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Positive solutions for a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters

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Abstract

In this paper, we study the existence of a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters. By using the properties of the Green's function and the Guo-Krasnosel'skii fixed point theorem, we obtain some existence results of positive solutions under some conditions concerning the nonlinear functions. The method of this paper is a unified method for establishing the existence of positive solutions for a large number of nonlinear differential equations with coupled boundary conditions. In the end, examples are given to demonstrate the validity of our main results.

MSC: 34B16; 34B18

Keywords: positive solutions; fractional differential system; coupled integral boundary conditions; singular

1 Introduction

Coupled boundary conditions arise in the study of reaction-diffusion equations, Sturm-Liouville problems, mathematical biology and so on; see [1–4]. Leung [5] studied the following reaction-diffusion system for prey-predator interaction:

$$u_t(t,x) = \sigma_1 \triangle u + u\big(a + f(u,v)\big), \quad t \ge 0, x \in \Omega \subset R^n,$$
$$v_t(t,x) = \sigma_2 \triangle v + v\big(-r + g(u,v)\big), \quad t \ge 0, x \in \Omega \subset R^n,$$

subject to the coupled boundary conditions

$$\frac{\partial u}{\partial \eta} = 0, \qquad \frac{\partial v}{\partial \eta} - p(u) - q(v) = 0 \quad \text{on } \partial\Omega,$$

where $\triangle = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, a, r, σ_1, σ_2 are positive constants, $f, g : R^2 \to R$ have Hölder continuous partial derivatives up to second order in compact sets, η is a unit outward normal at $\partial\Omega$ and p and q have Hölder continuous first derivatives in compact subsets of $[0, +\infty)$. The functions $u(t,x)$, $v(t,x)$ respectively represent the density of prey and predator at time

$t \geq 0$ and at position $x = (x_1, \ldots, x_n)$. Similar coupled boundary conditions are also studied in [6] for a biochemical system.

Existence theory for boundary value problems of ordinary differential equations is well studied. However, differential equations with fractional order are a generalization of the ordinary differential equations to non-integer order. This generalization is not a mere mathematical curiosity but rather has interesting applications in many areas of science and engineering such as electrochemistry, control, porous media, electromagnetism, *etc.* There has been a significant development in the study of fractional differential equations in recent years; see, for example, [7–13]. Wang *et al.* [14] researched a coupled system of nonlinear fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, v(t)) = 0, \\ D_{0^+}^{\beta} v(t) + g(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = v(0) = 0, \qquad u(1) = au(\xi), \qquad v(1) = bv(\xi), \end{cases}$$

where $1 < \alpha, \beta < 2$, $0 \leq a, b < 1$, $0 < \xi < 1$, $f, g : [0,1] \times [0, +\infty) \to [0, +\infty)$ are continuous functions, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are also two standard Riemann-Liouville fractional derivatives. By using the Banach fixed point theorem and nonlinear differentiation of Leray-Schauder type, the existence and uniqueness of positive solutions are obtained.

In [15], Yang considered the positive solutions to boundary values problem for a coupled system of nonlinear fractional differential equations as follows:

$$\begin{cases} D^{\alpha} u(t) + a(t)f(t, v(t)) = 0, \\ D^{\beta} v(t) + b(t)g(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \int_0^1 \kappa(t)u(t)\,dt, \qquad v(0) = 0, \qquad v(1) = \int_0^1 \mu(t)v(t)\,dt, \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $a, b : (0,1) \to [0, +\infty)$ are continuous, $\kappa, \mu : [0,1] \to [0, +\infty)$ are non-negative and integrable functions, $f, g : [0,1] \times [0, +\infty) \to [0, +\infty)$ are continuous, and D^{α}, D^{β} are standard Riemann-Liouville fractional derivatives. By applying the Banach fixed point theorem, nonlinear differentiation of Leray-Schauder type and the fixed point theorems of cone expansion and compression of norm type, sufficient conditions for the existence and nonexistence of positive solutions to a general class of integral boundary value problems for a coupled system of fractional differential equations are obtained.

Inspired by the above mentioned work and wide applications of coupled boundary conditions in various fields of sciences and engineering, in this paper, we research the existence result to a class of singular semipositone fractional differential systems with coupled integral boundary conditions of the type

$$\begin{cases} D_{0^+}^{\alpha_1} u(t) + \lambda_1 f_1(t, u(t), v(t)) = 0, \\ D_{0^+}^{\alpha_2} v(t) + \lambda_2 f_2(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \mu_1 \int_0^1 v(s)\,dA_1(s), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \qquad v(1) = \mu_2 \int_0^1 u(s)\,dA_2(s), \end{cases} \tag{1.1}$$

where $\lambda_i > 0$ is a parameter, $n - 1 < \alpha_i \leq n$, $n \geq 2$, $D_{0^+}^{\alpha_i}$ is the standard Riemann-Liouville derivative. $\mu_i > 0$ is a constant, A_i is right continuous on $[0,1)$, left continuous at $t = 1$, and nondecreasing on $[0,1]$, $A_i(0) = 0$, $\int_0^1 x(s)\,dA_i(s)$ denotes the Riemann-Stieltjes integrals of x with respect to A_i, $f_i : (0,1) \times [0, +\infty) \times [0, +\infty) \to (-\infty, +\infty)$ is a continuous function and may be singular at $t = 0, 1$ for $i = 1, 2$. By a positive solution of system (1.1), we mean that $(u, v) \in C[0,1] \times C[0,1]$, (u, v) satisfies (1.1) and $u(t) > 0$, $v(t) > 0$ for all $t \in (0, 1]$.

To the best knowledge of the authors, there is seldom earlier literature studying fractional differential system with coupled integral boundary conditions like system (1.1), especially when $f_i(t, u, v)$ $(i = 1, 2)$ may be sign-changing, and may be singular at $t = 0$ and $t = 1$. Motivated by the results mentioned above, this paper attempts to fill part of this gap in the literature.

2 Preliminaries and lemmas

For convenience of the reader, we present some necessary definitions about fractional calculus theory.

Definition 2.1 [16, 17] Let $\alpha > 0$ and let u be piecewise continuous on $(0, +\infty)$ and integrable on any finite subinterval of $[0, +\infty)$. Then, for $t > 0$, we call

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds$$

the Riemann-Liouville fractional integral of u of order α.

Definition 2.2 [16, 17] The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 \leq \alpha < n$, $n \in \mathbb{N}$ is defined as

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

where $\mathbb{N}$ denotes the natural number set, the function $u(t)$ is n times continuously differentiable on $[0, +\infty)$.

Lemma 2.1 [16, 17] *Let $\alpha > 0$, if the fractional derivatives $D_{0^+}^{\alpha-1} u(t)$ and $D_{0^+}^{\alpha} u(t)$ are continuous on $[0, +\infty)$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_1, c_2, \ldots, c_n \in (-\infty, +\infty)$, n is the smallest integer greater than or equal to α.

Lemma 2.2 *Assume that the following condition* $(\mathbf{H}_0)$ *holds.*

$(\mathbf{H}_0)$

$$k_1 = \int_0^1 t^{\alpha_2-1}\, dA_1(t) > 0, \qquad k_2 = \int_0^1 t^{\alpha_1-1}\, dA_2(t) > 0, \quad 1 - \mu_1\mu_2 k_1 k_2 > 0.$$

Let $h_i \in C(0,1) \cap L(0,1)$ $(i = 1, 2)$, then the system with the coupled boundary conditions

$$\begin{cases} D_{0^+}^{\alpha_1} u(t) + h_1(t) = 0, \qquad D_{0^+}^{\alpha_2} v(t) + h_2(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \mu_1 \int_0^1 v(s)\, dA_1(s), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \qquad v(1) = \mu_2 \int_0^1 u(s)\, dA_2(s) \end{cases} \tag{2.1}$$

has a unique integral representation

$$\begin{cases} u(t) = \int_0^1 K_1(t,s)h_1(s)\,ds + \int_0^1 H_1(t,s)h_2(s)\,ds, \\ v(t) = \int_0^1 K_2(t,s)h_2(s)\,ds + \int_0^1 H_2(t,s)h_1(s)\,ds, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} K_1(t,s) &= \frac{\mu_1\mu_2k_1t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t) + G_1(t,s), \\ H_1(t,s) &= \frac{\mu_1t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_2(t,s)\,dA_1(t), \\ K_2(t,s) &= \frac{\mu_2\mu_1k_2t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_2(t,s)\,dA_1(t) + G_2(t,s), \\ H_2(t,s) &= \frac{\mu_2t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t), \end{aligned} \tag{2.3}$$

and

$$G_i(t,s) = \frac{1}{\Gamma(\alpha_i)}\begin{cases} [t(1-s)]^{\alpha_i-1} - (t-s)^{\alpha_i-1}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha_i-1}, & 0 \le t \le s \le 1, \end{cases} \qquad i = 1,2.$$

Proof System (2.1) is equivalent to the system of integral equations

$$u(t) = u(1)t^{\alpha_1-1} + \int_0^1 G_1(t,s)h_1(s)\,ds, \tag{2.4}$$

$$v(t) = v(1)t^{\alpha_2-1} + \int_0^1 G_2(t,s)h_2(s)\,ds. \tag{2.5}$$

Integrating (2.4) and (2.5) with respect to $dA_2(t)$ and $dA_1(t)$ respectively, we have

$$\begin{aligned} \int_0^1 u(t)\,dA_2(t) &= u(1)\int_0^1 t^{\alpha_1-1}\,dA_2(t) + \int_0^1\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t), \\ \int_0^1 v(t)\,dA_1(t) &= v(1)\int_0^1 t^{\alpha_2-1}\,dA_1(t) + \int_0^1\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t). \end{aligned}$$

Therefore, we can get

$$\frac{1}{\mu_1}u(1) - k_1v(1) = \int_0^1\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t), \tag{2.6}$$

$$-k_2u(1) + \frac{1}{\mu_2}v(1) = \int_0^1\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t). \tag{2.7}$$

Note that

$$\begin{vmatrix} \frac{1}{\mu_1} & -k_1 \\ -k_2 & \frac{1}{\mu_2} \end{vmatrix} = \frac{1-\mu_1\mu_2k_1k_2}{\mu_1\mu_2} \neq 0.$$

Thus, system (2.6) and (2.7) has a unique solution for $u(1)$ and $v(1)$. By Cramer's rule and simple calculations, it follows that

$$\begin{aligned} u(1) &= \frac{\mu_1}{1-\mu_1\mu_2k_1k_2}\biggl(\int_0^1\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t) \\ &\quad + \mu_2k_1\int_0^1\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t)\biggr), \end{aligned} \tag{2.8}$$

$$\begin{aligned} v(1) &= \frac{\mu_2}{1-\mu_1\mu_2k_1k_2}\biggl(\int_0^1\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t) \\ &\quad + \mu_1k_2\int_0^1\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t)\biggr). \end{aligned} \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.4) and (2.5), we have

$$\begin{aligned} u(t) &= \frac{\mu_1t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\biggl(\int_0^1\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t) \\ &\quad + \mu_2k_1\int_0^1\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t)\biggr) + \int_0^1 G_1(t,s)h_1(s)\,ds \\ &= \int_0^1 K_1(t,s)h_1(s)\,ds + \int_0^1 H_1(t,s)h_2(s)\,ds, \\ v(t) &= \frac{\mu_2t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\biggl(\int_0^1\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t) \\ &\quad + \mu_1k_2\int_0^1\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t)\biggr) + \int_0^1 G_2(t,s)h_2(s)\,ds \\ &= \int_0^1 K_2(t,s)h_2(s)\,ds + \int_0^1 H_2(t,s)h_1(s)\,ds. \end{aligned}$$

So (2.2) holds. The proof is completed. □

Lemma 2.3 *For $t,s\in[0,1]$, the functions $K_i(t,s)$ and $H_i(t,s)$ $(i=1,2)$ defined as* (2.3) *satisfy*

$$K_1(t,s),H_2(t,s)\le \rho s(1-s)^{\alpha_1-1},\qquad K_2(t,s),H_1(t,s)\le \rho s(1-s)^{\alpha_2-1}, \tag{2.10}$$

$$K_1(t,s),H_1(t,s)\le \rho t^{\alpha_1-1},\qquad K_2(t,s),H_2(t,s)\le \rho t^{\alpha_2-1}, \tag{2.11}$$

$$K_1(t,s)\ge \varrho t^{\alpha_1-1}s(1-s)^{\alpha_1-1},\qquad H_2(t,s)\ge \varrho t^{\alpha_2-1}s(1-s)^{\alpha_1-1}, \tag{2.12}$$

$$K_2(t,s)\ge \varrho t^{\alpha_2-1}s(1-s)^{\alpha_2-1},\qquad H_1(t,s)\ge \varrho t^{\alpha_1-1}s(1-s)^{\alpha_2-1}, \tag{2.13}$$

where

$$\rho = \max\left\{\begin{array}{l} \frac{1}{\Gamma(\alpha_1-1)}(\frac{\mu_1\mu_2k_1}{1-\mu_1\mu_2k_1k_2}\int_0^1 dA_2(t)+1),\ \frac{\mu_1}{\Gamma(\alpha_2-1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 A_1(t), \\ \frac{1}{\Gamma(\alpha_2-1)}(\frac{\mu_2\mu_1k_2}{1-\mu_1\mu_2k_1k_2}\int_0^1 dA_1(t)+1),\ \frac{\mu_2}{\Gamma(\alpha_1-1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 dA_2(t) \end{array}\right\},$$

$$\varrho = \min\left\{\begin{array}{l} \frac{\mu_1\mu_2k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1(1-t)t^{\alpha_1-1}\,dA_2(t),\ \frac{\mu_1}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1(1-t)t^{\alpha_2-1}\,dA_1(t), \\ \frac{\mu_2\mu_1k_2}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1(1-t)t^{\alpha_2-1}\,dA_1(t),\ \frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1(1-t)t^{\alpha_1-1}\,dA_2(t) \end{array}\right\}.$$

Proof By [18, Lemma 3.2], for any $t,s \in [0,1]$, we have

$$\frac{(1-t)t^{\alpha_i-1}s(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \le G_i(t,s) \le \frac{s(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i-1)}, \quad i=1,2. \tag{2.14}$$

So, by (2.3) and (2.14), we have

$$\begin{aligned}
K_1(t,s) &= \frac{\mu_1\mu_2k_1t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t) + G_1(t,s)\\
&\le \frac{\mu_1\mu_2k_1t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1-1)}\,dA_2(t) + \frac{s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1-1)}\\
&\le \frac{1}{\Gamma(\alpha_1-1)}\left(\frac{\mu_1\mu_2k_1t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 dA_2(t)+1\right)s(1-s)^{\alpha_1-1}\\
&\le \frac{1}{\Gamma(\alpha_1-1)}\left(\frac{\mu_1\mu_2k_1}{1-\mu_1\mu_2k_1k_2}\int_0^1 dA_2(t)+1\right)s(1-s)^{\alpha_1-1}\\
&\le \rho s(1-s)^{\alpha_1-1},
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
H_2(t,s) &= \frac{\mu_2t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t)\\
&\le \frac{s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1-1)}\frac{\mu_2t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 dA_2(t)\\
&\le \left(\frac{\mu_2}{\Gamma(\alpha_1-1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 dA_2(t)\right)s(1-s)^{\alpha_1-1}\\
&\le \rho s(1-s)^{\alpha_1-1}.
\end{aligned} \tag{2.16}$$

By a similar proof as (2.15) and (2.16), we also obtain

$$K_2(t,s), H_1(t,s) \le \rho s(1-s)^{\alpha_2-1}, \quad t,s \in [0,1],$$

then we know that (2.10) holds.

By [18, Lemma 3.2], for any $t,s \in [0,1]$, we also have

$$\frac{(1-t)t^{\alpha_i-1}s(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \le G_i(t,s) \le \frac{t^{\alpha_i-1}(1-t)}{\Gamma(\alpha_i-1)}, \quad i=1,2. \tag{2.17}$$

So, by (2.3) and (2.17), we have

$$\begin{aligned}
K_1(t,s) &= \frac{\mu_1\mu_2k_1t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t) + G_1(t,s)\\
&\le \frac{\mu_1\mu_2k_1t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{t^{\alpha_1-1}(1-t)}{\Gamma(\alpha_1-1)}\,dA_2(t) + \frac{t^{\alpha_1-1}(1-t)}{\Gamma(\alpha_1-1)}\\
&\le \frac{\mu_1\mu_2k_1t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{1}{\Gamma(\alpha_1-1)}\,dA_2(t) + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1-1)}\\
&\le \frac{1}{\Gamma(\alpha_1-1)}\left(\frac{\mu_1\mu_2k_1}{1-\mu_1\mu_2k_1k_2}\int_0^1 dA_2(t)+1\right)t^{\alpha_1-1}\\
&\le \rho t^{\alpha_1-1},
\end{aligned} \tag{2.18}$$

$$\begin{aligned} H_2(t,s) &= \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t) \\ &\le \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{t^{\alpha_1-1}(1-t)}{\Gamma(\alpha_1-1)}\,dA_2(t) \\ &\le \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{1}{\Gamma(\alpha_1-1)}\,dA_2(t) \le \rho t^{\alpha_2-1}. \end{aligned} \tag{2.19}$$

By a similar proof as (2.18) and (2.19), we also obtain

$$K_2(t,s) \le \rho t^{\alpha_2-1}, \qquad H_1(t,s) \le \rho t^{\alpha_1-1}, \quad t \in [0,1],$$

then we know that (2.11) holds.

On the other hand, by (2.3) and (2.14), we also have

$$\begin{aligned} K_1(t,s) &= \frac{\mu_1\mu_2k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t) + G_1(t,s) \\ &\ge \frac{\mu_1\mu_2k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{(1-t)t^{\alpha_1-1}s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\,dA_2(t) \\ &\ge \left(\frac{\mu_1\mu_2k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 (1-t)t^{\alpha_1-1}\,dA_2(t)\right)t^{\alpha_1-1}s(1-s)^{\alpha_1-1} \\ &\ge \varrho t^{\alpha_1-1}s(1-s)^{\alpha_1-1}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} H_2(t,s) &= \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t) \\ &\ge \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{(1-t)t^{\alpha_1-1}s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\,dA_2(t) \\ &\ge \left(\frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 (1-t)t^{\alpha_1-1}\,dA_2(t)\right)t^{\alpha_2-1}s(1-s)^{\alpha_1-1} \\ &\ge \varrho t^{\alpha_2-1}s(1-s)^{\alpha_1-1}. \end{aligned} \tag{2.21}$$

So, we can get that (2.12) holds. By a similar proof as (2.20) and (2.21), we also obtain

$$K_2(t,s) \ge \varrho t^{\alpha_2-1}s(1-s)^{\alpha_2-1}, \qquad H_1(t,s) \ge \varrho t^{\alpha_1-1}s(1-s)^{\alpha_2-1}, \quad t \in [0,1],$$

which implies that (2.13) holds. The proof is completed. □

Remark 2.1 From Lemma 2.3, for $t, \tau, s \in [0,1]$, we have

$$\begin{aligned} &K_i(t,s) \ge \omega t^{\alpha_i-1}K_i(\tau,s), \qquad H_i(t,s) \ge \omega t^{\alpha_i-1}H_i(\tau,s), \quad i=1,2, \\ &K_1(t,s) \ge \omega t^{\alpha_1-1}H_2(\tau,s), \qquad H_2(t,s) \ge \omega t^{\alpha_2-1}K_1(\tau,s), \\ &K_2(t,s) \ge \omega t^{\alpha_2-1}H_1(\tau,s), \qquad H_1(t,s) \ge \omega t^{\alpha_1-1}K_2(\tau,s), \end{aligned}$$

where $\omega = \frac{\varrho}{\rho}$, ϱ, ρ are defined as Lemma 2.3, $0 < \omega < 1$.

In the rest of the paper, we always suppose that the following assumption holds:

($\mathbf{H}_1$) $f_i : (0,1) \times [0,+\infty) \times [0,+\infty) \to (-\infty,+\infty)$ is continuous and satisfies

$$-q_i(t) \le f_i(t,x,y) \le a_i(t)p_i(t,x,y), \quad (t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty), i = 1,2,$$

where $a_i, q_i : (0,1) \to [0,+\infty)$ are continuous and may be singular at $t = 0,1$, $p_i : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous and

$$0 < \int_0^1 q_i(s)\,ds < +\infty, \qquad 0 < \int_0^1 a_i(s)\,ds < +\infty, \quad i = 1,2.$$

Lemma 2.4 *Assume that* ($\mathbf{H}_0$), ($\mathbf{H}_1$) *hold. Then the system with the coupled boundary conditions*

$$\begin{cases} D_{0^+}^{\alpha_1}\varpi_1(t) + \lambda_1 q_1(t) = 0, \\ D_{0^+}^{\alpha_2}\varpi_2(t) + \lambda_2 q_2(t) = 0, \quad 0 < t < 1, \\ \varpi_1(0) = \varpi_1'(0) = \cdots = \varpi_1^{(n-2)}(0) = 0, \qquad \varpi_1(1) = \mu_1 \int_0^1 \varpi_2(s)\,dA_1(s), \\ \varpi_2(0) = \varpi_2'(0) = \cdots = \varpi_2^{(n-2)}(0) = 0, \qquad \varpi_2(1) = \mu_2 \int_0^1 \varpi_1(s)\,dA_2(s) \end{cases}$$

has a unique solution

$$\begin{cases} \varpi_1(t) = \lambda_1 \int_0^1 K_1(t,s)q_1(s)\,ds + \lambda_2 \int_0^1 H_1(t,s)q_2(s)\,ds, \\ \varpi_2(t) = \lambda_2 \int_0^1 K_2(t,s)q_2(s)\,ds + \lambda_1 \int_0^1 H_2(t,s)q_1(s)\,ds, \end{cases} \tag{2.22}$$

which satisfies

$$\varpi_i(t) \le \lambda_1 \rho t^{\alpha_i - 1} \int_0^1 q_1(s)\,ds + \lambda_2 \rho t^{\alpha_i - 1} \int_0^1 q_2(s)\,ds, \quad 0 \le t \le 1, i = 1,2. \tag{2.23}$$

Proof It follows from Lemmas 2.2, 2.3 and conditions ($\mathbf{H}_0$), ($\mathbf{H}_1$) that (2.22) and (2.23) hold. The proof is completed. □

Next we consider the following singular nonlinear system:

$$\begin{cases} D_{0^+}^{\alpha_1}u(t) + \lambda_1(f_1(t,[u(t)-\varpi_1(t)]^*,[v(t)-\varpi_2(t)]^*) + q_1(t)) = 0, \\ D_{0^+}^{\alpha_2}v(t) + \lambda_2(f_2(t,[u(t)-\varpi_1(t)]^*,[v(t)-\varpi_2(t)]^*) + q_2(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \mu_1 \int_0^1 v(s)\,dA_1(s), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \qquad v(1) = \mu_2 \int_0^1 u(s)\,dA_2(s), \end{cases} \tag{2.24}$$

where $\lambda_i > 0$, $\varpi_i(t)$ is defined as (2.22), $[z(t)]^* = \max\{z(t),0\}$ ($i = 1,2$).

Lemma 2.5 *If* (u,v) *is a solution of system* (2.24) *with* $u(t) > \varpi_1(t)$, $v(t) > \varpi_2(t)$ *for any* $t \in (0,1]$, *then* $(u - \varpi_1, v - \varpi_2)$ *is a positive solution of system* (1.1).

Proof In fact, if (u, v) is a positive solution of system (2.24) such that $u(t) > \varpi_1(t)$, $v(t) > \varpi_2(t)$ for any $t \in (0,1]$, then from system (2.24) and the definition of $[z(t)]^*$, we have

$$\begin{cases} D_{0^+}^{\alpha_1} u(t) + \lambda_1 (f_1(t, u(t) - \varpi_1(t), v(t) - \varpi_2(t)) + q_1(t)) = 0, \\ D_{0^+}^{\alpha_2} v(t) + \lambda_2 (f_2(t, u(t) - \varpi_1(t), v(t) - \varpi_2(t)) + q_2(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \mu_1 \int_0^1 v(s)\, dA_1(s), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \qquad v(1) = \mu_2 \int_0^1 u(s)\, dA_2(s). \end{cases} \tag{2.25}$$

Let $x(t) = u(t) - \varpi_1(t)$, $y(t) = v(t) - \varpi_2(t)$, $t \in (0,1)$, then

$$D_{0^+}^{\alpha_1} x(t) = D_{0^+}^{\alpha_1} u(t) - D_{0^+}^{\alpha_1} \varpi_1(t), \qquad D_{0^+}^{\alpha_2} y(t) = D_{0^+}^{\alpha_2} v(t) - D_{0^+}^{\alpha_2} \varpi_2(t),$$

which implies that

$$D_{0^+}^{\alpha_1} u(t) = D_{0^+}^{\alpha_1} x(t) - \lambda_1 q_1(t), \qquad D_{0^+}^{\alpha_2} v(t) = D_{0^+}^{\alpha_2} y(t) - \lambda_2 q_2(t).$$

Thus, system (2.25) becomes

$$\begin{cases} D_{0^+}^{\alpha_1} x(t) + \lambda_1 f_1(t, x(t), y(t)) = 0, \\ D_{0^+}^{\alpha_2} y(t) + \lambda_2 f_2(t, x(t), y(t)) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \qquad x(1) = \mu_1 \int_0^1 y(s)\, dA_1(s), \\ y(0) = y'(0) = \cdots = y^{(n-2)}(0) = 0, \qquad y(1) = \mu_2 \int_0^1 x(s)\, dA_2(s). \end{cases} \tag{2.26}$$

Then, by (2.26), $(u - \varpi_1, v - \varpi_2)$ is a positive solution of system (1.1). The proof is completed. □

Let $X = C[0,1] \times C[0,1]$, then X is a Banach space with the norm

$$\|(u,v)\| = \max\{\|u\|, \|v\|\}, \qquad \|u\| = \max_{t\in[0,1]} |u(t)|, \qquad \|v\| = \max_{t\in[0,1]} |v(t)|.$$

Let

$$K = \{(u,v) \in X : u(t) \geq \omega t^{\alpha_1 - 1} \|(u,v)\|,\ v(t) \geq \omega t^{\alpha_2 - 1} \|(u,v)\|, t \in [0,1]\},$$

where ω is defined as Remark 2.1. It is easy to see that K is a positive cone in X. Under the above conditions $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, for any $(u,v) \in K$, we can define an integral operator $T : K \to X$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \quad 0 \leq t \leq 1, \tag{2.27}$$

$$\begin{aligned} T_i(u,v)(t) = {} & \lambda_i \int_0^1 K_i(t,s)\big(f_i\big(s, [u(s) - \varpi_1(s)]^*, [v(s) - \varpi_2(s)]^*\big) + q_i(s)\big)\, ds \\ & + \lambda_j \int_0^1 H_i(t,s)\big(f_j\big(s, [u(s) - \varpi_1(s)]^*, [v(s) - \varpi_2(s)]^*\big) + q_j(s)\big)\, ds, \\ & 0 \leq t \leq 1, i = 1,2, i + j = 3. \end{aligned} \tag{2.28}$$

We know that (u,v) is a positive solutions of system (1.1) if and only if (u,v) is a fixed point of T in K.

Lemma 2.6 *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ *hold. Then* $T: K \to K$ *is a completely continuous operator.*

Proof By a routine discussion, we know that $T: K \to X$ is well defined, so we only prove $T(K) \subseteq K$. For any $(u,v) \in K$, $0 \le t, \tau \le 1$, by Remark 2.1, we have

$$\begin{aligned}
T_1(u,v)(t) &= \lambda_1 \int_0^1 K_1(t,s)\bigl(f_1\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_1(s)\bigr)\,ds \\
&\quad + \lambda_2 \int_0^1 H_1(t,s)\bigl(f_2\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_2(s)\bigr)\,ds \\
&\ge \lambda_1 \int_0^1 \omega t^{\alpha_1-1} K_1(\tau,s)\bigl(f_1\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_1(s)\bigr)\,ds \\
&\quad + \lambda_2 \int_0^1 \omega t^{\alpha_1-1} H_1(\tau,s)\bigl(f_2\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_2(s)\bigr)\,ds \\
&\ge \omega t^{\alpha_1-1}\biggl(\lambda_1 \int_0^1 K_1(\tau,s)\bigl(f_1\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_1(s)\bigr)\,ds \\
&\quad + \lambda_2 \int_0^1 H_1(\tau,s)\bigl(f_2\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_2(s)\bigr)\,ds\biggr) \\
&\ge \omega t^{\alpha_1-1} T_1(u,v)(\tau).
\end{aligned} \tag{2.29}$$

On the other hand,

$$\begin{aligned}
T_1(u,v)(t) &= \lambda_1 \int_0^1 K_1(t,s)\bigl(f_1\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_1(s)\bigr)\,ds \\
&\quad + \lambda_2 \int_0^1 H_1(t,s)\bigl(f_2\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_2(s)\bigr)\,ds \\
&\ge \lambda_1 \int_0^1 \omega t^{\alpha_1-1} H_2(\tau,s)\bigl(f_1\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_1(s)\bigr)\,ds \\
&\quad + \lambda_2 \int_0^1 \omega t^{\alpha_1-1} K_2(\tau,s)\bigl(f_2\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_2(s)\bigr)\,ds \\
&\ge \omega t^{\alpha_1-1}\biggl(\lambda_1 \int_0^1 H_2(\tau,s)\bigl(f_1\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_1(s)\bigr)\,ds \\
&\quad + \lambda_2 \int_0^1 K_2(\tau,s)\bigl(f_2\bigl(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\bigr)+q_2(s)\bigr)\,ds\biggr) \\
&\ge \omega t^{\alpha_1-1} T_2(u,v)(\tau).
\end{aligned} \tag{2.30}$$

Then we have

$$T_1(u,v)(t) \ge \omega t^{\alpha_1-1}\bigl\|T_1(u,v)\bigr\|, \qquad T_1(u,v)(t) \ge \omega t^{\alpha_1-1}\bigl\|T_2(u,v)\bigr\|,$$

i.e.,

$$T_1(u,v)(t) \ge \omega t^{\alpha_1-1}\bigl\|\bigl(T_1(u,v),T_2(u,v)\bigr)\bigr\|.$$

In the same way as (2.29) and (2.30), we can prove that

$$T_2(u,v)(t) \geq \omega t^{\alpha_2-1} \big\| \big(T_1(u,v), T_2(u,v)\big) \big\|.$$

Therefore, we have $T(K) \subseteq K$.

According to the Ascoli-Arzela theorem, we can easily get that $T : K \to K$ is completely continuous. The proof is completed. □

In order to obtain the existence of the positive solutions of system (1.1), we will use the following cone compression and expansion fixed point theorem.

Lemma 2.7 [19] *Let P be a positive cone in a Banach space E, Ω_1 and Ω_2 are bounded open sets in E, $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $A : P \cap \overline{\Omega}_2 \backslash \Omega_1 \to P$ is a completely continuous operator. If the following conditions are satisfied*:

$$\|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \qquad \|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_2,$$

or

$$\|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \qquad \|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_2,$$

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Main results

Theorem 3.1 *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ *hold and that for any fixed $\lambda_1, \lambda_2 \in (0,+\infty)$, the following conditions are satisfied*:

$(\mathbf{H}_2)$ *There exists a constant*

$$r_1 > \max\left\{L_1, L_2, \omega^{-1}\rho\left(\lambda_1 \int_0^1 q_1(s)\,ds + \lambda_2 \int_0^1 q_2(s)\,ds\right)\right\}$$

such that

$$p_i(t,x,y) \leq \frac{r_1}{L_i} - 1, \quad (t,x,y) \in [0,1] \times [0,r_1] \times [0,r_1], i = 1,2.$$

$(\mathbf{H}_3)$

$$0 < l_1 < \liminf_{x\to+\infty} \inf_{\substack{t\in[a,b]\subset(0,1)\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{x} \leq +\infty, \quad \textit{or}$$

$$0 < l_1 < \liminf_{y\to+\infty} \inf_{\substack{t\in[a,b]\subset(0,1)\\ x\in[0,+\infty)}} \frac{f_1(t,x,y)}{y} \leq +\infty,$$

where ω is defined as Remark 2.1, ρ is defined as Lemma 2.3,

$$L_i = 3\left(\lambda_i \rho \int_0^1 \big(a_i(s) + q_i(s)\big)\,ds\right)^{-1}, \quad i = 1,2,$$

$$l_1 = \frac{3}{2}\left(\lambda_1 \varrho \theta^2 \omega \int_a^b s(1-s)^{\alpha_1-1}\,ds\right)^{-1}, \quad \theta = \min_{t\in[a,b]}\big\{t^{\alpha_1-1}, t^{\alpha_2-1}\big\}.$$

Then system (1.1) *has at least one positive solution* $(\overline{u},\overline{v})$. *Moreover*, $(\overline{u},\overline{v})$ *satisfies* $\overline{u}(t) \geq \overline{l}t^{\alpha_1-1}$, $\overline{v}(t) \geq \overline{l}t^{\alpha_2-1}$, $t \in [0,1]$ *for some positive constant* $\overline{l}$.

Proof Let $K_{r_1} = \{(u,v) \in K : \|(u,v)\| < r_1\}$. For any $(u,v) \in \partial K_{r_1}$, $t \in [0,1]$, by the definition of $\|\cdot\|$, we know that

$$\begin{aligned} &[u(t) - \varpi_1(t)]^* \leq |u(t)| \leq \|u\| \leq \|(u,v)\| \leq r_1, \\ &[v(t) - \varpi_2(t)]^* \leq |v(t)| \leq \|v\| \leq \|(u,v)\| \leq r_1. \end{aligned}$$

So, for any $(u,v) \in \partial K_{r_1}$, by condition $(\mathbf{H}_2)$ and Lemma 2.3, we have

$$\begin{aligned} \|T_1(u,v)\| &= \max_{t\in[0,1]}\bigg|\lambda_1 \int_0^1 K_1(t,s)\big(f_1\big(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\big)+q_1(s)\big)\,ds \\ &\quad + \lambda_2 \int_0^1 H_1(t,s)\big(f_2\big(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\big)+q_2(s)\big)\,ds\bigg| \\ &\leq \max_{t\in[0,1]}\bigg|\lambda_1 \int_0^1 \rho t^{\alpha_1-1}\big(a_1(s)p_1\big(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\big)+q_1(s)\big)\,ds \\ &\quad + \lambda_2 \int_0^1 \rho t^{\alpha_1-1}\big(a_2(s)p_2\big(s,[u(s)-\varpi_1(s)]^*,[v(s)-\varpi_2(s)]^*\big)+q_2(s)\big)\,ds\bigg| \\ &\leq \lambda_1\rho \int_0^1 \left(a_1(s)\left(\frac{r_1}{L_1}-1\right)+q_1(s)\right)ds + \lambda_2\rho \int_0^1 \left(a_2(s)\left(\frac{r_1}{L_2}-1\right)+q_2(s)\right)ds \\ &\leq \left(\frac{r_1}{L_1}-1+1\right)\lambda_1\rho\int_0^1 \big(a_1(s)+q_1(s)\big)\,ds \\ &\quad + \left(\frac{r_1}{L_2}-1+1\right)\lambda_2\rho\int_0^1 \big(a_2(s)+q_2(s)\big)\,ds \\ &= \frac{2r_1}{3} < r_1 = \|(u,v)\|. \end{aligned} \tag{3.1}$$

Similarly as (3.1), for any $(u,v) \in \partial K_{r_1}$, by condition $(\mathbf{H}_2)$, we also have

$$\|T_2(u,v)\| < r_1 = \|(u,v)\|.$$

Consequently, we have

$$\|T(u,v)\| = \max\{\|T_1(u,v)\|, \|T_2(u,v)\|\} < r_1 = \|(u,v)\| \quad \text{for any } (u,v) \in \partial K_{r_1}. \tag{3.2}$$

On the other hand, by the first inequality in $(\mathbf{H}_3)$, there exists $\varepsilon_0 > 0$ such that $l_1 + \varepsilon_0 > 0$, and also there exists $r_0 > 0$ such that

$$|f_1(t,x,y)| \geq (l_1+\varepsilon_0)x, \quad x \geq r_0, y \geq 0, t \in [a,b]. \tag{3.3}$$

Choose $r_2 = \max\{3r_1, \frac{3r_0}{2\omega\theta}\}$. Let $K_{r_2} = \{(u,v) \in K : \|(u,v)\| < r_2\}$. For any $(u,v) \in \partial K_{r_2}$, by the definition of $\|\cdot\|$ and (2.23), we have

$$\begin{aligned}
u(t) - \varpi_1(t) &\geq \omega t^{\alpha_1-1} r_2 - \left(\lambda_1 \rho t^{\alpha_1-1} \int_0^1 q_1(s)\,ds + \lambda_2 \rho t^{\alpha_1-1} \int_0^1 q_2(s)\,ds\right) \\
&= t^{\alpha_1-1}\left(\omega r_2 - \left(\lambda_1 \rho \int_0^1 q_1(s)\,ds + \lambda_2 \rho \int_0^1 q_2(s)\,ds\right)\right) \\
&\geq \theta\left(\omega r_2 - \rho\left(\lambda_1 \int_0^1 q_1(s)\,ds + \lambda_2 \int_0^1 q_2(s)\,ds\right)\right) \\
&\geq \omega\theta(r_2 - r_1) \geq \frac{2\omega\theta r_2}{3} \geq r_0, \quad t \in [a,b], \qquad (3.4)\\
v(t) - \varpi_2(t) &\geq \omega t^{\alpha_2-1} r_2 - \left(\lambda_1 \rho t^{\alpha_2-1} \int_0^1 q_1(s)\,ds + \lambda_2 \rho t^{\alpha_2-1} \int_0^1 q_2(s)\,ds\right) \\
&= t^{\alpha_2-1}\left(\omega r_2 - \left(\lambda_1 \rho \int_0^1 q_1(s)\,ds + \lambda_2 \rho \int_0^1 q_2(s)\,ds\right)\right) \\
&\geq \theta\left(\omega r_2 - \rho\left(\lambda_1 \int_0^1 q_1(s)\,ds + \lambda_2 \int_0^1 q_2(s)\,ds\right)\right) \\
&\geq \omega\theta(r_2 - r_1) \geq \frac{2\omega\theta r_2}{3} \geq r_0 > 0, \quad t \in [a,b]. \qquad (3.5)
\end{aligned}$$

Thus, for any $(u,v) \in \partial K_{r_2}$, by (3.3)-(3.5), we have

$$f_1\big(t, [u(t) - \varpi_1(t)]^*, [v(t) - \varpi_2(t)]^*\big) \geq (l_1 + \varepsilon_0)[u(t) - \varpi_1(t)]^*, \quad t \in [a,b]. \qquad (3.6)$$

Hence, for any $(u,v) \in \partial K_{r_2}$, by (3.6) and Lemma 2.3, we conclude that

$$\begin{aligned}
\|T_1(u,v)\| &= \max_{t\in[0,1]} \bigg|\lambda_1 \int_0^1 K_1(t,s)\big(f_1\big(s, [u(s) - \varpi_1(s)]^*, [v(s) - \varpi_2(s)]^*\big) + q_1(s)\big)\,ds \\
&\quad + \lambda_2 \int_0^1 H_1(t,s)\big(f_2\big(s, [u(s) - \varpi_1(s)]^*, [v(s) - \varpi_2(s)]^*\big) + q_2(s)\big)\,ds\bigg| \\
&\geq \max_{t\in[0,1]} \lambda_1 \int_0^1 K_1(t,s)\big(f_1\big(s, [u(s) - \varpi_1(s)]^*, [v(s) - \varpi_2(s)]^*\big) + q_1(s)\big)\,ds \\
&\geq \max_{t\in[0,1]} \lambda_1 \int_a^b K_1(t,s) f_1\big(s, [u(s) - \varpi_1(s)]^*, [v(s) - \varpi_2(s)]^*\big)\,ds \\
&\geq \min_{t\in[a,b]} \lambda_1 \int_a^b \varrho t^{\alpha_1-1} s(1-s)^{\alpha_1-1}(l_1 + \varepsilon_0)[u(s) - \varpi_1(s)]^*\,ds \\
&\geq \frac{2\lambda_1 \varrho \theta^2 (l_1 + \varepsilon_0)\omega r_2}{3} \int_a^b s(1-s)^{\alpha_1-1}\,ds \\
&\geq r_2 = \|(u,v)\|.
\end{aligned}$$

Consequently,

$$\|T(u,v)\| = \max\{\|T_1(u,v)\|, \|T_2(u,v)\|\} \geq r_2 = \|(u,v)\| \quad \text{for any } (u,v) \in \partial K_{r_2}. \qquad (3.7)$$

Obviously, by the second inequality in $(\mathbf{H}_3)$, (3.7) is still valid.

It follows from the above discussion, (3.2), (3.7), Lemmas 2.6 and 2.7, that for any fixed $\lambda_1, \lambda_2 \in (0, +\infty)$, T has a fixed point $(u, v) \in \overline{K}_{r_2} \setminus K_{r_1}$ and $r_1 \leq \|(u, v)\| \leq r_2$. Since $\|(u, v)\| \geq r_1$, we have

$$\begin{aligned} u(t) - \varpi_1(t) &\geq u(t) - \left(\lambda_1 \rho t^{\alpha_1 - 1} \int_0^1 q_1(s)\,ds + \lambda_2 \rho t^{\alpha_1 - 1} \int_0^1 q_2(s)\,ds\right) \\ &\geq \omega t^{\alpha_1 - 1} r_1 - \left(\lambda_1 \rho t^{\alpha_1 - 1} \int_0^1 q_1(s)\,ds + \lambda_2 \rho t^{\alpha_1 - 1} \int_0^1 q_2(s)\,ds\right) \\ &\geq t^{\alpha_1 - 1}\left(\omega r_1 - \lambda_1 \rho \int_0^1 q_1(s)\,ds - \lambda_2 \rho \int_0^1 q_2(s)\,ds\right) > 0, \quad t \in (0,1], \\ v(t) - \varpi_2(t) &\geq v(t) - \left(\lambda_1 \rho t^{\alpha_2 - 1} \int_0^1 q_1(s)\,ds + \lambda_2 \rho t^{\alpha_2 - 1} \int_0^1 q_2(s)\,ds\right) \\ &\geq \omega t^{\alpha_2 - 1} r_1 - \left(\lambda_1 \rho t^{\alpha_2 - 1} \int_0^1 q_1(s)\,ds + \lambda_2 \rho t^{\alpha_2 - 1} \int_0^1 q_2(s)\,ds\right) \\ &\geq t^{\alpha_2 - 1}\left(\omega r_1 - \lambda_1 \rho \int_0^1 q_1(s)\,ds - \lambda_2 \rho \int_0^1 q_2(s)\,ds\right) > 0, \quad t \in (0,1]. \end{aligned}$$

Let $\bar{l} = \omega r_1 - \lambda_1 \rho \int_0^1 q_1(s)\,ds - \lambda_2 \rho \int_0^1 q_2(s)\,ds$, $\overline{u}(t) = u(t) - \varpi_1(t)$, $\overline{v}(t) = v(t) - \varpi_2(t)$, then we have

$$\overline{u}(t) \geq \bar{l} t^{\alpha_1 - 1} > 0, \qquad \overline{v}(t) \geq \bar{l} t^{\alpha_2 - 1} > 0, \quad t \in (0,1].$$

By Lemma 2.5, we know that for any fixed $\lambda_1, \lambda_2 \in (0, +\infty)$, system (1.1) has at least one positive solution $(\overline{u}, \overline{v})$; moreover, $(\overline{u}, \overline{v})$ satisfies $\overline{u}(t) \geq \bar{l} t^{\alpha_1 - 1}$, $\overline{v}(t) \geq \bar{l} t^{\alpha_2 - 1}$, $t \in [0,1]$. The proof is completed. □

Remark 3.1 From the proof of Theorem 3.1, we know that the conclusion of Theorem 3.1 is valid if condition $(\mathbf{H}_3)$ is replaced by

$$0 < l_2 < \liminf_{x \to +\infty} \inf_{\substack{t \in [a,b] \subset (0,1) \\ y \in [0,+\infty)}} \frac{f_2(t,x,y)}{x} \leq +\infty, \quad \text{or}$$

$$0 < l_2 < \liminf_{y \to +\infty} \inf_{\substack{t \in [a,b] \subset (0,1) \\ x \in [0,+\infty)}} \frac{f_2(t,x,y)}{y} \leq +\infty,$$

where

$$l_2 = \frac{3}{2}\left(\lambda_2 \varrho \theta^2 \omega \int_a^b s(1-s)^{\alpha_2 - 1}\,ds\right)^{-1}.$$

Theorem 3.2 *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ *hold and that for any fixed* $\lambda_1, \lambda_2 \in (0, +\infty)$, *the following conditions are satisfied*:

$(\mathbf{H}_4)$ *There exists a constant*

$$R_1 > \omega^{-1} \rho \left(\lambda_1 \int_0^1 q_1(s)\,ds + \lambda_2 \int_0^1 q_2(s)\,ds\right)$$

such that

$$f_1(t,x,y) \geq \frac{R_1}{l_1}, \quad (t,x,y) \in [a,b] \times [0,R_1] \times [0,R_1]. \tag{3.8}$$

$(\mathbf{H}_5)$

$$0 \leq \limsup_{x\to+\infty} \sup_{\substack{t\in[0,1]\\ y\in[0,+\infty)}} \frac{p_i(t,x,y)}{x} < L_i, \quad \text{or} \quad 0 \leq \limsup_{y\to+\infty} \sup_{\substack{t\in[0,1]\\ x\in[0,+\infty)}} \frac{p_i(t,x,y)}{y} < L_i, \quad i = 1,2,$$

where $[a,b] \subset (0,1)$, L_i $(i = 1,2)$, l_1 *are defined in Theorem* 3.1. *Then system* (1.1) *has at least one positive solution* $(\overline{u}^0, \overline{v}^0)$. *Moreover,* $(\overline{u}^0, \overline{v}^0)$ *satisfies* $\overline{u}^0(t) \geq \overline{l}^0 t^{\alpha_1-1}$, $\overline{v}^0(t) \geq \overline{l}^0 t^{\alpha_2-1}$, $t \in [0,1]$ *for some positive constant* $\overline{l}^0$.

The proof of Theorem 3.2 is similar to that of Theorem 3.1, and so we omit it.

Remark 3.2 The conclusion of Theorem 3.2 is valid if inequality (3.8) in condition $(\mathbf{H}_6)$ is replaced by

$$f_2(t,x,y) \geq \frac{R_1}{l_2}, \quad (t,x,y) \in [a,b] \times [0,R_1] \times [0,R_1],$$

where l_2 is defined in Remark 3.1.

Theorem 3.3 *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ *hold and that the following is satisfied*:

$(\mathbf{H}_6)$

$$\lim_{x\to+\infty} \inf_{\substack{t\in[c,d]\subset(0,1)\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{x} = +\infty, \quad \text{or} \quad \lim_{y\to+\infty} \inf_{\substack{t\in[c,d]\subset(0,1)\\ x\in[0,+\infty)}} \frac{f_1(t,x,y)}{y} = +\infty.$$

Then there exist $\overline{\lambda}_1 > 0$, $\overline{\lambda}_2 > 0$ *such that system* (1.1) *has at least one positive solution* $(\overline{u}', \overline{v}')$ *provided* $\lambda_1 \in (0, \overline{\lambda}_1)$, $\lambda_2 \in (0, \overline{\lambda}_2)$. *Moreover,* $(\overline{u}', \overline{v}')$ *satisfies* $\overline{u}'(t) \geq \overline{l}' t^{\alpha_1-1}$, $\overline{v}'(t) \geq \overline{l}' t^{\alpha_2-1}$, $t \in [0,1]$ *for some positive constant* $\overline{l}'$.

Proof Choose $R > \omega^{-1}\rho(\int_0^1 q_1(s)\,ds + \int_0^1 q_2(s)\,ds)$. Let

$$\overline{\lambda}_i = \min\left\{1, \frac{R}{2\rho\int_0^1 (a_i(s)S_{i,R} + q_i(s))\,ds}\right\}, \quad i = 1,2,$$

where ω is defined as Remark 2.1, ρ is defined as Lemma 2.3, $S_{i,R} := \sup\{p_i(t,x,y) : 0 \leq t \leq 1, 0 \leq x,y \leq R\}$ $(i = 1,2)$.

Let $K_R = \{(u,v) \in K : \|(u,v)\| < R\}$. For any $(u,v) \in \partial K_R$, $t \in [0,1]$, by the definition of $\|\cdot\|$, we know that

$$\begin{aligned} &\left[u(t) - \varpi_1(t)\right]^* \leq \left|u(t)\right| \leq \|u\| \leq \left\|(u,v)\right\| \leq R, \\ &\left[v(t) - \varpi_2(t)\right]^* \leq \left|v(t)\right| \leq \|v\| \leq \left\|(u,v)\right\| \leq R. \end{aligned}$$

So, for $\lambda_i \in (0, \overline{\lambda}_i)$, $(u,v) \in \partial K_R$, by Lemma 2.3, we have

$$\begin{aligned}
\|T_1(u,v)\| &= \max_{t\in[0,1]}\bigg|\lambda_1 \int_0^1 K_1(t,s)\big(f_1\big(s,\big[u(s)-\varpi_1(s)\big]^*,\big[v(s)-\varpi_2(s)\big]^*\big)+q_1(s)\big)\,ds \\
&\quad + \lambda_2 \int_0^1 H_1(t,s)\big(f_2\big(s,\big[u(s)-\varpi_1(s)\big]^*,\big[v(s)-\varpi_2(s)\big]^*\big)+q_2(s)\big)\,ds\bigg| \\
&\le \max_{t\in[0,1]}\bigg|\lambda_1 \int_0^1 \rho t^{\alpha_1-1}\big(a_1(s)p_1\big(s,\big[u(s)-\varpi_1(s)\big]^*,\big[v(s)-\varpi_2(s)\big]^*\big)+q_1(s)\big)\,ds \\
&\quad + \lambda_2 \int_0^1 \rho t^{\alpha_1-1}\big(a_2(s)p_2\big(s,\big[u(s)-\varpi_1(s)\big]^*,\big[v(s)-\varpi_2(s)\big]^*\big)+q_2(s)\big)\,ds\bigg| \\
&\le \lambda_1\rho \int_0^1 \big(a_1(s)S_{1,R}+q_1(s)\big)\,ds + \lambda_2\rho \int_0^1 \big(a_2(s)S_{2,R}+q_2(s)\big)\,ds \\
&\le R = \big\|(u,v)\big\|.
\end{aligned} \tag{3.9}$$

Similarly as (3.9), for any $(u,v) \in \partial K_R$, by condition $(\mathbf{H}_2)$, we also have

$$\|T_2(u,v)\| \le R = \big\|(u,v)\big\|.$$

Consequently, we have

$$\|T(u,v)\| = \max\big\{\|T_1(u,v)\|, \|T_2(u,v)\|\big\} \le R = \big\|(u,v)\big\| \quad \text{for any } (u,v) \in \partial K_R. \tag{3.10}$$

On the other hand, by the first inequality in $(\mathbf{H}_6)$, choose M_1 such that

$$\lambda_1 \varrho \theta'^2 M_1 \omega \int_c^d s(1-s)^{\alpha_1-1}\,ds > 2, \quad \theta' = \min_{t\in[c,d]}\big\{t^{\alpha_1-1}, t^{\alpha_2-1}\big\},$$

where ω is defined as Remark 2.1, ϱ is defined as Lemma 2.3. Then there exists $N^* > 0$ such that

$$f_1(t,x,y) \ge M_1 x, \quad x \ge N^*, y \ge 0, t \in [c,d]. \tag{3.11}$$

Let

$$K_{R'} = \big\{(x,y) \in K : \big\|(x,y)\big\| < R'\big\}, \quad R' > \max\left\{2R, \frac{2N^*}{\omega\theta'}\right\}.$$

For any $(x,y) \in \partial K_{R'}$, by (2.23), we have

$$\begin{aligned}
u(t)-\varpi_1(t) &\ge \omega t^{\alpha_1-1}R' - \left(\lambda_1 \rho t^{\alpha_1-1}\int_0^1 q_1(s)\,ds + \lambda_2 \rho t^{\alpha_1-1}\int_0^1 q_2(s)\,ds\right) \\
&= t^{\alpha_1-1}\left(\omega R' - \rho\left(\lambda_1 \int_0^1 q_1(s)\,ds + \lambda_2 \int_0^1 q_2(s)\,ds\right)\right) \\
&\ge \theta'\left(\omega R' - \rho\left(\int_0^1 q_1(s)\,ds + \int_0^1 q_2(s)\,ds\right)\right) \\
&\ge \omega\theta'\big(R'-R\big) \ge \frac{\omega\theta' R'}{2} \ge N^*, \quad t \in [c,d],
\end{aligned} \tag{3.12}$$

$$\begin{aligned} v(t)-\varpi_2(t) &\geq \omega t^{\alpha_2-1}R'-\left(\lambda_1\rho t^{\alpha_2-1}\int_0^1 q_1(s)\,ds+\lambda_2\rho t^{\alpha_2-1}\int_0^1 q_2(s)\,ds\right) \\ &= t^{\alpha_2-1}\left(\omega R'-\left(\lambda_1\rho\int_0^1 q_1(s)\,ds+\lambda_2\rho\int_0^1 q_2(s)\,ds\right)\right) \\ &\geq \theta'\left(\omega R'-\rho\left(\int_0^1 q_1(s)\,ds+\int_0^1 q_2(s)\,ds\right)\right) \\ &\geq \omega\theta'\left(R'-R\right)\geq \frac{\omega\theta' R'}{2}\geq N^*>0, \quad t\in[c,d]. \end{aligned} \tag{3.13}$$

Thus, for any $(u,v)\in\partial K_{R'}$, by (3.11)-(3.13), we have

$$f_1\left(t,\left[u(t)-\varpi_1(t)\right]^*,\left[v(t)-\varpi_2(t)\right]^*\right)\geq M_1\left[u(t)-\varpi_1(t)\right]^*, \quad t\in[a,b]. \tag{3.14}$$

Hence, for any $(u,v)\in\partial K_{R'}$, by (3.14) and Lemma 2.3, we have

$$\begin{aligned} \left\|T_1(u,v)\right\| &= \max_{t\in[0,1]}\left|\lambda_1\int_0^1 K_1(t,s)\left(f_1\left(s,\left[u(s)-\varpi_1(s)\right]^*,\left[v(s)-\varpi_2(s)\right]^*\right)+q_1(s)\right)ds\right. \\ &\quad \left.+\lambda_2\int_0^1 H_1(t,s)\left(f_2\left(s,\left[u(s)-\varpi_1(s)\right]^*,\left[v(s)-\varpi_2(s)\right]^*\right)+q_2(s)\right)ds\right| \\ &\geq \max_{t\in[0,1]}\lambda_1\int_0^1 K_1(t,s)\left(f_1\left(s,\left[u(s)-\varpi_1(s)\right]^*,\left[v(s)-\varpi_2(s)\right]^*\right)+q_1(s)\right)ds \\ &\geq \max_{t\in[0,1]}\lambda_1\int_c^d K_1(t,s)f_1\left(s,\left[u(s)-\varpi_1(s)\right]^*,\left[v(s)-\varpi_2(s)\right]^*\right)ds \\ &\geq \min_{t\in[c,d]}\lambda_1\int_c^d \varrho t^{\alpha_1-1}s(1-s)^{\alpha_1-1}M_1\left[u(s)-\varpi_1(s)\right]^*\,ds \\ &\geq \frac{\lambda_1\varrho\theta'^2M_1\omega R'}{2}\int_c^d s(1-s)^{\alpha_1-1}\,ds \\ &\geq R'=\left\|(u,v)\right\|. \end{aligned}$$

Consequently,

$$\left\|T(u,v)\right\|=\max\left\{\left\|T_1(u,v)\right\|,\left\|T_2(u,v)\right\|\right\}\geq R'=\left\|(u,v)\right\| \quad \text{for any } (u,v)\in\partial K_{R'}. \tag{3.15}$$

Obviously, by the second inequality in ($\mathbf{H}_6$), (3.15) is still valid.

It follows from the above discussion, (3.10), (3.15), Lemmas 2.6 and 2.7, that for any $\lambda_1\in(0,\overline{\lambda}_1)$, $\lambda_2\in(0,\overline{\lambda}_2)$, T has a fixed point $(u,v)\in\overline{K}_{R'}\setminus K_R$ and $R\leq\|(u,v)\|\leq R'$. Since $\|(u,v)\|\geq R$, we have

$$\begin{aligned} u(t)-\varpi_1(t) &\geq u(t)-\left(\lambda_1\rho t^{\alpha_1-1}\int_0^1 q_1(s)\,ds+\lambda_2\rho t^{\alpha_1-1}\int_0^1 q_2(s)\,ds\right) \\ &\geq \omega t^{\alpha_1-1}R-\left(\lambda_1\rho t^{\alpha_1-1}\int_0^1 q_1(s)\,ds+\lambda_2\rho t^{\alpha_1-1}\int_0^1 q_2(s)\,ds\right) \\ &\geq t^{\alpha_1-1}\left(\omega R-\rho\int_0^1 q_1(s)\,ds-\rho\int_0^1 q_2(s)\,ds\right)>0, \quad t\in(0,1], \end{aligned}$$

$$\begin{aligned} v(t)-\varpi_2(t) &\geq v(t)-\left(\lambda_1\rho t^{\alpha_2-1}\int_0^1 q_1(s)\,ds+\lambda_2\rho t^{\alpha_2-1}\int_0^1 q_2(s)\,ds\right)\\ &\geq \omega t^{\alpha_2-1}R-\left(\lambda_1\rho t^{\alpha_2-1}\int_0^1 q_1(s)\,ds+\lambda_2\rho t^{\alpha_2-1}\int_0^1 q_2(s)\,ds\right)\\ &\geq t^{\alpha_2-1}\left(\omega R-\rho\int_0^1 q_1(s)\,ds-\rho\int_0^1 q_2(s)\,ds\right)>0,\quad t\in(0,1]. \end{aligned}$$

Let $\overline{l}' = \omega R - \rho\int_0^1 q_1(s)\,ds - \rho\int_0^1 q_2(s)\,ds$, $\overline{u}'(t) = u(t) - \varpi_1(t)$, $\overline{v}'(t) = v(t) - \varpi_2(t)$, then we have

$$\overline{u}'(t) \geq \overline{l}' t^{\alpha_1-1} > 0, \qquad \overline{v}'(t) \geq \overline{l}' t^{\alpha_2-1} > 0, \quad t\in[0,1].$$

By Lemma 2.5 we know that for any $\lambda_1\in(0,\overline{\lambda}_1)$, $\lambda_2\in(0,\overline{\lambda}_2)$, system (1.1) has at least one positive solution $(\overline{u}',\overline{v}')$; moreover, $(\overline{u}',\overline{v}')$ satisfies $\overline{u}'(t)\geq\overline{l}' t^{\alpha_1-1}$, $\overline{v}'(t)\geq\overline{l}' t^{\alpha_2-1}$, $t\in[0,1]$. The proof is completed. □

Remark 3.3 From the proof of Theorem 3.3, we know that the conclusion of Theorem 3.3 is valid if condition $(\mathbf{H}_6)$ is replaced by

$$\lim_{x\to+\infty}\inf_{\substack{t\in[c,d]\subset(0,1)\\ y\in[0,+\infty)}}\frac{f_2(t,x,y)}{x}=+\infty,\quad\text{or}\quad \lim_{y\to+\infty}\inf_{\substack{t\in[c,d]\subset(0,1)\\ x\in[0,+\infty)}}\frac{f_2(t,x,y)}{y}=+\infty.$$

Theorem 3.4 *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ *hold and that the following condition is satisfied*:

$(\mathbf{H}_7)$

$$\limsup_{x\to+\infty}\sup_{\substack{t\in[0,1]\\ y\in[0,+\infty)}}\frac{p_i(t,x,y)}{x}=0,\quad i=1,2,$$

and

$$\liminf_{x\to+\infty}\inf_{\substack{t\in[\tilde{a},\tilde{b}]\subset(0,1)\\ y\in[0,+\infty)}} f_1(t,x,y)>\wedge,\quad \textit{or}\quad \liminf_{y\to+\infty}\inf_{\substack{t\in[\tilde{a},\tilde{b}]\subset(0,1)\\ x\in[0,+\infty)}} f_1(t,x,y)>\wedge,$$

where $\wedge=\frac{4\int_0^1(q_1(s)+q_2(s))\,ds}{\omega^2\widetilde{\theta}\int_{\tilde{a}}^{\tilde{b}} s(1-s)^{\alpha_0-1}\,ds}$, $\alpha_0=\max\{\alpha_1,\alpha_2\}$. *Then there exist* $\tilde{\lambda}_1,\tilde{\lambda}_2>0$ *such that system* (1.1) *has at least one positive solution* $(\tilde{u},\tilde{v})$ *provided* $\lambda_1\in(\tilde{\lambda}_1,+\infty)$, $\lambda_2\in(\tilde{\lambda}_2,+\infty)$. *Moreover*, $(\tilde{u},\tilde{v})$ *satisfies* $\tilde{u}(t)\geq\tilde{l}t^{\alpha_1-1}$, $\tilde{v}(t)\geq\tilde{l}t^{\alpha_2-1}$, $t\in[0,1]$ *for some positive constant* $\tilde{l}$.

Proof It follows from

$$\liminf_{x\to+\infty}\inf_{\substack{t\in[\tilde{a},\tilde{b}]\subset(0,1)\\ y\in[0,+\infty)}} f_1(t,x,y)>\wedge$$

of $(\mathbf{H}_7)$, there exists $\widetilde{N}>0$ such that

$$f_1(t,x,y)\geq\wedge,\quad x\geq\widetilde{N}, y\geq 0, t\in[\tilde{a},\tilde{b}]. \tag{3.16}$$

Select

$$\tilde{\lambda}_i = \frac{\widetilde{N}}{2\rho\widetilde{\theta}\int_0^1 q_i(s)\,ds}, \quad i = 1, 2.$$

In proving the theorem, we assume

$$R_1 = \max\left\{\{\lambda_1 + \lambda_2, 2\lambda_1, 2\lambda_2\}2\omega^{-1}\rho\int_0^1 \bigl(q_1(s) + q_2(s)\bigr)\,ds\right\}$$

and

$$K_{R_1} = \bigl\{(u,v) \in K : \bigl\|(u,v)\bigr\| < R_1\bigr\},$$

where $\widetilde{\theta} = \min_{t\in[\widetilde{a},\widetilde{b}]}\{t^{\alpha_1-1}, t^{\alpha_2-1}\}$. For any $(u,v) \in \partial K_{R_1}$, by (2.23), we have

$$\begin{aligned}
u(t) - \varpi_1(t) &\geq \omega t^{\alpha_1-1}R_1 - \left(\lambda_1\rho t^{\alpha_1-1}\int_0^1 q_1(s)\,ds + \lambda_2\rho t^{\alpha_1-1}\int_0^1 q_2(s)\,ds\right) \\
&= t^{\alpha_1-1}\left(\omega R_1 - \left(\lambda_1\rho\int_0^1 q_1(s)\,ds + \lambda_2\rho\int_0^1 q_2(s)\,ds\right)\right) \\
&\geq \widetilde{\theta}\left(\omega R_1 - \left(\lambda_1\rho\int_0^1 q_1(s)\,ds + \lambda_2\rho\int_0^1 q_2(s)\,ds\right)\right) \\
&\geq \lambda_1\rho\widetilde{\theta}\int_0^1 q_1(s)\,ds + \lambda_2\rho\widetilde{\theta}\int_0^1 q_2(s)\,ds \geq \widetilde{N}, \quad t \in [\tilde{a}, \tilde{b}], \qquad (3.17)\\
v(t) - \varpi_2(t) &\geq \omega t^{\alpha_2-1}R_1 - \left(\lambda_1\rho t^{\alpha_2-1}\int_0^1 q_1(s)\,ds + \lambda_2\rho t^{\alpha_2-1}\int_0^1 q_2(s)\,ds\right) \\
&= t^{\alpha_2-1}\left(\omega R_1 - \left(\lambda_1\rho\int_0^1 q_1(s)\,ds + \lambda_2\rho\int_0^1 q_2(s)\,ds\right)\right) \\
&\geq \widetilde{\theta}\left(\omega R_1 - \left(\lambda_1\rho\int_0^1 q_1(s)\,ds + \lambda_2\rho\int_0^1 q_2(s)\,ds\right)\right) \\
&\geq \lambda_1\rho\widetilde{\theta}\int_0^1 q_1(s)\,ds + \lambda_2\rho\widetilde{\theta}\int_0^1 q_2(s)\,ds \geq \widetilde{N}, \quad t \in [\tilde{a}, \tilde{b}]. \qquad (3.18)
\end{aligned}$$

Thus, for any $(u,v) \in \partial K_{R_1}$, by (3.16)-(3.18), we have

$$f_1\bigl(t, \bigl[u(t) - \varpi_1(t)\bigr]^*, \bigl[v(t) - \varpi_2(t)\bigr]^*\bigr) \geq \wedge, \quad t \in [\tilde{a}, \tilde{b}]. \qquad (3.19)$$

Hence, for any $(u,v) \in \partial K_{R_1}$, by (3.19) and Lemma 2.3, we have

$$\begin{aligned}
\bigl\|T_1(u,v)\bigr\| &= \max_{t\in[0,1]}\left|\lambda_1\int_0^1 K_1(t,s)\bigl(f_1\bigl(s, \bigl[u(s) - \varpi_1(s)\bigr]^*, \bigl[v(s) - \varpi_2(s)\bigr]^*\bigr) + q_1(s)\bigr)\,ds\right. \\
&\quad \left. + \lambda_2\int_0^1 H_1(t,s)\bigl(f_2\bigl(s, \bigl[u(s) - \varpi_1(s)\bigr]^*, \bigl[v(s) - \varpi_2(s)\bigr]^*\bigr) + q_2(s)\bigr)\,ds\right| \\
&\geq \max_{t\in[0,1]}\lambda_1\int_0^1 K_1(t,s)\bigl(f_1\bigl(s, \bigl[u(s) - \varpi_1(s)\bigr]^*, \bigl[v(s) - \varpi_2(s)\bigr]^*\bigr) + q_1(s)\bigr)\,ds
\end{aligned}$$

$$\geq \min_{t\in[\tilde{a},\tilde{b}]} \lambda_1 \int_{\tilde{a}}^{\tilde{b}} \varrho t^{\alpha_1-1} s(1-s)^{\alpha_1-1} f_1\bigl(s,\bigl[u(s)-\varpi_1(s)\bigr]^*,\bigl[v(s)-\varpi_2(s)\bigr]^*\bigr)\,ds$$

$$\geq \lambda_1 \widetilde{\theta} \wedge \int_{\tilde{a}}^{\tilde{b}} \varrho s(1-s)^{\alpha_0-1}\,ds \geq R_1 = \bigl\|(u,v)\bigr\|.$$

Consequently,

$$\bigl\|T(u,v)\bigr\| = \max\bigl\{\bigl\|T_1(u,v)\bigr\|,\bigl\|T_2(u,v)\bigr\|\bigr\} \geq R_1 = \bigl\|(u,v)\bigr\| \quad \text{for any } (u,v)\in\partial K_{R_1}. \tag{3.20}$$

Obviously, by the inequality

$$\liminf_{y\to+\infty} \inf_{\substack{t\in[\tilde{a},\tilde{b}]\subset(0,1)\\ x\in[0,+\infty)}} f_1(t,x,y) > \wedge$$

in ($\mathbf{H}_7$), (3.20) is still valid.

On the other hand, choose $\varepsilon_i > 0$ such that

$$\varepsilon_i = \left(3\lambda_i\rho\int_0^1 a_i(s)\,ds\right)^{-1}, \quad i=1,2.$$

Then, for the above ε_i, by the first inequality in ($\mathbf{H}_7$), there exists $N' > 0$ such that for any $t\in[0,1]$, we have

$$p_i(t,x,y) \leq \varepsilon_i x, \quad x\geq N', y\geq 0, i=1,2.$$

Then we have

$$p_i(t,x,y) \leq \Phi + \varepsilon_i x, \quad t\in[0,1], x\geq 0, y\geq 0, i=1,2, \tag{3.21}$$

where $\Phi = \max\{p_i(t,x,y) : t\in[0,1], 0\leq x\leq N', 0\leq y\leq N', i=1,2\}$. Select

$$R_2 \geq \max\left\{2R_1, (\Phi+1)\left(\lambda_1\rho\int_0^1\bigl(a_1(s)+q_1(s)\bigr)\,ds + \lambda_2\rho\int_0^1\bigl(a_2(s)+q_2(s)\bigr)\,ds\right)\right\}.$$

Assume

$$K_{R_2} = \bigl\{(u,v)\in K : \bigl\|(u,v)\bigr\| < R_2\bigr\}.$$

For any $(u,v)\in\partial K_{R_2}$, by (3.21) and Lemma 2.3, we have

$$\begin{aligned}\bigl\|T_1(u,v)\bigr\| &= \max_{t\in[0,1]}\biggl|\lambda_1\int_0^1 K_1(t,s)\bigl(f_1\bigl(s,\bigl[u(s)-\varpi_1(s)\bigr]^*,\bigl[v(s)-\varpi_2(s)\bigr]^*\bigr)+q_1(s)\bigr)\,ds \\ &\quad + \lambda_2\int_0^1 H_1(t,s)\bigl(f_2\bigl(s,\bigl[u(s)-\varpi_1(s)\bigr]^*,\bigl[v(s)-\varpi_2(s)\bigr]^*\bigr)+q_2(s)\bigr)\,ds\biggr| \\ &\leq \max_{t\in[0,1]}\biggl|\lambda_1\int_0^1 \rho t^{\alpha_1-1}\bigl(a_1(s)p_1\bigl(s,\bigl[u(s)-\varpi_1(s)\bigr]^*,\bigl[v(s)-\varpi_2(s)\bigr]^*\bigr)+q_1(s)\bigr)\,ds \\ &\quad + \lambda_2\int_0^1 \rho t^{\alpha_1-1}\bigl(a_2(s)p_2\bigl(s,\bigl[u(s)-\varpi_1(s)\bigr]^*,\bigl[v(s)-\varpi_2(s)\bigr]^*\bigr)+q_2(s)\bigr)\,ds\biggr|\end{aligned}$$

$$\begin{aligned}
&\le \lambda_1 \rho \int_0^1 \bigl(a_1(s)\bigl(\Phi + \varepsilon_1\bigl[u(s) - \varpi_1(s)\bigr]^*\bigr) + q_1(s)\bigr)\,ds \\
&\quad + \lambda_2 \rho \int_0^1 \bigl(a_2(s)\bigl(\Phi + \varepsilon_2\bigl[u(s) - \varpi_1(s)\bigr]^*\bigr) + q_2(s)\bigr)\,ds \\
&\le (\Phi + 1)\biggl(\lambda_1 \rho \int_0^1 \bigl(a_1(s) + q_1(s)\bigr)\,ds + \lambda_2 \rho \int_0^1 \bigl(a_2(s) + q_2(s)\bigr)\,ds\biggr) \\
&\quad + \lambda_1 \rho \varepsilon_1 \|u\| \int_0^1 a_1(s)\,ds + \lambda_2 \rho \varepsilon_2 \|u\| \int_0^1 a_2(s)\,ds \\
&\le R_2 = \bigl\|(u,v)\bigr\|.
\end{aligned} \tag{3.22}$$

Similarly as (3.22), for any $(u,v) \in \partial K_{R_2}$, by (3.21) and Lemma 2.3, we also have

$$\bigl\|T_2(u,v)\bigr\| \le R_2 = \bigl\|(u,v)\bigr\|.$$

Consequently, we have

$$\bigl\|T(u,v)\bigr\| = \max\bigl\{\bigl\|T_1(u,v)\bigr\|, \bigl\|T_2(u,v)\bigr\|\bigr\} \le R_2 = \bigl\|(u,v)\bigr\| \quad \text{for any } (u,v) \in \partial K_{R_2}. \tag{3.23}$$

It follows from the above discussion, (3.20), (3.23), Lemmas 2.6 and 2.7, that for any $\lambda_1 \in (\tilde{\lambda}_1, +\infty)$, $\lambda_2 \in (\tilde{\lambda}_2, +\infty)$, T has a fixed point $(u,v) \in \overline{K}_{R_2} \setminus K_{R_1}$ and $R_1 \le \|(u,v)\| \le R_2$. Since $\|(u,v)\| \ge R_1$, by the same method as Theorem 3.3, we know that for any $\lambda_1 \in (\tilde{\lambda}_1, +\infty)$, $\lambda_2 \in (\tilde{\lambda}_2, +\infty)$, system (1.1) has at least one positive solution $(\tilde{u}, \tilde{v})$. Moreover, $(\tilde{u}, \tilde{v})$ satisfies $\tilde{u}(t) \ge \tilde{l} t^{\alpha_1 - 1}$, $\tilde{v}(t) \ge \tilde{l} t^{\alpha_2 - 1}$, $t \in [0,1]$. The proof is completed. □

Remark 3.4 From the proof of Theorem 3.4, we know that the conclusion of Theorem 3.4 is valid if the second equality of condition ($\mathbf{H}_7$) is replaced by

$$\lim_{x \to +\infty} \inf_{\substack{t \in [\tilde{a}, \tilde{b}] \subset (0,1) \\ y \in [0,+\infty)}} f_1(t,x,y) = +\infty, \quad \text{or} \quad \lim_{y \to +\infty} \inf_{\substack{t \in [\tilde{a}, \tilde{b}] \subset (0,1) \\ x \in [0,+\infty)}} f_1(t,x,y) = +\infty.$$

Remark 3.5 From the proof of Theorem 3.4, we know that the conclusion of Theorem 3.4 is valid if the second equality of condition ($\mathbf{H}_7$) is replaced by

$$\liminf_{x \to +\infty} \inf_{\substack{t \in [\tilde{a}, \tilde{b}] \subset (0,1) \\ y \in [0,+\infty)}} f_2(t,x,y) > \wedge, \quad \text{or} \quad \liminf_{y \to +\infty} \inf_{\substack{t \in [\tilde{a}, \tilde{b}] \subset (0,1) \\ x \in [0,+\infty)}} f_2(t,x,y) > \wedge,$$

where $\wedge$ is defined in Theorem 3.4.

Similarly as Remark 3.4, the conclusion of Theorem 3.4 is also valid if the second equality of condition ($\mathbf{H}_7$) is replaced by

$$\lim_{x \to +\infty} \inf_{\substack{t \in [\tilde{a}, \tilde{b}] \subset (0,1) \\ y \in [0,+\infty)}} f_2(t,x,y) = +\infty, \quad \text{or} \quad \lim_{y \to +\infty} \inf_{\substack{t \in [\tilde{a}, \tilde{b}] \subset (0,1) \\ x \in [0,+\infty)}} f_2(t,x,y) = +\infty.$$

4 Example

Consider the fractional differential system

$$\begin{cases} D_{0^+}^{\frac{5}{2}} u(t) + \lambda_1 f_1(t, u(t), v(t)) = 0, \\ D_{0^+}^{\frac{5}{2}} v(t) + \lambda_2 f_2(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad u(1) = \frac{1}{2}\int_0^1 v(t)\,dt, \\ v(0) = v'(0) = 0, \qquad v(1) = \int_0^1 u(t)\,dt^{\frac{1}{2}}, \end{cases} \tag{4.1}$$

where $\lambda_i > 0$ $(i = 1, 2)$ is a parameter, $\alpha_1 = \alpha_2 = \frac{5}{2}$, $\mu_1 = \frac{1}{2}$, $\mu_2 = 1$, $A_1(t) = t$, $A_2(t) = t^{\frac{1}{2}}$. Then we have

$$k_1 = \int_0^1 t^{\alpha_2 - 1}\,dA_1(t) = \int_0^1 t^{\frac{3}{2}}\,dt = \frac{2}{5} > 0,$$
$$k_2 = \int_0^1 t^{\alpha_1 - 1}\,dA_2(t) = \int_0^1 t^{\frac{3}{2}}\,dt^{\frac{1}{2}} = \frac{1}{2}\int_0^1 t\,dt = \frac{1}{4} > 0, \quad 1 - \mu_1\mu_2 k_1 k_2 = \frac{19}{20} > 0.$$

So, condition $(\mathbf{H}_0)$ holds.

Next, in order to demonstrate the application of our main results obtained in Section 3, we choose two different sets of functions $f_i(t, x, y)$ $(i = 1, 2)$ such that f_i satisfies the conditions of Theorems 3.3 and 3.4.

Case 1. Let $f_1(t, x, y) = \frac{x^2 + y^2}{\sqrt{t(1-t)}} + \ln t$, $f_2(t, x, y) = \frac{1 + e^x + e^y}{\sqrt{t(1-t)}} + \ln(1 - t)$, $(t, x, y) \in (0, 1) \times [0, +\infty) \times [0, +\infty)$. Take $a_1(t) = a_2(t) = \frac{1}{\sqrt{t(1-t)}}$, $q_1(t) = -\ln t$, $q_2(t) = -\ln(1 - t)$, $p_1(t, x, y) = x^2 + y^2$, $p_2(t, x, y) = 1 + e^x + e^y$, then

$$-q_i(t) \le f(t, x, y) \le a_i(t) p_i(t, x, y), \quad (t, x, y) \in (0, 1) \times [0, +\infty) \times [0, +\infty), i = 1, 2.$$

By a direct calculation, we have

$$\int_0^1 a_1(t)\,dt = \int_0^1 a_2(t)\,dt = \pi, \qquad \int_0^1 q_1(t)\,dt = \int_0^1 q_2(t)\,dt = 1.$$

So condition $(\mathbf{H}_1)$ holds.

In addition, choose $[\frac{1}{3}, \frac{2}{3}] \subset [0, 1]$, we know

$$\lim_{x \to +\infty} \inf_{\substack{t \in [\frac{1}{3}, \frac{2}{3}] \\ y \in [0, +\infty)}} \frac{f_1(t, x, y)}{x} = +\infty, \quad \text{or} \quad \lim_{y \to +\infty} \inf_{\substack{t \in [\frac{1}{3}, \frac{2}{3}] \\ x \in [0, +\infty)}} \frac{f_1(t, x, y)}{y} = +\infty,$$

so condition $(\mathbf{H}_6)$ of Theorem 3.3 is satisfied.

Therefore, by Theorem 3.3, we obtain that system (4.1) has at least one positive solution provided $\lambda_i > 0$ $(i = 1, 2)$ is small enough.

Case 2. Let $f_1(t, x, y) = \frac{\sqrt{2}(x+y)^{\frac{1}{2}}}{\sqrt[4]{t^3(1-t)}(1+t^2(1-t))} - \frac{2}{\sqrt{t}}$, $f_2(t, x, y) = \frac{1}{e^{x+y+1}\sqrt{t(1-t)}} - \frac{3}{\sqrt{1-t}}$, $(t, x, y) \in (0, 1) \times [0, +\infty) \times [0, +\infty)$. Take $a_1(t) = \frac{\sqrt{2}}{\sqrt[4]{t^3(1-t)}}$, $a_2(t) = \frac{1}{\sqrt{t(1-t)}}$, $q_1(t) = \frac{2}{\sqrt{t}}$, $q_2(t) = \frac{3}{\sqrt{1-t}}$, $p_1(t, x, y) = \frac{(x+y)^{\frac{1}{2}}}{1+t^2(1-t)}$, $p_2(t, x, y) = \frac{1}{e^{x+y+1}}$, then

$$-q_i(t) \le f(t, x, y) \le a_i(t) p_i(t, x, y), \quad (t, x, y) \in (0, 1) \times [0, +\infty) \times [0, +\infty), i = 1, 2.$$

By a direct calculation, we have

$$\int_0^1 a_1(t)\,dt = 2\pi, \qquad \int_0^1 a_2(t)\,dt = \pi, \qquad \int_0^1 q_1(t)\,dt = 4, \qquad \int_0^1 q_2(t)\,dt = 6.$$

So condition $(\mathbf{H}_1)$ holds.

In addition, choose $[\frac{1}{4}, \frac{3}{4}] \subset [0,1]$, we know

$$\limsup_{x\to+\infty} \sup_{\substack{t\in[0,1]\\ y\in[0,+\infty)}} \frac{p_i(t,x,y)}{x} = 0, \quad i = 1,2,$$

and

$$\liminf_{x\to+\infty} \inf_{\substack{t\in[\frac{1}{4},\frac{3}{4}]\\ y\in[0,+\infty)}} f_1(t,x,y) = +\infty, \quad \text{or} \quad \liminf_{y\to+\infty} \inf_{\substack{t\in[\frac{1}{4},\frac{3}{4}]\\ x\in[0,+\infty)}} f_1(t,x,y) = +\infty,$$

so condition $(\mathbf{H}_7)$ of Theorem 3.4 is satisfied.

Therefore, by Theorem 3.4, we obtain that system (4.1) has at least one positive solution provided $\lambda_i > 0$ $(i = 1,2)$ is sufficiently large.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The study was carried out in collaboration between all authors. YW completed the main part of this paper and gave two examples; LSL and YHW corrected the main theorems and polished the manuscript. All authors read and approved the final manuscript.

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