INCONGRUENT RESTRICTED DISJOINT COVERING SYSTEMS

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Abstract. We define a restricted disjoint covering system on $[1, n]$ as a set of congruence classes such that each integer in the interval $[1, n]$ belongs to exactly one class, and each class contains at least two members of the interval. In this paper we report some computational and structural results and present some open problems concerning such systems.

1. Introduction

We write $S(m, a)$ for the congruence class $\{x : x \equiv a \pmod{m}\}$. A *covering system of congruences* is a set of congruence classes with the property that every integer belongs to at least one class. If no integer belongs to more than one class then the system is *disjoint* (or *exact*), if the moduli of the classes are distinct then the system is *incongruent*. An example of a disjoint covering system is

$$\{S(2, 0), S(2, 1)\},$$

and an example of an incongruent system is

$$\{S(2, 0), S(3, 0), S(4, 1), S(6, 1), S(12, 11)\}.$$

It is not possible for a system to be both disjoint and incongruent. These systems were introduced by Erdös in [1] and have spawned a large literature (see the surveys [11] and section F13 in [4]). The most important unsolved problems in the area are (a) do there exist incongruent systems in which all moduli are odd? and (b) do there exist systems in which the least modulus is arbitrarily large? For (b) the strongest result so far obtained is by Morikawa [7], [8] who constructed an incongruent covering system with least modulus 24. A number of variations on these ideas have been studied: Fraenkel considered coverings by Beatty sequences and made a notable conjecture about them (see [3] and [13]), Jin and Myerson [5] considered systems based on homogeneous congruences, and other authors studied covering groups with cosets ([10] and [12]). In this paper we introduce a new variation on the theme, a *restricted* disjoint covering system.

We define a *restricted disjoint covering system on* $[1, n]$ as a set of congruence classes such that each integer in the interval $[1, n]$ belongs to exactly one class, and each class contains at least two members of the interval. The condition that the classes contain at least two members is included to avoid trivialities. As with standard covering systems we describe such a system as *incongruent* if the moduli are distinct. It is not obvious, at first sight, that any incongruent restricted disjoint covering systems (henceforth IRDCS) exist. However they do. Here is an example of an IRDCS on $[1, 11]$.

Example 1.1.

$$S(6, 1), S(9, 2), S(3, 0), S(4, 0), S(5, 0).$$

Rather than exhibiting an IRDCS in this way we can do so by writing down a sequence of n integers the ith of which equals the modulus of the unique congruence class to which i belongs. We call this the *alternate notation* for an IRDCS. Thus Example 1.1 becomes

$$6, 9, 3, 4, 5, 3, 6, 4, 3, 5, 9.$$

We see that an IRDCS on $[1, n]$ could also be defined as a sequence of integers $s_1, \ldots, s_n$ with the property that $s_i = m$ for some m if and only if $s_{i+km} = m$ for all k for which $i + km$ is in $[1, n]$, and further such that any integer appearing in the sequence appears at least twice. Defined in this way we see a parallel with Langford Sequences. A Langford Sequence [6] of order n is defined as a sequence $l_1, \ldots, l_{2n}$ of $2n$ integers in which each integer from 1 to n appears exactly twice, and such that if $l_i = l_j$ then $l_i = |i - j| - 1$; for example,

Example 1.2. $4, 1, 3, 1, 2, 4, 3, 2.$

Even the alternate notation for an IRDCS becomes unwieldy for large examples. For these we just list the moduli in order of their first appearance in the system, so that Example 1.1 becomes 6,9,3,4,5 from which the IRDCS can be easily constructed. We call this the *compact notation*.

We need three more definitions. If $\{S(m_1, a_1), \ldots, S(m_t, a_t)\}$ is an IRDCS on $[1, n]$ then n is the *length* of the system, t is its *order* and $\sum_{i=1}^{t} 1/m_i$ is its *heft*. In Example 1.1 the system has length 11, order 5 and heft 191/180=1.0611...

2. Computational and Structural Results

We have found all IRDCS with length not exceeding 32. Their properties are summarised in Table 1.

It is easily checked that if

$$\mathcal{A} = \{S(m_i, a_i), i = 1 \ldots t\}$$

is an IRDCS on $[1, n]$ then

$$\mathcal{A}' = \{S(2m_i, 2a_i), i = 1 \ldots t\} \cup \{S(2, 1)\}$$

is an IRDCS on $[1, 2n + 1]$. We call this process "doubling" the IRDCS. If $\mathcal{A}$ has length n, order t and heft h then $\mathcal{A}'$ has length $2n + 1$ and order $t + 1$. The heft of $\mathcal{A}'$ is

$$\sum_{i=1}^{t} \frac{1}{2m_i} + \frac{1}{2} = \frac{1}{2}(\sum_{i=1}^{t} \frac{1}{m_i} + 1) = \frac{1}{2}(h + 1).$$

Length	Number of IRDCS	Orders
11	2	5
17	4	7
18	6	7,8
19	18	7,8
20	14	7,8
21	26	6,7,8,9
22	84	6,8,9,10
23	88	6,8,9,10
24	46	8,9,10
25	176	8,9,10
26	380	8,9,10,11,12
27	812	8,9,10,11,12
28	844	8,9,10,11,12
29	1770	9,10,11,12,13
30	2164	9,10,11,12,13
31	5554	9,10,11,12,13,14
32	9244	9,10,11,12,13,14

TABLE 1. IRDCS with length not exceeding 32.

The doubling process can be iterated producing arbitrarily long IRDCS, with heft approaching 1 and order $O(\log n)$. Note that using the alternate notation the IRDCS so produced begins and ends with 2. We can contract it to one of length $2n$ or $2n-1$ by removing the initial or final 2. A consequence of this observation is the following theorem.

Theorem 2.1. *There exist IRDCS of all lengths greater than 16.*

Proof: We have found IRDCS of all lengths in the interval $[17, 32]$. Doubling these produces all IRDCS of odd length in the interval $[35, 65]$ and the even length IRDCS in the interval can be constructed by removing the final 2 from each of the odd length ones. An IRDCS of length 33 is constructed by removing both the initial and final 2 from the length 35 IRDCS. Iterating this process produces IRDCS of any greater length. □

If

$$\mathcal{A} = \{S(m_i, a_i), i = 1 \dots t\}$$

is an IRDCS on $[1, n]$ then so is

$$\mathcal{A}' = \{S(m_i, n + 1 - a_i), i = 1 \dots t\}$$

which we call the *reversal* of $\mathcal{A}$. In the alternate notation the reversal is obtained by simply reversing the order of the sequence. An IRDCS and its reversal are distinct (which follows from the following result) so that the number of IRDCS of any length is even.

Theorem 2.2. *No IRDCS equals its reversal.*

Proof: Say that such an IRDCS is *palindromic* and suppose such an IRDCS exists. The doubling process can be reversed - that is an IRDCS with moduli $2, m_2, \ldots, m_t$ can be replaced by one with moduli $m_2/2, \ldots, m_t/2$, and if the original IRDCS was palindromic so is its replacement. Suppose we have a palindromic IRDCS, and without loss of generality, that each of its moduli is greater than 2. If its length is even the two central numbers must belong to the same congruence class which is impossible, if it's odd then the numbers on each side of the centre belong to the same class. This class therefore has modulus 2 which is a contradiction. Hence no palindromic IRDCS exists. □

A natural question to ask concerning IRDCS is whether they exist with the least modulus arbitrarily large. As noted above the same question is asked about (unrestricted) incongruent covering systems. The best example we have found in this direction is the following length 47 IRDCS,

$$\begin{aligned}\{&S(26,1), S(30,2), S(10,3), S(17,4), S(12,5), S(24,6), S(19,7),\\ &S(14,8), S(11,9), S(15,10), S(13,11), S(16,12), S(21,14),\\ &S(18,16), S(29,18), S(20,19)\}\end{aligned}$$

whose least modulus is 10. One can also ask whether an IRDCS exists in which no modulus is even, or in which no modulus is divisible by any of the first k primes.

3. Families of IRDCS

The doubling process produces an infinite family of IRDCS. At the 2006 Western Number Theory Conference ([9]) it was asked whether other infinite families of IRDCS could be constructed. We now present a technique for producing such families.

The idea is to start with a special type of IRDCS on $[1, n]$ called a good IRDCS, defined below, remove one of its classes and use a mapping of the remaining classes to cover most of the 1 modulo 3 members of $[1, 3n]$. We then use a collection of classes with moduli of the form $3(2^i)$ to cover most of the 2 modulo 3 members of the interval. We use $S(3, 0)$ to cover all the members congruent to 0 mod 3. Each reference to "most" means all but 2, so that 4 members of the interval are not yet covered. We introduce two new classes which cover these, producing an IRDCS. This turns out also to be a good IRDCS so the process can be repeated.

We now define "good", then present the algorithm, then an example and finally a proof of correctness. The description of the algorithm contains a number of brazen assertions which are demonstrated in the proof of correctness.

Definition An IRDCS on $[1, n]$ is *good* if,
(a) n is an odd multiple of 3,
(b) if m_1 is the modulus of the class containing 1 then $m_1 > 2/3n$. With (a) this implies that

$$3m_1 \geq 2n + 3 \tag{3.1}$$

(c) $3m_1 - n - 1$ is not a power of 2,
(d) no modulus in the collection is a power of 2.

An example of a good IRDCS is given in the example below.

The Algorithm Let $\{S(m_i, a_i), i = 1 \dots t\}$ be a good IRDCS on $[1, n]$ where $a_1 = 1$, so that $m_1 > 2n/3$. We construct 4 collections of congruence classes, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, whose union is a good IRDCS on $[1, 3n]$.

Set $\mathcal{A} = \{S(3, 0)\}$.

Set $\mathcal{B} = \{S(3m_i, 3a_i - 2), i = 2 \dots t\}$. Label $x_1 = 1$ and $x_2 = 1 + 3m_1$. $\mathcal{B}$ covers all of $S(3, 1) \cap [1, 3n]$ except x_1 and x_2.

Let $\theta = \lfloor \log_2(n/3) \rfloor$ and $m = 3(2^\theta)$. It follows that $2m > n > m$ (since n is odd $3n/m$ is not an integer so the inequalities are strict), and since m and n are divisible by 3 we have

$$2m \geq n + 3 \geq m + 6. \tag{3.2}$$

We set

$$\begin{aligned} y_1 &= n + 2 \\ y_2 &= n + 2m + 2. \end{aligned} \tag{3.3}$$

Set $\mathcal{C} = \{S(3(2^i), y_1 + 3(2^{i-1})), i = 1 \dots \theta + 1\}$. It will be shown that $\mathcal{C}$ covers all of $S(3, 2) \cap [1, 3n]$ except y_1 and y_2.

Finally set $\mathcal{D} = \{S(y_2 - x_1, x_1), S(x_2 - y_1, y_1)\}$.

Then $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ is a good IRDCS on $[1, 3n]$.

Example 3.1. The collection $\{S(19, 1), S(13, 2), S(9, 3), S(5, 4), S(6, 5), S(10, 6), S(11, 7), S(17, 8), S(12, 10), S(14, 13)\}$ is a good IRDCS on $[1, 27]$. In the alternate notation it is

19 13 9 5 6 10 11 17 5 12 6 9 14 5 13 10 6 11 5 19 9 12 6 5 17 10 14

We have $n = 27$, $m_1 = 19$ (which exceeds $2n/3$, as required) so $x_1 = 1$ and $x_2 = 1 + 3(19) = 58$. Also, $\theta = \lfloor \log_2(27/3) \rfloor = 3$ and $m = 3(2^\theta) = 24$. Then $y_1 = n + 2 = 29$ and $y_2 = y_1 + 2m = 77$. Our four collections of congruence classes are then:

$$
\begin{aligned}
\mathcal{A} &= \{S(3,0)\},\\
\mathcal{B} &= \{S(39,4), S(27,7), S(15,10), S(18,13), S(30,16),\\
&\quad S(33,19), S(51,22), S(36,28), S(42,37)\},\\
\mathcal{C} &= \{S(6,32), S(12,35), S(24,41), S(48,53\},\\
\mathcal{D} &= \{S(76,1), S(29,29)\},
\end{aligned}
$$

and the new good IRDCS is their union.

Proof of Correctness: We first show that the algorithm produces an IRDCS on $[1,3n]$, then show that this IRDCS is good. To show that it's an IRDCS we must show that each member of $[1,3n]$ belongs to a congruence class in the collection, that these classes are disjoint, that their moduli are distinct and that each class contains at least two members of $[1,3n]$.

Clearly all integers congruent to 0 modulo 3 are covered by $\mathcal{C}$. Removing $S(m_1,a_1)$ from our original collection only uncovers 1 and m_1+1 since the goodness of this collection implies $m_1 > 2n/3$. It follows that $\mathcal{B}$ covers all members of $[1,3n]$ which are congruent to 1 modulo 3 except x_1 and x_2.

Now consider an integer z in $[1,3n]$ which is congruent to 2 modulo 3. This belongs to $S(3(2^i), y_1+3(2^{i-1}))$ if and only if $3(2^i)$ divides $y_1 - z + 3(2^{i-1})$, that is, if 2^i divides $(y_1-z)/3 + 2^{i-1}$.

We see that all integers congruent to 2 modulo 3 in $[1,3n]$ are covered except those for which $(y_1-z)/3$ does not belong to $S(2^i, 2^{i-1})$ for any positive $i \le \theta+1$. This happens when $(y_1-z)/3$ is divisible by $2^{\theta+1}$, that is, when

$$z_1 = y_1 + 3(2^{\theta+1})l = y_1 + 2lm$$

for some integer l. The cases $l=0$ and $l=1$ give us y_1 and y_2 respectively, which are covered by congruence classes in $\mathcal{D}$. We must show that if $l<0$ or $l>1$ we get integers outside $[1,3n]$.

If $l<0$ then, by (3.2) and (3.3), $y_1+2lm \le n+2-2m \le 0$ and if $l>1$ then

$$y_1 + 2lm \ge y_1 + 4m = n + 2 + 4m > 3n,$$

as required. So only $l=0$ and $l=1$ give integers in $[1,3n]$. It follows that every integer in $[1,3n]$ is in one of our classes.

We next show that the classes in $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}$ are disjoint.

The classes in $\mathcal{B}$ are disjoint from each other since they are derived from disjoint classes in the original collection. Each is a subset of $S(3,1)$ and so disjoint from the classes in $\mathcal{A}$ and $\mathcal{C}$ which are subsets of $S(3,0)$ and$S(3,2)$ respectively. For the same reason the classes in $\mathcal{C}$ are disjoint from that in $\mathcal{A}$. The classes in $\mathcal{C}$ are disjoint from each other by construction. We still need to consider the two classes in $\mathcal{D}$. These contain x_1, x_2, y_1 and y_2 which clearly do not belong to any class in $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$. It remains to show that these classes contain no other members of $[1,3n]$.

Consider $S(y_2 - x_1, x_1)$ and recall that $x_1 = 1$ and $y_2 = y_1 + 2m$. Clearly this class contains no member of $[1, 3n]$ less than x_1. Any member greater than y_2 is at least $2y_1 + 4m - 1$ which is greater than $3n$ by (3.2) and (3.3). Similar reasoning using (3.1) shows that $S(x_2 - y_1, y_1)$ contains exactly two members of $[1, 3n]$.

Next we show the moduli are distinct. It's clear that the moduli in $\mathcal{B}$ are distinct from each other. Similarly for the moduli in $\mathcal{C}$. The moduli in $\mathcal{B}$ have the form $3m_i$, while those in $\mathcal{C}$ have the form $3(2^i)$. These are distinct since goodness requires that no m_i is a power of 2. The moduli in $\mathcal{D}$ are $y_2 - x_1$ and $x_2 - y_1$ which are congruent to 1 and 2 modulo respectively, and so distinct from each other and from moduli in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Hence the moduli in our collection are distinct.

Finally we must show that each class has at least two members of $[1, 3n]$. This is immediate for all classes except the class $S(3(2^{\theta+1}), y_1 + 3(2^\theta)) = S(2m, y_1 + m)$ in $\mathcal{C}$, and it is easily checked that $1 \leq y_1 - m < y_1 + m \leq 3n$ so this class too contains at least 2 members of $[1, 3n]$.

We have shown our collection is an IRDCS. We now show it's good.

(a) The length $3n$ is an odd multiple of 3 since n was assumed odd.

(b) The class containing 1 is $S(y_2 - x_1, x_1)$, and

$$\begin{aligned} y_2 - x_1 &= n + 2 + 2m - 1 \\ &> 2n \end{aligned}$$

by (3.2), and so this modulus is greater than 2/3 the length of the IRDCS, as required.

(c) We must show that $3(y_2 - x_1) - 3n - 1$ is not a power of 2. After substituting we see find that this expression equals $9(2^\theta + 1) + 2$ which is clearly not a power of 2

(d) We must show that no modulus in our collection is a power of 2. This is clear for the moduli of classes in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ which are all multiples of 3. The two moduli from $\mathcal{D}$ are $y_2 - x_1$ and $x_2 - y_1$. Now $y_2 - x_1 = n + 2m + 1$ and by (3.2)

$$3m < n + 2m + 1 < 4m.$$

Now $m = 3(2^\theta)$ so $y_2 - x_1$ lies in the interval $(9(2^\theta), 12(2^\theta))$. This interval contains no power of 2. Finally $x_2 - y_1 = 3m_1 - n - 1$. This is not a power of 2 by part (c) of the definition of goodness. □

4. Bounds on Order and Heft

In this section we will write $n(\mathcal{A})$, $t(\mathcal{A})$ and $h(\mathcal{A})$ for, respectively, the length, order and heft of an IRDCS $\mathcal{A}$, and abbreviate these to n, t and h when there is no risk of confusion. We have seen that the doubling process produces IRDCS with $t(\mathcal{A}) = O(\log(n(\mathcal{A}))$, but how large can the order be in terms of n?

Theorem 4.1. *For any IRDCS $\mathcal{A}$,*

$$t(\mathcal{A}) \leq \frac{n(\mathcal{A})-1}{2} \tag{4.1}$$

with equality if and only if $n(\mathcal{A}) = 11$.

Proof: We first suppose that n is odd and write $n = 2r - 1$. The modulus covering r must be used at least 3 times. If it's used 5 or more times then the remaining $2r - 6$ or less members of the interval belong to at most $r - 3$ congruence classes and so

$$t \leq r - 2 < \frac{n}{2}$$

and we are done. That modulus can't be 3 or less, except in the case $n = 11$. We assume $n > 11$.

The modulus covering $r - 1$ must be $r - 1$ or r, ditto for the modulus covering $r + 1$. We may assume $r - 1$ covers $r - 1$, and r covers $r + 1$.

This forces the modulus covering $r + 2$ to be $r - 2$, and that forces the modulus covering $r - 2$ to be $r + 1$, and that forces the modulus covering $r - 3$ to be $r - 3$, and then there's no way to cover $r + 3$.

Now suppose that $n = 2r$. Either r or $r + 1$ must belong to a modulus less than r and so belong to a class of size at least 3. Suppose r belongs to the modulus r and that the class containing $r + 1$ is the only of size greater than r. Then $r - 1$ must belong to the modulus $r - 1$ and this leaves no modulus for $r + 2$ to belong to.

The only IRDCS of length 11 are that in Example 1.1 and its reverse. These have $t = 5$ so we get equality in (4.1). □

We now consider the heft of an IRDCS. In the case of an (unrestricted) covering systems an easy density argument shows that the heft is always at least 1 and equals 1 if and only if the system is disjoint.

Theorem 4.2. *For any IRDCS $\mathcal{A}$,*

$$\frac{n-t}{n-1} \leq h \leq \frac{n+t}{n+1}. \tag{4.2}$$

.

Proof: We consider an IRDCS $\mathcal{A} = \{S(m_i, a_i), i = 1 \ldots t\}$ and assume, without loss of generality, that $m_1 < m_2 < \ldots m_t$ and that $1 \leq a_i \leq m_i$ for each i so that a_i is the first member of $[1, n]$ belonging to $S(m_i, a_i)$. We let the last member of $[1, n]$ belonging to $S(m_i, a_i)$ to be $n + 1 - b_i$. This implies that the a_is are positive and distinct and so are the b_is. We see that the number of elements of $[1, n]$ belonging to $S(m_i, a_i)$ in $[1, n]$ is $(n + 1 - b_i - a_i)/m_i + 1$. Since each member of $[1, n]$ belongs to exactly one class we have:

$$\sum_{i=1}^{t} \{\frac{n+1-b_i-a_i}{m_i} + 1\} = n.$$

Thus

$$n\sum_{i=1}^{t}\frac{1}{m_i}=\sum_{i=1}^{t}\frac{b_i+a_i-1}{m_i}+n-t. \tag{4.3}$$

The right hand side is minimised when

$$a_i=b_i=i$$

for each i, so we have

$$n\sum_{i=1}^{t}\frac{1}{m_i}\geq\sum_{i=1}^{t}\frac{2i-1}{m_i}+n-t.$$

Using $2i\geq 2$ this leads to

$$\sum_{i=1}^{t}\frac{1}{m_i}\geq\frac{n-t}{n-1}. \tag{4.4}$$

In the other direction we note that $a_i\leq m_i$ and $b_i\leq m_i$ for each i. Applying this observation to (4.3) we get,

$$n\sum_{i=1}^{t}\frac{1}{m_i}\leq\sum_{i=1}^{t}\frac{2m_i-1}{m_i}+n-t=2t-\sum_{i=1}^{t}\frac{1}{m_i}+n-t$$

and then

$$\sum_{i=1}^{t}\frac{1}{m_i}\leq\frac{n+t}{n+1}. \tag{4.5}$$

Combining (4.4) and (4.5) completes the proof. □

Corollary 4.3. *For any IRDCS with length greater than 11,*

$$\frac{n+1}{2(n-1)}\leq h\leq\frac{3n-1}{2(n+1)}. \tag{4.6}$$

.

Proof: Substitute (4.1) into (4.2). □

The bounds obtained here are weak compared with the values we found by computation. For example the largest and smallest heft values we found were 1.061111 (in Example 1.1) and 0.989552 (from an IRDCS of length 28 on moduli $6,7,8,9,11,12,13,14,16,17$). Note that the Corollary doesn't tell us much more than that the heft lies between 0.5 and 1.5.

5. Open Problems

We end with some open problems.

(1) Do there exist IRDCS with all moduli odd? Can the smallest modulus of an IRDCS be arbitrarily large?

(2) Can we sharpen the inequalities relating order, length and heft?

(3) The following two IRDCS both have length 43 and their sets of moduli are disjoint.

$$\{S(24,1), S(2,2), S(4,3), S(36,5), S(12,9), S(16,13), S(20,17)\}$$

$$\{S(25,1), S(33,2), S(7,3), S(8,4), S(9,5), S(21,6), S(18,7), S(13,8), \\ S(10,9), S(11,11), S(27,13), S(15,15), S(26,16), S(2,1)\}.$$

Generally, if each of two sets of congruences $\{S(m_1, a_1), ..., S(m_s, a_s)\}$ and $\{S(n_1, b_1), ..., S(n_t, b_t)\}$ is an IRDCS for $[1, n]$ and their sets of moduli are disjoint, as in the case above, then

$$\{S(3m_i, 3a_i + 1) : i = 1...s\} \cup \{S(3n_i, 3b_i + 2) : i = 1...t\} \cup \{S(3, 0\}$$

is an IRDCS for $[1, 3n]$ in which every modulus is divisible by 3.

This suggests the question, does an IRDCS exist with every modulus divisible by k for any value of k? Doubling produces one for $k = 2$ and the example above for $k = 3$.

(4) Our definition of an IRDCS requires that each congruence is satisfied at least twice. Do analogous systems exist in which each congruence is satisfied at least k times for values of k exceeding 2?

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