m$ be the components of $G-S$.

Lemma 5.1.4. Let G be a maximal $3-\gamma_c-$vertex critical graph. For all $v \in V(G)$, if $m \geq 3$ or $v \in S \cup V(C_i)$ where $|V(C_i)| > 1$, then G satisfies the following two properties.

(1) $D_v^c \cap S \neq \emptyset$.

(2) v does not dominate S.

Proof. (1) Suppose that $D_v^c \cap S = \emptyset$. Since $(G-v)[D_v^c]$ is connected, $D_v^c \subseteq V(C_j)$ for some $1 \leq j \leq m$. Because $D_v^c \succ_c G-v$, $m = 2$ and $V(C_i) - \{v\} = \emptyset$ where $i \in \{1,2\} - \{j\}$. Thus $\{v\} = V(C_i)$, contradicting the assumption. This completes the proof of (1).

(2) Suppose that $v \succ S$. Consider $G-v$. Lemma 2.1.22(1) yields that $D_v^c \cap S = \emptyset$, contradicting (1) and this completes the proof of (2). $\square$

Lemma 5.1.5. Let G be a maximal $3-\gamma_c-$vertex critical graph. Let $a \in V(C_i)$ for some $i \in \{1, ..., m\}$. Then G has these following properties.

(1) Let $b \in V(C_j)$ for some $j \in \{1, ..., m\}$ such that $\{a,b\}$ does not dominate G, if $m \geq 3$ or $|V(C_i)|, |V(C_j)| > 1$, then $|D_{ab}^c \cap \{a,b\}| = 1$ and $|D_{ab}^c \cap S| = 1$.

(2) If $c \in D_a^c$ where c is an isolated vertex in S, then $m = 2$ and $\{w\} = V(C_j)$ for some $j \in \{1,2\}$ where $\{w\} = D_a^c - \{c\}$.

Proof. (1) Consider $G+ab$. Lemma 2.1.15(2) gives $D^c_{ab} \cap \{a,b\} \neq \emptyset$. Since $\{a,b\}$ does not dominate G, $|D^c_{ab} \cap \{a,b\}| = 1$. To dominate $\cup_{i'=1}^{m} V(C_{i'}) - \{a,b\}$ and by the connectedness of $(G+ab)[D^c_{ab}]$, $D^c_{ab} \cap S \neq \emptyset$. By Lemma 2.1.15(1), $|D^c_{ab} \cap S| = 1$, as required.

(2) Suppose that $c \in S \cap D^c_a$. Lemma 2.1.22(2) yields that $|D^c_a - \{c\}| = 1$. Let $\{w\} = D^c_a - \{c\}$. Thus $wc \in E(G)$. Since c is an isolated vertex in S, $w \succ S$. By Lemma 5.1.4(2), $m = 2$ and $\{w\} = V(C_j)$ for some $j \in \{1,2\}$, as required. □

5.2 Some Characterizations

In this section, we characterize all graphs in $\mathscr{M}(3,3,r) \cup \mathscr{M}(3,l,0)$ for any positive integer l. Ananchuen et al. [7] pointed out that a graph $2-\gamma_c-$vertex critical is $K_{2n} - M$ where $n \geq 2$ and M is a perfect matching. By Theorem 2.1.17, we have that a maximal $2-\gamma_c-$vertex critical graph is $K_{2n} - M$ where $n \geq 2$. We turn the attention to characterize maximal $3-\gamma_c-$vertex critical graphs. First of all, we establish a property of maximal $3-\gamma_c-$vertex critical graphs that they can be constructed from some graphs.

We observe that the connected domination number of the complement of every disconnected graph is at most 2.

Lemma 5.2.1. If H is a graph of order n such that $\gamma_c(H) > 2$ or H is disconnected, then there exists a maximal $3-\gamma_c-$vertex critical graph of order $2n+1$ containing H and $\overline{H}$ as induced subgraphs.

Proof. Let H be any graph of order n such that $\gamma_c(H) > 2$ or H is disconnected. Moreover, we let $V(H) = \{u_i | i = 1,2,...,n\}$ and we relabel u_i in H to v_i in $\overline{H}$ for $i = 1,2,...,n$. We next construct a maximal $3-\gamma_c-$vertex critical graph G from H and $\overline{H}$. Let $V(G) = V(H) \cup V(\overline{H}) \cup \{x\}$ and $E(G) = \{xu_i | 1 \leq i \leq n\} \cup \{u_i v_j | 1 \leq i \neq j \leq n\}$.

Firstly, we prove that $\gamma_c(G) = 3$. Clearly $\{x,u_1,u_2\} \succ_c G$ and so $\gamma_c(G) \leq 3$. Suppose there exists a γ_c-set D of two vertices. If $x \in D$, then $D = \{x,u_i\}$ for some $i \in \{1,...,n\}$. But D does not dominate v_i. Hence $x \notin D$. To dominate x, $D \cap V(H) \neq \emptyset$. Since $\gamma_c(H) > 2$ or H is not connected, D is not a subset of $V(H)$. Therefore $|D \cap V(H)| = 1$ and $|D \cap V(\overline{H})| = 1$. Since $G[D]$ is connected, we have $D = \{u_i, v_j\}$ where $i \neq j$. Moreover, the construction gives $|\{u_i u_j, v_i v_j\} \cap E(G)| = 1$. If $u_i u_j \in E(G)$, then D does not dominate v_i. Similarly, D does not dominate u_j if $v_i v_j \in E(G)$. Hence $\gamma_c(G) \geq 3$. Therefore $\gamma_c(G) = 3$.

Consider $G+ab$. If $\{a,b\} = \{x,v_i\}$, then $D^c_{ab} = \{x,u_i\}$. If $\{a,b\}$ is either $\{u_i,u_j\}$ or $\{v_i,v_j\}$, then $D^c_{ab} = \{u_i,v_j\}$. Finally, if $\{a,b\} = \{u_i,v_i\}$, then $D^c_{ab} = \{u_i,v_i\}$. Thus

G is a $3-\gamma_c-$edge critical graph. We next show that G is $3-\gamma_c-$vertex critical. As $\gamma_c(\overline{H}) \leq 2$, we can suppose that $\{v_1, v_2\} \succ_c \overline{H}$. Because $\gamma_c(H) > 2$, every vertex u_i is not adjacent at least one vertex u_j in H. Let $D^c_x = \{v_1, v_2\}$, $D^c_{u_i} = \{u_j, v_i\}$ and $D^c_{v_i} = \{u_i, x\}$. So G is a $3-\gamma_c-$vertex critical graph. Therefore G is a maximal $3-\gamma_c-$vertex critical graph containing H and $\overline{H}$ as induced subgraphs, thus establishing this lemma. □

Moreover, if H is an arbitrary graph of order n, we can construct a maximal $3-\gamma_c-$vertex critical graph containing H and $\overline{H}$ as induced subgraphs by using slightly more vertices.

Theorem 5.2.2. For any graph H of order n, there exists a maximal $3-\gamma_c-$vertex critical graph of order $2n+3$ containing H and $\overline{H}$ as induced subgraphs.

Proof. Let H' be a graph H union a vertex y. So $V(H') = V(H) \cup \{y\}$ and $E(H') = E(H)$. Further, we let $\overline{H}$ be the complement of H and we label y in H' to z in $\overline{H'}$. Hence $\overline{H'}$ is $\overline{H}$ joining every vertex to a vertex z.

Obviously, H' is not connected and $\{z, v\} \succ_c \overline{H'}$ for some $v \in V(\overline{H'}) - \{z\}$. By Lemma 5.2.1, this completes the proof of Theorem 5.2.2. □

In view of Theorem 5.2.2, there is no characterization of maximal $3-\gamma_c-$vertex critical graphs in terms of forbidden graphs.

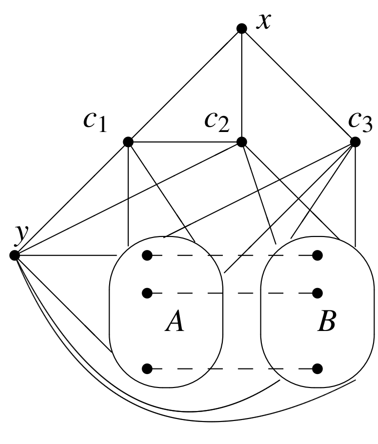

Figure 5.3 : G_3

Ananchuen et al. [7] established that $3-\gamma_c-$vertex critical graphs of connectivity two are isomorphic to a cycle C_5 on five vertices (see Theorem 5.1.1). So we start our study of maximal $3-\gamma_c-$vertex critical graphs of connectivity three by using Theorems 5.1.2 and 5.1.3. Ananchuen et al. [7], further, claimed that a minimum cut set of size 3 of every $3-\gamma_c-$vertex critical graph contains at most one edge. That is $\mathscr{M}(3,3,r) = \mathscr{M}(3,3,1)$. Let $H_1, H_2, A, B, c_1, c_2, c_3, x$ and y be defined as in Theorem 5.1.3, clearly, $A = N_{H_2}(c_1) \cap N_{H_2}(c_3) = \{u_i | i = 1,...,t\}$ and $B = N_{H_2}(c_2) \cap N_{H_2}(c_3) = \{v_i | i = 1,...,t\}$.

Theorem 5.2.3. Let $G \in \mathscr{M}(3,3,1)$. Then G is G_3 where G_3 satisfies the following statements.

(1) $V(G_3) = \{c_1, c_2, c_3, x, y\} \cup A \cup B$ where A and B are cliques.

(2) c_1 is not adjacent to any vertex in B and c_2 is not adjacent to any vertex in A.

(3) $y \succ A \cup B$.

(4) $\{c_1c_2, xc_1, xc_2, xc_3, yc_1, yc_2\} \subseteq E(G_i)$, moreover, $yc_3 \notin E(G_i)$.

(5) Every vertex in A is adjacent to each vertex in B accept one vertex and vice versa.

Proof. Theorems 5.1.2 and 5.1.3 imply that either $G \in \{G_0, G'_0\}$ or G has G_1 or G_2 as a spanning subgraph. It is easy to see that G_0 is not a $3-\gamma_c-$edge critical graph. Therefore G is G'_0 or has G_1 or G_2 as a spanning subgraph. We first consider the case when G has G_1 as a spanning subgraph. As a consequence of Theorem 5.1.3(7), we establish (5).

Claim 1 : c_1 and c_2 are not adjacent to any vertex in B and A, respectively.

As otherwise, $N_G(x) \subseteq N_G(u_i)$ when $u_ic_2 \in E(G)$ and $N_G(x) \subseteq N_G(v_i)$ when $v_ic_1 \in E(G)$. These contradict Corollary 2.1.23, thus establishing Claim 1.

So Claim 1 gives (2).

Claim 2 : $G[B]$ and $G[A]$ are cliques.

Suppose there are non-adjacent vertices $v_i, v_j \in B$. Consider $G + v_iv_j$. Lemma 2.1.15(3) yields that $c_2, c_3 \notin D^c_{v_iv_j}$, moreover, the connectedness of $(G + v_iv_j)[D^c_{v_iv_j}]$ yields that $c_1 \notin D^c_{v_iv_j}$. These contradict Lemma 5.1.5(1). Thus $G[B]$ is complete, similarly, $G[A]$ is complete, thus establishing Claim 2.

Therefore Claim 2 gives (1). Theorem 5.1.3(6) yields that y is adjacent to at least one vertex of c_1 or c_2. We let without loss of generality that $yc_1 \in E(G)$.

Claim 3 : $y \succ A$.

Suppose there exists $u_i \in A$ which is not adjacent to y. Consider $G + yu_i$. Lemma 5.1.5(1) implies that $|D^c_{yu_i} \cap \{y, u_i\}| = 1$ and $|D^c_{yu_i} \cap \{c_1, c_2, c_3\}| = 1$. If $y \in D^c_{yu_i}$, then $c_1 \in D^c_{yu_i} - \{y\}$ by the connectedness of $(G + yu_i)[D^c_{yu_i}]$. Thus $D^c_{yu_i}$ does not dominate

c_3. So $u_i \in D^c_{yu_i}$. Lemma 2.1.15(3) yields that $c_1 \notin D^c_{yu_i}$ and the connectedness of $(G+yu_i)[D^c_{yu_i}]$ yields that $c_2 \notin D^c_{yu_i}$. Therefore $c_3 \in D^c_{yu_i}$. Claim 1 yields that $D^c_{yu_i}$ does not dominate c_2, a contradiction, thus establishing Claim 3.

Hence Claim 3 together with Theorem 5.1.3(8) imply (3).

Claim 4 : $yc_2 \in E(G)$.

Suppose $yc_2 \notin E(G)$. Consider $G+yc_2$. Lemma 2.1.15(2) gives $D^c_{yc_2} \cap \{y, c_2\} \neq \emptyset$. If $D^c_{yc_2} = \{y, c_2\}$, then $D^c_{yc_2}$ does not dominate c_3. If $\{c_2\} = D^c_{yc_2} \cap \{c_2, y\}$, then by Lemma 2.1.15(3), $(\{c_1\} \cup B) \cap D^c_{yc_2} = \emptyset$. As $(G+yc_2)[D^c_{yc_2}]$ is connected, we obtain $x \in D^c_{yc_2}$. Claim 2 implies that $D^c_{yc_2}$ does not dominate A, contradicting Lemma 2.1.15(1). Therefore $\{y\} = D^c_{yc_2} \cap \{c_2, y\}$. Since $(G+yc_2)[D^c_{yc_2}]$ is connected, to dominate x, $c_1 \in D^c_{yc_2}$. This contradicts Lemma 2.1.15(3), thus establishing Claim 4.

Claim 4 and Theorem 5.1.3(4) give (4). We now have that G is G_3 if G has G_1 as a spanning subgraph.

We then consider when G has G_2 as a spanning subgraph. Theorem 5.1.3(7) implies (5). By similar arguments as Claims 1, 2 and 3, we obtain (2), (1) and $y \succ A$. Moreover, Theorem 5.1.3(4) and (8) give (4). We need only show that $y \succ B$. Suppose there exists $v_i \in B$ such that $yv_i \notin E(G)$. Consider $G+yv_i$. Lemma 5.1.5(1) yields that $|D^c_{yv_i} \cap \{y, v_i\}| = 1$ and $|D^c_{yv_i} \cap \{c_1, c_2, c_3\}| = 1$. If $y \in D^c_{yv_i}$, then by Lemma 2.1.15(3), $c_1 \in D^c_{yv_i}$. But $D^c_{yv_i}$ does not dominate c_3. Therefore $v_i \in D^c_{yv_i}$. Lemma 2.1.15(3) gives that $c_3 \in D^c_{yv_i}$. Clearly $D^c_{yv_i}$ does not dominate c_1. Therefore $y \succ B$ and this implies that G is G_3.

We see that G'_0 is G_3 when $|A| = |B| = 1$. This completes the proof. □

Since we have characterizations of all maximal $3-\gamma_c-$vertex critical graphs of connectivity 2 and 3, we focus on maximal $3-\gamma_c-$vertex critical graphs which $\kappa \geq 4$ in the following studies. Hereafter, all graphs satisfy $\delta \geq \kappa \geq 4$. We will show that a graph G in $\mathscr{M}(3, l, 0)$ is isomorphic to the Mycielskian $\mu(K_l)$ of a complete graph K_l. We would like to mention that the graph $\mu(K_l)$ has been provided independently by Ananchuen et al. [7].

Theorem 5.2.4. Let $G \in \mathscr{M}(3, l, 0)$. Then G is isomorphic to the Mycielskian $\mu(K_l)$ of a clique of order l.

Proof. Let S be a minimum cut set of size $l \geq 4$ such that $G[S]$ contains no edge and $C_1, C_2, ..., C_m$ be components of $G-S$.

Claim 1 : There exists $j \in \{1, 2, ..., m\}$ such that $|V(C_j)| > 1$.

Suppose to the contrary that $|V(C_i)| = 1$ for all $i \in \{1, 2, ..., m\}$. Since S is a minimum cut set, every vertex in S is adjacent to at least one vertex in $V(C_i)$ for each $i \in \{1,2,...,m\}$. Because $|V(C_i)| = 1$ for all i, $c_i \succ S$ where $\{c_i\} = V(C_i)$ and $s \succ \cup_{i=1}^{m} V(C_i)$ where $s \in S$. Thus $\{c_1, s\} \succ_c G$, contradicting $\gamma_c(G) = 3$, thus establishing Claim 1.

Claim 2 : $m = 2$ and $|V(C_i)| = 1$ for some $i \in \{1,2\}$, moreover, $x \succ S$ where $\{x\} = V(C_i)$.

By Claim 1, let C_j be a non-singleton component and $c_j \in V(C_j)$. Consider $G - c_j$. Lemma 5.1.4(1) gives that $D^c_{c_j} \cap S \neq \emptyset$. Let $s \in D^c_{c_j} \cap S$. Since s is an isolated vertex in S, by Lemma 5.1.5(2), $m = 2$ and $D^c_{c_j} = \{x, s\}$ where $\{x\} = V(C_i)$ for some $i \in \{1,2\}$. Because $|V(C_j)| > 1$, $i = 3 - j$. As S is an independent set, we must have $x \succ S$, thus establishing Claim 2.

Without loss of generality, let $V(C_1) = \{x\}$. Hence $|V(C_2)| > 1$.

Claim 3 : For all $s_i \in S, i \in \{1,2,...,l\}$, there exists $t_i \in V(C_2)$ such that $t_i \succ S - \{s_i\}$.

Consider $G - s_i$. This implies, by Lemma 5.1.4(1), that there exists $s_j \in D^c_{s_i} \cap S$ where $j \neq i$. Moreover, Lemma 2.1.22(2) yields $|D^c_{s_i} - \{s_j\}| = 1$. Let $\{t_i\} = D^c_{s_i} - \{s_j\}$. Clearly $t_i s_j \in E(G)$. As $(G - s_i)[D^c_{s_i}]$ is connected, we obtain $t_i \in \cup_{q=1}^{2} V(C_q)$. By Lemma 2.1.22(1), $t_i \in V(C_2)$. Since s_j is an isolated vertex in S, $t_i \succ S - \{s_i\}$, thus establishing Claim 3.

Claim 4 : $V(C_2) = \{t_i | 1 \leq i \leq l\}$.

Suppose there exists $t_{l+1} \in V(C_2) - \{t_1, t_2, ..., t_l\}$. Consider $G - t_{l+1}$. Lemma 5.1.4(1) gives that there exists $s_i \in D^c_{t_{l+1}} \cap S$. Since s_i is an isolated vertex in S, by Lemma 5.1.5(2), $\{x\} = D^c_{t_{l+1}} - \{s_i\}$. Clearly $D^c_{t_{l+1}}$ does not dominate t_i, a contradiction. Hence $V(C_2) = \{t_1, t_2, ..., t_l\}$, and thus establishing Claim 4.

Claim 5 : C_2 is complete.

Suppose that $ab \notin E(G)$ for $a, b \in V(C_2)$. Consider $G + ab$. We see that $\{a, b\}$ does not dominate G. Lemma 5.1.5(1) implies that $|D^c_{ab} \cap \{a, b\}| = 1$ and $|D^c_{ab} \cap S| = 1$. Without loss of generality, let $D^c_{ab} = \{a, s\}$ for some $s \in S$. Thus $as \in E(G)$. Since S is an independent set, $a \succ S$, contradicting Lemma 5.1.4(2). This establishes Claim 5.

In view of Claims $1, 2, 3, 4$ and 5, we have that G is $\mu(K_l)$ and this completes the proof. □

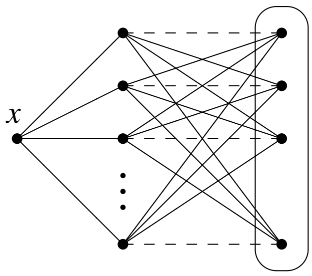

Figure 5.4(a) : $\mu(K_l)$

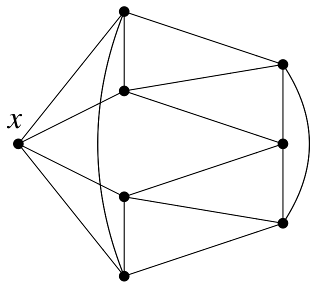

Figure 5.4(b) : G_4

We now show that every maximal $3-\gamma_c-$vertex critical graph with $\delta \geq 4$ contains at least eight vertices. We also characterize that all maximal $3-\gamma_c-$vertex critical graphs with $\delta \geq 4$ containing eight vertices are isomorphic to G_4 (see the Figure 5.4(b)). Let x be a vertex of degree δ and $A = V(G) - N[x]$. Since G is connected, a is adjacent to a vertex x' in $N(x)$ for all vertex a in A. We have that $|A| > 1$ because $\gamma_c(G) = 3$.

Lemma 5.2.5. Let G be a maximal $3-\gamma_c-$vertex critical graph with $\delta \geq 4$. Then it contains at least eight vertices.

Proof. Since $|A| > 1$, it is easy to see that a maximal $3-\gamma_c-$vertex critical graph contains at least eight vertices when $\delta \geq 5$. We assume that there exists a maximal $3-\gamma_c-$vertex critical graph G with $\delta = 4$ containing seven vertices. So $|A| = 2$. Let $A = \{u, v\}$. As $\delta = 4$, we obtain $|N(v) \cap N(x)| \geq 3$ and $|N(u) \cap N(x)| \geq 3$. Since $|N(x)| = 4$, it is not difficult to show that there exists a vertex y such that $yu, yv \in E(G)$. Thus $\{x, y\} \succ_c G$ contradicting $\gamma_c(G) = 3$ and this completes the proof. □

Theorem 5.2.6. If G is a maximal $3-\gamma_c-$vertex critical graph with $\delta \geq 4$ containing eight vertices, then G is G_4.

Proof. Since $|A| > 1$ and G has eight vertices, it follows that δ is equal to either 4 or 5. Suppose that $\delta = 5$. So $|A| = 2$. Because $\delta = 5$, there exists a vertex x' in $N(x)$ which is a common neighbor of the two vertices in A. Thus $\{x, x'\} \succ_c G$ contradicting $\gamma_c(G) = 3$. Therefore $\delta = 4$ and $|A| = 3$. Consider $G - x$. Lemma 2.1.22(1) implies that $D_x^c \subseteq A$. Lemma 2.1.22(2) implies that $|D_x^c| = 2$. Let $D_x^c = \{u, v\}$ and $A - D_x^c = \{w\}$. Hence $vu \in E(G)$ and w is adjacent to u or v, without loss of generality, $wv \in E(G)$. Since $\delta = 4$, v is adjacent to at least two vertices x_3 and x_4 in $N(x)$.

Claim 1 : For all $a \in A$, $|N(a) \cap N(x)| \leq 3$.

As otherwise $N(x) \subseteq N(a)$ contradicting Corollary 2.1.23, thus establishing Claim 1.

Claim 2 : For all $a \in A$ such that $a \succ A$, a is adjacent to at most two vertices in $N(x)$.

Suppose a is adjacent at least three vertices in $N(x)$. Claim 1 yields that a is adjacent to exactly three vertices in $N(x)$, x_1, x_2 and x_3 says, moreover, a is not adjacent to x_4 where $\{x_4\} = N(x) - \{x_1, x_2, x_3\}$. Since $\delta = 4$, x_4 is adjacent to at least one vertex in $\{x_1, x_2, x_3\}$, without loss of generality, $x_4x_1 \in E(G)$. Because $a \succ A$, $\{a, x_1\} \succ_c G$ contradicting $\gamma_c(G) = 3$ thus establishing Claim 2.

As a consequence of Claim 2, $N(v) = \{u, w, x_3, x_4\}$. Because $D^c_x = \{v, u\}$, u is adjacent to the two vertices in $N(x) - \{x_3, x_4\}$, x_1, x_2 say. We distinguish two cases according to the existence of an edge uw.

Case 1 : $uw \notin E(G)$.

By $\delta = 4$ and Claim 1, u is adjacent to exactly one vertex of $\{x_3, x_4\}$, without loss of generality, let $ux_3 \in E(G)$. We first suppose that w is adjacent to at most one vertex of $\{x_1, x_2\}$. By $\delta = 4$, $wx_3, wx_4 \in E(G)$. Thus $x_3 \succ A$, moreover, $\{x, x_3\} \succ_c G$ contradicting $\gamma_c(G) = 3$. Therefore $wx_1, wx_2 \in E(G)$. By $\delta = 4$ and Claim 1, either $wx_3 \in E(G)$ or $wx_4 \in E(G)$. Since $x_3u, x_3v \in E(G), x_3w \notin E(G)$ as otherwise $\{x, x_3\} \succ_c G$ contradicting $\gamma_c(G) = 3$. Therefore $wx_4 \in E(G)$.

We next show that $x_1x_3, x_2x_4, x_1x_4, x_2x_3 \notin E(G)$. Suppose that $x_1x_3 \in E(G)$. Clearly $\{x_1, w\} \succ_c G$ contradicting $\gamma_c(G) = 3$. Similarly, $x_2x_4 \notin E(G)$. We then suppose that $x_2x_3 \in E(G)$. Thus $\{x_2, w\} \succ_c G$, again a contradiction. Similarly, $x_1x_4 \notin E(G)$. Hence $x_1x_3, x_2x_4, x_1x_4, x_2x_3 \notin E(G)$.

Clearly $deg_G(x_1) < \delta$, a contradiction. Hence Case 1 cannot occur.

Case 2 : $uw \in E(G)$.

Therefore $u \succ A$ and $w \succ A$. By Claim 2, $N(u) \cap N(x) = \{x_1, x_2\}$. Moreover, $\delta = 4$ implies that w is adjacent to two vertices in $N(x)$. Lemma 2.1.22(3) yields w is adjacent only one vertex of $\{x_1, x_2\}$ and only one vertex of $\{x_3, x_4\}$, without loss of generality, $wx_2, wx_3 \in E(G)$. We now have that $N(u) \cap N(x) = \{x_1, x_2\}, N(w) \cap N(x) = \{x_2, x_3\}$ and $N(v) \cap N(x) = \{x_3, x_4\}$.

Claim 3 : $|\{x_4x_2, x_4x_1\} \cap E(G)| \leq 1$ and $|\{x_4x_3, x_4x_2\} \cap E(G)| \leq 1$.

Suppose $x_4x_2, x_4x_1 \in E(G)$. Clearly $\{x_4, v\} \succ_c G$ contradicting $\gamma_c(G) = 3$. Therefore $|\{x_4x_2, x_4x_1\} \cap E(G)| \leq 1$. We next suppose that $x_4x_3, x_4x_2 \in E(G)$. If $x_2x_1 \in E(G)$, then $\{x_2, w\} \succ_c G$ contradicting $\gamma_c(G) = 3$. So $x_2x_1 \notin E(G)$. Clearly $x_3x_1 \notin E(G)$ as otherwise $\{x_3, w\} \succ_c G$. These imply that $deg_G(x_1) \leq 3 < \delta$, a contradiction, and thus establishing Claim 3.

If $x_4x_2 \in E(G)$, then $x_4x_3 \in E(G)$ or $x_4x_1 \in E(G)$ because $\delta = 4$. This contradicts Claim 3. Thus $x_4x_2 \notin E(G)$ and $x_4x_3, x_4x_1 \in E(G)$. Similarly, $x_1x_3 \notin E(G)$ as otherwise $\{x_1, u\} \succ_c G$. By $\delta = 4$, $x_1x_2 \in E(G)$. If $x_2x_3 \in E(G)$, then $\{x_2, x_3\} \succ_c G$

contradicting $\gamma_c(G)=3$. Thus $x_2x_3 \notin E(G)$. We see that G is now isomorphic to G_4 and this completes the proof of Theorem 5.2.6. $\square$

5.3 Matching Properties

In this section, G will always denote a maximal $3-\gamma_c-$vertex critical graph. We let S be a vertex cut set such that $|S|=s$. Note that S, in this section, need not be a minimum cut set. Let $\omega(G-S)=m$ and $C_1,C_2,...,C_m$ be the components of $G-S$. We will show that every maximal $3-\gamma_c-$vertex critical graph G satisfies $m+2\leq s$ or G is isomorphic to $\mu(K_l)$. In view of Theorems 5.1.1 and 5.2.3, we concentrate only when $s\geq 4$. We first provide the following lemma.

Lemma 5.3.1. Let G be a maximal $3-\gamma_c-$vertex critical graph. For all $a\in V(C_i), b\in V(C_j)$ such that $i\neq j$ and $m\geq 3$, $N_S(a)\neq N_S(b)$.

Proof. Suppose that $N_S(a)=N_S(b)$. Consider $G+ab$. Lemma 5.1.5(1) implies that $|D^c_{ab}\cap\{a,b\}|=1$ and $|D^c_{ab}\cap S|=1$. Suppose that $\{a,c\}=D^c_{ab}$ where $c\in S$. Thus $ac\in E(G)$. Lemma 2.1.15(3) implies that $cb\notin E(G)$ contradicting $N_S(a)=N_S(b)$. We have a contradiction by the same arguments when $b\in D^c_{ab}$. This completes the proof. $\square$

Suppose that $m+1\geq s$. Since $s\geq 4$, $m\geq 3$. Let I' be a maximum independent set of $\cup_{i=1}^m V(C_i)$. We see that $I'\cap V(C_i)\neq\emptyset$ for $i=1,2,...,m$. Therefore $|I'|\geq m\geq 3$. The next lemma gives the upper bound of $|I'|$.

Lemma 5.3.2. $|I'|+1\leq s$.

Proof. Suppose to the contrary that $|I'|=t\geq s$. Since $s\geq 4$, $|I'|\geq 4$. Lemma 2.1.16 implies that the vertices in I' can be ordered as $x_1,x_2...,x_t$ and there exists a path $y_1,y_2...,y_{t-1}$ with $\{x_i,y_i\}\succ_c G-x_{i+1}$ for $i=1,2,...,t-1$. As $m\geq 3$, to dominate $\cup_{i=1}^m V(A_i)$, we must have $\{y_1,y_2,...,y_{t-1}\}\subseteq S$. Thus $t-1\leq s$. If $t-1=s$, then $\{y_1,y_2,..., y_{t-1}\}=S$. It follows by Lemma 2.1.16 that $x_1\succ S$. This contradicts Lemma 5.1.4(2). Therefore $t-1<s$. By the assumption that $t\geq s$, we have $t=s$. Let $\{y_s\}=S-\{y_1,y_2...,y_{t-1}\}$. Since $x_1\succ\{y_1,y_2...,y_{s-1}\}$, by Lemma 5.1.4(2), $x_1y_s\notin E(G)$. Thus $y_1y_s\in E(G)$ because $\{x_1,y_1\}\succ_c G-x_2$. Consider $G+x_2x_i$ for $2<i\leq t$. Lemma 5.1.5(1) thus implies $D^c_{x_2x_i}=\{x_2,y\}$ or $D^c_{x_2x_i}=\{x_i,y'\}$ where $y,y'\in S$. To dominate x_1, we have $y,y'\notin\{y_s\}$. Suppose first that $D^c_{x_2x_i}=\{x_2,y\}$. By Lemma 2.1.16, $y_jx_i\in E(G)$ for all $j\neq i-1,s$. Lemma 2.1.15(3) thus implies $y=y_{i-1}$. Since $x_2y_1\notin E(G)$, $y_{i-1}y_1\in E(G)$. In the case $D^c_{x_2x_i}=\{x_i,y'\}$, similarly, we have $y'=y_1$ and

$y_{i-1}y_1 \in E(G)$. In both cases, $y_1 y_{i-1}$ for $2 < i \le t$. Therefore $y_1 \succ \{y_2, y_3, ..., y_{t-1}\}$. Since $y_1 y_s \in E(G)$, $y_1 \succ S$, contradicting Lemma 5.1.4(2). This completes the proof. □

As a consequence of Lemma 5.3.2, we have the following corollary.

Corollary 5.3.3. For any maximal $3-\gamma_c-$vertex critical graph, $m+1 \le s$, moreover, if $m+1 = s$, then C_i is complete for all $i = 1, 2, ..., m$.

Proof. Since $I' \cap V(C_i) \neq \emptyset$ for $i = 1, 2, ..., m$, $m \le |I'|$. By Lemma 5.3.2, $m+1 \le |I'|+1 \le s$. If $m+1 = s$, then $m+1 \le |I'|+1 \le s = m+1$. Clearly $|I'| = m$. Therefore the independent number of every component is equal to 1. That is C_i is complete. This completes the proof of Corollary 5.3.3. □

We can use Corollary 5.3.3 to investigate matching properties of these graphs. Recall that, for any vertex cut set S of G, $\omega_o(G-S)$ denotes the number of odd components of $G-S$.

Corollary 5.3.4. Let G be a maximal $3-\gamma_c-$vertex critical graph of even order with $\delta \ge 3$. Then G is bi-critical.

Proof. Suppose G is not bi-critical. Thus there exists a vertex cut set S' of size two such that $G' = G - S'$ does not contain a perfect matching. Tutte's Theorem and parity then implies that there exists a vertex cut set S'' such that $\omega_o(G'-S'') \ge |S''|+2$. Let $\tilde{S} = S' \cup S''$. We have $|\tilde{S}| = |S''|+2$ and $\omega_o(G'-S'') = \omega_o(G-\tilde{S})$. Therefore $\omega_o(G-\tilde{S}) \ge |\tilde{S}|$. Corollary 5.3.3 then yields that $\omega(G-\tilde{S})+1 \le |\tilde{S}| \le \omega_o(G-\tilde{S}) \le \omega(G-\tilde{S})$, a contradiction. Thus G is bi-critical. This completes the proof of Corollary 5.3.4. □

We will show that if a maximal $3-\gamma_c-$vertex critical graph G satisfies $m+1 = s$, then G is isomorphic to $\mu(K_l)$.

In the following lemmas, we suppose that $m = s-1$ and let $c_i \in V(C_i)$ for $i = 1, 2, ..., s-1$. Since $s \ge 4$, $s-1 \ge 3$. By Lemma 2.1.16, the vertices $c_1, c_2, ..., c_{s-1}$ can be ordered as $x_1, x_2, ..., x_{s-1}$ and there exists a path $y_1, y_2, ..., y_{s-2}$ with $\{x_i, y_i\} \succ_c G - x_{i+1}$ for $i \in \{1, 2, ..., s-2\}$. The components $C_1, C_2, .., C_{s-1}$ are also ordered as $A_1, A_2, ..., A_{s-1}$ where $x_i \in V(A_i)$ for $i = 1, 2, ..., s-1$. We see that $x_1 \succ \{y_1, y_2, ..., y_{s-2}\}$. By Lemma 5.1.4(2), x_1 is not adjacent to at least one vertex in S, y_{s-1} says. Thus $y_1 y_{s-1} \in E(G)$.

In the following, we let

$$x_1 y_{s-1} \notin E(G) \text{ and } y_1 y_{s-1} \in E(G).$$

Let $\{y_s\} = S - \{y_1, y_2, ..., y_{s-1}\}$.

Lemma 5.3.5. Let $i, j \in \{1, 2, ..., s-2\}$. We have these following properties.

(1) $y_i \succ \cup_{j=1}^{s-1} V(A_j) - (V(A_i) \cup \{x_{i+1}\})$, in particular, $\{y_1, y_2, ..., y_{s-2}\} - \{y_i\} \subseteq N_S(x_{i+1})$ and $\{y_1, y_2, ..., y_{s-2}\} - \{y_i\} \subseteq N_S(a_i)$ where $a_i \in V(A_i) - \{x_i\}$.

(2) If $y_i y_j \in E(G)$, then $\{y_i, y_j\} \succ_c \cup_{i'=1}^{s-1} V(A_{i'})$.

Proof. (1) Because x_i is not adjacent to any vertex of a component A_j where $j \neq i$, By Lemma 2.1.16, we must have $y_i \succ \cup_{j=1}^{s-1} V(A_j) - (V(A_i) \cup \{x_{i+1}\})$. This implies that $\{y_1, y_2, ..., y_{s-2}\} - \{y_i\} \subseteq N_S(x_{i+1})$ and $\{y_1, y_2, ..., y_{s-2}\} - \{y_i\} \subseteq N_S(a_i)$ where $a_i \in V(A_i) - \{x_i\}$, thus establishing (1).

(2) If $j = i-1$, then, by (1), $y_j \succ (V(A_i) - \{x_i\}) \cup \{x_{i+1}\}$. This implies by (1) that $\{y_i, y_j\} \succ_c \cup_{i'=1}^{s-1} V(A_{i'})$. If $j = i+1$, then, similarly, $\{y_i, y_j\} \succ_c \cup_{i'=1}^{s-1} V(A_{i'})$. Suppose that $j \neq i+1, i-1$. Therefore (1) implies $y_j \succ V(A_i) \cup \{x_{i+1}\}$ and so $\{y_i, y_j\} \succ_c \cup_{i'=1}^{s-1} V(A_{i'})$, thus establishing (2). □

Lemma 5.3.6. For $i, j \in \{2, 3, ..., s-1\}$, if $x_i y_s, x_j y_s \in E(G)$, then $y_{i-1} y_{j-1} \in E(G)$.

Proof. Consider $G + x_i x_j$. Lemma 5.1.5(1) gives $|D^c_{x_i x_j} \cap \{x_i, x_j\}| = 1$ and $|D^c_{x_i x_j} \cap S| = 1$. Suppose without loss of generality that $D^c_{x_i x_j} = \{x_i, y\}$ where $y \in S$. To dominate x_1, $y \neq y_{s-1}$. As $x_j y_s \in E(G)$, by Lemma 5.3.5(1), we must have $\{y_1, y_2, ..., y_{s-2}, y_s\} - \{y_{j-1}\} \subseteq N_S(x_j)$. By Lemma 2.1.15(3), $y \notin \{y_1, y_2, ..., y_{s-2}, y_s\} - \{y_{j-1}\}$. Thus $y = y_{j-1}$. Since $x_i y_{i-1} \notin E(G)$, it follows that $y_{j-1} y_{i-1} \in E(G)$. This completes the proof. □

Lemma 5.3.7. For $i, j \in \{2, ..., s-1\}$, $y_s \notin D^c_{x_i x_j}$ and $y_{i-1} y_{j-1} \in E(G)$.

Proof. Consider $G + x_i x_j$. By Lemma 5.1.5(1), $|D^c_{x_i x_j} \cap \{x_i, x_j\}| = 1$ and $|D^c_{x_i x_j} \cap S| = 1$. Without loss of generality let $x_j \in D^c_{x_i x_j}$. Suppose to the contrary that $y_s \in D^c_{x_i x_j}$. Thus $D^c_{x_i x_j} = \{y_s, x_j\}$. Since x_j is not adjacent to $(\cup_{i' \neq j} V(A_{i'})) \cup \{y_{j-1}\}$, $y_s \succ ((\cup_{i' \neq j} V(A_{i'})) \cup \{y_{j-1}, x_j\}) - \{x_i\}$, in particular, $y_s \succ \{x_1, x_2, ..., x_{s-1}\} - \{x_i\}$. As a consequence of Lemma 5.3.6, $G[\{y_1, y_2, ..., y_{s-2}\} - \{y_{i-1}\}]$ is complete. Since $\{x_i, y_i\} \succ_c G - x_{i+1}$, $y_i y_s \in E(G)$. We distinguish three cases according to i.

Case 1 : $i = 2$.

Clearly $y_2 y_s \in E(G)$ and $G[\{y_2, y_3, ..., y_{s-2}\}]$ is complete. Since $y_1 y_{s-1} \in E(G)$, by Lemma 5.3.5(2), $\{y_1, y_2\} \succ_c G$, contradicting $\gamma_c(G) = 3$. So Case 1 cannot occur.

Case 2 : $3 \leq i \leq s-2$.

Because $G[\{y_1, y_2, ..., y_{s-2}\} - \{y_{i-1}\}]$ is complete, it follows that $S - \{y_{i-1}, y_s\} \subseteq$

$N_S(y_1)$. As $y_iy_{s-1}, y_iy_{i-1} \in E(G)$, we obtain $\{y_1, y_i\} \succ_c G$, again, a contradiction. So Case 2 cannot occur.

Case 3 : $i = s-1$.

Clearly $y_s \succ \{x_1, x_2, ..., x_{s-2}\}$. Lemma 5.3.6 gives that $G[\{y_1, y_2, ..., y_{s-3}\}]$ is complete. Since $\{y_s, x_j\} = D^c_{x_ix_j}$ and $x_jy_{j-1} \notin E(G)$, it follows that $y_sy_{j-1} \in E(G)$.

We show that $y_1y_{s-2} \in E(G)$. If $s = 4$, then, clearly, $y_1y_{s-2} \in E(G)$. Suppose that $s \geq 5$. Consider $G + x_2x_{s-1}$. Lemma 5.1.5(1) thus implies $|D^c_{x_2x_{s-1}} \cap \{x_2, x_{s-1}\}| = 1$ and $|D^c_{x_2x_{s-1}} \cap S| = 1$. Suppose that $x_2 \in D^c_{x_2x_{s-1}}$. Since $(G + x_2x_{s-1})[D^c_{x_2x_{s-1}}]$ is connected, $y_1 \notin D^c_{x_2x_{s-1}}$. To dominate x_1, $y_{s-1} \notin D^c_{x_2x_{s-1}}$. To dominate $x_3, x_4, ..., x_{s-2}$, we have by Lemma 5.3.5(1) that $y_2, y_3, ..., y_{s-3} \notin D^c_{x_2x_{s-1}}$. Thus $y_s \in D^c_{x_2x_{s-1}}$ or $y_{s-2} \in D^c_{x_2x_{s-1}}$. If $y_s \in D^c_{x_2x_{s-1}}$, then $D^c_{x_2x_{s-1}} = \{y_s, x_2\}$. Since $x_2y_1 \notin E(G)$, $y_1y_s \in E(G)$ and so $y_1 \succ \{y_2, y_3, ..., y_s\} - \{y_{s-2}\}$. Because $y_{s-3}y_{s-2} \in E(G)$, by Lemma 5.3.5(2), $\{y_1, y_{s-3}\} \succ_c G$, contradicting $\gamma_c(G) = 3$. So $D^c_{x_2x_{s-1}} = \{x_2, y_{s-2}\}$. Since $x_2y_1 \notin E(G)$, $y_1y_{s-2} \in E(G)$, as required. We then suppose that $x_{s-1} \in D^c_{x_2x_{s-1}}$. By the connectedness of $(G + x_2x_{s-1})[D^c_{x_2x_{s-1}}]$, $y_{s-2} \notin D^c_{x_2x_{s-1}}$. As $y_sx_2 \in E(G)$, by Lemma 2.1.15(3), we must have $y_s \notin D^c_{x_2x_{s-1}}$. To dominate x_1, $y_{s-1} \notin D^c_{x_2x_{s-1}}$. To dominate $x_3, x_4, ..., x_{s-2}$, we have by Lemma 5.3.5(1) that $y_2, y_3, ..., y_{s-3} \notin D^c_{x_2x_{s-1}}$. Therefore $y_1 \in D^c_{x_2x_{s-1}}$ and $D^c_{x_2x_{s-1}} = \{y_1, x_{s-1}\}$. Since $x_{s-1}y_{s-2} \notin E(G)$, $y_1y_{s-2} \in E(G)$, as required.

We now have $y_1 \succ \{y_2, y_3, ..., y_{s-1}\}$. Since $y_{j-1}y_s \in E(G)$, by Lemma 5.1.4(2), $j \neq 2$. Moreover, Lemma 5.3.5(2) yields $\{y_1, y_{j-1}\} \succ_c G$ contradicting $\gamma_c(G) = 3$ and Case 3 cannot occur.

Hence $y_s \notin D^c_{x_ix_j}$ for all $i, j \in \{2, 3, ..., s-1\}$. Moreover, to dominate x_1, $y_{s-1} \notin D^c_{x_ix_j}$. As a consequence of Lemma 5.3.5(1), $\{y_1, y_2, ..., y_{s-2}\} - \{y_{i-1}\} \subseteq N_S(x_i)$. Lemma 2.1.15(3) thus implies $(\{y_1, y_2, ..., y_{s-2}\} - \{y_{i-1}\}) \cap D^c_{x_ix_j} = \emptyset$. Hence $D^c_{x_ix_j} = \{x_j, y_{i-1}\}$. Because $x_jy_{j-1} \notin E(G)$, it follows that $y_{i-1}y_{j-1} \in E(G)$. This completes the proof. □

Lemma 5.3.8. $G[\{y_1, y_2, ..., y_{s-2}\}]$ is complete, moreover, for all $i = 1, 2, ..., s-2$, $y_sy_i \notin E(G)$ and $y_sx_i \in E(G)$.

Proof. Lemma 5.3.7 yields that $G[\{y_1, y_2, ..., y_{s-2}\}]$ is complete. Because $y_1y_{s-1} \in E(G)$, by Lemma 5.1.4(2), $y_1y_s \notin E(G)$. If $y_sy_i \in E(G)$ for some $i \in \{2, 3, ..., s-2\}$, then Lemma 5.3.5(2) implies $\{y_1, y_i\} \succ_c G$ contradicting $\gamma_c(G) = 3$. Thus $y_sy_i \notin E(G)$ for all $i = 1, 2, ..., s-2$. Because $\{x_i, y_i\} \succ_c G - x_{i+1}$, $x_iy_s \in E(G)$ for $i \in \{1, 2, ..., s-2\}$. This completes the proof. □

Lemma 5.3.9. $N_S(x_1) = S - \{y_{s-1}\}$.

Proof. Recall that $x_1 \succ \{y_1, y_2, ..., y_{s-2}\}$ and $x_1y_{s-1} \notin E(G)$. These imply, by Lemma 5.3.8, that $N_S(x_1) = S - \{y_{s-1}\}$. This completes the proof. □

Lemma 5.3.10. $y_{s-1}y_i \in E(G)$ for $i = 1, 2, ..., s-3$.

Proof. For $i \in \{2, 3, ..., s-2\}$, consider $G + x_1x_i$. Lemma 5.1.5(1) implies that $|D^c_{x_1x_i} \cap \{x_1, x_i\}| = 1$ and $|D^c_{x_1x_i} \cap S| = 1$. Lemma 5.3.8 gives that $y_s \succ \{x_1, x_2, ..., x_{s-2}\}$. Thus, by Lemma 2.1.15(3), $y_s \notin D^c_{x_1x_i}$. We consider the case $\{x_1\} = D^c_{x_1x_i} \cap \{x_1, x_i\}$. As a consequence of Lemma 5.3.5(1), $\{y_1, y_2, ..., y_{s-2}\} - \{y_{i-1}\} \subseteq N_S(x_i)$. This implies that $(\{y_1, y_2, ..., y_{s-2}\} - \{y_{i-1}\}) \cap D^c_{x_1x_i} = \emptyset$ by Lemma 2.1.15(3). Because $(G + x_1x_i)[D^c_{x_1x_i}]$ is connected, $y_{s-1} \notin D^c_{x_1x_i}$. Lemma 5.1.4(1) thus implies $D^c_{x_1x_i} = \{x_1, y_{i-1}\}$. Hence $y_{i-1}y_{s-1} \in E(G)$ because $x_1y_{s-1} \notin E(G)$. We now consider the case when $\{x_i\} = D^c_{x_1x_i} \cap \{x_1, x_i\}$. Lemma 5.3.9 yields that $N_S(x_1) = S - \{y_{s-1}\}$. By Lemma 2.1.15(3), we must have $(S - \{y_{s-1}\}) \cap D^c_{x_1x_i} = \emptyset$. That is $D^c_{x_1x_i} = \{x_i, y_{s-1}\}$. Similarly, $y_{i-1}y_{s-1} \in E(G)$. Therefore $y_{s-1}y_i \in E(G)$ for $i = 1, 2, ..., s-3$. This completes the proof. □

We next establish the following lemmas which describe the structure of a maximal $3-\gamma_c-$vertex critical graph satisfying $m + 1 = s$ and will be used directly to proof our main result.

Lemma 5.3.11. y_s is an isolated vertex of S.

Proof. In view of Lemma 5.3.8, we need only show that $y_{s-1}y_s \notin E(G)$. Suppose $y_{s-1}y_s \in E(G)$. Lemmas 5.1.4(2) and 5.3.10 imply that $y_{s-1}y_{s-2} \notin E(G)$. Thus $y_{s-1} \succ S - \{y_{s-2}\}$.

Consider $G - y_{s-1}$. Lemma 5.1.4(1) gives that $D^c_{y_{s-1}} \cap S \neq \emptyset$. By Lemma 2.1.22(1), $D^c_{y_{s-1}} \cap S = \{y_{s-2}\}$. As $(G - y_{s-1})[D^c_{y_{s-1}}]$ is connected, to dominate x_{s-1}, we obtain $D^c_{y_{s-1}} \cap (V(A_{s-1}) - \{x_{s-1}\}) \neq \emptyset$. Moreover, Lemma 2.1.22(2) yields that $|D^c_{y_{s-1}} \cap (V(A_{s-1}) - \{x_{s-1}\})| = 1$. Let $\{b\} = D^c_{y_{s-1}} - \{y_{s-2}\}$. As Lemma 5.3.8 gives that $y_{s-2}y_s \notin E(G)$, we must have $by_s \in E(G)$. This together with Lemma 5.3.5(1) yields $S - \{y_{s-1}\} \subseteq N_S(b)$. So $N_S(b) = S - \{y_{s-1}\}$ by Lemma 5.1.4(2). It follows from Lemma 5.3.9 that $N_S(x_1) = N_S(b)$, contradicting Lemma 5.3.1. Hence $y_{s-1}y_s \notin E(G)$. This completes the proof. □

Lemma 5.3.12. $G[S - \{y_s\}]$ is complete.

Proof. In view of Lemma 5.3.10, we need only show that $y_{s-2}y_{s-1} \in E(G)$. Consider $G + x_1x_{s-1}$. Lemma 5.1.5(1) yields $|D^c_{x_1x_{s-1}} \cap \{x_1, x_{s-1}\}| = 1$ and $|D^c_{x_1x_{s-1}} \cap S| = 1$.

We first consider the case when $\{x_1\} = D^c_{x_1x_{s-1}} \cap \{x_1, x_{s-1}\}$. By the connectedness of $(G + x_1x_{s-1})[D^c_{x_1x_{s-1}}]$, $y_{s-1} \notin D^c_{x_1x_{s-1}}$. Lemma 5.3.5(1) yields that $\{y_1, y_2, ..., y_{s-3}\} \subseteq$

$N_S(x_{s-1})$. Lemma 2.1.15(3) thus implies $\{y_1, y_2, ..., y_{s-3}\} \cap D^c_{x_1x_{s-1}} = \emptyset$. Hence $y_s \in D^c_{x_1x_{s-1}}$ or $y_{s-2} \in D^c_{x_1x_{s-1}}$. We note by Lemma 5.3.11 that $y_sy_{s-1} \notin E(G)$. Thus the first case is impossible because $D^c_{x_1x_{s-1}} = \{x_1, y_s\}$ does not dominate y_{s-1}. Therefore $D^c_{x_1x_{s-1}} = \{x_1, y_{s-2}\}$. Since $x_1y_{s-1} \notin E(G)$, $y_{s-2}y_{s-1} \in E(G)$.

We now consider the case when $\{x_{s-1}\} = D^c_{x_1x_{s-1}} \cap \{x_1, x_{s-1}\}$. We note by Lemma 5.3.9 that $N_S(x_1) = S - \{y_{s-1}\}$. Lemma 2.1.15(3) then yields that $(S - \{y_{s-1}\}) \cap D^c_{x_1x_{s-1}} = \emptyset$. Therefore $D^c_{x_1x_{s-1}} = \{x_{s-1}, y_{s-1}\}$. So $y_{s-1}y_{s-2} \in E(G)$. Hence $G[S - \{y_s\}]$ is complete. This completes the proof. □

Lemma 5.3.13. $y_s \succ \{x_1, x_2, ..., x_{s-1}\}$.

Proof. By Lemma 5.3.8, we need only show that $y_sx_{s-1} \in E(G)$. Suppose $y_sx_{s-1} \notin E(G)$ and then consider $G + x_{s-1}y_s$. Lemma 2.1.15(2) yields that $D^c_{x_{s-1}y_s} \cap \{x_{s-1}, y_s\} \neq \emptyset$. To dominate y_{s-2}, $D^c_{x_{s-1}y_s} \neq \{x_{s-1}, y_s\}$. By Lemma 2.1.15(1) and (2), $|D^c_{x_{s-1}y_s} - \{x_{s-1}, y_s\}| = 1$ and $|D^c_{x_{s-1}y_s} \cap \{x_{s-1}, y_s\}| = 1$. Suppose $y_s \in D^c_{x_{s-1}y_s}$. Let $\{y\} = D^c_{x_{s-1}y_s} - \{y_s\}$. Since y_s is an isolated vertex in S, $y \succ S$ contradicting Lemma 5.1.4(2). So $x_{s-1} \in D^c_{x_{s-1}y_s}$. To dominate x_1, $\{y_1, y_2, ..., y_{s-2}\} \cap D^c_{x_{s-1}y_s} \neq \emptyset$. By the connectedness of $(G + x_{s-1}y_s)[D^c_{x_{s-1}y_s}]$, $y_{s-2} \notin D^c_{x_{s-1}y_s}$. Thus $y_i \in D^c_{x_{s-1}y_s}$ for some $i \in \{1, 2, ..., s-3\}$. So $D^c_{x_{s-1}y_s}$ does not dominate x_{i+1}, a contradiction. Hence $y_sx_{s-1} \in E(G)$. This completes the proof. □

Lemma 5.3.14. $y_{s-1} \succ \cup_{i=2}^{s-1} V(A_i)$.

Proof. Suppose $y_{s-1}c \notin E(G)$ for some $c \in \cup_{i=2}^{s-1} V(A_i)$. Consider $G + y_{s-1}c$. Lemma 2.1.15(1) and (2) then imply that $|D^c_{y_{s-1}c}| = 2$ and $D^c_{y_{s-1}c} \cap \{y_{s-1}, c\} \neq \emptyset$. To dominate x_1, $D^c_{y_{s-1}c} \neq \{y_{s-1}, c\}$.

Case 1 : $c \in D^c_{y_{s-1}c}$.

This imply, by Lemmas 5.3.12 and 2.1.15(3), that $y_i \notin D^c_{y_{s-1}c}$ for $i \in \{1, 2, ..., s-2\}$. To dominate $\cup_{i=1}^{s-1} V(A_i)$, $y_s \in D^c_{y_{s-1}c}$. Clearly $cy_s \in E(G)$. Since y_s is an isolated vertex in S, $N_S(c) = S - \{y_{s-1}\}$. It follows from Lemma 5.3.9 that $N_S(x_1) = N_S(c)$. This contradicts Lemma 5.3.1. So Case 1 cannot occur.

Case 2 : $y_{s-1} \in D^c_{y_{s-1}c}$.

By Lemma 2.1.15(1), $|D^c_{y_{s-1}c} - \{y_{s-1}\}| = 1$. Let $D^c_{y_{s-1}c} - \{y_{s-1}\} = \{a\}$. Remind that $x_1y_{s-1} \notin E(G)$. As Lemma 5.3.11 gives that y_s is an isolated vertex in S, to dominate $\{x_1, y_s\}$, we must have $a \in V(A_1)$. Thus $y_{s-1} \succ \cup_{i=2}^{s-1} V(A_i) - \{c\}$. Since $(G + y_{s-1}c)[D^c_{y_{s-1}c}]$ is connected and $y_{s-1}y_s \notin E(G)$, it follows that $ay_{s-1}, ay_s \in E(G)$. This, together with Lemma 5.3.5(1), gives that $S - \{y_1\} \subseteq N_S(a)$. By Lemma 5.1.4(2), $N_S(a) = S - \{y_1\}$. Consider $G - a$. Lemmas 2.1.22(1) and 5.1.4(1) yield that $\{y_1\} = D^c_a \cap S$. By Lemma 2.1.22(2), $|D^c_a - \{y_1\}| = 1$. Let $\{b\} = D^c_a - \{y_1\}$. To dominate

$\{x_2, y_s\}$, $b \in V(A_2)$. Lemma 5.3.11 gives $by_s \in E(G)$. Lemma 5.3.5(1) then implies that $S - \{y_2, y_{s-1}\} \subseteq N_S(b)$.

Subcase 2.1 : $b = c$.

Thus $y_{s-1}b \notin E(G)$. Because $S - \{y_2, y_{s-1}\} \subseteq N_S(b)$, it follows that $y_2 b \notin E(G)$ as otherwise $N_S(b) = N_S(x_1)$ contradicting Lemma 5.3.1. Therefore $N_S(b) = S - \{y_2, y_{s-1}\}$. Consider $G + x_1c$. Lemma 5.1.5(1) gives that $|D^c_{x_1c} \cap \{x_1, c\}| = 1$ and $|D^c_{x_1c} \cap S| = 1$. We note by Lemma 5.3.9 that $S - \{y_{s-1}\} = N_S(x_1)$. Thus $N_S(x_1) \cap N_S(b) = S - \{y_2, y_{s-1}\}$. By Lemma 2.1.15(3), $(S - \{y_2, y_{s-1}\}) \cap D^c_{x_1c} = \emptyset$. By the connectedness of $(G + x_1c)[D^c_{x_1c}]$, $y_{s-1} \notin D^c_{x_1c}$. Therefore $y_2 \in D^c_{x_1c}$. Clearly $D^c_{x_1c}$ does not dominate x_3, a contradiction. So Subcase 2.1 cannot occur.

Subcase 2.2 : $b \neq c$.

Since $y_{s-1} \succ \cup_{i=2}^{s-1} V(A_i) - \{c\}$, it follows that $y_{s-1}b \in E(G)$. By Lemma 5.1.4(2), $N_S(b) = S - \{y_2\}$. Lemmas 5.3.13 and 5.3.5(1) give $S - \{y_2, y_{s-1}\} \subseteq N_S(x_3)$. If $x_3 \neq c$, then $x_3 y_{s-1} \in E(G)$ because $y_{s-1} \succ \cup_{i=2}^{s-1} V(A_i) - \{c\}$. This implies by Lemma 5.1.4(2) that $N_S(x_3) = S - \{y_2\}$. Thus $N_S(x_3) = N_S(b)$ contradicting Lemma 5.3.1. Therefore $x_3 = c$, moreover, $N_S(x_3) = S - \{y_2, y_{s-1}\}$.

We finally consider $G + bx_3$. Lemma 5.1.5(1) yields that $|D^c_{bx_3} \cap \{b, x_3\}| = 1$ and $|D^c_{bx_3} \cap S| = 1$. Suppose first that $b \in D^c_{bx_3}$. As $N_S(x_3) = S - \{y_2, y_{s-1}\}$, by Lemma 2.1.22(1), we obtain $(S - \{y_2, y_{s-1}\}) \cap D^c_{bx_3} = \emptyset$. Thus y_2 or y_{s-1} is in $D^c_{bx_3}$. Since $(G + bx_3)[D^c_{bx_3}]$ is connected, $D^c_{bx_3} = \{b, y_{s-1}\}$. But $D^c_{bx_3}$ does not dominate x_1, a contradiction. Therefore $x_3 \in D^c_{bx_3}$ and $b \notin D^c_{bx_3}$. By Lemma 2.1.22(1), $(S - \{y_2\}) \cap D^c_{bx_3} = \emptyset$ and so $D^c_{bx_3} = \{x_3, y_2\}$. Clearly $(G + bx_3)[D^c_{bx_3}]$ is not connected, a contradiction and Case 2 cannot occur.

Hence $y_{s-1} \succ \cup_{i=2}^{s-1} V(A_i)$ and we finish proving this lemma. □

Lemma 5.3.15. $|V(A_i)| = 1$ for all $i = 1, 2, ..., s-1$.

Proof. Let A_i be a non-singleton component and $b_i \in V(A_i) - \{x_i\}$. We distinguish three cases according to i.

Case 1 : $1 \leq i \leq s-3$.

By Lemma 5.3.5(1), $S - \{y_i, y_{s-1}, y_s\} \subseteq N_S(b_i)$. Consider $G - b_i$. Lemma 2.1.22(1) implies that $(S - \{y_i, y_{s-1}, y_s\}) \cap D^c_{b_i} = \emptyset$, moreover, by Lemma 5.1.5(2), $y_s \notin D^c_{b_i}$. Thus Lemma 5.1.4(1) implies that either $y_{s-1} \in D^c_{b_i}$ or $y_i \in D^c_{b_i}$. Suppose that $y_{s-1} \in D^c_{b_i}$. As a consequence of Lemma 2.1.22(1), $y_{s-1}b_i \notin E(G)$. This implies, by Lemma 5.3.14, that $i = 1$. We let $\{z\} = D^c_{b_i} - \{y_{s-1}\}$ by Lemma 2.1.22(2). To dominate $\{x_1, y_s\}$, $z \in V(A_1)$. Lemma 2.1.22(1) yields that $zb_i \notin E(G)$. Thus A_1 is not complete contradicting Corollary 5.3.3. Therefore $y_i \in D^c_{b_i}$. Let $\{w\} = D^c_{b_i} - \{y_i\}$. To dominate $\{x_{i+1}, y_s\}$,

$w \in V(A_{i+1})$. So $wy_i, wy_s \in E(G)$. Lemmas 5.3.14 and 5.3.5(1) imply that $S - \{y_{i+1}\} \subseteq N_S(w)$. Moreover, Lemma 5.1.4(2) gives $N_S(w) = S - \{y_{i+1}\}$. Lemma 5.3.5(1) together with Lemmas 5.3.13 and 5.3.14 yield that $S - \{y_{i+1}\} \subseteq N_S(x_{i+2})$. In fact, $N_S(x_{i+2}) = S - \{y_{i+1}\}$ by Lemma 5.1.4(2). Clearly $N_S(w) = N_S(x_{i+2})$ contradicting Lemma 5.3.1. So Case 1 cannot occur.

Case 2 : $i = s-2$.

Consider $G - b_i$. Lemmas 5.3.14 and 5.3.5(1) yield that $S - \{y_{s-2}, y_s\} \subseteq N_S(b_i)$. Thus Lemma 2.1.22(1) gives $(S - \{y_{s-2}, y_s\}) \cap D^c_{b_i} = \emptyset$. Moreover, $y_s \notin D^c_{b_i}$ by Lemma 5.1.5(2). Thus $y_{s-2} \in D^c_{b_i}$ by Lemma 5.1.4(1). As a consequence of Lemma 2.1.22(2), we have $D^c_{b_i} = \{y_{s-2}, z'\}$ for some z'. To dominate x_{s-1}, $z' \in V(A_{s-1})$. It follows from Lemmas 5.3.14 and 5.3.5(1) that $z' \succ \{y_1, y_2, ..., y_{s-1}\}$. Since $y_{s-2}y_s \notin E(G)$, $z'y_s \in E(G)$. Therefore $z' \succ S$ contradicting Lemma 5.1.4(2). So Case 2 cannot occur.

Case 3 : $i = s-1$.

Clearly $b_i \succ \{y_1, y_2, ..., y_{s-1}\}$ by Lemmas 5.3.5(1) and 5.3.14. Consider $G - b_i$. Lemma 5.1.4(1) yields $D^c_{b_i} \cap S \neq \emptyset$. Since Lemma 2.1.22(1) gives $D^c_{b_i} \cap \{y_1, y_2, ..., y_{s-1}\} = \emptyset$, it follows that $y_s \in D^c_{b_i}$ contradicting Lemma 5.1.5(2). Hence $|V(A_i)| = 1$ for all $i = 1, 2, ..., s-1$. This completes the proof. □

Theorem 5.3.16. Let G be a maximal $3-\gamma_c-$vertex critical graph and let $\omega(G-S) = m$ and $|S| = s$. If $m+1 = s$, then G is isomorphic to $\mu(K_{s-1})$.

Proof. In view of Lemmas 5.3.11 - 5.3.15, we have y_s is a vertex such that $N_G(y_s) = \{x_1, x_2, ..., x_{s-1}\}$ is an independent set and $G[\{y_1, y_2, ..., y_{s-1}\}]$ is complete. Moreover, let $y'_1 = x_2, y'_2 = x_3, y'_3 = x_4, ..., y'_{s-2} = x_{s-1}$ and $y'_{s-1} = x_1$. Thus $y_i \succ N_G(y_s) - \{y'_i\}$ for $1 \leq i \leq s-1$. Clearly G is isomorphic to $\mu(K_{s-1})$. This completes the proof. □

We conclude this section by establishing the next corollary.

Corollary 5.3.17. Let G be a maximal $3-\gamma_c-$vertex critical graph of odd order with $\delta \geq 4$. Then G is $3-$factor critical or isomorphic to $\mu(K_l)$.

Proof. Suppose G is not $3-$factor critical. Thus there exists a vertex cut set of size three S' such that $G' = G - S'$ does not contain a perfect matching. Tutte's Theorem and parity then imply that there exists a vertex cut set S'' such that $\omega_o(G' - S'') \geq |S''| + 2$. Let $\tilde{S} = S' \cup S''$. We have $|\tilde{S}| = |S''| + 3$ and $\omega_o(G' - S'') = \omega_o(G - \tilde{S})$. Therefore $\omega_o(G - \tilde{S}) + 1 \geq |\tilde{S}|$. Corollary 5.3.3 yields that

$$\omega(G - \tilde{S}) + 1 \leq |\tilde{S}| \leq \omega_o(G - \tilde{S}) + 1 \leq \omega(G - \tilde{S}) + 1.$$

Thus $\omega(G - \tilde{S}) + 1 = |\tilde{S}|$. By Theorem 5.3.16, G is isomorphic to $\mu(K_l)$. □

CHAPTER 6

Hamiltonicities of Critical Graphs with respect to Domination Numbers

The problem of interest is to investigate the structures of $k-\mathscr{D}-$edge critical graphs where $\mathscr{D} \in \{\gamma_c, \gamma_t, \gamma, i\}$. In particular, we consider the hamiltonicity of $k-\gamma_c-$edge critical graphs. Observe that the cycle C_{k+2} on $k+2$ vertices is a $k-\gamma_c-$edge critical hamiltonian graph. There are a number of results on the hamiltonian properties of $k-\mathscr{D}-$edge critical graphs when $\mathscr{D} \in \{\gamma_t, \gamma, i\}$. Sumner and Blitch [138] conjectured that :

Conjecture B1. Every $3-\gamma-$edge critical graph with more than six vertices contains a hamiltonian path.

This conjecture was proved by Wojcicka [151] who made the further conjecture that :

Conjecture B2. Every connected $3-\gamma-$edge critical graph with $\delta \geq 2$ is hamiltonian.

Favaron et al. [65] showed that every connected $3-\gamma-$edge critical graph satisfies $\alpha \leq \delta + 2$ where α is the independence number. Further, when this inequality is strict and $\delta \geq 2$, the graph is hamiltonian. Conjecture B2 was finally established by Tian et al. [139] when they proved that every connected $3-\gamma-$edge critical graph with $\delta \geq 2$ and $\alpha = \delta + 2$ is hamiltonian. Moreover, in Chapter 16 of [81], Sumner and Wojcicka proposed the following conjecture.

Conjecture B3. Every $(k-1)-$connected $k-\gamma-$edge critical graph is hamiltonian.

Yuansheng et al. [153] gave, for $k=4$, a smallest counter example to this conjecture, a graph with thirteen vertices.

Ao et al. [22] established that $2-$connected $3-i-$edge critical graphs are hamiltonian when $\delta \geq 3$. They also showed that there is only one family of $2-$connected

$3-i-$edge critical non-hamiltonian graphs when $\delta = 2$.

Simmons [136] showed that every $3-\gamma_t-$edge critical graph with a cut vertex contains a hamiltonian path. She, further, studied the hamiltonicity of $2-$connected $3-\gamma_t-$edge critical graphs according to the diameter. She proved that $2-$connected $3-\gamma_t-$edge critical graphs of diameter three contain a hamiltonian path. When the diameter is two, she focused on $2-$connected $3-\gamma_t-$edge critical graphs with $\delta \in \{2,3\}$ and showed that they contain a hamiltonian cycle. To investigate the hamiltonicity of $2-$connected $3-\gamma_t-$edge critical graphs remains unsolved.

This gives rise to the following problem :

When are $l-$connected $k-\gamma_c-$edge critical graphs hamiltonian?

The chapter is organized as the following. In Section 6.1, we prepare the notation and terminology as well as results that we make use in this chapter. In Section 6.2, we prove that every $2-$connected $k-\gamma_c-$edge critical graph is hamiltonian for $k = 1,2$ or 3. It is not difficult to show that every $3-\gamma_c-$edge critical graph is $3-\gamma_t-$edge critical. Hence, this implies that every $2-$connected $3-\gamma_t-$edge critical graph is also hamiltonian. In Section 6.3, we provide a class of $l-$connected $k-\gamma_c-$edge critical non-hamiltonian graphs for $k \geq 4$ and $2 \leq l \leq \frac{n-3}{k-1}$. Hence, for $n \geq (k-1)l+3$, the class of $l-$connected $k-\gamma_c-$edge critical non-hamiltonian graphs of order n is empty if and only if $k = 1,2$ or 3. For $k-\gamma_t-$edge critical graphs, we provide classes of $2-$connected $k-\gamma_t-$edge critical non-hamiltonian graphs for $k = 4$ or 5. For $k-i-$edge critical graphs, we give a construction for a class of $2-$connected $k-i-$edge critical non-hamiltonian graphs for $k \geq 3$.

6.1 Preliminaries

In this section we set up the notation and terminology that we make use of in our work. Suppose G is a $2-$connected $3-\gamma_c-$edge critical non-hamiltonian graph. Let C be a longest cycle of G. We write $\overrightarrow{C}$ to indicate the clockwise orientation of C. Similarly, we denote the anticlockwise orientation of C by $\overleftarrow{C}$. In particular, for vertices u and v of C we denote the $(u,v)-$directed segment of $\overrightarrow{C}(\overleftarrow{C})$ by $u\overrightarrow{C}v$ $(u\overleftarrow{C}v)$. These two segments are illustrated in the following figure.

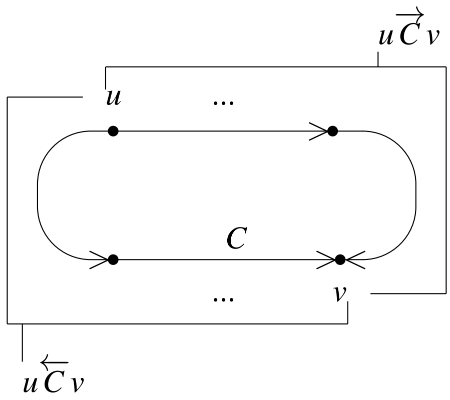

Figure 6.1 : The clockwise and anticlockwise orientations.

The successor (predecessor) of a vertex v of C in $\overrightarrow{C}$ is denoted by $v^+(v^-)$. Note that we always use the orientation $\overrightarrow{C}$ when we mention about the successor and the predecessor of any vertex of C. Let H be a component of $G-C$ and $X = N_C(H)$. Suppose $|X| = d$. We may order the vertices of the set X as $x_1, x_2, ..., x_d$ according to $\overrightarrow{C}$. We, further, let

$X^+ = \{a_1, a_2, ..., a_d\}$ where $a_i = x_i^+$,
$X^- = \{b_1, b_2, ..., b_d\}$ where $b_i = x_{i+1}^-$,
$C_i = V(a_i \overrightarrow{C} b_i)$.

All subscripts are taken modulo d throughout. The following figure illustrates our notation and terminology.

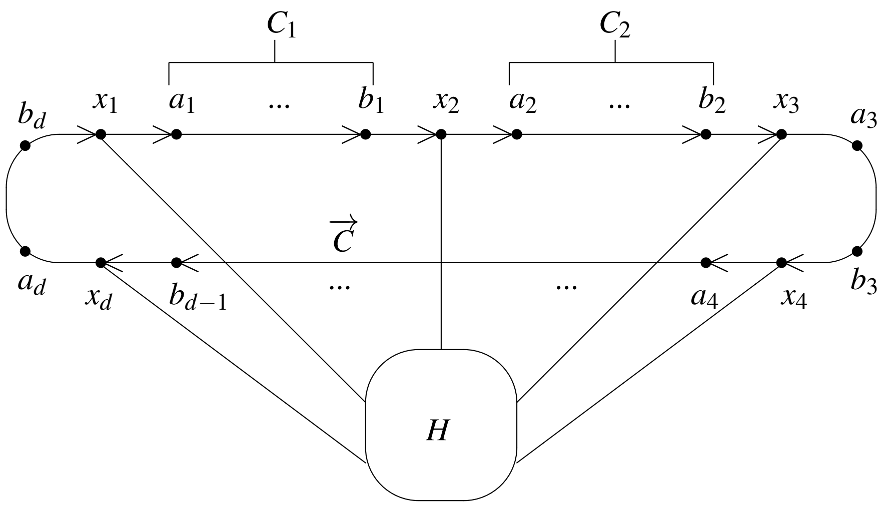

Figure 6.2

A vertex $v \in C_i$ is called an $X^+-vertex$ if $v^+a_i \in E(G)$ and an $X^- -vertex$ if $v^-b_i \in E(G)$. Hence, every $a_i(b_i)$ is an $X^+(X^-)-$vertex. Favaron et al. [65] and Tian et al. [139] provided some structural properties described in Lemmas 6.1.1 - 6.1.10. These lemmas are consequences of the assumption that C is a longest cycle of a non-hamiltonian graph G and do not require the property that G is $3-\gamma_c-$edge critical.

Lemma 6.1.1. [65] $X^+ \cap X = \emptyset$ and $X^- \cap X = \emptyset$.

Lemma 6.1.2. [65] If $u_i \in C_i$ and $u_j \in C_j$ are two X^+-vertices($X^- -$vertices) with $i \neq j$, then there is no $u_i P_{G-C} u_j$ path, in particular, $u_i u_j \notin E(G)$.

Lemma 6.1.3. [65]

(1) Let $u_i \in C_i, u_j \in C_j$ be two X^+-vertices($X^- -$vertices) with $i \neq j$. For any vertex $v \in V(u_i^+ \overrightarrow{C} u_j^-)$, if $vu_i \in E(G)$, then $v^- u_j \notin E(G)$.

(2) Let $a_i \in X^+$ and $b_j \in X^-$ be such that $i \neq j+1$. For any vertex $v \in V(a_{j+1} \overrightarrow{C} x_i)$, if $vb_j \in E(G)$, then $v^- a_i \notin E(G)$.

Lemma 6.1.4. [65] Suppose $b_{j-1} a_j \in E(G)$. If $w \in C_i$ is an $X^- -$vertex with $i \neq j-1$, then $wx_j \notin E(G)$. Further, if $w \in C_i$ is an X^+-vertex with $i \neq j$, then $wx_j \notin E(G)$.

Lemma 6.1.5. [139] Let $a_i \in X^+$ and $b_j \in X^-$ be such that $i \neq j+1$. For any vertex $v \in V(x_{j+1} \overrightarrow{C} b_{i-1})$, if $vb_j \in E(G)$, then $v^+ a_i \notin E(G)$.

Lemma 6.1.6. [139] Suppose $a_i b_j \in E(G)$ for some i and j with $i \neq j+1$, and $u, u^+ \in V(a_i \overrightarrow{C} b_j)$, $v, v^+ \in V(b_j^+ \overrightarrow{C} a_i^-)$. Then $uv \in E(G)$ implies $u^+ v^+ \notin E(G)$; $uv^+ \in E(G)$ implies $u^+ v \notin E(G)$.

Lemma 6.1.7. [139] If $a_i b_j \in E(G)$ for some i and j with $i \neq j+1$ and $i \neq j$ and $G[X]$ is a clique, then $N(a_{j+1}) \cap \{b_i, b_{i+1}, ..., b_{j-1}\} = \emptyset$.

Moreover, from Favaron et al. [65], we can establish, using similar arguments, the following three lemmas when $V(H) = \{v\}, deg_G(v) = d$ and $\alpha \leq d+1$.

Lemma 6.1.8. [65] If $u \in C_i$ is an X^+-vertex, then all the vertices of $V(a_i \overrightarrow{C} u)$ are X^+-vertices. Similarly, if $u \in C_i$ is an $X^- -$vertex, then all the vertices of $V(u \overrightarrow{C} b_i)$ are $X^- -$vertices.

Lemma 6.1.9. [65] Let $u_i \in C_i$ be an X^+-vertex. If $N(u_i) \cap C_{i-1} \neq \emptyset$, then $u_i b_{i-1} \in E(G)$. Similarly, let $v_i \in C_i$ be an $X^- -$vertex. If $N(v_i) \cap C_{i+1} \neq \emptyset$, then $v_i a_{i+1} \in E(G)$.

Lemma 6.1.10. [65] For each vertex $a_i \in X^+ - X^-$, $N(a_i) \cap X^- \neq \emptyset$ and for each vertex $b_i \in X^- - X^+$, $N(b_i) \cap X^+ \neq \emptyset$.

6.2 Hamiltonicity of $3-\mathscr{D}-$Edge Critical Graphs where $\mathscr{D} \in \{\gamma_c, \gamma_t\}$

In this section, we prove that $3-\gamma_c-$edge critical graphs are hamiltonian. We will do this by considering two cases according to the value of δ. Our first case is that of $\delta = 2$ and the other case is $\delta \geq 3$.

6.2.1 $\delta = 2$

Our main aim in this subsection is to prove the following theorem.

Theorem 6.2.1. Let G be a $3-\gamma_c-$edge critical graph. If $\delta = 2$, then G is hamiltonian.

We establish this theorem by proving a number of lemmas. Throughout we make use of the following notation and terminology. Let G be a $3-\gamma_c-$edge critical graph containing a vertex c of degree $\delta = 2$. Let $N(c) = \{c_1, c_2\}$, $V_i = N(c_i) - N[c_{3-i}]$ for $i = 1, 2$, $V_3 = N_{G-c}(c_1) \cap N_{G-c}(c_2)$ and $V_4 = V(G) - (\cup_{i=1}^{3} V_i \cup \{c, c_1, c_2\})$.

Lemma 6.2.2. Let G be a $3-\gamma_c-$edge critical graph with $\delta = 2$. If $a \in V_i, b \in V_j$ and $ab \notin E(G)$, then $D^c_{ab} = \{x, y\}$ where $x \in \{a, b\}$, $y \in \{c_1, c_2\}$ and $xy \in E(G)$.

Proof. Consider $G + ab$. By Lemma 2.1.15(1), $|D^c_{ab}| = 2$. Let $D^c_{ab} = \{x, y\}$. We have $D^c_{ab} \cap \{a, b\} \neq \emptyset$ by Lemma 2.1.15(2). Without loss of generality, let $x \in \{a, b\}$. To dominate c, $y \in \{c, c_1, c_2\}$. Since $(G + ab)[D^c_{ab}]$ is connected, $y \neq c$ and $xy \in E(G)$. Therefore $y \in \{c_1, c_2\}$. This completes the proof. □

Lemma 6.2.3. Let G be a $3-\gamma_c-$edge critical graph with $\delta = 2$. Then either $V_i = \emptyset$ or $G[V_i]$ is complete for $i = 1, ..., 4$.

Proof. Suppose there exist $a, b \in V_i$ such that $ab \notin E(G)$. Consider $G + ab$. As a consequence of Lemma 6.2.2, $|D^c_{ab} \cap \{a, b\}| = 1, |D^c_{ab} \cap \{c_1, c_2\}| = 1$ and $i \neq 4$. This implies that either $V_4 = \emptyset$ or $G[V_4]$ is complete. Without loss of generality let $\{a\} = D^c_{ab} \cap \{a, b\}$ and $c_j \in D^c_{ab}$ for some $j \in \{1, 2\}$. As $ac_j \in E(G)$, we must have $a \in V_{j'}$ for some $j' = j$ or 3. By the assumption $b \in V_{j'}$. Clearly $bc_j \in E(G)$ contradicting Lemma 2.1.15(3). Thus $G[V_i]$ is complete and this completes the proof. □

Lemma 6.2.4. Let G be a $3-\gamma_c-$edge critical graph with $\delta = 2$. For all $u \in V_i, i \in \{1, 2, 3\}$, we have $|N_{V_4}(u)| \geq |V_4| - 1$. If $|N_{V_4}(u)| = |V_4| - 1$, then

(1) for $i = 3$, $u \succ V_j$ for some $j = 1$ or 2,

(2) for $i \neq 3$, $u \succ V_{3-i}$ and $c_1c_2 \in E(G)$.

Proof. Suppose there exist $a, b \in V_4$ such that $au, bu \notin E(G)$. Consider $G + au$. Lemma 6.2.2 yields that $|\{c_1, c_2\} \cap D^c_{au}| = 1$ and $\{u\} = D^c_{au} \cap \{a, u\}$. Thus D^c_{au} does not dominate b, a contradiction. So $|N_{V_4}(u)| \geq |V_4| - 1$. Suppose that $u \succ V_4 - a$ where $u \in V_i$ and $a \in V_4$. If $i \in \{1, 2\}$, then, by Lemma 6.2.2, $D^c_{au} = \{u, c_i\}$. Thus $u \succ V_{3-i}$ and $c_i c_{3-i} \in E(G)$. If $i = 3$, then, similarly, $D^c_{au} = \{u, c_j\}$ for some $j \in \{1, 2\}$. We have $u \succ V_{3-j}$, thus completing the proof of this lemma. □

From now on, let $F = G[\cup_{i=1}^4 V_i]$ and $\alpha_1 = \alpha(F)$. The following lemma shows that G is hamiltonian when $\alpha_1 \geq 3$.

Lemma 6.2.5. Let G be a $3-\gamma_c-$edge critical graph with $\delta = 2$. If $\alpha_1 \geq 3$, then G is hamiltonian.

Proof. Let I be an independent set of F of size $\alpha_1 \geq 3$. Lemma 2.1.16 implies that there exists an ordering $w_1, ..., w_{\alpha_1}$ of the vertices of I and a path $z_1, ..., z_{\alpha_1 - 1}$ satisfying $\{w_i, z_i\} \succ_c G - w_{i+1}$ for $1 \leq i \leq \alpha_1 - 1$. Since $I \subseteq \cup_{i=1}^4 V_i$, $w_i c \notin E(G)$. To dominate c, $z_i \in \{c, c_1, c_2\}$. By the connectedness of $G[\{w_i, z_i\}]$, $z_i \neq c$. Therefore $\{z_1, ..., z_{\alpha_1 - 1}\} \subseteq \{c_1, c_2\}$. Thus $\alpha_1 - 1 \leq 2$ and so $\alpha_1 = 3$.

Suppose without loss of generality that $z_1 = c_1$ and $z_2 = c_2$. Lemma 2.1.16 yields that $w_1 \in V_3, w_2 \in V_2, w_3 \in V_1$. Moreover, $z_1 z_2 \in E(G)$. Thus $V_4 \neq \emptyset$, as otherwise $\{z_1, z_2\} \succ_c G$, contradicting $\gamma_c(G) = 3$. Since $\{w_1, w_2, w_3\}$ is an independent set, w_1 does not dominate V_i for $i \in \{1, 2\}$. Further, w_2 and w_3 do not dominate V_1 and V_2, respectively. It follows from Lemma 6.2.4 that $w_i \succ V_4$ for $i = 1, 2, 3$. Consider $G + cw_3$. Lemma 2.1.15(1) implies that $|D^c_{cw_3}| = 2$. Moreover, Lemma 2.1.15(2) gives $|\{c, w_3\} \cap D^c_{cw_3}| \geq 1$. To dominate w_2, $|\{c, w_3\} \cap D^c_{cw_3}| = 1$. If $c \in D^c_{cw_3}$, then by the connectedness of $(G + cw_3)[D^c_{cw_3}]$, $D^c_{cw_3} = \{c, c_i\}$ for some $i \in \{1, 2\}$. Thus $D^c_{cw_3}$ does not dominate V_4 contradicting $D^c_{cw_3} \succ_c G + cw_3$. Therefore $c \notin D^c_{cw_3}$ and $w_3 \in D^c_{cw_3}$. Let $\{a\} = D^c_{cw_3} - \{w_3\}$. Clearly, $w_3 a \in E(G)$. Thus $a \neq w_1, w_2$. To dominate c_2, $a \in V_2 \cup V_3 \cup \{c_1\}$. Lemma 2.1.15(3) yields that $a \neq c_1$. Suppose that $a \in V_2$. For $1 \leq i \leq 4$, we let $U_i = G[V_i]$. Lemma 6.2.3 together with $w_i \succ V_4$ for $1 \leq i \leq 3$ then imply that there are U_i-hamiltonian paths $c_1 P_{U_1} w_3, w_3 P_{U_2} w_2, w_1 P_{U_3} c_2$ and $w_2 P_{U_4} w_1$. Thus

$$C'_1 = c, c_1 P_{U_1} w_3 P_{U_2} w_2 P_{U_4} w_1 P_{U_3} c_2, c$$

is a hamiltonian cycle. We also have, using similar arguments, a hamiltonian cycle when $a \in V_3$. Hence if $\alpha_1 \geq 3$, then G is hamiltonian. This completes the proof. □

Lemma 6.2.6. Let G be a $3-\gamma_c-$edge critical graph with $\delta = 2$. If F has the connectivity less than two, then G is hamiltonian.

Proof. By Lemma 6.2.5, it suffices to consider the case $\alpha_1 \le 2$. Suppose F is not connected. Let H_1 and H_2 be components of F. Therefore $G-\{c_1,c_2\}$ has at least three components H_1,H_2 and H_3 with $V(H_3)=\{c\}$, contradicting Theorem 2.1.18. Hence F is connected.

Suppose that F has a cut vertex u. Since $\alpha_1 \le 2$, $F-u$ has only two components H_1,H_2 and they are complete. Thus, for $i\in\{1,2\}$ and $x,y\in V(H_i)$, there exists an H_i-hamiltonian path $xP_{H_i}y$. We have that $G-\{u,c_1,c_2\}$ contains three components H_1,H_2 and H_3 with $V(H_3)=\{c\}$. Let $h_i\in N_{H_i}(u)$. If $|V(H_i)|\ge 2$, then, as G is $2-$connected, there is $k_i\in N_{H_i}(u)-\{h_i\}$ adjacent to c_1 or c_2. But if $|V(H_i)|=1$, then let $k_i=h_i$. In both cases k_i is adjacent to c_1 or c_2. If $k_ic_1,k_{3-i}c_2\in E(G)$, then

$$C_2' = u,h_iP_{H_i}k_i,c_1,c,c_2,k_{3-i}P_{H_{3-i}}h_{3-i},u$$

is a hamiltonian cycle. If $c_jk_1,c_jk_2\in E(G)$ for some $j\in\{1,2\}$, we then consider $G+k_1k_2$. Since $k_1,k_2\in V(F)$, by Lemma 6.2.2, $|\{k_1,k_2\}\cap D^c_{k_1k_2}|=1$ and $|D^c_{k_1k_2}\cap\{c_j,c_{3-j}\}|=1$. Without loss of generality, let $k_1\in D^c_{k_1k_2}$. Lemma 2.1.15(3) implies that $c_{3-j}\in D^c_{k_1k_2}$. Therefore $c_{3-j}k_1\in E(G)$. Thus $c_jk_2,c_{3-j}k_1\in E(G)$ and the above argument again establishes a hamiltonian cycle. This completes the proof. □

We are now ready to prove Theorem 6.2.1.

Proof. In view of Lemmas 6.2.5 and 6.2.6, it remains to consider the case $\alpha_1\le 2$ and F is $2-$connected. Suppose there exists a cut set of size two of F, say $\{a,b\}$. Since $\alpha_1\le 2$, there are only two components H_1,H_2 of $F-\{a,b\}$. Moreover, H_i is complete for $i=1,2$ and there exists an H_i-hamiltonian path $xP_{H_i}y$ for any $x,y\in V(H_i)$. We have that $G-\{a,b,c_1,c_2\}$ contains only H_1,H_2 and H_3 as three components with $V(H_3)=\{c\}$. Since F is $2-$connected, there are vertices $a'\in N_{H_1}(a)$, $a''\in N_{H_2}(a)$. Moreover, if $|V(H_1)|>1$, then there exists $b'\in N_{H_1}(b)-\{a'\}$. Otherwise, if $|V(H_1)|=1$, then let $b'=a'$. Consider $G+a'a''$. By Lemma 6.2.2 we can let without loss of generality that $D^c_{a'a''}=\{a',c_1\}$. Lemma 2.1.15(3) yields that $a''c_1\notin E(G)$. We now distinguish two cases.

Case 1 : $|V(H_2)|=1$.

Therefore $ba''\in E(G)$ because F is $2-$connected. Consider $G+a''c$. Thus $|D^c_{a''c}|=2$ by Lemma 2.1.15(1) and $D^c_{a''c}\cap\{a'',c\}\ne\emptyset$ by Lemma 2.1.15(2). To dominate H_1, $D^c_{a''c}\ne\{a'',c\}$.

Suppose first that $c\in D^c_{a''c}$. By the connectedness of $(G+a''c)[D^c_{a''c}]$, either $D^c_{a''c}=\{c,c_1\}$ or $D^c_{a''c}=\{c,c_2\}$. Suppose that $D^c_{a''c}=\{c,c_1\}$. Thus $c_1\succ V(H_1)\cup\{a,b\}$. Let $J_1=\{c_2a'',c_2a,c_2b,c_2a'\}$. If $J_1\cap E(G)\ne\emptyset$, then

$C_3'=c_2,a'',a,a'P_{H_1}b',b,c_1,c,c_2$ or $C_4'=c_2,a,a'',b,b'P_{H_1}a',c_1,c,c_2$ or

$C'_5 = c_2, b, a'', a,\ a'P_{H_1}b', c_1, c, c_2$ or $C'_6 = c_2, a'P_{H_1}b', b, a'', a, c_1, c, c_2$

is a hamiltonian cycle. We now consider the case when $J_1 \cap E(G) = \emptyset$. Then there exists $c' \in N_{H_1}(c_2) - \{a'\}$ otherwise c_1 is a cut vertex of G contradicting G is $2-$connected. Hence

$C'_7 = c_2, c'P_{H_1}a', a, a'', b, c_1, c, c_2$

is a hamiltonian cycle. Thus, in either case, G is hamiltonian. By similar arguments, G has hamiltonian cycle when $D^c_{a''c} = \{c, c_2\}$.

We next suppose that $a'' \in D^c_{a''c}$. This implies, by Lemma 2.1.15(3), that $c_1, c_2 \notin D^c_{a''c} - \{a''\}$. By the connectedness of $(G + a''c)[D^c_{a''c}]$, either $D^c_{a''c} = \{a'', a\}$ or $D^c_{a''c} = \{a'', b\}$. Without loss of generality let $D^c_{a''c} = \{a'', a\}$. Since $a''c_1 \notin E(G)$, $ac_1 \in E(G)$. If $a''c_2 \in E(G)$, then

$C'_8 = c_2, a'', b, b'P_{H_1}a', a, c_1, c, c_2$

is a hamiltonian cycle. Suppose $a''c_2 \notin E(G)$. Thus $ac_2 \in E(G)$ and $a'' \in V_4$. Since $a'a'' \notin E(G)$, by Lemma 6.2.2, $a' \notin V_4$. Therefore a' is adjacent to c_1 or c_2. Hence

$C'_9 = c_1, a'P_{H_1}b', b, a'', a, c_2, c, c_1$ or $C'_{10} = c_2, a'P_{H_1}b', b, a'', a, c_1, c, c_2$

is a hamiltonian cycle. This completes Case 1.

Case 2 : $|V(H_2)| \geq 2$.

Since F is $2-$connected, we can choose $b'' \in N_{H_2}(b) - \{a''\}$. Recall that $D^c_{a'a''} = \{a', c_1\}$. Thus $a'c_1, b''c_1 \in E(G)$. Consider $G + a'b''$. Since $a', b'' \in V(F)$, by Lemmas 6.2.2 and 2.1.15(3), $c_2 \in D^c_{a'b''}$ and $|D^c_{a'b''} \cap \{a', b''\}| = 1$.

We first suppose that $D^c_{a'b''} = \{b'', c_2\}$. Clearly $c_2b'' \in E(G)$. Thus $c_2 \succ H_1 - a'$. If $c_2a \in E(G)$, then

$C'_{11} = c_2, a, a''P_{H_2}b'', b, b'P_{H_1}a', c_1, c, c_2$

is a hamiltonian cycle. Suppose that $c_2a \notin E(G)$. As $D^c_{a'b''} = \{b'', c_2\}$, we must have $b''a \in E(G)$. If $|V(H_1)| = 1$, then $V(H_1) = \{a'\}$ and $c_1 \succ H_1$. Moreover, $\{c_1, b'\} \succ_c G$ contradicting $\gamma_c(G) = 3$. Clearly $|V(H_1)| \geq 2$. As $c_2 \succ H_1 - a'$, there exists an $(H_1 - a')-$hamiltonian path $c_2P_{H_1-a'}b'$. Hence

$C'_{12} = c_2P_{H_1-a'}b', b, b''P_{H_2}a'', a, a', c_1, c, c_2$

is a hamiltonian cycle. Therefore if $D^c_{a'b''} = \{b'', c_2\}$, then G is hamiltonian.

We now consider the case when $D^c_{a'b''} = \{a', c_2\}$. Thus $c_2a' \in E(G)$ and $c_2 \succ H_2 - b''$. Clearly there exists an $(H_2 - b'')-$hamiltonian path $c_2P_{H_2-b''}a''$. Thus

$C'_{13} = c_2P_{H_2-b''}a'', a, a'P_{H_1}b', b, b'', c_1, c, c_2$

is a hamiltonian cycle. This completes Case 2. Therefore if F has the connectivity two, then G is hamiltonian.

We finally suppose that F is $3-$connected. Since $\alpha_1 = 2$, by Theorem 2.2.1, F is hamiltonian connected. Since G is $2-$connected, $V_1 \cup V_3$ and $V_2 \cup V_3$ are not empty. We then have an $F-$hamiltonian path aP_Fb with $a \in V_1 \cup V_3, b \in V_2 \cup V_3$. We form

$C'_{14} = c, c_1, aP_Fb, c_2, c$

as a hamiltonian cycle, completing the proof of Theorem 6.2.1. □

6.2.2 $\delta \geq 3$

In the following results we use the notation and terminology introduced in Section 6.1. More specifically, let G be a $2-$connected $3-\gamma_c-$edge critical graph with $\delta \geq 3$. Suppose that G is non-hamiltonian. Recall that C is a longest cycle of G and H is a component of $G-C$. Moreover, $X = N_C(H) = \{x_1, x_2, ..., x_d\}$, $X^+ = \{a_1, a_2, ..., a_d\}$ where $a_i = x_i^+, X^- = \{b_1, b_2, ..., b_d\}$ where $b_i = x_{i+1}^-$ and $C_i = V(a_i \overrightarrow{C} b_i)$ for $i = 1, ..., d$. All subscripts are all taken modulo d.

Lemma 6.2.7. Let G be a $3-\gamma_c-$edge critical non-hamiltonian graph. Let a, b and v be pairwise non-adjacent vertices such that $a, b \in X^+ \cup X^-$ and $v \in V(H)$. Then $|D^c_{ab} \cap \{a, b\}| = 1$ and $|D^c_{ab} \cap X| = 1$, moreover, if $v \in D^c_{av}$, then $|D^c_{av} \cap X| = 1$.

Proof. Consider $G + ab$. Lemma 2.1.15(2) implies that $\{a, b\} \cap D^c_{ab} \neq \emptyset$. Without loss of generality let $a \in D^c_{ab}$. To dominate H and by Lemma 6.1.1, $D^c_{ab} \neq \{a, b\}$. By the connectedness of $(G + ab)[D^c_{ab}]$, $X \cap (D^c_{ab} - \{a\}) \neq \emptyset$. Lemma 2.1.15(1) thus implies $|D^c_{ab} \cap \{a, b\}| = 1$ and $|D^c_{ab} \cap X| = 1$. We next consider $G + av$. Suppose that $v \in D^c_{av}$. Lemma 6.1.1 yields v is not adjacent to any vertex in $X^+ \cup X^-$. If $a \in D^c_{av}$, then by Lemma 2.1.15(1), $D^c_{av} = \{a, v\}$. Lemma 6.1.2 implies that D^c_{av} does not dominate $X^+ \cup X^-$, a contradiction. Thus $a \notin D^c_{av}$. In view of Lemma 6.1.2, to dominate X^+, we must have $D^c_{av} - \{v\} \subseteq V(C)$. By the connectedness of $(G + av)[D^c_{av}]$, $D^c_{av} - \{v\} \subseteq X$. Lemma 2.1.15(1) thus implies $|D^c_{av} \cap X| = 1$, as required. □

Lemma 6.2.8. Let G be a $3-\gamma_c-$citical non-hamiltonian graph. Then $|V(H)| = 1$.

Proof. Suppose that $|V(H)| > 1$. We distinguish two cases according to the cardinality of X.

Case 1 : $d = 2$.

We have that $a_1a_2 \notin E(G)$ by Lemma 6.1.2. Consider $G + a_1a_2$. Lemma 6.2.7 yields that either a_1 or a_2 is in $D^c_{a_1a_2}$. Without loss of generality, let $a_1 \in D^c_{a_1a_2}$. Lemma

6.2.7 gives also that x_1 or x_2 is in $D^c_{a_1a_2}$. Clearly $x_1 \in D^c_{a_1a_2}$ by Lemma 2.1.15(3). Thus $x_1 \succ H$ and there exists $x_1P_Hx_2$ such that $|V(x_1P_Hx_2)| > 3$. If $|C_1| = 1$, then

$$C' = x_1P_Hx_2\overrightarrow{C}x_1$$

is a cycle longer than C, a contradiction. Thus $|C_1| \geq 2$. Similarly, $|C_2| \geq 2$. Let $v \in N_H(x_2)$. Therefore $vx_1, vx_2 \in E(G)$. Consider $G + a_1v$. It follows from Lemma 2.1.15(1) that $|D^c_{a_1v}| = 2$. By Lemmas 2.1.15(2) and 6.2.7, $|\{a_1, v\} \cap D^c_{a_1v}| = 1$. Suppose that $a_1 \in D^c_{a_1v}$. Let $\{a\} = D^c_{a_1v} - \{a_1\}$. Lemma 6.1.2 implies that $a \succ X^+$ and $a \in V(C)$. Thus $a \in V(C) - X$ by Lemma 2.1.15(3). Clearly $D^c_{a_1v}$ dominates only one vertex v of H. Since $D^c_{a_1v} \succ_c G + a_1v$, $|V(H)| = |\{v\}| = 1$, a contradiction. Therefore $v \in D^c_{a_1v}$ and hence, by Lemma 6.2.7, x_1 or x_2 is in $D^c_{a_1v}$. Now, by Lemma 2.1.15(3), $x_1 \notin D^c_{a_1v}$. So $D^c_{a_1v} = \{x_2, v\}$. Lemma 6.1.1 yields that $vb_2 \notin E(G)$. Therefore $x_2b_2 \in E(G)$. Consider $G + vb_2$. By the same arguments, $v \in D^c_{vb_2}$ and $b_2 \notin D^c_{vb_2}$, moreover, x_1 or x_2 is in $D^c_{vb_2}$. Because $b_2x_1, b_2x_2 \in E(G)$, this contradicts Lemma 2.1.15(3). Thus $d \neq 2$.

Case 2 : $d \geq 3$.

Lemma 2.1.16 implies that there exists an ordering $w_1, w_2, ..., w_d$ of the vertices of X^+ and a path $z_1, z_2, ..., z_{d-1}$ such that $\{w_i, z_i\} \succ_c G - w_{i+1}$ for all $1 \leq i \leq d-1$. Thus w_i is not adjacent to any vertex in H by Lemma 6.1.1. To dominate H, $z_i \in X \cup V(H)$. By the connectedness of $G[\{w_i, z_i\}]$, $z_i \in X$ for all $1 \leq i \leq d-1$, moreover, $z_i \succ H$. Let $z_d \in X - \{z_1, ..., z_{d-1}\}$ and $v \in N_H(z_d)$. We have that $v \succ X$, implying there exists a path $x_iP_Hx_j$ of length at least three for $1 \leq i \neq j \leq d$. By the same argument as Case 1, $|C_i| \geq 2$ for $1 \leq i \leq d$. Therefore $a_i \neq b_i$ for $1 \leq i \leq d$.

Claim : $b_1 \succ X$.

For $i \in \{1, ..., d\}$, consider $G + w_iv$. By Lemmas 2.1.15(2) and 6.2.7, $|D^c_{w_iv} \cap \{w_i, v\}| = 1$. Lemma 6.1.2 yields that w_i is not adjacent to any vertex in X^+. If $w_i \in D^c_{w_iv}$, then $D^c_{w_iv} - \{w_i\} \subseteq V(C)$ to dominate X^+. Since $v \succ X$, it follows by Lemma 2.1.15(3) that $D^c_{w_iv} - \{w_i\} \subseteq V(C) - X$. Clearly $D^c_{w_iv}$ dominates only one vertex v of H. As $D^c_{w_iv} \succ_c G + w_iv$, we must have $V(H) = \{v\}$, a contradiction. So $v \in D^c_{w_iv}$. Lemma 6.2.7 yields that there exists $x_{i'} \in X \cap (D^c_{w_iv} - \{v\})$. We note by Lemma 6.1.1 that v is not adjacent to any vertex in $(X^+ \cup X^-) - \{w_i\}$. Hence $x_{i'} \succ (X^+ \cup X^-) - \{w_i\}$. Since $a_1 \neq b_1$, $x_{i'}b_1 \in E(G)$. Let $j \in \{1, ..., d\} - \{i\}$. Consider $G + w_jv$. By the same arguments, there exists $x_{j'}$ such that $x_{j'} \succ (X^+ \cup X^-) - \{w_j\}$. Clearly $x_{i'} \neq x_{j'}$ for $1 \leq i \neq j \leq d$. Since $a_1 \neq b_1$, $x_{j'}b_1 \in E(G)$ for $1 \leq j' \neq i' \leq d$. As i and j are arbitrary, we obtain $b_1 \succ X$. This settles the claim.

We consider $G + vb_1$. Similarly, $\{v\} = \{v, b_1\} \cap D^c_{vb_1}$. Lemma 6.2.7 yields that $D^c_{vb_1} - \{v\} \subseteq X$. By the claim, the vertex in $D^c_{vb_1} - \{v\}$ is adjacent to b_1 contradicting

Lemma 2.1.15(3). Hence $|V(H)| = 1$. $\square$

In view of Lemma 6.2.8, hereafter, we let $V(H) = \{v\}$. Since $N_C(H) = X$ and $|X| = d$, it follows that $N_C(v) = X$ and $deg_C(v) = deg_G(v) = d$. Recall that, for vertex subsets A and B of $V(G)$, we use $E(A,B)$ to denote the set of all edges having one end vertex in A and the other one in B.

Lemma 6.2.9. Let G be a $3-\gamma_c-$edge critical non-hamiltonian graph. Then $d+1 < \alpha$.

Proof. Suppose to the contrary that $d+1 \geq \alpha$. By Lemmas 6.1.1 and 6.1.2, $X^+ \cup \{v\}$ is independent. Therefore $\alpha \geq |X^+ \cup \{v\}| = d+1$. It follows that $\alpha = d+1$.

As a consequence of Theorem 6.2.1, $d \geq 3$. Lemma 2.1.16 implies that there exists an ordering $w_1,...,w_d$ of the vertices of X^+ and a path $P = z_1,...,z_{d-1}$ such that $\{w_i, z_i\} \succ_c G - w_{i+1}$ for all $1 \leq i \leq d-1$. Lemma 6.1.1 yields that w_i is not adjacent to v. To dominate v, $V(P) \subseteq X \cup \{v\}$. By the connectedness of $G[\{w_i, z_i\}]$, $V(P) \subseteq X$. Let $\{z_d\} = X - V(P)$. Similarly, there exists an ordering $h_1,...,h_d$ of the vertices of X^- and a path $P' = y_1,...,y_{d-1}$ of X such that $\{h_i, y_i\} \succ_c G - h_{i+1}$ for all $1 \leq i \leq d-1$. Let $\{y_d\} = X - V(P')$. Since $z_d \in X$, there is $p \in \{1,...,d\}$ such that $z_d = x_p$. Lemma 2.1.16 yields that, for all $i \in \{1,2,...,d\} - \{p\}$, there is $j \neq i$ such that $x_i a_j \in E(G)$. If $|C_j| = 1$ and $j = i-1$, then $a_{i-1} = b_{i-1}$ and $a_i b_{i-1} \notin E(G)$ by Lemma 6.1.2. Otherwise if $|C_j| > 1$ or $j \neq i-1$, then $a_i b_{i-1} \notin E(G)$ by Lemma 6.1.4. Thus $a_i b_{i-1} \notin E(G)$ for all $1 \leq i \neq p \leq d$.

Claim 1 : If $D^c_{a_i v} = \{a_i, y\}$ for some $y \in V(C)$, then $y \in C_{p-1}$.

Clearly, $y \succ X^+$. Suppose that $y \in C_r$ such that $r \neq p-1$. Since $ya_{r+1} \in E(G)$ and a_{r+1} is an X^+-vertex, it follows by Lemma 6.1.9 that $a_{r+1} b_r \in E(G)$, a contradiction, thus establishing Claim 1.

Claim 2 : $D^c_{a_{p-1} v} = \{v, x_j\}$ for some $j \in \{1,...,d\}$.

Lemma 2.1.15(1) yields that $|D^c_{a_{p-1} v}| = 2$. By Lemmas 2.1.15(2) and 6.2.7, $|D^c_{a_{p-1} v} \cap \{a_{p-1}, v\}| = 1$. Suppose $D^c_{a_{p-1} v} = \{a_{p-1}, y'\}$ for some $y' \in V(G) - \{v\}$. Lemma 6.1.2 yields that $y' \succ X^+$ and $y' \in V(C)$. This implies, by Claim 1, that $y' \in C_{p-1}$. Because $a_{p-1} b_{p-2} \notin E(G)$, $y' b_{p-2} \in E(G)$. As b_{p-2} is an X^--vertex, by Lemma 6.1.9, we obtain $a_{p-1} b_{p-2} \in E(G)$ which is a contradiction. Therefore $\{v\} = D^c_{a_{p-1} v} \cap \{a_{p-1}, v\}$. As a consequence of Lemma 6.2.7, $D^c_{a_{p-1} v} = \{v, x_j\}$ for some $j \in \{1,...,d\}$, thus establishing Claim 2.

Claim 3 : $D^c_{a_p v} = \{v, x_j\}$ for some $j \in \{1,...,d\}$.

Similarly, $|D^c_{a_p v} \cap \{a_p, v\}| = 1$. Suppose $D^c_{a_p v} = \{a_p, y''\}$ for some $y'' \in V(G) - \{v\}$. Lemma 6.1.2 yields that $y'' \succ X^+$ and $y'' \in V(C)$. By Claim 1, $y'' \in C_{p-1}$. Lemma 6.1.9

thus implies $a_pb_{p-1}\in E(G)$. It follows by Lemma 6.1.2 that $|C_p|,|C_{p-1}|>1$. Clearly $x_pb_i\notin E(G)$ for all $i\neq p-1$ by Lemma 6.1.4. Therefore $x_p\notin V(P')$ and $x_p=y_d$.

Consider $G+a_{p-1}v$. By Claim 2, let $x_r\in D^c_{a_{p-1}v}-\{v\}$. Lemma 6.1.1 implies that v is not adjacent to any vertex in X^-. Thus $x_r\succ X^-$ since $a_{p-1}\neq b_{p-1}$. As $x_pb_i\notin E(G)$ for $i\neq p-1$, we obtain $x_r\neq x_p$. Since $x_r\succ X^-$, $x_r\notin V(P')$. Clearly $V(P')\subseteq X-\{x_r,x_p\}$. Thus $|V(P')|\leq d-2$, a contradiction. Therefore $\{v\}=D^c_{a_pv}\cap\{a_p,v\}$. Lemma 6.2.7 thus implies $D^c_{a_pv}=\{v,x_j\}$ for some $j\in\{1,...,d\}$, thus establishing Claim 3.

Claim 4 : $|C_p|=1$ or $|C_{p-1}|=1$ and $a_ib_{i-1}\notin E(G)$ for all $1\leq i\leq d$.

Suppose that $|C_p|\geq 2$ and $|C_{p-1}|\geq 2$. By Claims 2 and 3, let $x_i\in D^c_{a_{p-1}v}-\{v\}$, $x_j\in D^c_{a_pv}-\{v\}$. By Lemma 6.1.1, v is not adjacent to any vertex in $X^+\cup X^-$. Lemma 2.1.15(3) then implies that $x_i\succ(X^+\cup X^-)-\{a_{p-1}\}$ and $x_j\succ(X^+\cup X^-)-\{a_p\}$. Therefore $x_i\neq x_j$. Because $a_p\neq b_p$ and $a_{p-1}\neq b_{p-1}$, $x_i\succ X^-$ and $x_j\succ X^-$. It follows that $V(P')\subseteq X-\{x_i,x_j\}$. So $|V(P')|\leq d-2$, a contradiction. Hence $|C_i|=1$ for some $i\in\{p,p-1\}$.

Since $a_ib_{i-1}\notin E(G)$ for all $1\leq i\neq p\leq d$, we need only show that $a_pb_{p-1}\notin E(G)$. If $|C_p|=1$, then $a_p=b_p$. Lemma 6.1.2 yields $a_pb_{p-1}\notin E(G)$. Moreover, if $|C_{p-1}|=1$, then $a_{p-1}=b_{p-1}$. As a consequence of Lemma 6.1.2, $a_pb_{p-1}\notin E(G)$, thus establishing Claim 4.

Claim 5 : For all $i\in\{1,...,d\}$, $D^c_{a_iv}=\{v,x_j\}$ for some $j\in\{1,...,d\}$

Suppose to the contrary that there exists $j\in\{1,...,d\}$ such that $D^c_{a_jv}=\{a_j,z\}$ for some $z\in V(G)$. Lemma 6.2.7 implies that $z\neq v$. To dominate X^+, $z\in V(C)$. Claim 1 yields that $z\in C_{p-1}$. Because $\{a_j,z\}\succ_c G-v$, by Lemma 6.1.2, $za_p\in E(G)$. By Lemma 6.1.9, $a_pb_{p-1}\in E(G)$, contradicting Claim 4. As a consequence of Lemma 6.2.7, $D^c_{a_iv}=\{v,x_j\}$ for some $j\in\{1,...,d\}$, thus establishing Claim 5.

Claim 6 : $|C_r|\geq 2$ and $|C_s|\geq 2$ for some $1\leq r\neq s\leq d$.

Suppose first that $|C_i|=1$ for all $1\leq i\leq d$. So $a_i=b_i$. Thus, by Lemma 6.1.2, there is no $a_iP_{G-C}a_j$ path for $1\leq i\neq j\leq d$, in particular, $E(C_i,C_j)=\emptyset$. Hence X is a cut set with $|X|=d$ but $G-X$ contains at least $d+1$ components, contradicting Theorem 2.1.18. We then suppose that C_r is only one component such that $|C_r|\geq 2$. Since $a_i=b_i$ for all $i\neq r$, by Lemma 6.1.2, there is no $a_iP_{G-C}b_r$, in particular, $a_ib_r\notin E(G)$. Lemma 6.1.10 yields that $b_ra_r\in E(G)$. As a consequence of Lemma 6.1.8, all vertices of $C_r-\{b_r\}$ are X^+-vertices. This implies, by Lemma 6.1.2, that there is no $u_iP_{G-C}u_j$ for $u_i\in C_i,u_j\in C_j$ and $1\leq i\neq j\leq d$. Therefore $|X|<\omega(G-X)$ contradicting Theorem 2.1.18, thus establishing Claim 6.

In view of Claim 6, we let C_r and C_s be such that $|C_r|\geq 2$ and $|C_s|\geq 2$. Moreover, by Claim 5, we let $x_{r'}\in D^c_{a_rv}-\{v\}, x_{s'}\in D^c_{a_sv}-\{v\}$. Lemma 6.1.1 implies that v is not adjacent to any vertex in $X^+\cup X^-$. Thus $x_{r'}\succ(X^+\cup X^-)-\{a_r\}$ and $x_{s'}\succ(X^+\cup X^-)-\{a_s\}$ by Lemma 2.1.15(3). So $x_{r'}\neq x_{s'}$, moreover, $x_{r'}\succ X^-$ and $x_{s'}\succ X^-$ because $a_r\neq b_r$ and $a_s\neq b_s$. So $V(P')\subseteq X-\{x_{r'},x_{s'}\}$. Thus $|V(P')|\leq d-2$, a contradiction.

Hence $d+1<\alpha$. This completes the proof of Lemma 6.2.9. □

Lemma 6.2.10. Let G be a $3-\gamma_c-$edge critical non-hamiltonian graph. Then $deg_G(v)=\delta=\alpha-2$.

Proof. Lemma 6.2.9 yields that $\alpha>d+1$. Suppose that $\alpha\neq\delta+2$. As a consequence of Theorem 2.1.19, $\alpha\leq\delta+1$. Since $d\geq\delta$, $d+1\geq\delta+1\geq\alpha>d+1$, a contradiction. Hence $\alpha=\delta+2$.

Finally, suppose that $d\neq\delta$. Therefore $d\geq\delta+1$ and it follows that $d+1\geq(\delta+1)+1=\delta+2=\alpha>d+1$, a contradiction. So $deg_G(v)=d=\delta=\alpha-2$, as required. □

Lemmas 2.1.20 and 6.2.10 give the following corollary.

Corollary 6.2.11. Let G be a $3-\gamma_c-$edge critical graph and C be a longest cycle of G. Then $|V(G)-V(C)|\leq 1$, moreover, if $\{v\}=V(G)-V(C)$, then $|C_i|\geq 2$ for $1\leq i\leq\delta$ where C_i is defined as in Section 6.1.

Proof. Suppose that there exist $v,w\in V(G)-V(C)$. Thus $deg_G(v)=deg_G(w)=\delta=\alpha-2$ by Lemma 6.2.10. Since $\alpha=\delta+2$, by Lemma 2.1.20, $v=w$. Therefore $|V(G)-V(C)|=1$. We next suppose that $\{v\}=V(G)-V(C)$ and $|C_i|=1$. Let $\{a_i\}=C_i$. We then have

$$C'=x_i,v,x_{i+1}\overrightarrow{C}x_i$$

is a longest cycle that does not contain a_i. Lemma 6.2.10 yields that $deg_G(a_i)=deg_G(v)=\delta$, contradicting Lemma 2.1.20. This completes the proof. □

It follows from Lemma 6.2.10 and Corollary 6.2.11 that $\{v\}=V(G)-V(C)$ and $deg_G(v)=d=\delta=\alpha-2$. Further, by Lemma 2.1.20, $G[X]=G[N(v)]$ is a clique.

Lemma 6.2.12. Let G be a $3-\gamma_c-$edge critical non-hamiltonian graph. For any $a\in X^+\cup X^-$, there exists a vertex $y\in V(C)-X$ such that $D^c_{av}=\{a,y\}$.

Proof. Let $a\in X^+\cup X^-$. Consider $G+va$. Lemma 6.2.7 yields that $|D^c_{va}\cap\{v,a\}|=1$. If $v\in D^c_{va}$, then, by Lemma 6.2.7, $D^c_{va}=\{x,v\}$ for some $x\in X$. Lemma 6.2.10 thus implies $deg_G(v)=\delta$. If follows from Lemma 2.1.20 that $G[N[v]]$ is a clique. Because, $x\in N[v]$, $N[v]\subseteq N[x]$. Hence $x\succ G-a$. But $a=x_i^+$ for some $x_i\in X$ when $a\in X^+$

and $a = x_j^-$ for some $x_j \in X$ when $a \in X^-$. Therefore $\{x, x_i\} \succ_c G$ or $\{x, x_j\} \succ_c G$ contradicting $\gamma_c(G) = 3$. So $a \in D^c_{av}$. Let $y \in D^c_{av} - \{a\}$. By Lemma 2.1.15(3), $y \in V(C) - X$, as required. □

The next two lemmas show that at least one of the sets $X^+ \cup \{b\}$ or $X^- \cup \{a\}$ for some $b \in X^-$ and $a \in X^+$ is independent.

Lemma 6.2.13. Let G be a $3-\gamma_c-$edge critical non-hamiltonian graph. If $N(b_i) \cap X^+ \neq \emptyset$ for all $b_i \in X^-$ and $N(a_i) \cap X^- \neq \emptyset$ for all $a_i \in X^+$, then $X^+ \nsubseteq I$ and $X^- \nsubseteq I$ for any maximum independent set I of G.

Proof. By Corollary 6.2.11, $|C_i| \geq 2$ for all $1 \leq i \leq \delta$. We suppose to the contrary that there exists a maximum independent set I of G which $X^+ \subseteq I$ or $X^- \subseteq I$.

Without loss of generality let $X^+ \subseteq I$. Theorem 2.1.19 thus implies $v \in I$ and $|I| = \delta + 2$. Let $J = I - \{v\}$. Clearly $X^+ \subseteq J$ and $|J| = \delta + 1$. Since $\delta \geq 3$, $|J| \geq 4$. Lemma 2.1.16 yields that there exists an ordering $w_1, ..., w_{\delta+1}$ of the vertices of J and a path $P = y_1, ..., y_\delta$ such that $\{w_i, y_i\} \succ_c G - w_{i+1}$ for $1 \leq i \leq \delta$. Since $\{v\} \cup J$ is independent, to dominate v, $y_i \in X \cup \{v\}$ for $1 \leq i \leq \delta$. By the connectedness of $G[\{w_i, y_i\}]$, $y_i \neq v$. Therefore $y_i \in X$. Because $|X| = \delta$, $X = V(P)$. We now establish the following claims.

Claim 1 : $a_i b_{i-1} \notin E(G)$ for all $1 \leq i \leq \delta$.

By the ordering, each of y_i is adjacent to w_j for $i \in \{1, ..., \delta\}, j \in \{1, ..., \delta+1\} - \{i+1\}$. Because $\delta + 1 \geq 4$ and $V(P) = X$, it follows that x_i is adjacent to at least one of a_j where $j \neq i$. Lemma 6.1.4 yields that $a_i b_{i-1} \notin E(G)$ for all $1 \leq i \leq \delta$, thus establishing Claim 1.

Claim 2 : $|N(a) \cap X| \geq \delta - 1$ for all $a \in J$.

We note that by the ordering, there is $j \in \{1, ..., \delta+1\}$ such that $a = w_j$. Therefore $N(a) \cap X = \{y_1, ..., y_\delta\} - \{y_{j-1}\}$, if $j \geq 2$ and otherwise $N(a) = X$. Thus establishing Claim 2.

Claim 3 : $N(a_i) \cap X^- = \{b_i\}$ for all $i = 1, ..., \delta$.

Without loss of generality, we consider a_1. By our assumption, $a_1 b_j \in E(G)$ for some $j \in \{1, ..., \delta\}$. Suppose that $j > 1$. By Claim 1, $x_{j+1} \neq x_1$. Lemma 6.1.3(1) and (2) thus imply $a_j x_{j+1}, a_j x_1 \notin E(G)$, contradicting Claim 2. Hence $j = 1$, thus establishing Claim 3.

Claim 4 : $a_i \succ C_i$ and $b_i \succ C_i$ for $1 \leq i \leq \delta$.

Suppose without loss of generality that a_1 does not dominate C_1. Let $a \in C_1$ such that $a_1 a \notin E(G)$ and $a_1 \succ V(a^+ \overrightarrow{C_1} b_1)$. This choice is possible because $a_1 \succ \{b_1\}$ by Claim 3. Therefore a is an X^+-vertex. Lemma 6.1.2 yields that $X^+ \cup \{a\}$ is independent. Moreover, if there exists $j \in \{2, ..., \delta\}$ such that a_j does not dominate C_j, we then choose $z \in C_j$ such that $a_j z \notin E(G)$ and $a_j \succ V(z^+ \overrightarrow{C}_j b_j)$. Thus z is an X^+-vertex. Lemmas 6.1.1 and 6.1.2 then imply that $X^+ \cup \{z, a, v\}$ is an independent set of size $\delta + 3$, a contradiction. So $a_i \succ C_i$ for all $2 \leq i \leq \delta$. It follows that every vertex in $C_i - \{b_i\}$ is an X^+-vertex.

Consider $G + a_1 v$. Lemma 6.2.12 yields that $D^c_{a_1 v} = \{a_1, y\}$ for some $y \in V(C) - X$. Therefore $ya_1 \in E(G)$. Claim 3 together with Lemma 6.1.2 imply that $y \notin C_i$ for $2 \leq i \leq \delta$. Moreover, $y \neq b_1$ to dominate b_2. Thus $y \in C_1 - \{a_1, b_1\}$.

As $a_1 b_\delta, a_1 a_2 \notin E(G)$, we obtain $yb_\delta, ya_2 \in E(G)$. Since $yb_\delta, a_1 b_1 \in E(G)$, it follows by Lemmas 6.1.2 and 6.1.3(1) that $y^- \neq a_1$ and $y^- b_1 \notin E(G)$. Lemma 6.1.5 and $ya_2 \in E(G)$ then imply that $y^- b_i \notin E(G)$ for $2 \leq i \leq \delta$. Therefore $X^- \cup \{y^-\}$ is independent.

Lemma 2.1.16 implies that there exists an ordering $q_1, ..., q_{\delta+1}$ of the vertices of $X^- \cup \{y^-\}$ and path $P' = r_1, r_2, ..., r_\delta$ such that $\{q_i, r_i\} \succ_c G - q_{i+1}$ for $1 \leq i \leq \delta$. By similar arguments as Claim 2, $V(P') = X$ and $|N(a) \cap X| \geq \delta - 1$ for $a \in X^- \cup \{y^-\}$. This implies that y^- is adjacent to x_1 or x_2. Since $yb_\delta, ya_2, a_1 b_1 \in E(G)$, this contradicts Lemma 6.1.6. So $a_1 \succ C_1$. Therefore $a_i \succ C_i$ for all $1 \leq i \leq \delta$. By similar arguments $b_i \succ C_i$, thus establishing Claim 4.

Claims 3 and 4 together with Lemma 6.1.2 imply that $E(C_i, C_j) = \emptyset$ for $1 \leq i \neq j \leq \delta$. We now have X as a cut set with $|X| = \delta$ but $G - X$ contains $\delta + 1$ components. This contradicts Theorem 2.1.18. Thus $X^+ \nsubseteq I$ and $X^- \nsubseteq I$. This completes the proof of Lemma 6.2.13. □

Lemma 6.2.14. Let G be a $3 - \gamma_c-$edge critical non-hamiltonian graph. Then at least one of $X^+ \cup \{b\}$ or $X^- \cup \{a\}$ for some $b \in X^-$ and for some $a \in X^+$ is independent.

Proof. Suppose to the contrary that $N(b_i) \cap X^+ \neq \emptyset$ for all $b_i \in X^-$ and $N(a_i) \cap X^- \neq \emptyset$ for all $a_i \in X^+$. Therefore, we are in the situation of Lemma 6.2.13. We first establish the following claims.

Claim 1 : For $1 \leq i \leq \delta$, if an X^+-vertex u belongs to C_i, then every vertex in $V(a_i \overrightarrow{C} u)$ is an X^+-vertex.

Suppose to the contrary that there exists a vertex in $V(a_i^+ \overrightarrow{C} u)$ which is not an X^+-vertex. Let y be the last vertex in $V(a_i^+ \overrightarrow{C} u)$ which is not adjacent to a_i. Thus $y^+ a_i \in E(G)$. Therefore y is an X^+-vertex. By Lemmas 6.1.1 and 6.1.2, $X^+ \cup \{y, v\}$ is an independent set of size $\delta + 2$. This contradicts Lemma 6.2.13. Clearly every vertex in $V(a_i \overrightarrow{C} u)$ is an X^+-vertex. Thus establishing Claim 1.

Claim 2 : For $1 \leq i \leq \delta$, if $N(a_i) \cap C_{i-1} \neq \emptyset$, then $a_i b_{i-1} \in E(G)$. Similarly, if

$N(b_i) \cap C_{i+1} \neq \emptyset$, then $a_{i+1}b_i \in E(G)$.

Let $w \in N(a_i) \cap C_{i-1}$. Suppose $a_ib_{i-1} \notin E(G)$ and let y be the first vertex in $V(w^+\overrightarrow{C_{i-1}}b_{i-1})$ which is not adjacent to a_i. Clearly, $y \neq a^+_{i-1}$ and $y^-a_i \in E(G)$. Lemma 6.1.3(1) yields that $X^+ \cup \{y\}$ is independent. Similarly, $X^+ \cup \{y, v\}$ is an independent set of size $\delta + 2$, a contradiction. Hence $a_ib_{i-1} \in E(G)$. We can prove that if $N(b_i) \cap C_{i+1} \neq \emptyset$, then $a_{i+1}b_i \in E(G)$ by similar arguments. Thus establishing Claim 2

Since $\delta \geq 3$ and X^+ is independent, by Lemma 2.1.16, there exists an ordering $w_1, ..., w_\delta$ of the vertices of X^+ and a path $P'' = y_1, ..., y_{\delta-1}$ such that $\{w_i, y_i\} \succ_c G - w_{i+1}$ for $1 \leq i \leq \delta - 1$. To dominate v and by the connectedness of $G[\{w_i, y_i\}]$, $V(P'') \subseteq X$. Without loss of generality let $\{x_1\} = X - V(P'')$.

Claim 3 : $a_{i+1}b_i \notin E(G)$ for $1 \leq i \leq \delta - 1$.

Since each of y_i is adjacent to w_j for $i \in \{1, ..., \delta - 1\}, j \in \{1, ..., \delta\} - \{i + 1\}$ and $\{y_1, ..., y_{\delta-1}\} = \{x_2, .., x_\delta\}$, it follows by Lemma 6.1.4 that $a_{i+1}b_i \notin E(G)$ for $1 \leq i \leq \delta - 1$, thus establishing Claim 3.

Consider $G + va_1$. Lemma 6.2.12 implies that there exists $y \in V(C) - X$ such that $D^c_{va_1} = \{a_1, y\}$. Suppose that $y \notin C_\delta$. Hence $y \in C_j$ for some $j < \delta$. So $a_{j+1} \neq a_1$. Since $a_1a_{j+1} \notin E(G)$, $ya_{j+1} \in E(G)$. By Claim 2, $a_{j+1}b_j \in E(G)$ but this contradicts Claim 3. So $y \in C_\delta$.

We next suppose that $a_1b_{\delta-1} \in E(G)$. Lemma 6.1.7 implies that $a_\delta b_i \notin E(G)$ for $1 \leq i \leq \delta - 2$. Moreover, Claim 3 gives $a_\delta b_{\delta-1} \notin E(G)$. Because $N(a_\delta) \cap X^- \neq \emptyset$, $a_\delta b_\delta \in E(G)$. Claim 1 yields that every vertex in $C_\delta - \{b_\delta\}$ is an X^+-vertex. Since $ya_1 \in E(G)$, it follows by Lemma 6.1.2 that $y = b_\delta$. Because $ya_{\delta-1} \in E(G)$, $b_\delta a_{\delta-1} \in E(G)$. So

$$C' = a_1, b_{\delta-1}\overleftarrow{C}a_{\delta-1}, b_\delta\overleftarrow{C}x_\delta, v, x_1, x_{\delta-1}\overleftarrow{C}a_1$$

is a hamiltonian cycle, a contradiction. So $a_1b_{\delta-1} \notin E(G)$. Since $\{y, a_1\} \succ_c G - v$, $yb_{\delta-1} \in E(G)$. Because $y \in C_\delta$, it follows by Claim 2 that $b_{\delta-1}a_\delta \in E(G)$. This contradicts Claim 3. Therefore at least one of $X^+ \cup \{b\}$ or $X^- \cup \{a\}$ for some $b \in X^-$ and for some $a \in X^+$ is independent. This completes the proof of Lemma 6.2.14. □

We now prove our main result.

Theorem 6.2.15. Every $2-$connected $3-\gamma_c-$edge critical graph is hamiltonian.

Proof. Suppose G is non-hamiltonian. Let C be a longest cycle of G and $v \in V(G) - V(C)$. Lemma 6.2.10 and Corollary 6.2.11 then imply that $V(G) - V(C) = \{v\}$ and $deg_G(v) = \delta = \alpha - 2$. By Lemma 6.2.14, without loss of generality, let $X^+ \cup \{b_1, v\}$ be a maximum independent set. Theorem 6.2.1 yields that $|X^+ \cup \{b_1, v\}| \geq 5$.

Consider $G+vb_1$. Lemma 6.2.12 implies that there is $y\in V(C)-X$ such that $D^c_{vb_1}=\{y,b_1\}$. So $y\succ X^+\cup X^-$. We have by Lemma 6.1.2 that $y\notin X^+\cup X^-$. Therefore $y\in C_p-\{a_p,b_p\}$ for some $p\in\{1,...,\delta\}$.

Suppose $y=a_p^+$ or $y=b_p^-$. Hence

$C_1'=x_p,v,x_{p+1}\overleftarrow{C}y,a_{p+1}\overrightarrow{C}x_p$ or $C_2'=x_p,v,x_{p+1}\overrightarrow{C}b_{p-1},y\overleftarrow{C}x_p$

are longest cycles that do not contain a_p or b_p, respectively. Lemma 6.2.10 yields that $deg_G(a_p)=\delta=deg_G(b_p)$, contradicting Lemma 2.1.20. Hence $y\neq a_p^+$ and $y\neq b_p^-$.

Because $y\succ X^+\cup X^-$, it follows by Lemmas 6.1.3(1) and 6.1.5 that $X^+\cup\{y^+\}$ and $X^-\cup\{y^-\}$ are independent sets of size $\delta+1\geq 4$. This implies, by Lemma 2.1.16, that there is an ordering $u_1,...,u_{\delta+1}$ of the vertices of $X^+\cup\{y^+\}$ and a path $P=z_1,...,z_\delta$ such that $\{u_i,z_i\}\succ_c G-u_{i+1}$ for $1\leq i\leq\delta$. Since $y\notin\{a_p,b_p\}$, by Lemma 6.1.1, v is not adjacent to y^+ and y^-. Moreover, $\{u_1,...,u_{\delta+1},v\}$ is an independent set. To dominate v, $z_i\in X\cup\{v\}$. By the connectedness of $G[\{u_i,z_i\}]$, $z_i\in X$. Thus $V(P)=X$. Similarly, there is an ordering $q_1,...,q_{\delta+1}$ of the vertices of $X^-\cup\{y^-\}$ and a path $P'=r_1,...,r_\delta$ such that $\{q_i,r_i\}\succ_c G-q_{i+1}$ for $1\leq i\leq\delta$ and $V(P')=X$.

Claim : For $x\in X$, if $x\succ X^+$, then $xy^+\notin E(G)$. Similarly, if $x\succ X^-$, then $xy^-\notin E(G)$.

By the ordering, there is z_j such that $z_j=x$. Since $\{z_j,u_j\}\succ_c G-u_{j+1}$, it follows that $z_j\succ(X^+\cup\{y^+\})-\{u_{j+1}\}$. As $x\succ X^+$, we must have $u_{j+1}=y^+$ and $xy^+\notin E(G)$. By similar arguments, if $x\succ X^-$, then $xy^-\notin E(G)$. This settles our claim.

We finally suppose that $y^+y^-\in E(G)$. Thus

$C_3'=y,b_p\overleftarrow{C}y^+,y^-\overleftarrow{C}x_p,v,x_{p+1}\overrightarrow{C}b_{p-1},y$

is a hamiltonian cycle. So $y^+y^-\notin E(G)$. Consider $G+y^+y^-$. By Lemma 2.1.15(2), $D^c_{y^+y^-}\cap\{y^+,y^-\}\neq\emptyset$. But $D^c_{y^+y^-}\neq\{y^+,y^-\}$ to dominate v. Lemma 2.1.15(1) yields that $D^c_{y^+y^-}=\{y^+,x\}$ or $D^c_{y^+y^-}=\{y^-,x\}$ for some $x\in X\cup\{v\}$. If $D^c_{y^+y^-}=\{y^+,x\}$, then $x\succ X^+$ because $X^+\cup\{y^+\}$ is independent. As a consequence of Lemma 6.1.1, $x\neq v$. Therefore $x\in X$. By the claim, $xy^+\notin E(G)$, contradicting $(G+y^+y^-)[D^c_{y^+y^-}]$ is connected. We can show that $D^c_{y^+y^-}=\{y^-,x\}$ can not occur by the same arguments. Hence, G is hamiltonian as required. □

Since every $3-\gamma_c-$edge critical graph is $3-\gamma_t-$edge critical, this together with Theorem 6.2.15, we have the following result on $3-\gamma_t-$edge critical graphs.

Corollary 6.2.16. Every $2-$connected $3-\gamma_t-$edge critical graph is hamiltonian.

6.3 $k-\mathscr{D}-$Edge Critical Non-Hamiltonian Graphs where $\mathscr{D} \in \{\gamma_c, \gamma_t, \gamma, i\}$

In this section, we are interested in the existence of $k-\mathscr{D}-$edge critical non-hamiltonian graphs where $\mathscr{D} \in \{\gamma_c, \gamma_t, \gamma, i\}$.

For $k \geq 1$ and $\mathscr{D} \in \{\gamma_c, \gamma_t, \gamma, i\}$, let

$\mathscr{G}_k^{\mathscr{D}}$: class of 2−connected $k-\mathscr{D}-$edge critical non-hamiltonian graphs.

Sumner and Blitch [138] proved that a $1-\gamma-$edge critical graph is a K_n and a graph G is $2-\gamma-$edge critical graph if and only if $\overline{G} = \cup_{i=1}^{n} K_{1,n_i}$ where $n, n_i \geq 1$ (see Theorem 2.1.2). Clearly, every 2−connected $k-\gamma-$edge critical graph is hamiltonian when $k = 1$ or 2. Note that the class of 2−connected graphs is a subclass of graphs with $\delta \geq 2$. As Favaron et al. [65] and Tian et al. [139] proved that every $3-\gamma-$edge critical graph with $\delta \geq 2$ is hamiltonian, these imply that

$\mathscr{G}_k^{\gamma} = \emptyset$ for $k = 1, 2$ or 3, moreover, $\mathscr{G}_4^{\gamma} \neq \emptyset$

because Conjecture B3 was disproved for $k = 4$.

6.3.1 $k-\gamma_c-$Edge Critical Graphs

For integers $k, l \geq 2$, let

$\mathscr{C}_k^l$: class of $k-\gamma_c-$edge critical non-hamiltonian graphs of connectivity l.

Obviously, for given integers $k, l \geq 2$, $\mathscr{C}_k^l \subseteq \mathscr{G}_k^{\gamma_c}$, in fact, $\cup_{l\geq 2}\mathscr{C}_k^l = \mathscr{G}_k^{\gamma_c}$. Moreover, every $1-\gamma_c-$edge critical graph is K_n. Chen et al. [55] showed that a graph G is $2-\gamma_c-$edge critical if and only if $\overline{G} = \cup_{i=1}^{n} K_{1,n_i}$ for $n_i \geq 1$ and $n \geq 2$. It is not difficult to see that 2−connected $2-\gamma_c-$edge critical graphs are hamiltonian. Further, by Theorem 6.2.15, $\mathscr{G}_k^{\gamma_c} = \mathscr{C}_k^l = \emptyset$ for all $k = 1, 2$ or 3. We now consider the case when $k \geq 4$.

Define the graph $B_k^l(n_1, n_2, ..., n_{k-1})$ as

$$B_k^l(n_1, n_2, ..., n_{k-1}) = x \vee H_1 \vee H_2 \vee ... \vee H_{k-2} \vee H_{k-1} \vee y$$

where $H_i \cong K_{n_i}, n_1 = l, n_i \geq l$ for all $2 \leq i \leq k-2$ and $H_{k-1} \cong \overline{K}_{n_{k-2}+1}$. Let $M = \{e_1, e_2, ..., e_{n_{k-2}}\}$ be a matching in $B_k^l(n_1, n_2, ..., n_{k-1})$ of size n_{k-2} between the sets $V(H_{k-2})$ and $V(H_{k-1})$. Define the graph $H_k^l(n_1, n_2, ..., n_{k-1})$ as

$$H_k^l(n_1, n_2, ..., n_{k-1}) = B_k^l(n_1, n_2, ..., n_{k-1}) - e_1 - e_2 - e_3 ... - e_{n_{k-2}}.$$

Figure 6.3 illustrates our construction. Note that the *dash lines* represent the edges of M. We remark also that the vertices of H_{k-1} may be ordered as $w_1, ..., w_p$ in such a way that the vertices $z_1, ..., z_{p-1}$ of H_{k-2} satisfy $z_i \succ H_{k-1} - w_{i+1}$ for $1 \le i \le p-1$ and $p = n_{k-2} + 1$.

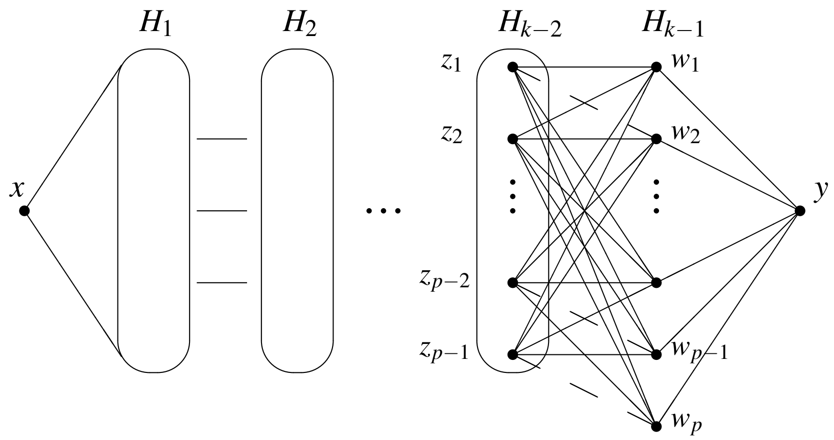

Figure 6.3 : A graph $H_k^l(n_1, n_2, ..., n_{k-1})$

We define the class $\mathscr{H}_k^l$ as

$$\mathscr{H}_k^l = \cup\{H_k^l(n_1, n_2, ..., n_{k-1}) : l \ge 2, k \ge 4 \text{ and } n_i \ge l \text{ for } 1 \le i \le k-1\}.$$

Theorem 6.3.1. $\mathscr{C}_k^l \neq \emptyset$ for $k \ge 4$ and $2 \le l \le \frac{n-3}{k-1}$, in particular, $\mathscr{G}_k^{\gamma_c} \neq \emptyset$.

Proof. As $\mathscr{C}_k^l \subseteq \mathscr{G}_k^{\gamma_c}$, it suffices to show that $\mathscr{H}_k^l \subseteq \mathscr{C}_k^l$. Let $G = H_k^l(n_1, n_2, ..., n_{k-1}) \in \mathscr{H}_k^l$ for $k \ge 4$ and $n_i \ge l \ge 2$. By the construction, G is an $l-$connected graph on n vertices with $n = 2 + \Sigma_{i=1}^{k-1} n_i$. Since $n_{k-1} = n_{k-2} + 1$, it follows that

$$n = 2 + \Sigma_{i=1}^{k-1} n_i \ge 2 + (k-1)l + 1 = 3 + (k-1)l.$$

So $l \le \frac{n-3}{k-1}$. Let $h_i \in V(H_i)$ for $1 \le i \le k-3$. We first show that $\gamma_c(G) = k$. We see that $\{h_1, h_2, ..., h_{k-3}, z_1, w_1, y\} \succ_c G$. Thus $\gamma_c(G) \le k$. Since $G[D^c]$ is connected, to dominate x and y, $|D^c \cap V(H_i)| \ge 1$ for all $1 \le i \le k-1$. But there is no subset of two vertices $\{a, b\}$ of $V(H_{k-1}) \cup V(H_{k-2})$ such that $\{a, b\} \succ_c V(H_{k-1}) \cup V(H_{k-2}) \cup \{y\}$. Thus $|D^c \cap (V(H_{k-1}) \cup V(H_{k-2}) \cup \{y\})| \ge 3$, and so $k \le \gamma_c(G)$. Clearly $\gamma_c(G) = k$.

We now consider the connected domination number of $G + uv$ where $uv \in E(\overline{G})$. We first suppose that $x \in \{u, v\}$, without loss of generality let $x = u$. If $v \in \cup_{i=2}^{k-3} V(H_i)$, then there exists $j \in \{2,...,k-3\}$ such that $v \in V(H_j)$. Without loss of generality let $v = h_j$. Thus $\{h_2,...,h_{j-1},v,h_{j+1},...,h_{k-3},z_1,w_1,y\} \succ_c G$. If $v \in V(H_{k-2})$, then $\{h_2,...,h_{k-3},v,w_1,y\} \succ_c G$. If $v \in V(H_{k-1})$, then $\{h_2,...,h_{k-3},z_1,v,y\} \succ_c G$ or $\{h_2,..., h_{k-3},z_2,v,y\} \succ_c G$. If $v = y$, then $\{x,h_1,...,h_{k-3},y\} \succ_c G$. Therefore $\gamma_c(G+uv) < k$ when $x \in \{u,v\}$.

We next consider when $\{u,v\} \subseteq \cup_{i=1}^{k-3} V(H_i)$. By the construction, $k \geq 6$. Thus $|\{u,v\} \cap V(H_j)| = 1$ and $|\{u,v\} \cap V(H_p)| = 1$ where $1 \leq j \leq p-2 \leq k-5$. Without loss of generality let $u = h_j$ and $v = h_p$. If $j+2 < p$, then $\{h_1,...,h_{j-1},u,h_{j+1},..., h_{p-3},v,h_{p+1},...,h_{k-3},z_1,w_1,y\} \succ_c G$. If $j+2 = p$, then $\{h_1,...,h_{j-1},u,v,h_{p+1},..., h_{k-3},z_1,w_1,y\} \succ_c G$. Clearly $\gamma_c(G+uv) < k$ if $\{u,v\} \subseteq \cup_{i=1}^{k-3} V(H_i)$.

Suppose that $|\{u,v\} \cap \cup_{i=1}^{k-3} V(H_i)| = 1$, without loss of generality let $u = h_j$ for some $j \in \{1,...,k-3\}$. Suppose that $v \in V(H_{k-2})$. Therefore $1 \leq j \leq k-4$ and we have $\{h_1,...,h_{j-1},u,h_{j+1},...,h_{k-4},v,w_1,y\} \succ_c G$. If $v \in V(H_{k-1})$, then $\{h_1,...,h_{j-1},u,..., h_{k-3},v,y\} \succ_c G$. If $v = y$, then $\{h_1,...,h_{j-1},u,...,h_{k-3},y\} \succ_c G$. Hence $\gamma_c(G+uv) < k$ when $|\{u,v\} \cap \cup_{i=1}^{k-3} V(H_i)| = 1$.

We finally suppose that $\{u,v\} \subseteq (\cup_{i=k-2}^{k-1} V(H_i)) \cup \{y\}$. Thus $|\{u,v\} \cap V(H_{k-2})| \leq 1$ because H_{k-2} is complete. Consider when $|\{u,v\} \cap V(H_{k-2})| = 1$, without loss of generality let $u \in V(H_{k-2})$. If $v \in V(H_{k-1})$, then there exists $q \in \{1,...,p\}$ such that $u \in z_q$ and $v = w_{q+1}$. Since $z_q \succ H_{k-1} - w_{q+1}$ and $w_q y \in E(G)$, it follows that $\{h_1,h_2,...,h_{k-3},z_q,w_q\} \succ_c G$. If $v = y$, then $\{h_1,h_2,...,h_{k-3},u,v\} \succ_c G$. We then consider when $|\{u,v\} \cap V(H_{k-2})| = 0$. Thus $\{u,v\} \subseteq V(H_{k-1})$ and there exist $q,r \in \{1,...,p\}$ such that $u = w_q$ and $v = w_r$, without loss of generality let $1 \leq q < r$. Since $z_{r-1} \succ H_{k-1} - w_r$, $z_{r-1}u \in E(G)$. Thus $\{h_1,...,h_{k-3},z_{r-1},u\} \succ_c G$. Therefore if $\{u,v\} \subseteq (\cup_{i=k-2}^{k-1} V(H_i)) \cup \{y\}$, then $\gamma_c(G+uv) < k$. These imply that G is a $k-\gamma_c-$edge critical graph.

Let $S = V(H_{k-2}) \cup \{y\}$. We see that S is a cut set of size p of G such that $G-S$ contains $G[\{x\} \cup (\cup_{i=1}^{k-3} V(H_i))], w_1,...,w_{p-1}$ and w_p as $p+1$ different components. Thus $\frac{|S|}{\omega(G-S)} < 1$. Proposition 2.2.2 implies that G is not a hamiltonian graph and, hence, $G \in \mathscr{C}_k^l$. Therefore $\mathscr{H}_k^l \subseteq \mathscr{C}_k^l$ as required. This completes the proof of Theorem 6.3.1. □

We have shown that $\mathscr{C}_k^l = \emptyset$ for $k = 1,2$ or 3. This together with Theorem 6.3.1 imply the following result.

Corollary 6.3.2. For $n \geq (k-1)l+3$, $\mathscr{C}_k^l = \emptyset$ if and only if $k = 1,2$ or 3.

6.3.2 $k-\gamma_t-$Edge Critical Graphs

We begin this subsection by providing a class of $2-$connected $5-\gamma_t-$edge critical non-hamiltonian graphs.

The Class $\mathscr{T}$:

Let x and y be two isolated vertices. For $1\leq i\leq 3$, let $K^i_{n_i}$ be a copy of a complete graph of order $n_i\geq 3$. Further, we let K^i_x, K^i_y and K^i_z be three induced subgraphs of $K^i_{n_i}$ such that $|V(K^i_z)|=1$ and $\{V(K^i_x),V(K^i_y),V(K^i_z)\}$ is a partition of $V(K^i_{n_i})$. A graph G in the class $\mathscr{T}$ can be obtained from $\{x,y\},K^1_{n_1},K^2_{n_2}$ and $K^3_{n_3}$ by adding edges according to the join operations :

- $x\vee K^i_x$ and
- $y\vee K^i_y$

for $1\leq i\leq 3$. A graph G in the class $\mathscr{T}$ is illustrated by the following figure.

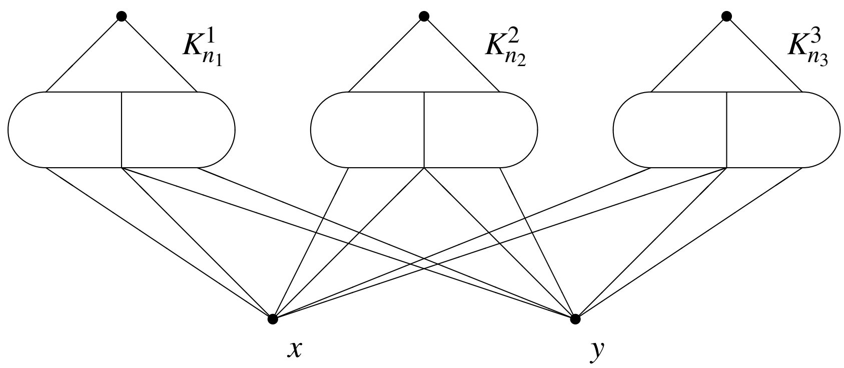

Figure 6.4 : A graph in the class $\mathscr{T}$

Lemma 6.3.3. $\mathscr{T}\subseteq\mathscr{G}^{\gamma_t}_5$.

Proof. We need to show that, for a graph $G\in\mathscr{T}$, $G\in\mathscr{G}^{\gamma_t}_5$. So we need to prove that G is a $2-$connected $5-\gamma_t-$edge critical non-hamiltonian graph. Let $G\in\mathscr{T}$. For $i\in\{1,2,3\}$, we let x^i and y^i be vertices of $K^i_{n_i}$. We see that $\{x,x^1,x^2,x^3,y^1\}\succ_t G$. Thus $\gamma_t(G)\leq 5$. We need to show that $5\leq\gamma_t(G)$. Let D be a γ_t-set of G. To dominate K^i_z, $V(K^i_{n_i})\cap D\neq\emptyset$. If $\{x,y\}\subseteq D$, then $5\leq|D|\leq\gamma_t(G)$. But if $\{x,y\}\cap D=\emptyset$, then $|D\cap V(K^i_{n_i})|\geq 2$ for $1\leq i\leq 3$. Thus $6\leq|D|\leq\gamma_t(G)\leq 5$, a contradiction. Therefore, $|\{x,y\}\cap D|=1$. Without loss of generality let $x\in D$. To dominate y, $y^{i'}\in D$ for some $1\leq i'\leq 3$. Thus $|V(K^{i'}_{n_{i'}})\cap D|\geq 2$. Therefore $5\leq|D|\leq\gamma_t(G)\leq 5$.

Let $u,v \in V(G)$ be any non-adjacent vertices. Clearly, $|\{u,v\} \cap V(K^i_{n_i})| \leq 1$. If $u \in V(K^i_{n_i})$ and $v \in V(K^{i'}_{n_{i'}})$ for $i \neq i'$, then $\{x^j, y^j, u, v\} \succ_t G+uv$ where $\{j\} = \{1,2,3\} - \{i,i'\}$. Suppose $u \in \{x,y\}$ and $v \in V(K^{i'}_{n_{i'}})$. Let $\{w\} = \{x,y\} - \{u\}$ and $\{i,j\} = \{1,2,3\} - \{i'\}$. If $v \in V(K^{i'}_w)$, then $\{u,v,u^i,u^j\} \succ_t G+uv$. If $v \in V(K^{i'}_z)$, then $\{u,u^i,w,w^j\} \succ_t G+uv$. Finally if $\{u,v\} = \{x,y\}$, then $\{x,x^1,x^2,x^3\} \succ_t G+uv$. Therefore, $\gamma_t(G+uv) < \gamma_t(G)$ and G is a $5-\gamma_t-$edge critical graph.

We see that $\{x,y\}$ is a cut set of size 2 such that $G-\{x,y\}$ has $K^1_{n_1}, K^2_{n_2}$ and $K^3_{n_3}$ as the three components. Thus there is a cut set $S = \{x,y\}$ such that $\frac{|S|}{\omega(G-S)} < 1$. By Proposition 2.2.2, G is a non-hamiltonian graph. Thus $G \in \mathscr{G}^{\gamma_t}_5$ and this completes the proof. □

Corollary 6.3.4. If $k = 2$ or 3, then $\mathscr{G}^{\gamma_t}_k = \emptyset$, but if $k = 4$ or 5, then $\mathscr{G}^{\gamma_t}_k \neq \emptyset$.

Proof. By Theorem 2.1.5, it is easy to prove that $\mathscr{G}^{\gamma_t}_2 = \emptyset$, moreover, Corollary 6.2.16 implies that $\mathscr{G}^{\gamma_t}_3 = \emptyset$. By Theorems 3.1.1 and 6.3.1, we see that $\mathscr{G}^{\gamma_t}_4 \neq \emptyset$. Finally, Lemma 6.3.3 yields that $\mathscr{G}^{\gamma_t}_5 \neq \emptyset$. This completes the proof. □

Although $2-$connected $4-\gamma_t-$edge critical graphs and $2-$connected $5-\gamma_t-$edge critical graphs need not be hamiltonian, we can show that $2-$connected $4-\gamma_t-$edge critical graphs and $3-$connected $5-\gamma_t-$edge critical graphs are hamiltonian when they are claw-free in the next chapter.

6.3.3 $k-i-$Edge Critical Graphs

We first give a construction of $k-i-$edge critical non-hamiltonian graphs for $k \geq 3$.

The Class $\mathscr{I}_k$:

For $k \geq 3$, let $\overline{K}_k$ be the complement of K_k and $\overline{K}_{k-1}$ the complement of K_{k-1}. Let K_{n_1} be a copy of a complete graph of order $n_1 \geq 2$. These three graphs $\overline{K}_k, \overline{K}_{k-1}$ and K_{n_1} are disjoint. We, further, let $V(\overline{K}_k) = X = \{x_1, x_2, ..., x_k\}$ and $V(\overline{K}_{k-1}) = Y = \{y_1, y_2, ..., y_{k-1}\}$. A graph G in the class $\mathscr{I}_k$ is obtained from $\overline{K}_k, \overline{K}_{k-1}$ and K_{n_1} by adding edges according to the join operations :

- $(\overline{K}_k - x_k) \vee \overline{K}_{k-1}$ and
- $\overline{K}_k \vee K_{n_1}$.

A graph in this class is illustrated by the following figure. It is worth noting that, for $k = 3$, the class $\mathscr{I}_3$ was found independently by Ao et al. [22].

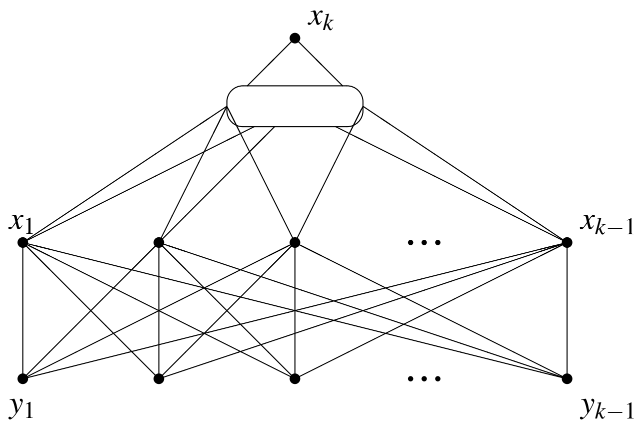

Figure 6.5 : A graph in the class $\mathscr{I}_k$

Lemma 6.3.5. $\mathscr{I}_k \subseteq \mathscr{G}^i_k$.

Proof. We need to show that, for a graph $G \in \mathscr{I}_k$, $G \in \mathscr{G}^i_k$. Therefore, we have to show that G is a $2-$connected $k-i-$edge critical non-hamiltonian graph. Let $G \in \mathscr{I}_k$. Clearly, $X \succ_i G$. Thus $i(G) \leq k$. We show that $k \leq i(G)$. Let D be a smallest independent dominating set. Suppose that $x_1 \in D$. Since $x_1 \succ V(K_{n_1}) \cup Y$, $(V(K_{n_1}) \cup Y) \cap D = \emptyset$. Clearly $X \subseteq D$ to dominate X. Thus $k \leq |D| \leq i(G)$. Suppose that $x_1 \notin D$. To dominate x_1, we have that $V(K_{n_1})$ or Y intersects D. We first consider the case when $Y \cap D \neq \emptyset$. Thus $(X - \{x_k\}) \cap D = \emptyset$ because D is an independent set. We have $Y \subseteq D$ for dominating Y. Moreover, to dominate $V(K_{n_1}) \cup \{x_k\}$, we have $D \cap (V(K_{n_1}) \cup \{x_k\}) \neq \emptyset$. Thus $k \leq |D| \leq i(G)$. We now consider the case when $V(K_{n_1}) \cap D \neq \emptyset$. Thus $X \cap D = \emptyset$ because $x_j \succ V(K_{n_1})$ for all $1 \leq j \leq k$. To dominate Y, we have $Y \subseteq D$. Therefore $k \leq |D| \leq i(G)$. These imply that $i(G) = k$.

Let u, v be any non-adjacent vertices of G. If $\{u, v\} \subseteq X$, then $X - \{v\} \succ_i G + uv$. If $\{u, v\} \subseteq Y$, then $(Y - \{v\}) \cup \{x_k\} \succ_i G + uv$. If $u \in Y$ and $v \in V(K_{n_1}) \cup \{x_k\}$, then $\{v\} \cup (Y - \{u\}) \succ_i G + uv$. Therefore $i(G + uv) < i(G)$ and G is a $k-i-$edge critical graphs. We see that $X - \{x_k\}$ is a cut set of size $k-1$ but $G - (X - \{x_k\})$ has $G[V(K_{n_1} \cup \{x_k\})], y_1, y_2, ..., y_{k-1}$ as the k components and so $\frac{|X - \{x_k\}|}{\omega(G - (X - \{x_k\}))} < 1$. Proposition 2.2.2 thus implies G is a non-hamiltonian graph. Therefore $G \in \mathscr{G}^i_k$ and this completes the proof. $\square$

On $k-i-$edge critical graphs, it is obvious that every $1-i-$edge critical graph is K_n for $n \geq 1$. Moreover, Ao [3] proved that a graph G is $2-i-$edge critical if and only if $\overline{G} \cong \cup_{i=1}^n K_{1,n_i}$ where $n \geq 1$. It is easy to see that every $2-$connected $k-i-$edge critical graph when $k = 1$ or 2 is hamiltonian. Moreover, Lemma 6.3.5 yields that $\mathscr{I}_k \subseteq \mathscr{G}^i_k$ for $k \geq 3$. These imply the following corollary.

Corollary 6.3.6. $\mathscr{G}^i_k = \emptyset$ if and only if $k = 1$ or 2.

CHAPTER 7

Hamiltonicities of Claw-Free Graphs that are Critical with respect to Domination Numbers

In the previous chapter, we showed that $2-$connected $k-\gamma_c-$edge critical graphs need not be hamiltonian for $k \geq 4$. We observed that those non-hamiltonian graphs contain a claw as an induced subgraph, for example, a graph G in the class $\mathscr{H}_k^l$ in Chapter 6 contains claw as an induced subgraph. Moreover, we have noticed that $k-\gamma_c-$edge critical claw-free graphs are hamiltonian, for example the graph $x \vee K_{n_1} \vee K_{n_2} \vee ... \vee K_{n_{k-1}} \vee K_{n_k} - e_1 - e_2 - ... - e_{n_k}$, where $n_1 \geq n_2 \geq ... \geq n_{k-1} = n_k \geq 2$ and $\{e_1, e_2, ...e_{n_k}\}$ is a matching of size n_k in the graph between the vertices in $V(K_{n_{k-1}})$ and $V(K_{n_k})$.

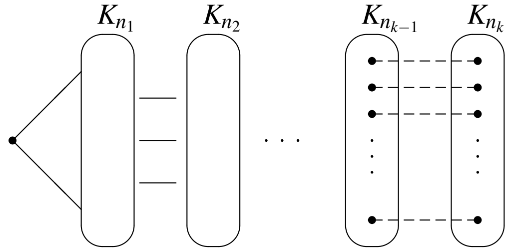

Figure : 7.1 A $k-\gamma_c-$edge critical claw-free graph which is hamiltonian

Hence, this gives rise to the following problem : Is every $4-\gamma_c-$edge critical claw-free graph hamiltonian. We further study hamiltonian properties of edge critical graphs with respect to domination number, independent domination number and total domination number when they are claw-free.

The chapter is organized as the following. We show that every $2-$connected $4-\gamma_c-$edge critical claw-free graph is hamiltonian and show that the claw-free condition cannot be relaxed. We further prove that the class of $k-\gamma_c-$edge critical claw-free non-hamiltonian graphs of connectivity two is empty if and only if $k = 1, 2, 3$ or 4. We show that every 3−connected $k-\gamma_c-$edge critical claw-free graph is hamiltonian for $1 \leq k \leq 6$. For $k-\gamma_t-$critical graphs, we show that every 3−connected $k-\gamma_t-$edge critical claw-free graph is hamiltonian for $2 \leq k \leq 5$. For every 3−connected $4-\mathscr{D}-$critical claw-free graph where $\mathscr{D} \in \{\gamma, i\}$, we show that it is hamiltonian.

7.1 $k-\gamma_c-$Edge Critical Claw-free Graphs

In this section, we use the claw-freeness to determine when 2−connected $k-\gamma_c-$edge critical claw-free graphs are hamiltonian. Firstly, we give two optional proofs for the following theorem

Theorem 7.1.1. Let G be a 2−connected $4-\gamma_c-$edge critical claw-free graph. Then G is hamiltonian.

Our first proof utilizes the classical way by suppose to the contrary that a $4-\gamma_c-$edge critical claw-free graph is not hamiltonian. We then have a longest cycle which does not contain some vertices of the graph. We can apply the results in Favaron et al. [65], Tian et al. [139] and Brousek [30] as the tool to work on this proof. For the second proof, we have observed that a $4-\gamma_c-$edge critical claw-free graph cannot contain the nets $N_{1,2,2}$ and $N_{1,1,3}$ as an induced subgraph. From Theorem 2.2.4, if we can show that G is neither isomorphic to $P_{3,3,3}$ where $P_{3,3,3} \in \mathscr{P}$ nor the closure $cl(G)$ is in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, then we will obtain that G is hamiltonian.

7.1.1 The First Proof of Theorem 7.1.1

Suppose G is a 2−connected $4-\gamma_c-$edge critical non-hamiltonian graph. Let C be a longest cycle of G. As in the previous chapter, we write $\overrightarrow{C}$ to indicate the clockwise orientation of C. Similarly, we can denote the anticlockwise orientation of C by $\overleftarrow{C}$. In particular, for vertices u and v of C we denote the $(u,v)-$directed segment of $\overrightarrow{C}$ ($\overleftarrow{C}$) by $u\overrightarrow{C}v$ ($u\overleftarrow{C}v$), moreover, we let $\overrightarrow{C}[u,v] = V(u\overrightarrow{C}v)$. The successor (predecessor) of a vertex v of C in $\overrightarrow{C}$ is denoted by $v^+(v^-)$. Furthermore, for $i \geq 1$, $v^{(i+1)+} = (v^{i+})^+$ and $v^{(i+1)-} = (v^{i-})^-$ where $v^{1+} = v^+$ and $v^{1-} = v^-$. This notation is illustrated by the following figure. Note that we always use an orientation $\overrightarrow{C}$ when we mention about the successor and the predecessor of any vertex of C.

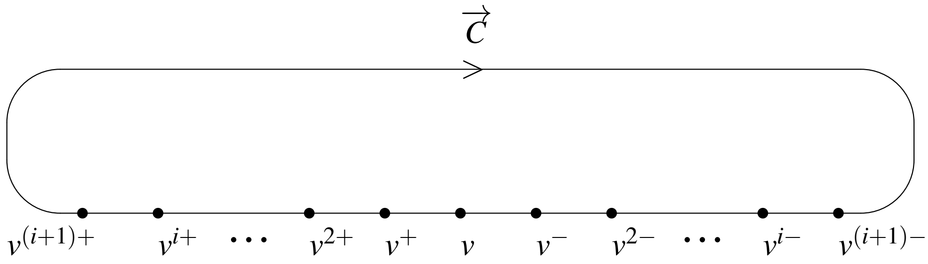

Figure 7.2

Let H be a component of $G-C$ and $X = N_C(H)$. Suppose $|X| = d$. We may order the vertices of the set X as $x_1, x_2, ...,$ and x_d according to the orientation $\overrightarrow{C}$. Lemmas 7.1.2 and 7.1.3 are well known (see Brousek [30]) and proved under the condition that G is claw-free non-hamiltonian graph. For completeness, we also provide the proofs. Recall that, for $1 \le i \ne j \le d$, $x_i P_H x_j$ denotes a path from x_i to x_j which the internal vertices are in $V(H)$.

Lemma 7.1.2. For all $1 \le i \le d$, $x_i^+ x_i^- \in E(G)$.

Proof. Suppose to the contrary that $x_i^+ x_i^- \notin E(G)$. Let $h \in N_H(x_i)$. The maximality of C yields that $hx_i^+, hx_i^- \notin E(G)$. Thus $G[\{x_i, x_i^+, x_i^-, h\}]$ is a claw centered at x_i, a contradiction. Thus $x_i^+ x_i^- \in E(G)$, as required. □

Lemma 7.1.3. For $1 \le i \ne j \le d$, $\{x_i x_j^{2+}, x_i^+ x_j^{2+}, x_i x_j^{2-}, x_i^- x_j^{2-}\} = \emptyset$.

Proof. Lemma 7.1.2 implies that $x_i^+ x_i^- \in E(G)$. By symmetry, it suffices to show that $x_i x_j^{2+}, x_i^+ x_j^{2+} \notin E(G)$. If $x_i x_j^{2+} \in E(G)$, then

$$x_i, x_j^{2+} \overrightarrow{C} x_i^-, x_i^+ \overrightarrow{C} x_j^-, x_j^+, x_j P_H x_i$$

is a cycle longer than C. Moreover if $x_i^+ x_j^{2+} \in E(G)$, then

$$x_i^+, x_j^{2+} \overrightarrow{C} x_i P_H x_j, x_j^+, x_j^- \overleftarrow{C} x_i^+$$

is a cycle longer than C. In both cases, the maximality of C is contradicted. Thus $x_i x_j^{2+}, x_i^+ x_j^{2+} \notin E(G)$ and this completes the proof. □

Finally, we prove Lemmas 7.1.4-7.1.9 which we use to prove our main results.

Lemma 7.1.4. For all $1 \le i \le d$, $|\overrightarrow{C}[x_i^+, x_{i+1}^-]| \ge 3$.

Proof. Lemma 7.1.2 yields $x_i^+ x_i^- \in E(G)$. Clearly $x_i^+ \ne x_{i+1}$. Suppose to the contrary that x_i^{2+} or x_i^{3+} is x_{i+1}. Thus

$x_iP_Hx_{i+1}\overrightarrow{C}x_i^-, x_i^+, x_i$ or

$x_i, x_i^+, x_i^-\overleftarrow{C}x_{i+1}^+, x_{i+1}^-, x_{i+1}P_Hx_i$

is a cycle longer than C which is a contradiction. Hence, $|\overrightarrow{C}[x_i^+, x_{i+1}^-]| \geq 3$. This completes the proof. □

The proof of Lemmas 7.1.5-7.1.9 are under the assumption that $x_i^+x_i^- \in E(G)$ and $|\overrightarrow{C}[x_i^+, x_{i+1}^-]| \geq 3$ for all $1 \leq i \leq d$.

Lemma 7.1.5. For all $1 \leq i \neq j \leq d$, $x_i^{2+}x_j^{2+}, x_i^{2-}x_j^{2-} \notin E(G)$.

Proof. By symmetry, it suffices to show that $x_i^{2+}x_j^{2+} \notin E(G)$. If $x_i^{2+}x_j^{2+} \in E(G)$, then a cycle

$x_i^{2+}, x_j^{2+}\overrightarrow{C}x_i^-, x_i^+, x_iP_Hx_j, x_j^+, x_j^-\overleftarrow{C}x_i^{2+}$

has length longer than C. This contradiction establishes the lemma. □

Lemma 7.1.6. For all $v \in \overrightarrow{C}[x_i^+, x_j^{3-}]$, if $x_iv \in E(G)$, then $x_j^{2-}v^- \notin E(G)$.

Proof. Let $x_iv \in E(G)$. Suppose to the contrary that $x_j^{2-}v^- \in E(G)$. If $v = x_i^+$, then $v^- = x_i$. It follows that $x_j^{2-}x_i \in E(G)$ contradicting Lemma 7.1.3. Therefore, $v \in \overrightarrow{C}[x_i^{2+}, x_j^{3-}]$. Clearly a cycle

$x_i, v\overrightarrow{C}x_j^{2-}, v^-\overleftarrow{C}x_i^+, x_i^-\overleftarrow{C}x_j^+, x_j^-, x_jP_Hx_i$

has length longer than C. This contradiction gives $x_j^{2-}v^- \notin E(G)$, thus establishing the lemma. □

Lemma 7.1.7. For all $v \in \overrightarrow{C}[x_i^+, x_j^{3-}]$, if $x_i^-v \in E(G)$, then $x_j^{2-}v^- \notin E(G)$.

Proof. Let $x_i^-v \in E(G)$. Suppose to the contrary that $x_j^{2-}v^- \in E(G)$. Thus

$x_i^-, v\overrightarrow{C}x_j^{2-}, v^-\overleftarrow{C}x_iP_Hx_j, x_j^-, x_j^+\overrightarrow{C}x_i^-$

is a cycle longer than C, a contradiction. Therefore $x_j^{2-}v^- \notin E(G)$ and establishing this lemma. □

Lemma 7.1.8. For all $v \in \overrightarrow{C}[x_i^{2+}, x_j^-]$, if $x_j^{2+}v \in E(G)$, then $x_i^-v^- \notin E(G)$.

Proof. Let $x_j^{2+}v \in E(G)$. Suppose to the contrary that $x_i^-v^- \in E(G)$. Thus a cycle

$x_i^-, v^-\overleftarrow{C}x_iP_Hx_j, x_j^+, x_j^-\overleftarrow{C}v, x_j^{2+}\overrightarrow{C}x_i^-$

is longer than C which is a contradiction. Hence $x_i^-v^- \notin E(G)$ and establishing this lemma. □

Lemma 7.1.9. For all $v \in \overrightarrow{C}[x_i^{2+}, x_j^{2-}]$, if $x_i^{2-}v \in E(G)$, then $x_j^{2-}v^- \notin E(G)$.

Proof. Let $x_i^{2-}v \in E(G)$. Suppose to the contrary that $x_j^{2-}v^- \in E(G)$. Lemma 7.1.5 yields that $v \neq x_j^{2-}$. Therefore

$$x_i^{2-}, v\overrightarrow{C}x_j^{2-}, v^-\overleftarrow{C}x_i^+, x_i^-, x_iP_Hx_j, x_j^-, x_j^+\overrightarrow{C}x_i^{2-}$$

is a cycle longer than C. This implies that $x_j^{2-}v^- \notin E(G)$ and completes the proof. □

Note that Lemmas 7.1.2 and 7.1.4 are used in almost every proof in this section. For the sake of convenience, from now on, we let

$$x_i^+x_i^- \in E(G) \text{ and } |\overrightarrow{C}[x_i^+, x_{i+1}^-]| \geq 3$$

for all $1 \leq i \leq d$. We note also that Lemmas 6.1.2-6.1.5 from Chapter 6 may be used in the proofs of some lemmas.

Lemma 7.1.10. $d = 2$.

Proof. Suppose to the contrary that $d \geq 3$. Lemma 6.1.2 yields $x_1^+x_2^+ \notin E(G)$. Consider $G + x_1^+x_2^+$. Lemma 2.1.15(2) implies that $D^c_{x_1^+x_2^+} \cap \{x_1^+, x_2^+\} \neq \emptyset$. We distinguish three cases.

Case 1 : $\{x_1^+, x_2^+\} \subseteq D^c_{x_1^+x_2^+}$.

As $(G + x_1^+x_2^+)[D^c_{x_1^+x_2^+}]$ is connected, to dominate H, we obtain $x_i \in D^c_{x_1^+x_2^+}$ for some $i \in \{1, 2, 3, ..., d\}$. Therefore $x_ix_1^+ \in E(G)$ or $x_ix_2^+ \in E(G)$. Lemma 6.1.4 gives that either $x_i = x_1$ or $x_i = x_2$. This imply by Lemmas 6.1.2 and 6.1.4 that $D^c_{x_1^+x_2^+}$ does not dominate x_3^+ contradicting $D^c_{x_1^+x_2^+} \succ_c G + x_1^+x_2^+$. Thus Case 1 cannot occur.

Case 2 : $|D^c_{x_1^+x_2^+} \cap \{x_1^+, x_2^+\}| = 1$.

Without loss of generality let $x_1^+ \in D^c_{x_1^+x_2^+}$. Suppose that $x_1 \in D^c_{x_1^+x_2^+}$. In view of Lemma 7.1.3, x_1^+ and x_1 are not adjacent to any vertex in $\{x_2^{2+}, x_3^{2+}\}$. To dominate $\{x_2^{2+}, x_3^{2+}\}$, we have that $D^c_{x_1^+x_2^+} - \{x_1^+, x_1\} \neq \emptyset$. In fact, $|D^c_{x_1^+x_2^+} - \{x_1^+, x_1\}| = 1$ as a consequence of Lemma 2.1.15(1). Let $\{a\} = D^c_{x_1^+x_2^+} - \{x_1^+, x_1\}$. By the connectedness of $(G + x_1^+x_2^+)[D^c_{x_1^+x_2^+}]$, $a \notin \{x_2^{2+}, x_3^{2+}\}$. Thus $ax_2^{2+}, ax_3^{2+} \in E(G)$. Lemma 7.1.5 yields that $x_3^{2+}x_2^{2+} \notin E(G)$. Because $(G + x_1^+x_2^+)[D^c_{x_1^+x_2^+}]$ is connected, a is adjacent to a vertex $v \in \{x_1^+, x_1\}$. Clearly $G[\{a, v, x_2^{2+}, x_3^{2+}\}]$ is a claw centered at a, a contradiction. Hence, $x_1 \notin D^c_{x_1^+x_2^+}$.

Because $D^c_{x_1^+x_2^+} \succ_c H$ and $(G + x_1^+x_2^+)[D^c_{x_1^+x_2^+}]$ is connected, it follows that $x_i \in D^c_{x_1^+x_2^+}$ for some $i \in \{2, 3, ..., d\}$. Moreover, $x_1^+x_i \notin E(G)$ by Lemma 6.1.4. Thus $D^c_{x_1^+x_2^+} - \{x_i, x_1^+\} \neq \emptyset$. In fact, Lemma 2.1.15(1) yields $|D^c_{x_1^+x_2^+} - \{x_1^+, x_1\}| = 1$. Let $\{a\} = D^c_{x_1^+x_2^+} - \{x_i, x_1^+\}$. Clearly $ax_i, ax_1^+ \in E(G)$. We note by Lemma 2.1.15(3) that $i \neq 2$. Lemma 7.1.3 thus implies $x_1^+x_2^{2+}, x_ix_2^{2+} \notin E(G)$. Because $ax_1^+ \in E(G)$, $a \neq x_2^{2+}$.

Clearly $ax_2^{2+} \in E(G)$. We have $G[\{a, x_i, x_1^+, x_2^{2+}\}]$ is a claw centered at a, a contradiction. Therefore Case 2 cannot occur. This together with Case 1 implies that $d = 2$. This completes the proof. □

Let $\{i, j\} = \{1, 2\}$. Lemmas 7.1.11-7.1.16 are characterizations of a cycle C according to the existence of edges $x_i^+ x_j^-$ and $x_j^+ x_i^-$. Let $h_l \in N_H(x_l)$ for $l \in \{i, j\}$.

Lemma 7.1.11. For all $x \in N_C(x_i) - \{x_j\}$, $N_C(x_i) - \{x, x_j\} \subseteq N_C(x)$.

Proof. Suppose there exists $y \in (N_C(x_i) - \{x, x_j\}) - N_C(x)$. Therefore $yx \notin E(G)$. Since $y \neq x_j$, $yh_i \notin E(G)$. Thus $G[\{x_i, x, y, h_i\}]$ is a claw centered at x_i, a contradiction. Hence $N_C(x_i) - \{x, x_j\} \subseteq N_C(x)$ and this completes the proof. □

Lemma 7.1.12. If $x_i^+ x_j^- \notin E(G)$, then $|\{x_i^+, x_j^-\} \cap D^c_{x_i^+ x_j^-}| = 1$.

Proof. Suppose to the contrary that $\{x_i^+, x_j^-\} \subseteq D^c_{x_i^+ x_j^-}$. To dominate H, $D^c_{x_i^+ x_j^-} - \{x_i^+, x_j^-\} \neq \emptyset$, in fact, Lemma 2.1.15(1) yields $|D^c_{x_i^+ x_j^-} - \{x_i^+, x_j^-\}| = 1$. Moreover, by the connectedness of $(G + x_i^+ x_j^-)[D^c_{x_i^+ x_j^-}]$, $x_i \in D^c_{x_i^+ x_j^-}$ or $x_j \in D^c_{x_i^+ x_j^-}$. Without loss of generality, we let $x_i \in D^c_{x_i^+ x_j^-}$. Lemma 7.1.11 thus implies $N_C(x_i) - \{x_j, x_i^+\} \subseteq N_C(x_i^+)$. As $D^c_{x_i^+ x_j^-} = \{x_i, x_i^+, x_j^-\}$, we obtain $\{x_i^+, x_j^-\} \succ_c \overrightarrow{C}[x_j^+, x_i^-]$. Because $x_i^+ x_i^-, x_j^+ x_j^- \in E(G)$, there exists an integer l such that $x_j^- x_j^{l+}$, $x_i^+ x_j^{(l+1)+} \in E(G)$. This contradicts Lemma 6.1.5. Therefore $|\{x_i^+, x_j^-\} \cap D^c_{x_i^+ x_j^-}| = 1$. This completes the proof. □

Lemma 7.1.13. Suppose that $x_i^+ x_j^- \notin E(G)$. If $D^c_{x_i^+ x_j^-} = \{x_i^+, x_i, a\}$ or $D^c_{x_i^+ x_j^-} = \{x_j^-, x_j, a\}$ for some $a \in V(G)$, then $a \in \overrightarrow{C}[x_j^+, x_i^-]$.

Proof. By symmetry, it suffices to consider the case $D^c_{x_i^+ x_j^-} = \{x_j^-, x_j, a\}$. To dominate x_i^-, we have $a \in V(C)$. Suppose to the contrary that $a \notin \overrightarrow{C}[x_j^+, x_i^-]$. Therefore $a \in \overrightarrow{C}[x_i, x_j]$. Lemma 2.1.15(3) gives $a \neq x_i$.

We note by Lemmas 6.1.2 and 6.1.4 that $x_j^- x_i^-, x_j x_i^- \notin E(G)$. Moreover, Lemma 7.1.3 yields $x_j^- x_i^{2-}, x_j x_i^{2-} \notin E(G)$. Hence $ax_i^-, ax_i^{2-} \in E(G)$. By the connectedness of $(G + x_i^+ x_j^-)[D^c_{x_i^+ x_j^-}]$, $ax_j \in E(G)$ or $ax_j^- \in E(G)$. Since $a \neq x_i$, by Lemma 7.1.11, $ax_j^- \in E(G)$. As $ax_i^- \in E(G)$, by Lemma 6.1.3(1), we obtain $a^- x_j^- \notin E(G)$. Thus a^- is adjacent to x_i^- and x_i^{2-}, as otherwise $G[\{a, x_i^-, a^-, x_j^-\}]$ and $G[\{a, x_i^{2-}, a^-, x_j^-\}]$ are claws centered at a.

Let $\overrightarrow{C}[x_j^+, x_i^{3-}] = \{x_j^+, x_j^{2+}, \ldots, x_j^{p+} = x_i^{3-}\}$.

Claim : If $x_j^- x_j^{l+} \in E(G)$ for $1 \leq l \leq p-1$, then $x_j^- x_j^{(l+1)+} \in E(G)$.

We prove the claim by induction on l. Clearly, $x_j^- x_j^+ \in E(G)$. We first show that

$x_j^- x_j^{2+}$. Since $a^- x_i^- \in E(G)$, it follows from Lemma 7.1.8 that $ax_j^{2+} \notin E(G)$. As $D^c_{x_i^+ x_j^-} \succ x_j^{2+}$, by Lemma 7.1.11, we obtain $x_j^- x_j^{2+}$.

Assume that, for all $1 \le l_0 < l$, if $x_j^- x_j^{l_0+}$, then $x_j^- x_j^{(l_0+1)+} \in E(G)$. Choose $l_0 = l-1$, by induction, $x_j^- x_j^{l+} \in E(G)$. We have to show that $x_j^- x_j^{(l+1)+} \in E(G)$. If $ax_j^{(l+1)+} \in E(G)$, then we have a contradiction that

$$a, x_j^{(l+1)+} \overrightarrow{C} x_i^-, a^- \overleftarrow{C} x_i P_H x_j \overrightarrow{C} x_j^{l+}, x_j^- \overleftarrow{C} a$$

is a cycle longer than C. So $ax_j^{(l+1)+} \notin E(G)$. Therefore $x_j x_i^{(l+1)+} \in E(G)$ or $x_j^- x_j^{(l+1)+} \in E(G)$. Lemma 7.1.11 yields $x_j^- x_j^{(l+1)+} \in E(G)$, thus establishing the claim.

As a consequence of the claim, $x_j^- x_i^{3-} \in E(G)$. Thus

$$x_j \overrightarrow{C} x_i^{3-}, x_j^- \overleftarrow{C} a, x_i^{2-}, a^- \overleftarrow{C} x_i^+, x_i^-, x_i P_H x_j$$

is a cycle longer than C, a contradiction. Therefore $a \in \overrightarrow{C}[x_j^+, x_i^-]$. This completes the proof. □

Lemma 7.1.14. If $x_i^+ x_j^- \notin E(G)$, then $x_i^+ x_j^{2-} \in E(G)$ or $x_j^- x_i^{2+} \in E(G)$.

Proof. Lemma 7.1.12 implies that $|D^c_{x_i^+ x_j^-} \cap \{x_i^+, x_j^-\}| = 1$. Suppose first that $x_i^+ \in D^c_{x_i^+ x_j^-}$. To dominate H and by the connectedness of $(G + x_i^+ x_j^-)[D^c_{x_i^+ x_j^-}]$, $D^c_{x_i^+ x_j^-} \cap X \neq \emptyset$. Lemma 2.1.15(3) yields that $x_j \notin E(G)$. Thus $x_i \in D^c_{x_i^+ x_j^-}$. Moreover, Lemma 2.1.15(1) gives $|D^c_{x_i^+ x_j^-} - \{x_i, x_i^+\}| \le 1$. By Lemmas 6.1.2 and 6.1.4, $x_i x_j^+, x_i^+ x_j^+ \notin E(G)$. Thus, to dominate x_j^+, $|D^c_{x_i^+ x_j^-} - \{x_i, x_i^+\}| = 1$. Let $\{a\} = D^c_{x_i^+ x_j^-} - \{x_i, x_i^+\}$. Therefore $ax_j^+ \in E(G)$. Lemma 7.1.13 thus implies $a \in \overrightarrow{C}[x_j^+, x_i^-]$.

Because $(G + x_i^+ x_j^-)[D^c_{x_i^+ x_j^-}]$ is connected, $ax_i^+ \in E(G)$ or $ax_i \in E(G)$. As a consequence of Lemma 7.1.11, $ax_i^+ \in E(G)$.

By claw-freeness $a^+ x_j^+ \in E(G)$ or $a^+ x_i^+ \in E(G)$, as otherwise $G[\{a, a^+, x_i^+, x_j^+\}]$ is a claw centered at a. Because $ax_i^+ \in E(G)$, it follows by Lemma 6.1.3(1) that $a^+ x_j^+ \notin E(G)$. Thus $a^+ x_i^+ \in E(G)$. Lemma 7.1.8 implies that $x_j^{2-} a \notin E(G)$. By Lemma 7.1.3, $x_i x_j^{2-} \notin E(G)$. Therefore $x_i^+ x_j^{2-} \in E(G)$ as required. For the case $x_j^- \in D^c_{x_i^+ x_j^-}$, we can show that $x_j^- x_i^{2+} \in E(G)$ by the similar arguments. This completes the proof. □

Lemma 7.1.15. If $x_i^+ x_j^{2-} \in E(G)$, then x_i^{2+} is not adjacent to any vertex in $\{x_i, x_i^-, x_j, x_j^+\}$. If $x_j^- x_i^{2+} \in E(G)$, then x_j^{2-} is not adjacent to any vertex in $\{x_j, x_j^+, x_i, x_i^-\}$.

Proof. Suppose first that $|\overrightarrow{C}[x_i^+, x_j^-]| = 3$. Clearly $x_i^{2+} = x_j^{2-}$. Lemma 7.1.3 yields x_i^{2+} is not adjacent to any vertex in $\{x_i, x_i^-, x_j, x_j^+\}$ as required. Suppose that $|\overrightarrow{C}[x_i^+, x_j^-]| \ge 4$. If $x_i^+ x_j^{2-} \in E(G)$, then, by Lemmas 7.1.6 and 7.1.7, $x_i^{2+} x_i^-, x_i^{2+} x_i \notin E(G)$. Moreover, Lemma 7.1.3 gives $x_i^{2+}x_j^+, x_i^{2+}x_j \notin E(G)$. The case $x_j^- x_i^{2+} \in E(G)$ can be proved by the same arguments. This completes the proof. □

Lemma 7.1.16. If $x_i^+ x_j^- \in E(G)$, then $x_i^{2+}x_j^-, x_j^{2-}x_i^+ \in E(G)$.

Proof. Lemma 6.1.4 implies that $x_i x_j^- \notin E(G)$. If $x_i^{2+}x_i, x_i^{2+}x_j^- \notin E(G)$, then $G[\{x_i^+, x_i^{2+}, x_i, x_j^-\}]$ is a claw centered at x_i^+. Hence, $x_i^{2+}x_i \in E(G)$ or $x_i^{2+}x_j^- \in E(G)$. If $x_i^{2+}x_i \in E(G)$, then

$$x_i, x_i^{2+}\overrightarrow{C}x_j^-, x_i^+, x_i^-\overleftarrow{C}x_j P_H x_i$$

is a cycle longer than C, a contradiction. Thus $x_i^{2+}x_j^- \in E(G)$. Similarly, $x_j^{2-}x_i^+ \in E(G)$ and this completes the proof. □

We next define a pair of non-adjacent vertices on C with a situation which is, in fact, prohibited as it will give a claw as an induced subgraph.

A pair of non-adjacent vertices (x,y) is called an *extremal pair of vertices* if

(i) $x \in \overrightarrow{C}[x_i^{2+}, x_j^{2-}]$,

(ii) $y \in \overrightarrow{C}[x_j^{2+}, x_i^{2-}]$,

(iii) x is not adjacent to any vertex in $\{x_i, x_i^-, x_j, x_j^+\}$ and

(iv) y is not adjacent to any vertex in $\{x_i, x_i^+, x_j, x_j^-\}$.

Figure 7.3 illustrates the extremal pair of vertices. Observe that Lemma 7.1.4 implies that x and y are well defined, that is, there always exists x and y satisfying *(i)* and *(ii)*. However, if x and y are not adjacent and satisfy *(iii)* and *(iv)*, i.e. C contains an extremal pair of vertices, then G must contains a claw as induced subgraph as we will detail in the following lemma.

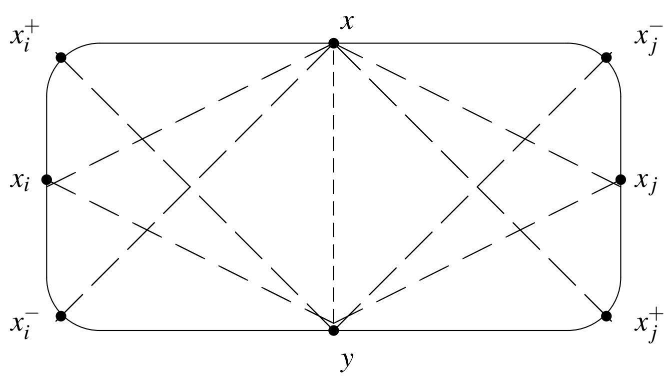

Figure 7.3 An extremal pair of vertices

Lemma 7.1.17. If C contains an extremal pair of vertices, then G contains a claw as an induced subgraph.

Proof. Consider $G+xy$. Lemma 2.1.15(2) yields that $\{x,y\}\cap D^c_{xy}\neq\emptyset$. Suppose first that $\{x,y\}\subseteq D^c_{xy}$. Hence $|D^c_{xy}-\{x,y\}|\leq 1$ by Lemma 2.1.15(1). As $(G+xy)[D^c_{xy}]$ is connected, we obtain $D^c_{xy}\cap(V(H)\cup\{x_i,x_j\})=\emptyset$. Thus D^c_{xy} does not dominate H, a contradiction. Therefore, $|\{x,y\}\cap D^c_{xy}|=1$. Without loss of generality, let $\{x\}=D^c_{xy}\cap\{x,y\}$. To dominate H and by the connectedness of $(G+xy)[D^c_{xy}]$, $D^c_{xy}\cap\{x_i,x_j\}\neq\emptyset$ and there exists $a\in V(G)$ such that $D^c_{xy}=\{x,x_i,a\}$ or $D^c_{xy}=\{x,x_j,a\}$.

We now consider the case when $D^c_{xy}=\{x,x_i,a\}$. Since $xx_i\notin E(G)$, we have that $ax,ax_i\in E(G)$. By the assumption, $xx_j^+\notin E(G)$. Lemma 6.1.4 thus implies that $x_ix_j^+\notin E(G)$. By the connectedness of $(G+xy)[D^c_{xy}]$, $a\neq x_j^+$. Since $D^c_{xy}\succ_c x_j^+$, $ax_j^+\in E(G)$. Hence, $G[\{a,x_i,x,x_j^+\}]$ is a claw centered at a.

We now consider the case when $D^c_{xy}=\{x,x_j,a\}$. Similarly, $ax,ax_j\in E(G)$. By the assumption, $xx_i^-\notin E(G)$. Lemma 6.1.4 implies that $x_jx_i^-\notin E(G)$. By the connectedness of $(G+xy)[D^c_{xy}]$, $a\neq x_i^-$. Since $D^c_{xy}\succ_c x_i^-$, $ax_i^-\in E(G)$. Hence, $G[\{a,x_j,x,x_i^-\}]$ is a claw centered at a. This completes the proof. □

We further define a pair of edge for which is prohibited as it also gives a claw as an induced subgraph.

For any edges $e_1',e_2'\in E(G)$, (e_1',e_2') is called an *extremal pair of edges* if

(i) $e_1\in\{x_i^+x_j^{2-},\ x_i^{2+}x_j^-\}$ and

(ii) $e_2\in\{x_j^+x_i^{2-},x_j^{2+}x_i^-\}$.

This extremal pair of edges is the main tool to prove by the contradiction.

Lemma 7.1.18. If C contains an extremal pair of edges, then G contains a claw as an induced subgraph.

Proof. Suppose that $e_1=x_i^+x_j^{2-}$. This implies by Lemma 7.1.15 that x_i^{2+} is not adjacent to any vertex in $\{x_i,x_i^-,x_j,x_j^+\}$. As a consequence of Lemma 7.1.17, it suffices to show that C contains an extremal pair of vertices.

We first consider the case when $e_2=x_j^{2+}x_i^-$. Thus Lemma 7.1.15 implies x_i^{2-} is not adjacent to any vertex in $\{x_i,x_i^+,x_j,x_j^-\}$. Since $x_i^+x_j^{2-}\in E(G)$, it follows by Lemma 7.1.9 that $x_i^{2-}x_i^{2+}\notin E(G)$. Clearly (x_i^{2+},x_i^{2-}) is an extremal pair of vertices.

We now consider the case when $e_2=x_j^+x_i^{2-}$. Lemma 7.1.15 implies that x_j^{2+} is not adjacent to any vertex in $\{x_i,x_i^+,x_j,x_j^-\}$. Moreover, Lemma 7.1.5 yields $x_i^{2+}x_j^{2+}\notin E(G)$. Therefore (x_i^{2+},x_j^{2+}) is an extremal pair of vertices. We can show that C contains an extremal pair of vertices when $e_1 = x_i^{2+}x_j^-$ by the same arguments. We finish proving this lemma. □

We now ready to prove Theorem 7.1.1.

Proof. Suppose to the contrary that G is not hamiltonian. Let C be a longest cycle in G and H a component of $G-C$ such that $X = N_C(H) \neq \emptyset$. We, further, let $|X| = d$ and order the vertices of the set X as $x_1, x_2, ...,$ and x_d according the orientation $\overrightarrow{C}$. As a consequence of Lemma 7.1.10, $d = 2$. Let $\{i, j\} = \{1, 2\}$.

Our proof is a contradiction. In view of Lemma 7.1.18, it suffices to prove that C contains an extremal pair of edges. We first consider the case when $x_i^+x_j^- \in E(G)$. Thus Lemma 7.1.16 implies $x_i^{2+}x_j^-, x_i^+x_j^{2-} \in E(G)$. We now consider the case when $x_i^+x_j^- \notin E(G)$. Therefore $x_i^+x_j^{2-} \in E(G)$ or $x_i^{2+}x_j^- \in E(G)$ by Lemma 7.1.14. Hence $\{x_i^+x_j^{2-}, x_i^{2+}x_j^-\} \neq \emptyset$. Similarly, if $x_j^+x_i^- \in E(G)$, then Lemma 7.1.16 yields $x_j^{2+}x_i^-$, $x_j^+x_i^{2-} \in E(G)$. Finally, if $x_j^+x_i^- \notin E(G)$, then by Lemma 7.1.14, $x_j^+x_i^{2-} \in E(G)$ or $x_j^{2+}x_i^- \in E(G)$. Clearly $\{x_j^+x_i^{2-}, x_j^{2+}x_i^-\} \neq \emptyset$ and so C contains an extremal pair of edges, as required. As a consequence of Lemma 7.1.18, G contains a claw as an induced subgraph, a contradiction. Therefore, G is hamiltonian. This complete the first proof of Theorem 7.1.1. □

7.1.2 The Second Proof of Theorem 7.1.1

We next prepare some results to establish the second proof of theorem 7.1.1. Recall the classes $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{F}_3$ from Chapter 2. We moreover establish the following lemma which provides the relationship of the cardinalities of a connected dominating set and an independent set on claw-free graphs.

Lemma 7.1.19. Let G be a claw-free graph, X be a vertex subset of G and I be an independent set such that $X \succ_c I$. Then $|I| \leq |X| + 1$. Moreover if $X \cap I \neq \emptyset$, then $|I| \leq |X|$.

Proof. Our proof is by induction on the number of vertices in X. If $|X| = 1$, then, by claw-freeness, the vertex in X is adjacent to at most two vertices in I. Thus $|I| \leq 2 = |X| + 1$. Assume that for all $|X| < p$, $|I| \leq |X| + 1$. Consider the case $|X| = p$. We will prove that $|I| \leq p + 1$. Suppose to the contrary that $|I| \geq p + 2$. Since $G[X]$ is connected, there exists a spanning tree T of $G[X]$. Choose a vertex $v \in X$ which is a leaf of T. By claw-fredom, $|PN_I(v, X)| \leq 1$. Since v is a leaf, $G[X - \{v\}]$ is connected. Thus $X - \{v\} \succ_c I - PN_I(v, X)$. We see that $|X - \{v\}| = p - 1$ but $|I - PN_I(v, X)| \geq p + 1$ violates the assumption. Therefore $|I| \leq |X| + 1$.

Suppose that $X \cap I \neq \emptyset$. If $|X| = 1$, then $X \subseteq I$. Because $X \succ_c I$ and I is an independent set, it follows that $|I| = 1$. Assume that for all $|X| < p$, $|I| \leq |X|$. Consider the case $|X| = p$. We will prove that $|I| \leq p$. Suppose to the contrary that $|I| \geq p+1$. Let T' be a spanning tree of $G[X]$ and v' be a leaf of T'. If $v' \in X \cap I$, then, by independence of I, $PN_I(v', X) = \emptyset$. But if $v' \in X - I$, then, by claw-freeness, $|PN_I(v', X)| \leq 1$. Hence $|PN_I(v', X)| \leq 1$ whether $v' \in I$ or not. Since v' is a leaf of T', $X - \{v'\} \succ_c I - PN_I(v', X)$. We see that $|X - \{v'\}| = p-1$ but $|I - PN_I(v', X)| \geq p$ contradicts the assumption. Thus $|I| \leq |X|$ and this completes the proof of the lemma. □

Our next observation is the structure of the induced subgraph $G[D^c_{xy}]$ of G when $\{x, y\} \subseteq D^c_{xy}$.

Observation 7.1.20. If $\{x, y\} \subseteq D^c_{xy}$, then $G[D^c_{xy}]$ contains exactly two components C_1 and C_2 containing x and y respectively.

Proof. By Lemma 2.1.15(1), $|D^c_{xy}| < k$. If $G[D^c_{xy}]$ is connected, then $D^c_{xy} \succ_c G$ contradicting the minimality of k. Since all components of $G[D^c_{xy}]$ are joined by an edge xy, there are exactly two components C_1 and C_2 containing x and y respectively. □

Our next lemma is to establish the minimum number of k from $k-\gamma_c-$edge critical claw-free graphs when some independent set is given.

Lemma 7.1.21. Let G be a $k-\gamma_c-$edge critical claw-free graph and x, y be any pair of non-adjacent vertices of G. If $\{x\} = D^c_{xy} \cap \{x, y\}$ and there exists an independent set I_1 of order p of $G - y$ containing x or $\{x, y\} \subseteq D^c_{xy}$ and there exists an independent set I_2 of order p of G containing both x and y, then $k \geq p+1$.

Proof. As a consequence of Lemma 2.1.15(1) and (2), $|D^c_{xy}| < k$ and $D_{xy} \cap \{x, y\} \neq \emptyset$. Clearly it suffices to show $|D^c_{xy}| \geq p$. We first consider the case when $|D^c_{xy} \cap \{x, y\}| = 1$. Without loss of generality, let $x \in D^c_{xy}$. By the assumption there exists an independent set I_1 of order p of $G - y$ containing x. Because $D^c_{xy} \succ_c G + xy$, $D^c_{xy} \succ_c I_1$. Since $D^c_{xy} \cap I_1 \neq \emptyset$, it follows by Lemma 7.1.19 that $|D^c_{xy}| \geq p$.

We next consider the case $\{x, y\} \subseteq D^c_{xy}$. Thus there exists an independent set I_2 of order p of G containing both x and y. Observation 7.1.20 gives $G[D^c_{xy}]$ consists of two components C_1 and C_2. Therefore $D^c_{xy} = V(C_1) \cup V(C_2)$. Since $D^c_{xy} \succ_c I_2$ and $x, y \in D^c_{xy} \cap I_2$, it follows by Lemma 7.1.19 that $|N_{I_2}(C_1)| \leq |V(C_1)|$ and $|N_{I_2}(C_2)| \leq |V(C_2)|$. Since $D^c_{xy} \succ_c I_2$, $|I_2| \leq |N_{I_2}(C_1)| + |N_{I_2}(C_2)|$. Thus

$$p = |I_2| \leq |N_{I_2}(C_1)| + |N_{I_2}(C_2)| \leq |V(C_1)| + |V(C_2)| \leq |D^c_{xy}|$$

as required. This completes the proof. □

Let $P_{s_1} = u_1, u_2, u_3, ..., u_{s_1+1}, P_{s_2} = v_1, v_2, v_3\ , ...v_{s_2+1}$ and $P_{s_3} = w_1, w_2, w_3, ..., w_{s_3+1}$ be three disjoint paths of length s_1, s_2 and s_3 respectively. Hereafter, let N_{s_1,s_2,s_3} be the net constructed by adding so that the vertices in $\{u_{s_1+1}, v_{s_2+1}, w_{s_3+1}\}$ form a clique K_3. We also use the graph $P_{3,3,3}$ in the class $\mathscr{P}$ from Section 2.2 of Chapter 2.

We are ready to give the second proof of Theorem 7.1.1.

Proof. We first show that G is $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}-$free graph. Clearly G is a claw-free graph. Suppose to the contrary that G contains $N_{1,2,2}$ as an induced subgraph. Consider $G + v_1w_1$. By Lemma 2.1.15(1), $|D^c_{v_1w_1}| \leq 3$. Moreover, Lemma 2.1.15(2) gives $\{v_1, w_1\} \cap D^c_{v_1w_1} \neq \emptyset$. If $\{v_1, w_1\} \subseteq D^c_{v_1w_1}$, then $\{w_3, u_1, v_1, w_1\}$ is an independent set of size 4 of G containing both v_1 and w_1. By Lemma 7.1.21, $k \geq 5$, a contradiction. Thus $|\{v_1, w_1\} \cap D^c_{v_1w_1}| = 1$. By symmetry let $v_1 \in D^c_{v_1w_1}$. Thus $\{v_1, v_3, u_1, w_2\}$ is an independent set of size 4 of G containing v_1. Lemma 7.1.21 thus implies $k \geq 5$, a contradiction. Thus G is an $N_{1,2,2}-$free graph. By similar arguments, G is an $N_{1,1,3}-$free graph. Therefore G is a $\{K_{1,3}, N_{1,2,2}, N_{1,1,3}\}-$free graph. It is not difficult to see that $\gamma_c(P_{3,3,3}) = 5$. Thus G is not isomorphic to $P_{3,3,3}$. To show that G is a hamiltonian graph, by Theorem 2.2.4, it remains to show that $cl(G) \notin \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$.

Suppose to the contrary that $cl(G) \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$. Consider the case $cl(G) \in \mathscr{F}_1$. Let $q_{i,j} \in V(Q_i) \cap V(Q_j)$ where $1 \leq i \leq 2$ and $3 \leq j \leq 5$. Since $|V(Q_j)| \geq 3$ for all $3 \leq j \leq 5$, there exist $x \in V(Q_3) - \{q_{1,3}, q_{2,3}\}, y \in V(Q_4) - \{q_{1,4}, q_{2,4}\}$ and $z \in V(Q_5) - \{q_{1,5}, q_{2,5}\}$. Clearly, $E(G) \subseteq E(cl(G))$. Thus $xy \notin E(G)$ and z is not adjacent to any vertex in $V(G) - V(Q_5)$. Consider $G + xy$. Lemma 2.1.15(1) yields $|D^c_{xy}| \leq 3$, moreover, Lemma 2.1.15(2) implies that $\{x, y\} \cap D^c_{xy} \neq \emptyset$. By symmetry let $x \in D^c_{xy}$. To dominate z, $D^c_{xy} \cap V(Q_5) \neq \emptyset$. By the connectedness of $(G + xy)[D^c_{xy}]$, $\{q_{i,3}, q_{i,5}\} \subseteq D^c_{xy}$ where $1 \leq i \leq 2$. Thus $D^c_{xy} = \{x, q_{i,3}, q_{i,5}\}$ for some i. We see that D^c_{xy} does not dominate $q_{3-i,4}$, a contradiction. Thus $cl(G) \notin \mathscr{F}_1$.

We now consider the case when $cl(G) \in \mathscr{F}_2$. Since $|V(R_i)| \geq 3$ for all $1 \leq i \leq 2$, there exist $x \in V(R_1) - \{c_1, c'_1\}$ and $y \in V(R_2) - \{c_2, c'_2\}$. Since $E(G) \subseteq E(cl(G))$, $xy \notin E(G)$ and r is not adjacent to any vertex in $V(G) - (V(R_3) \cup \{c'_3\})$. Consider $G + xy$. Similarly, $|D^c_{xy}| \leq 3$ and $\{x, y\} \cap D^c_{xy} \neq \emptyset$. By symmetry let $x \in D^c_{xy}$. Thus $|D^c_{xy} - \{x\}| \leq 2$. To dominate r, $D^c_{xy} \cap (V(R_3) \cup \{c'_3\}) \neq \emptyset$. By the connectedness of $(G + xy)[D^c_{xy}]$, D^c_{xy} is $\{x, c_1, c_3\}$ or $\{x, c'_1, c'_3\}$. But $\{x, c_1, c_3\}$ does not dominate c'_2 and $\{x, c'_1, c'_3\}$ does not dominate c_2, a contradiction. Thus $cl(G) \notin \mathscr{F}_2$.

For the case $cl(G) \in \mathscr{F}_3$, let $s'' \in V(F) - \{s, s'\}$. Since $E(G) \subseteq E(cl(G))$, $c_2c_5 \notin E(G)$ and s'' is not adjacent to any vertex in $V(G) - V(F)$. Consider $G + c_2c_5$. Similarly, Lemma 2.1.15(1) and (2) thus implies $|D^c_{c_2c_5}| \leq 3$ and $\{c_2, c_5\} \cap D^c_{c_2c_5} \neq \emptyset$. Without loss of generality let $c_2 \in D^c_{c_2c_5}$. We have that $V(F) \cap D^c_{c_2c_5} \neq \emptyset$ to dominate s''. Thus

$D^c_{c_2c_5}$ is $\{c_2,c_1,s\}$ or $\{c_2,c_3,s'\}$. But $\{c_2,c_1,s\}$ does not dominate c_4 and $\{c_2,c_3,s'\}$ does not dominate c_6, a contradiction. Thus $cl(G) \notin \mathscr{F}_3$.

Clearly $cl(G) \notin \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ and so, Theorem 2.2.4 yields that G is a hamiltonian graph. This completes the second proof of Theorem 7.1.1. □

By Theorems 3.1.1 and 7.1.1, we have the following corollary.

Corollary 7.1.22. Let G be a $2-$connected $4-\gamma_t-$edge critical claw-free graph. Then G is hamiltonian.

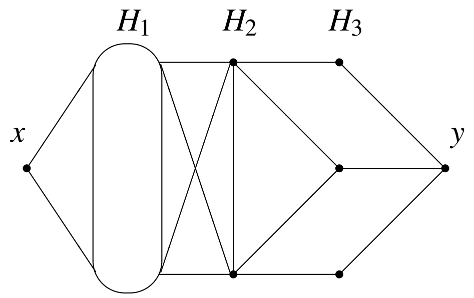

Figure 7.4 : A graph $H^2_4(n_1,2,3)$

Note that a graph $H^2_4(n_1,2,3)$ (in Section 6.3.1 of Chapter 6) is a $2-$connected $4-\gamma_c-$edge critical $K_{1,4}-$free graph which is non-hamiltonian but contain $K_{1,3}$. Hence, the claw-free condition in Theorem 7.1.1 and Corollary 7.1.22 are best possible.

7.1.3 $k-\gamma_c-$Edge Critical Claw-free Graphs for $k \geq 5$

We have observed that the graph P_{n_1,n_2,n_3} in the class $\mathscr{P}$ in Section 2.2 of Chapter 2 is a $k-\gamma_c-$edge critical claw-free graph.

Let

$\mathscr{K}^{\gamma_c}_k$: the class of $k-\gamma_c-$edge critical claw-free non-hamiltonian graphs of connectivity two.

The following theorem shows that $\mathscr{K}^{\gamma_c}_k \neq \emptyset$ for $k \geq 5$.

Theorem 7.1.23. For $n_1+n_2+n_3-4=k \geq 5$ and $n_i \geq 3$ for all $1 \leq i \leq 3$, $P_{n_1,n_2,n_3} \in \mathscr{K}^{\gamma_c}_k$.

Proof. For a fixed $k \geq 5$, let $G = P_{n_1,n_2,n_3}$ be such that $n_1+n_2+n_3-4 = k$ and $n_i \geq 3$ for all $1 \leq i \leq 3$. We, further, let $P = x_1, x_2, ..., x_{n_1}, P' = y_1, y_2, ..., y_{n_2}$ and $P'' = z_1, z_2, ..., z_{n_3}$.

We first show that $\gamma_c(G) = k$. We see that $V(P) \cup (V(P') - \{y_{n_2-1}, y_{n_2-2}\}) \cup (V(P'') - \{z_{n_3-1}, z_{n_3-2}\}) \succ_c G$. Thus $\gamma_c(G) \leq n_1+n_2+n_3-4 = k$. Let D be a γ_c-set of G. We distinguish two cases according to D.

Case 1 : $\{x_1, x_{n_1}, y_1, y_{n_2}, z_1, z_{n_3}\} \not\subseteq D$.

Without loss of generality let $x_{n_1} \notin D$. To dominate x_{n_1-1} and by the connectedness of $G[D]$, we have that $\{x_1, ..., x_{n_1-2}\} \subseteq D$. If neither y_{n_2} nor z_{n_3} is in D, then, to dominate y_{n_2} and z_{n_3}, $\{y_1, y_2, ..., y_{n_2-1}\} \cup \{z_1, z_2, ..., z_{n_3-1}\} \subseteq D$. Thus $|D| \geq (n_1-2)+(n_2-1)+(n_3-1) = k$. We now consider the case when $y_{n_2} \in D$ or $z_{n_3} \in D$. Since $x_1 \in D$ and $G[D]$ is connected, it follows that $V(P') \subseteq D$ or $V(P'') \subseteq D$. Suppose that $V(P') \subseteq D$. Clearly $V(P'') \cap D \neq \emptyset$ to dominate $\{z_2, z_3, ..., z_{n_3-1}\}$. As $G[D]$ is connected, we obtain z_1 or z_{n_3} is in D. We have three more subcases.

Subcase 1.1 : $z_1 \in D$ but $z_{n_3} \notin D$.

Thus $z_{n_3-2} \in D$ to dominate z_{n_3-1}. By the connectedness of $G[D]$, $\{z_1, z_2, ..., z_{n_3-2}\} \subseteq D$. Clearly $\{x_1, ..., x_{n_1-2}\} \cup V(P') \cup \{z_1, z_2, ..., z_{n_3-2}\} \subseteq D$. Therefore $|D| \geq (n_1-2)+n_2+(n_3-2) = k$. Thus $|D| = k$ as required. This proves Subcase 1.1.

Subcase 1.2 : $z_1 \notin D$ but $z_{n_3} \in D$.

Thus $z_3 \in D$ to dominate z_2. Similarly, $\{z_3, z_4, ..., z_{n_3}\} \subseteq D$ and so $|D| \geq k$. Thus $|D| = k$ as required. This proves Subcase 1.2.

Subcase 1.3 : $z_1, z_{n_3} \in D$.

Let j be the maximum integer that $G[D \cap \{z_1, z_2, ..., z_j\}]$ is connected. As $z_{n_3} \in D$, by the minimality of $|D|$, we obtain $j \leq n_3 - 3$. Moreover, $z_{j+3} \in D$ to dominate z_{j+2}. The connectedness of $G[D]$ yields that $z_i \in D$ for all $j+3 \leq i \leq n_3$. We have $\{x_1, x_2, ..., x_{n_1-2}\} \cup V(P') \cup \{z_1, z_2, ..., z_j\} \cup \{z_{j+3}, z_{j+4}, ..., z_{n_3}\} \subseteq D$. Therefore $|D| \geq (n_1-2)+n_2+j+(n_3-(j+2)) = k$ and $|D| = k$.

We can show that $\gamma_c(G) = k$ when $V(P'') \subseteq D$ by the same arguments, thus establishing Case 1.

Case 2 : $\{x_1, x_{n_1}, y_1, y_{n_2}, z_1, z_{n_3}\} \subseteq D$.

Since $G[D]$ is connected, at least one of the sets $V(P), V(P')$ or $V(P'')$ is contained in D. Without loss of generality let $V(P) \subseteq D$. Let l and j be the maximum integers such that $G[D \cap \{y_1, y_2, ..., y_l\}]$ and $G[D \cup \{z_1, z_3, ..., z_j\}]$ are both connected. By the similar arguments as Subcase 1.3, $V(P) \cup \{y_1, y_2, ..., y_l\} \cup \{y_{l+3}, y_{l+4}, ..., y_{n_2}\} \cup \{z_1, z_2, ..., z_j\} \cup \{z_{j+3}, z_{j+4}, ..., z_{n_3}\} \subseteq D$. Therefore $|D| \geq n_1 + l + (n_2-(l+2)) + j + (n_3-(j+2)) = k$. We have that $|D| = k$, thus establishing Case 2.

Cases 1 and 2 give $\gamma_c(G)=4$.

We next establish the criticality. Let $u,v\in V(G)$ such that $uv\notin E(G)$. Clearly $u\in V(P)\cup V(P')\cup V(P'')$. Without loss of generality let $u\in V(P)$. Thus $x_i=u$ for some $1\leq i\leq n_1$. If $v\in V(P)$, then $v\in x_j$ for some $1\leq j\leq n_1$ and $|i-j|\geq 2$. By symmetry, we suppose that $i<j$. Let $Y=V(P')-\{y_1,y_2\}$ and $Z=V(P'')-\{z_1,z_2\}$. Clearly $(V(P)-\{x_{i+1}\})\cup Y\cup Z\succ_c G+uv$. Thus $\gamma_c(G+uv)\leq k-1=(n_1-1)+(n_2-2)+(n_3-2)$. We then suppose that $v\in V(P')\cup V(P'')$. Without loss of generality let $v\in V(P')$. If $v=y_1$, then $vx_1\in E(G)$. Because $uv\notin E(G)$, it follows that $i>1$. Thus $(V(P)-\{x_1\})\cup(V(P')-\{y_2,y_3\})\cup Z\succ_c G+uv$. Thus $\gamma_c(G+uv)\leq k-1=(n_1-1)+(n_2-2)+(n_3-2)$. When $v=y_{n_2}$, similarly, Thus $\gamma_c(G+uv)\leq k-1=(n_1-1)+(n_2-2)+(n_3-2)$. We now consider the case when $v\in\{y_2,y_3,...,y_{n_2-1}\}$. If $n_2=3$, then $V(P)\cup Z\succ_c G$. We then suppose that $n_2\geq 4$. We, further, let $Y_1=\{y_2,y_3,...,y_{n_2-2}\}$ if $v=y_2$ and $Y_1=\{y_3,y_4,...,y_{n_2-1}\}$ otherwise. This implies that $V(P)\cup Y_1\cup Z\succ G$, that is, $\gamma_c(G+uv)\leq k-1=n_1+(n_2-3)+(n_3-2)$.

Therefore G is a $k-\gamma_c-$edge critical graphs. By Theorem 2.2.3, $G\in\mathscr{K}_k^{\gamma_c}$, as required. □

We note by Corollary 6.3.2 that $\mathscr{G}_k^{\gamma_c}=\emptyset$ if and only if $k=1,2$ or 3. Thus $\mathscr{K}_k^{\gamma_c}=\emptyset$ when $k=1,2$ or 3 because $\mathscr{K}_k^{\gamma_c}\subseteq\mathscr{G}_k^{\gamma_c}$. Theorem 7.1.1 also implies that $\mathscr{K}_4^{\gamma_c}=\emptyset$. Thus, Theorem 7.1.23 implies the following corollary.

Corollary 7.1.24. $\mathscr{K}_k^{\gamma_c}=\emptyset$ if and only if $k=1,2,3$ or 4.

We have observed that the graph P_{n_1,n_2,n_3} has the connectivity two. We now turn attention to 3−connected graphs. We conclude this section with the following theorem.

Theorem 7.1.25. Let G be a 3−connected $k-\gamma_c-$edge critical claw-free graph. If $1\leq k\leq 6$, then G is hamiltonian.

Proof. Suppose that G contains $N_{3,3,3}$ as an induced subgraph. Consider $G+u_1v_1$. Lemma 2.1.15(2) yields that $\{u_1,v_1\}\cap D^c_{u_1v_1}\neq\emptyset$. If $\{u_1,v_1\}\subseteq D^c_{u_1v_1}$, then $\{u_1,v_1,u_3,v_3,w_1,w_3\}$ is an independent set in G. It follows from Lemma 7.1.21 that $k\geq 7$, a contradiction. Therefore $|\{u_1,v_1\}\cap D^c_{u_1v_1}|=1$. By symmetry let $u_1\in D^c_{u_1v_1}$. Clearly $\{u_1,u_3,v_2,v_4,w_1,w_3\}$ is an independent set. Lemma 7.1.21 gives that $k\geq 7$, a contradiction. Therefore G is an $N_{3,3,3}-$free graph. Theorem 2.2.5 thus implies G is a hamiltonian graph. □

7.2 $k-\mathscr{D}-$Edge Critical Claw-free Graphs where $\mathscr{D}\in\{\gamma_t,\gamma,i\}$

By Corollary 7.1.22, we have that all $2-$connected $4-\gamma_t-$edge critical claw-free graphs are hamiltonian. In Chapter 6, we have shown by giving the construction of the class $\mathscr{T}$ that $2-$connected $5-\gamma_t-$edge critical graphs need not be hamiltonian. However, if these graphs are $3-$connected and claw-free, then they are hamiltonian.

Observation 7.2.1. Let G be a $k-\gamma_t-$edge critical graph and let u,v be a pair of non-adjacent vertices of G. We, moreover, let D^t_{uv} be a γ_t-set of $G+uv$. If $\{u,v\}\subseteq D^t_{uv}$, then at least one of u or v is an isolated vertex of $G[D^t_{uv}]$.

Proof. Let $H_1,H_2,...,H_p$ be the components of $(G+vu)[D^t_{uv}]$. Thus $D^t_{uv}=\cup_{i=1}^{p}V(H_i)$. Without loss of generality let $\{u,v\}\subseteq V(H_1)$. If H_1 is connected in G, then $D^t_{uv}\succ_t G$ contradicting the minimality of k. Thus H_1 has two components $H_{1,1}$ and $H_{1,2}$ containing u and v, respectively. Clearly, $V(H_1)=V(H_{1,1})\cup V(H_{1,2})$. If $|V(H_{1,1})|>1$ and $|V(H_{1,2})|>1$, then $D^t_{uv}\succ_t G$, a contradiction. Thus at least one of them is a singleton component. This implies that at least one of u or v is an isolated vertex of $G[D^t_{uv}]$ and we finish proving this lemma. □

Lemma 7.2.2. Let G be a claw-free graph, Y be a vertex subset of G and I be an independent set such that $Y\succ_t I$ and $Y\cap I\neq\emptyset$. Then $|I|\leq|Y|+\lfloor\frac{|Y|}{2}\rfloor-1$.

Proof. Let $H_1,H_2,...,H_p$ be the components of $G[Y]$. Therefore $Y=\cup_{i=1}^{p}V(H_i)$ and $|Y|=\sum_{i=1}^{p}|V(H_i)|$. Because $Y\succ_t I$, $|V(H_i)|\geq 2$. Therefore $p\leq\lfloor\frac{|Y|}{2}\rfloor$. Without loss of generality let $V(H_1)\cap I\neq\emptyset$. Since H_i is connected, $V(H_i)\succ_c N_I(H_i)$. Lemma 7.1.19 thus implies $|N_I(H_1)|\leq|V(H_1)|$ and $|N_I(H_i)|\leq|V(H_i)|+1$ for $2\leq i\leq p$. Therefore

$$|I|\leq\sum_{i=1}^{p}|N_I(H_i)|\leq\sum_{i=1}^{p}|V(H_i)|+p-1\leq|Y|+\lfloor\frac{|Y|}{2}\rfloor-1$$

as required. □

Theorem 7.2.3. Let G be a $3-$connected $k-\gamma_t-$edge critical claw-free graph. If $2\leq k\leq 5$, then G is hamiltonian.

Proof. Suppose that G contains $N_{3,3,3}$ as an induced subgraph. Consider $G+u_1v_1$. Let $D^t_{u_1v_1}$ be a γ_t-set of $G+u_1v_1$.

Claim : $|D^t_{u_1v_1}|\geq 5$.

The minimality of k gives $D^t_{u_1v_1}\cap\{u_1,v_1\}\neq\emptyset$. We distinguish two cases.

Case 1 : $\{u_1,v_1\}\subseteq D^t_{u_1v_1}$.

Let $I_1=\{u_1,u_3,v_1,v_3,w_1,w_3\}$. We see that I_1 is an independent set in G, moreover,

$D^t_{u_1v_1} \succ_t I_1$ and $D^t_{u_1v_1} \cap I_1 \neq \emptyset$. By Observation 7.2.1, suppose without loss of generality that that u_1 is an isolated vertex of $G[D^t_{u_1v_1}]$.

Subcase 1.1 : v_1 is not an isolated vertex of $G[D^t_{u_1v_1}]$.

Therefore $D^t_{u_1v_1} - \{u_1\} \succ_t I_1 - \{u_1\}$ and $v_1 \in (D^t_{u_1v_1} - \{u_1\}) \cap (I_1 - \{u_1\})$. As $|I_1 - \{u_1\}| = 5$, by Lemma 7.2.2, we obtain $|D^t_{u_1v_1} - \{u_1\}| \geq 4$. This implies that $|D^t_{u_1v_1}| \geq 5$, as required.

Subcase 1.2 : v_1 is also an isolated vertex of $G[D^t_{u_1v_1}]$.

Thus $D^t_{u_1v_1} - \{u_1, v_1\} \succ_t I_1 - \{u_1, v_1\}$. If $(G+u_1v_1)[D^t_{u_1v_1} - \{u_1, v_1\}]$ is connected, then by Lemma 7.1.19, $|D^t_{u_1v_1} - \{u_1, v_1\}| \geq 3$. But if it is not connected, then each component of $(G+u_1v_1)[D^t_{u_1v_1} - \{u_1, v_1\}]$ has at least two vertices. Thus $|D^t_{u_1v_1} - \{u_1, v_1\}| \geq 4$. These imply that $|D^t_{u_1v_1}| \geq 5$, as required.

Case 2 : $|\{u_1, v_1\} \cap D^t_{u_1v_1}| = 1$.

Without loss of generality let $u_1 \in D^t_{u_1v_1}$. Let $I_2 = \{u_1, u_3, v_2, v_4, w_1, w_3\}$. Clearly I_2 is an independent set of six vertices and $D^t_{u_1v_1} \cap I_2 \neq \emptyset$. By Lemma 7.2.2, $|D^t_{u_1v_1}| \geq 5$. In view of Cases 1 and 2, $|D^t_{u_1v_1}| \geq 5$, thus establishing the claim.

Since $|D^t_{u_1v_1}| < k$, it follows by the claim that $k \geq 6$ violates $k \leq 5$. Hence G is an $N_{3,3,3}-$free graph. As a consequence of Theorem 2.2.5, G is a hamiltonian graph. □

On $k-i-$edge critical graphs, we have showed in Chapter 6 that $2-$connected $k-i-$edge critical graphs need not be hamiltonian for $k \geq 3$. We can prove in this section that, for $k = 4$, if these graphs are claw-free and $3-$connected, then they are hamiltonian. On the one hand, although Conjecture B3 (in Chapter 6) was disproved when $k = 4$, it can be proved that if those $4-\gamma-$edge critical graphs are claw-free, then they are hamiltonian.

Theorem 7.2.4. Let G be a $3-$connected $4-\gamma-$edge critical claw-free graph. Then G is hamiltonian.

Proof. By Theorem 2.2.5, it suffices to show that G is an $N_{3,3,3}-$free graph. Suppose to the contrary that G contains $N_{3,3,3}$ as an induced subgraph. Consider $G+u_1v_1$. By Lemma 2.1.3(1), $D_{u_1v_1} < k = 4$. Thus by Lemma 2.1.3(2), $|D_{u_1v_1} \cap \{u_1, v_1\}| = 1$. Suppose without loss of generality that $u_1 \in D_{u_1v_1}$. Note that $\{u_1, u_3, v_2, v_4, w_1, w_3\}$ is an independent set of size 6 of G. Thus $D_{u_1v_1} - \{u_1\} \succ \{u_3, v_2, v_4, w_1, w_3\}$. Since $|D_{u_1v_1} - \{u_1\}| \leq 2$, by the pigeonhole principle, there is a vertex $v \in D_{u_1v_1} - \{u_1\}$ adjacent to three vertices of $\{u_3, v_2, v_4, w_1, w_3\}$. Then G contains a claw as an induced subgraph, a contradiction. This completes the proof of Theorem 7.2.4. □

By Theorems 7.2.4 and 3.2.1, we have the following theorem.

Theorem 7.2.5. Let G be a $3-$connected $4-i-$edge critical claw-free graph. Then G is hamiltonian.

Finally, by Theorem 7.2.4, we have the following corollary.

Corollary 7.2.6. The Conjecture B3 is true when $k = 4$ under the condition that the graphs are claw-free.

CHAPTER 8

Future Work and Open Problems

In this final chapter, we would like to discuss more open problems which arise from results in chapters 3-7. We present this according to the chapters.

Chapter 3

In view of Theorem 3.1.2 that $\mathbb{T}_k^e - \mathbb{C}_k^e \neq \emptyset$, it would be interesting to characterize properties of graphs in the class $\mathbb{T}_k^e - \mathbb{C}_k^e$. In particular, when $k = 5$, could we characterize the structures of all graphs in $\mathbb{T}_k^e - \mathbb{C}_k^e$? We could also ask the same questions for those graphs in the class $\mathbb{T}_k^v - \mathbb{C}_k^v$.

A graph G is *domination perfect* if for each induced subgraph H of G, $\gamma(H) = i(H)$. Fulman [68] established a class of graphs, apart from $K_{1,3}$, for which if a graph G does not contain any of these graphs as an induced subgraph, then G is domination perfect. This gives rise to the following problem : Does there exists any graph H, apart from $K_{1,3}$, such that the class of $k-\gamma-$edge critical graphs and the class of $k-i-$edge critical graphs are the same when they are $H-$free?

Chapter 4

We have shown in Theorem 4.1.6 that the upper bound is sharp for all $k = 2, 3$ or 4 when Δ is even. We are interested in investigating the existence of $k-\gamma_c-$vertex critical graphs achieving the upper bound $(\Delta-1)(k-1)+3$ for $k \geq 5$ and $\Delta \geq 3$ is odd.

As a consequence of Theorems 4.1.5 and 4.1.7 that for $k \geq 5$ and $\Delta \geq 3$, the order of $k-\gamma_c-$vertex critical graphs with maximum degree Δ is bounded by $\Delta+k+1$ and $(\Delta-1)(k-1)+3$, it would be interesting to characterize all $k-\gamma_c-$vertex critical graphs of order $\Delta+k+1$.

We are, moreover, interested in the realizability of $k-\gamma_c-$vertex critical graphs whose order is between these bounds. This gives rise to the following problem : determine the relationship of n, Δ and k to guarantee the existences of $k-\gamma_c-$vertex critical graphs of order n where $\Delta+k+1 \leq n \leq (\Delta-1)(k-1)+3$ with maximum degree Δ.

Chapter 5

We would like to investigate whether or not there exists a maximal $3-\gamma_c-$vertex critical graph G of even order and a cut set S of G which $\omega_o(G-S)+2 \geq |S|$. This result together with Tutte's Theorem imply that every maximal $3-\gamma_c-$vertex critical graph of even order is $4-$factor critical with, probably, some exceptions.

Chapter 6

We first establish a link between $k-\gamma_c-$edge critical non-hamiltonian graphs and connectivity, order and independence number.

For our graph $H_k^l(n_1,n_2,...,n_{k-1})$, we observe that the set of vertices $\{w_1,w_2,w_3,...,w_p\} \cup \{h_1,h_3,h_5,...,h_{k-5},h_{k-3}\}$ and $\{w_1,w_2,w_3,...,w_p\} \cup \{x,h_2,h_4,...,h_{k-5},h_{k-3}\}$ where $h_i \in V(H_i)$ and $1 \leq i \leq k-3$ form independent sets of size $p+\lceil \frac{k}{2} \rceil -1$ for even and odd k, respectively. Hence

$$\alpha(H_k^l(n_1,n_2,...,n_{k-1})) \geq p+\lceil \tfrac{k}{2} \rceil -1.$$

Further, observe that

$$\kappa(H_k^l(n_1,n_2,...,n_{k-1})) = min\{n_i : 1 \leq i \leq k-2\}$$

because $n_{k-1} = n_{k-2}+1$.

For $r \geq 1$, we define the graph $H_k^l(n_1,n_2,...,n_{k-1}:r)$ on $n_1+n_2+...+n_{k-1}+r+1$ vertices as follows. We add a new component K_{r-1} to $H_k^l(n_1,n_2,...,n_{k-1})$ and join each of these $r-1$ new vertices to each vertex in $N[w_1]$. Clearly

$$\alpha(H_k^l(n_1,n_2,...,n_{k-1}:r)) = \alpha(H_k^l(n_1,n_2,...,n_{k-1})) \text{ and}$$

$$\kappa(H_k^l(n_1,n_2,...,n_{k-1}:r)) = \kappa(H_k^l(n_1,n_2,...,n_{k-1})).$$

Note that $H_k^l(n_1,n_2,...,n_{k-1}:r) = H_k^l(n_1,n_2,...,n_{k-1})$ when $r=1$.

Further, observe that if we take $n_2 = n_3 = ... = n_{k-3} \geq n_{k-2} = n_1 = p-1$ and $r \geq 1$, then $p-1 = n_1 = min\{n_i : 1 \leq i \leq k-2\}$. Therefore

$$\kappa(H_k^l(p-1,n_2,n_3,...,n_{k-1}:r)) = p-1.$$

Since $\alpha(H_k^l(p-1,n_2,n_3,...,n_{k-1}:r)) \geq p+\lceil \frac{k}{2} \rceil -1$, it follows that

$$\kappa(H_k^l(p-1,n_2,n_3,...,n_{k-1}:r)) \leq \alpha(H_k^l(p-1,n_2,n_3,...,n_{k-1}:r)) - \lceil \tfrac{k}{2} \rceil.$$

Moreover, for all $0 \leq i \leq p-3$, if we let $G_i = H_k^l(p-1-i,n_2,n_3,...,n_{k-1}:r+i)$, then

$\kappa(G_i) = p-1-i$ and $\alpha(G_0) = \alpha(G_1) = ... = \alpha(G_{p-3}) \geq p + \lceil \frac{k}{2} \rceil - 1.$

We must also have

$\kappa(G_i) \leq \alpha(G_i) - \lceil \frac{k}{2} \rceil.$

This implies that, for a fixed $n \geq \Sigma_{i=1}^{k-1} n_i + 2$ and $\alpha = p + \lceil \frac{k}{2} \rceil - 1$, there exists a $k - \gamma_c-$edge critical non-hamiltonian graph of order n with the independence number α and the connectivity κ for all $2 \leq \kappa \leq \alpha - \lceil \frac{k}{2} \rceil$. It was proved (see Theorem 2.2.1) by Chvátal and Erdös [57] that if a graph G satisfies $\kappa \geq \alpha$, then G is hamiltonian. Hence the case which is left for consideration is $\alpha - \lceil \frac{k}{2} \rceil + 1 \leq \kappa \leq \alpha - 1$.

On the other hand, since a graph $H_k^l(n_1, n_2, ..., n_{k-1} : 1)$ has the connectivity $\kappa = min\{n_i : 1 \leq i \leq k-2\}$, it follows that $n = \Sigma_{i=1}^{k-1} n_i + 2 \geq (k-1)\kappa + 3$. Thus

$\kappa \leq \lfloor \frac{n-3}{k-1} \rfloor.$

Note that the condition $n_1 = n_2 = ... = n_{k-2}$ implies $\kappa = \lfloor \frac{n-3}{k-1} \rfloor$, that is, $n_1 = n_2 = ... = n_{k-2} = \lfloor \frac{n-3}{k-1} \rfloor$. Thus, by selecting $n_2 = n_3 = ... = n_{k-2}$ and $n_1 = t, 2 \leq t \leq \lfloor \frac{n-3}{k-1} \rfloor$, then the graph $H_k^l(t, n_2, n_3, ..., n_{k-1} : 1 + \lfloor \frac{n-3}{k-1} \rfloor - t)$ has connectivity t and is $k - \gamma_c-$critical non-hamiltonian graph. So this leaves $\lfloor \frac{n-3}{k-1} \rfloor + 1 \leq \kappa$ as the unresolved case.

The question is :

Is every $k - \gamma_c-$critical graph with $\alpha - \lceil \frac{k}{2} \rceil + 1 \leq \kappa \leq \alpha - 1$ and $\lfloor \frac{n-3}{k-1} \rfloor + 1 \leq \kappa$ hamiltonian?

Observe that, when $k = 4$, we have that $\alpha - 1 = \alpha - \lceil \frac{k}{2} \rceil + 1 \leq \kappa \leq \alpha - 1$. So the only case to consider is $\alpha - 1 = \kappa$. That is :

Is every $4 - \gamma_c-$critical graph with $\lfloor \frac{n-3}{3} \rfloor + 1 \leq \kappa = \alpha - 1$ hamiltonian?

We, moreover, observed that all graphs in the class $\mathscr{H}_k^l$ contain a hamiltonian path. However, we have found $k - \gamma_c-$edge critical graphs for $k \geq 7$ which do not contain a hamiltonian path. For $k \geq 7$, let $P_{k-5} = a_1, a_2, ..., a_{k-5}$ be a path of length $k-6$, $C_5 = c_1, c_2, c_3, c_4, c_5, c_1$ and $C_5' = c_1', c_2', c_3', c_4', c_5', c_1'$ be two cycles of length five. A graph $\mathscr{R}(k)$ is obtained from P_{k-5}, C_5 and C_5' by adding edges c_2c_5 and $c_2'c_5'$ and inducing $\{a_1, c_1, c_1'\}$ as a clique. The following figure illustrates a graph $\mathscr{R}(k)$.

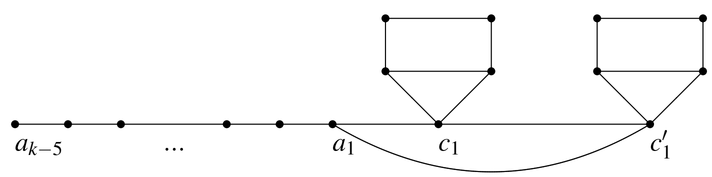

Figure 8.1 : A graph $\mathscr{R}(k)$

Observe that this graph is a $k-\gamma_c-$edge critical graph where $k \geq 7$ does not contain a hamiltonian path. This gives rise to the following problem : Does every $k-\gamma_c-$edge critical graph for $4 \leq k \leq 6$ contain a hamiltonian path.

For $k-\gamma_t-$edge critical graphs, we provided such graphs which are non-hamiltonian when $k = 4$ or 5. However, we could not determine if every $k-\gamma_t-$edge critical graph is hamiltonian or not for $k \geq 6$. It would be interesting to investigate the hamiltonian property of these graphs.

Chapter 7

We think that the idea of the proof in Theorem 7.1.25 still can be used to show that every $3-$connected $k-\gamma_c-$edge critical claw-free graph is hamiltonian when $k = 7, 8$ or 9.

When $k = 10$, we construct some class $3-$connected $k-\gamma_c-$edge critical claw-free graphs which cannot apply the idea of the Theorem 7.1.25. For integers $s_1, s_2 \geq 1, s_3 \geq 2$ and $s_1 + s_2 + s_3 = 9$, define

$$Z^1_{s_1} = K_{n^1_1} \vee K_{n^1_2} \vee ... \vee K_{n^1_{s_1+2}} \vee K_{n^1_{s_1+3}}$$

such that $n^1_j = 1$ if $j = \lfloor \frac{s_1+3}{2} \rfloor$ or $\lfloor \frac{s_1+3}{2} \rfloor + 1$ and $n^1_j \geq 2$ otherwise. Moreover, for $2 \leq i \leq 3$, define

$$Z^i_{s_i} = K_{n^i_1} \vee K_{n^i_2} \vee ... \vee K_{n^i_{s_i+1}} \vee K_{n^i_{s_i+2}}$$

such that $n^i_j = 1$ if $j = \lfloor \frac{s_i+2}{2} \rfloor$ or $\lfloor \frac{s_i+2}{2} \rfloor + 1$ and $n^i_j \geq 2$ otherwise.

For $s_1, s_2 \geq 1, s_3 \geq 2$ and $s_1 + s_2 + s_3 = 9$, let $\mathscr{X}(s_1, s_2, s_3)$ be the graph constructed from $Z^1_{s_1}, Z^2_{s_2}$ and $Z^3_{s_3}$ by adding edges so that the vertices in $V(K_{n^1_1}) \cup V(K_{n^1_{s_1+3}}) \cup V(K_{n^2_1}) \cup V(K_{n^2_{s_2+2}}) \cup V(K_{n^3_1}) \cup V(K_{n^3_{s_3+2}})$ form a clique.

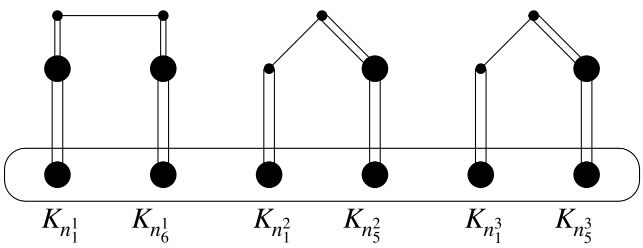

Figure 8.2 : The graph $\mathscr{X}(3,3,3)$

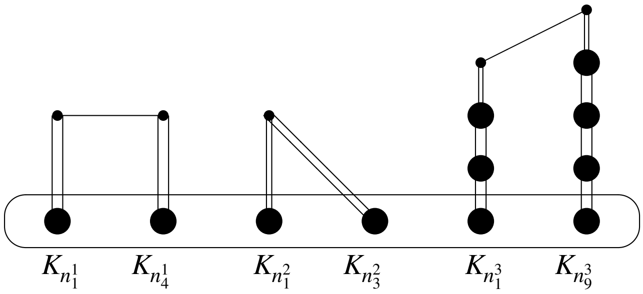

Figure 8.3 : The graph $\mathscr{X}(1,1,7)$

We can show that these are $3-$connected $10-\gamma_c-$edge critical claw-free graphs containing N_{s_1,s_2,s_3} as an induced subgraph. Although these graphs are hamiltonian, but the idea of Theorem 7.1.25 cannot be applied when $k \geq 10$.

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