An Analytical Study of PPP-RTK Corrections: Precision, Correlation and User-Impact

A. Khodabandeh · P.J.G. Teunissen

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Abstract PPP-RTK extends the PPP-concept by providing single-receiver users, next to orbits and clocks, also information about the satellite phase and code biases, thus enabling single-receiver ambiguity resolution. It is the goal of the present contribution to provide an analytical study of the quality of the PPP-RTK corrections as well as of their impact on the user ambiguity resolution performance. We consider the geometry-free (GF) and the geometry-based (GB) network derived corrections, as well as the impact of network ambiguity resolution on these corrections. Next to the insight that is provided by the analytical solutions, the closed form expressions of the variance matrices also demonstrate how the corrections depend on network parameters such as number of epochs, number of stations, number of satellites, and number of frequencies. As a result we are able to describe in a qualitative sense how the user ambiguity resolution performance is driven by the data from the different network scenarios.

Keywords Global Navigation Satellite Systems (GNSS), Precise Point Positioning (PPP), Integer Ambiguity Resolution (IAR), PPP-RTK corrections, Geometry-Free (GF), Geometry-Based (GB), Ambiguity Dilution of Precision (ADOP)

1 Introduction

PPP-RTK is integer ambiguity resolution enabled precise point positioning (PPP) (Wubbena et al, 2005; Mervart et al, 2008). It extends the PPP-concept (Heroux and Kouba, 1995; Zumberge et al, 1997) by providing single-receiver users, next to the orbits and clocks, also information about the satellite phase and code biases. This information, when properly provided, enables recovery of the integrerness of the user-ambiguities, thus enabling single-receiver ambiguity resolution, thereby reducing convergence times as compared to that of PPP.

Several PPP-RTK methods have been formulated in recent years, see e.g., Wubbena et al (2005); Laurichesse and Mercier (2007); Mervart et al (2008); Collins (2008); Ge et al (2008); Bertiger et al (2010); Teunissen et al (2010); Geng et al (2012); Loyer et al (2012); Geng and Bock (2013); Lannes and Prieur (2013); Banville et al (2014). For an overview and a critical comparison of these methods, see the review (Teunissen and Khodabandeh, 2015). As was demonstrated in (ibid), a careful interpretation of the estimable parameters involved is essential for obtaining a proper insight into the general mechanics of PPP-RTK. It is the goal of the present contribution to take this one step further by providing an analytical study of the multi-frequency PPP-RTK corrections themselves, thereby presenting a precision and correlation analysis that will enable us to demonstrate how the quality of these corrections, as well as their impact on the user parameters, are driven by the information content and adjustment of the external network.

PPP-RTK is founded on the idea of single-receiver integer ambiguity resolution (IAR). This idea, together with the estimability of the associated PPP-RTK corrections, is best described by starting with the single-
receiver user observation equations. Consider the user’s antenna \( u \) tracking \( f \)-frequency GNSS data that are transmitted by a satellite \( s \) and a chosen pivot satellite \( p \). The corresponding between-satellite single-difference (SD) observation equations read then (Tennissen and Kleusberg, 1998; Hofmann-Wellenhof et al, 2008)

\[
\Delta \phi_{u,j}^{ps} = g_{ps}^T \Delta x_u - \mu_j \phi_{u,j}^{ps} - dt^{ps} + \lambda_j (z_{u,j}^{ps} - \delta_{j}^{ps})
\]
\[
\Delta \phi_{j}^{ps} = g_{ps}^T \Delta x_u + \mu_j \phi_{j}^{ps} - dt^{ps} - d_{j}^{ps}
\]

where \( \Delta \phi_{u,j}^{ps} \) and \( \Delta \phi_{j}^{ps} \) denote the SD ‘observed-minus-computed’ phase and code observables on the frequency band \( f_j \) \( (j = 1, \ldots, f) \), respectively. Here and in the following, the precise orbital corrections are assumed included in the ‘observed-minus-computed’ observables. The \( \nu \)-vector \( \Delta x_u \) contains the user’s position increments and/or the zenith tropospheric delay (ZTD). Parameters \( \nu \) can take the values \( \nu = 3 \) (position-only model), \( \nu = 1 \) (ZTD-only model) or \( \nu = 4 \) (position-plus-ZTD model). Thus the \( \nu \)-vector \( g_{ps} \) contains the user-receiver satellite-direction vector and/or the SD slant ionospheric delay, expressed on the first frequency, is denoted by \( z_{u,j}^{ps} \). Thus the frequency-dependent coefficients are defined as the ratio \( \mu_j = (f_j^2/f_1^2) \). The SD integer ambiguity \( z_{u,j}^{ps} \in \mathbb{Z} \) and the SD satellite phase bias \( \delta_{j}^{ps} \) both expressed in cycles, are linked to the phase observables through the wavelength \( \lambda_j \). The SD satellite clocks are denoted by \( dt^{ps} \), while the SD satellite code biases are denoted by \( d_{j}^{ps} \). Apart from \( z_{u,j}^{ps} \) and \( \delta_{j}^{ps} \), the rest of the quantities are all expressed in units of range. We assume that \( m \) satellites are tracked and thus \( p, s \in \{h | h = 1, \ldots, m\} \), with \( p \neq s \).

If we make use of the more compact multi-frequency vector notation \( \Delta \phi_{u}^{ps} = [\Delta \phi_{u,1}^{ps}, \ldots, \Delta \phi_{u,f}^{ps}]^T \), \( \Delta \phi_{p}^{ps} = [\Delta \phi_{p,1}^{ps}, \ldots, \Delta \phi_{p,f}^{ps}]^T \), \( \mu = [\mu_1, \ldots, \mu_f]^T \), \( \mu_j = [z_{u,1}^{ps}, \ldots, z_{u,f}^{ps}]^T \), \( \delta_{j}^{ps} = [\delta_{j,1}^{ps}, \ldots, \delta_{j,f}^{ps}]^T \), and \( d_{j}^{ps} = [d_{j,1}^{ps}, \ldots, d_{j,f}^{ps}]^T \), we may write (1) as

\[
\Delta \phi_{u}^{ps} = e \Delta \phi_{u}^{ps} - \mu \tilde{v}_{u}^{ps} - e dt^{ps} + A (z_{u}^{ps} - \delta_{u}^{ps})
\]
\[
\Delta \phi_{p}^{ps} = e \Delta \phi_{p}^{ps} + \mu \tilde{v}_{p}^{ps} - e dt^{ps} - d_{p}^{ps}
\]

where \( e = [1, \ldots, 1]^T \), \( A = \text{diag}(\lambda_1, \ldots, \lambda_f) \), and \( \Delta \phi_{u}^{ps} = g_{ps}^T \Delta x_u \). The user observation equations (2) do not contain enough information to solve for an integer ambiguity resolved user position. This would become possible though, were information about the satellite clocks, \( dt^{ps} \), and satellite biases, \( \delta_{j}^{ps} \) and \( d_{j}^{ps} \), be given. Using such externally provided information to correct the observations as

\[
\Delta \phi_{u}^{ps} = \Delta \phi_{u}^{ps} + e dt^{ps} + A \delta_{u}^{ps}
\]
\[
\Delta \phi_{p}^{ps} = \Delta \phi_{p}^{ps} + e dt^{ps} + d_{p}^{ps}
\]

results in user-equations that take the form

\[
\Delta \phi_{u}^{ps} = e \Delta \phi_{u}^{ps} - \mu \tilde{v}_{u}^{ps} + A z_{u}^{ps}
\]
\[
\Delta \phi_{p}^{ps} = e \Delta \phi_{p}^{ps} + \mu \tilde{v}_{p}^{ps}
\]

This system is now in a form that can be used to solve for the integer ambiguity resolved user parameters \( \Delta \phi_{u,i} \) and \( \Delta \phi_{p,i} \). Hence, with externally provided corrections \( dt^{ps}, \delta_{j}^{ps}, \) and \( d_{j}^{ps} \), the user system of observation equations (4) can be solved as a mixed-integer system of equations, thereby profiting from the integerness of \( z_{u,i}^{ps} \in \mathbb{Z}^f \). This is the basic idea of single-receiver, IAR-enabled, positioning.

The operationalization of this basic idea is somewhat more involved however. This is due to the fact that the above needed parameters \( dt^{ps}, \delta_{j}^{ps}, d_{j}^{ps} \) cannot be determined as such. GNSS-data is namely not capable of providing these ‘absolute’ parameters, but instead only estimable functions that can act as such. These estimable parameters, denoted with a tilde as \( \tilde{d}^{ps}, \tilde{\delta}^{ps}, \tilde{\delta}^{ps} \), achieve the same goal, namely of enabling the construction of a user system of observation equations that is in mixed-integer form. Thus although they are not the original absolute parameters,

\[
d^{ps} \neq \tilde{d}^{ps}, \tilde{\delta}^{ps} \neq \delta^{ps}, \tilde{\delta}^{ps} \neq d^{ps}
\]

they still do the job in ensuring that the user can work with integer ambiguities. When they are used to correct the user-observations as

\[
\Delta \phi_{u,i}^{ps} = \Delta \phi_{u,i}^{ps} + e \tilde{d}^{ps} + A \tilde{\delta}^{ps}
\]
\[
\Delta \phi_{p,i}^{ps} = \Delta \phi_{p,i}^{ps} + e \tilde{d}^{ps} + \tilde{\delta}^{ps}
\]

the user equations take the form

\[
\Delta \phi_{u,i}^{ps} = e \Delta \phi_{u,i}^{ps} - \mu \tilde{v}_{u,i}^{ps} + A z_{u,i}^{ps}
\]
\[
\Delta \phi_{p,i}^{ps} = e \Delta \phi_{p,i}^{ps} + \mu \tilde{v}_{p,i}^{ps}
\]

with integer \( z_{u,i}^{ps} \in \mathbb{Z}^f \). Thus the structure of these equations is indeed identical to that of the mixed-integer system (4), be it that the interpretation of the estimable parameters in (7) is different from those of (4). This difference in parameter interpretation is important and it is due to the differences in (5).

It is the goal of the present contribution to provide an analytical study of the estimable PPP-RTK corrections \( d^{ps}, \delta^{ps}, \tilde{d}^{ps} \), and \( \tilde{d}^{ps} \), as well as of their impact on the user ambiguity resolution performance. In Sect. 2 we start by introducing single-station PPP-RTK, while in Sect. 3 we warn for the pitfalls that exist when evaluating the PPP-RTK corrections on an individual basis. The results of Sect. 2 are generalized in Sect. 4 and Sect. 5, respectively, to the geometry-free (GF) and geometry-based (GB) network case. This is done in network ambiguity-fixed mode as well as in network ambiguity-fixed mode. Next to the insight that is provided by the analytical solutions, the closed form expressions of the variance matrices also demonstrate how the corrections depend on network parameters such as number of epochs, number of stations, number of satellites, and
number of frequencies. As a result we are able to describe in a qualitative sense how the user ambiguity resolution performance is driven by the data from the different network scenarios. A summary with conclusions is finally provided for in Sect. 6.

2 Single-Station PPP-RTK

Although estimators of the PPP-RTK corrections \( \tilde{d}_{ps}, \tilde{\delta}_{ps}, \) and \( \tilde{d}_{ps} \) are usually computed from an external network, they can—as the below will show—be obtained from the data of a single station as well.

2.1 Single-station corrections

The observation equations of a single reference station \( r \) follow by replacing the user index \( u \) in (2) by \( r \),

\[
\Delta \phi^p_r = e (\Delta \rho^p_r - dt^p_r) - \mu \tau^p_r + A (z^p_r - \delta^p_r)
\]

\[
\Delta p^p_r = e (\Delta \rho^p_r - dt^p_r) + \mu \tau^p_r - d^p_r
\]

(8)

These equations are underdetermined as there are 2\( f \) equations in 3\( f \) + 3 unknowns. The rank defect is \( f + 3 \). There are many different ways to eliminate a rank defect, each with a different interpretation of the resulting estimable parameters. These different sets of estimable parameters are linked by \( S \)-transformations (Baarda, 1973; Teunissen, 1985). Examples of such different sets in the context of PPP-RTK can be found in (Teunissen et al., 2010; Zhang et al., 2011; Lannes and Teunissen, 2011; Odijk et al., 2012) and (Teunissen and Khodabandeh, 2015).

To eliminate the rank defect of the above system (8), we first lump the parameters that have common coefficients,

\[
\rho^p_r := \Delta \rho^p_r - dt^p_r, \quad \tau^p_r := z^p_r - \delta^p_r
\]

(9)

As this takes care of \( f + 1 \) rank defects, there are still 2 defects that need to be taken care of. This will be done by applying the ionosphere-free/geometry-free decomposition of the code bias \( d^p_s \),

\[
d^p_s = [e, \mu, E] \begin{bmatrix} \mu_T^p_r \\ \mu_G^p_r \\ E^- \end{bmatrix}, \text{ with } [e, \mu, E]^{-1} = \begin{bmatrix} \mu_T^p_r \\ \mu_G^p_r \\ E^- \end{bmatrix}
\]

(10)

where

\[
\mu_T^p_r = \frac{1}{\mu_2 - \mu_1} [\mu_2, \mu_1, 0, \ldots, 0]^T
\]

\[
\mu_G^p_r = \frac{1}{\mu_2 - \mu_1} [-1, 1, 0, \ldots, 0]^T
\]

\[
E^- = E^T (I_f - e \mu_T^p_r - \mu \mu_T^p_r)
\]

(11)

The \( f \times (f - 2) \) matrix \( E \) is structured by eliminating the first two columns of \( I_f \).

The above decomposition shows that the ionosphere-free and geometry-free combinations \( d_{Tps}^p \) and \( d_{Gps}^p \) have the same coefficients as \( \rho_{ps}^p \) and \( \tau_{ps}^p \), namely \( e \) and \( \mu \), respectively. Hence, a further lumping can take place, thus taking care of the remaining two rank defects. Substitution of (10) into (8) gives therefore, together with (9), the full-rank single-station model as

\[
E(\rho^p_r) = e \tilde{\rho}_r^p - \mu \tilde{\tau}_r^p + A \tilde{\delta}_r^p
\]

\[
E(\rho^p_s) = e \tilde{\rho}_s^p + \mu \tilde{\tau}_s^p - E \tilde{d}_s^p
\]

(12)

where

\[
\tilde{\rho}_r^p = \rho^p_r - d_{Tps}^p
\]

\[
\tilde{\delta}_r^p = \delta^p_r + A^{-1} \left( \mu \delta^p_r - \mu \delta^p_r \right)
\]

\[
\tilde{\delta}_s^p = \delta^p_s + A^{-1} \left( \mu \delta^p_s - \mu \delta^p_s \right)
\]

(13)

The estimable parameters in the above system can be interpreted as a biased range \( \tilde{\rho}_r^p \), a biased ambiguity \( \tilde{\delta}_s^p \), and a biased ionospheric delay \( \tilde{d}_s^p \). However, note that with the definitions

\[
d_{Tps}^p := - \tilde{\rho}_r^p \quad \text{and} \quad \tilde{d}_s^p := - \tilde{\delta}_s^p
\]

(14)

the ‘range’ and ‘ambiguity’ can likewise be interpreted as a biased clock \( \tilde{d}_s^p \) and a biased phase-bias \( \tilde{\delta}_s^p \), thus giving instead of (13), the estimable parameters

\[
d_{Tps}^p := d_{Tps}^p + \delta^p_s - \Delta \rho^p_r
\]

\[
\Delta \rho^p_r = \Delta \rho^p_r - \Delta \rho^p_r - A^{-1} \left( \mu \delta^p_r - \mu \delta^p_r \right) - z^p_s
\]

\[
\tilde{\delta}_s^p = \tilde{\delta}_s^p + A^{-1} \left( \mu \delta^p_s - \mu \delta^p_s \right)
\]

(15)

The corresponding system of observation equations now reads instead of (12),

\[
E(\rho^p_r) = - e \tilde{d}_s^p - \mu \tilde{\tau}_s^p - A \tilde{\delta}_s^p
\]

\[
E(\rho^p_s) = - e \tilde{d}_s^p + \mu \tilde{\tau}_s^p - E \tilde{d}_s^p
\]

(16)

As this is an invertible system of 2\( f \) equations in 2\( f \) unknowns per satellite pair \( ps \), its solution follows after inversion as (cf. 10)

\[
\tilde{d}_s^p = - \mu T^p_r \tilde{\rho}_s^p - \delta^p_s + (-1) [\tilde{d}_s^p + (\mu T^p_r - e \mu_T^p_r)p^r_s] \]

\[
\tilde{\delta}_s^p = \tilde{\delta}_s^p + A^{-1} \left( \mu \delta^p_s - \mu \delta^p_s \right)
\]

\[
\tilde{d}_s^p = - E^T \tilde{d}_s^p + \mu \mu_T^p_r \tilde{\rho}_s^p + \mu \mu_T^p_r \tilde{\rho}_s^p
\]

(17)

In the following the user-aiding functionality of each of these estimators is described.

2.2 The individual corrections applied

To show the effect that each of the PPP-RTK corrections \( \tilde{d}_{ps}^p \), \( \tilde{\delta}_{ps}^p \), and \( \tilde{d}_{ps}^p \) has, we apply them sequentially to the user observation equations

\[
E(\rho^p_r) = e \tilde{\rho}_r^p + \mu \tilde{\tau}_r^p + A \tilde{\delta}_r^p
\]

\[
E(\rho^p_s) = e \tilde{\rho}_s^p + \mu \tilde{\tau}_s^p - E \tilde{d}_s^p
\]

(19)
2.2.1 Clock $\delta \nu$ provides positional link

For the expectation of the clock correction we have (cf. 15),

$$E(d \nu) = d \nu = dt \nu + d \nu_p - \Delta \nu_p$$

(20)

Application to (19) gives

$$E(\delta \nu_u + e dt \nu) = e \Delta \nu_p - \mu \tilde{v}_u + \Lambda \hat{\delta}_u$$

$$E(p \nu_u + e dt \nu) = e \Delta \nu_p + \mu \tilde{v}_u - E \tilde{d} \nu$$

(21)

with $\Delta \nu_p = \tilde{\nu}_p - \nu_p = \Delta \nu_p - \Delta \nu_p$ being the double-differenced (DD) geometric/tropospheric delay’s increment. This shows, when comparing (21) with (19), that the satellite clock correction has the function to establish a positional link between user $u$ and reference $r$.

2.2.2 Phase-bias $\delta \nu$ provides ambiguity link

Although (21) is solvable for the user’s position, it is not yet in mixed-integer form, since $\tilde{\delta}_u \notin \mathbb{Z}^f$. To enable user integer ambiguity resolution, the satellite phase-bias $\tilde{\delta} \nu$ is needed. For the expectation of the phase-bias correction we have (cf. 15)

$$E(\delta \nu) = \delta \nu = \delta \nu + \Lambda^{-1}(\mu d \nu_p - e d \nu_p) - z \nu_p$$

(22)

Application to (21) gives

$$E(\delta \nu_u + e dt \nu + \Lambda \delta \nu) = e \Delta \nu_p - \mu \tilde{v}_u + \Lambda z \nu_p$$

$$E(p \nu_u + e dt \nu + \Lambda \nu_p) = e \Delta \nu_p + \mu \tilde{v}_u - E \tilde{d} \nu$$

(23)

with $z \nu_p = \tilde{\nu}_p - \nu_p = z \nu_p - z \nu_p$ being the integer-valued double-differenced (DD) ambiguities. This shows that the satellite phase-bias correction has the function of replacing the noninteger user ambiguity $\tilde{\delta}_u \nu$ by the integer DD ambiguity between user $u$ and reference $r, z \nu_p \in \mathbb{Z}^f$.

2.2.3 Code-bias $\tilde{d} \nu$ exploits multi-frequency code data

Although (23) is in mixed-integer form, it does not yet fully exploit all information in case $f > 2$. The reason being that in each of the last $(f - 2)$ code equations of (23), the code-biases $\tilde{d} \nu$ are treated as unknown parameters. Hence, to have the multi-frequency user-data properly contribute to the user-solution, the code-bias corrections need to be provided as well. The resulting user-equations then finally read

$$E(\nu_p - c_p \nu_p) = c \Delta \nu_p - \mu \tilde{v}_u + \Lambda z \nu_p$$

$$E(p \nu_u - c_p \nu_p) = c \Delta \nu_p + \mu \tilde{v}_u$$

(24)

with the combined PPP-RTK phase and code corrections, $c \nu_p$ and $c_p \nu_p$, given as

$$c \nu_p = -e d \nu_p - \Lambda \tilde{\nu}_p$$

$$c_p \nu_p = -e d \nu_p - E \tilde{d} \nu$$

(25)

2.3 Single-Baseline RTK

The above has shown that the PPP-RTK corrected user-model is in fact a DD-like model. The clock correction establishes the geometry in DD-form and the phase-bias correction establishes the ambiguity in DD-form. The question that comes to the fore is therefore how this DD-like model of the PPP-RTK user compares to the more traditional single-baseline model. The latter is given as

$$\Delta \nu_f = e \Delta \nu_p - \mu \nu_f + \Lambda z \nu_p$$

$$\Delta \nu_p = e \Delta \nu_p + \mu \nu_p$$

with $\Delta \nu_f = \nu_f - \nu_p$ and $\Delta \nu_p = p \nu_p - p \nu_p$. It follows from subtracting (8) from its user version, i.e. with $r$ replaced by $u$.

A comparison of the PPP-RTK user model (24) with the single-baseline RTK model (26) shows that the two models are identical except for their ionospheric delay parametrization, $\tilde{\nu} \nu_p$ vs $\nu_f$. For users that are interested in positioning, the performance of the two models (24) and (26) will be the same, both in ambiguity-float as well as in ambiguity-fixed mode. Also their ambiguity-resolution performance will be the same. The ambiguity convergence times of the PPP-RTK user-model (24), i.e. its time-to-first-fix, will therefore be comparable to what one is used to with long baseline ambiguity resolution (Blewitt, 1989; Jonkman et al, 2000; Teunissen et al, 2000; Yu et al, 2011; Li et al, 2014).

The difference in ionospheric delay parametrization between (24) and (26) is essential for those users that are interested in ionospheric delay estimation. With the PPP-RTK user-model (24) a biased ionospheric delay $\nu_f = \nu_p - \nu_f$ is obtained, whereas an unbiased DD delay $\nu_f$ is estimated with the single baseline model (26). In contrast to the single-baseline model, the PPP-RTK user-model would thus be able to provide absolute ionospheric delays if $d \nu_f$ would be available, e.g. through calibration (Schaer, 1999).

2.3.1 Ionospheric delay $\tilde{\nu}$ to allow for improved IAR

From a positioning perspective, the single-baseline model (26) has the advantage over (24) in that it is parametrized in the relative ionospheric delay $\nu_f$. Hence, it allows for a further strengthening by making use of the spatial correlation of the ionospheric delays (Odiijk, 2002; Grejner-Brzezinska et al, 2007; Wielgosz et al, 2008). This is not possible with the PPP-RTK user-model (24).

To make this possible, an additional ionospheric correction is needed. As

$$E(\nu_f) = \nu_f - \nu_f - d \nu_f$$

(27)
provision of this ionospheric correction to the user gives
\[ E(\phi_u^{ps} - \phi_p^{ps} + \mu \delta_r^p) = e \Delta \rho_r^{ps} - \mu \delta_r^p + A z_r^{ps} \]
\[ E(p_u^{ps} - p_p^{ps} - \mu \delta_r^p) = e \Delta \rho_r^{ps} + \mu \delta_r^p \]  
(28)
in which the DD ionospheric delay is recognized as
\[ \rho_r^{ps} = \rho_r^{\text{std}} - \rho_r^{\text{ps}} = \rho_r^{ps} - \rho_r^{\text{std}}. \]
Thus with the ionospheric delay provided as an extra user-correction, the PPP-RTK user-model (28) now has the same capabilities as the single baseline model (26). In this contribution though, we restrict attention to the currently more customary case of providing only the PPP-RTK corrections \( \delta_r^p \), \( \delta_r^s \), and \( \delta_r^d \).

3 Pitfalls in analyzing an individual correction

3.1 PPP-RTK corrections in combined-form

The above has shown what roles are taken up by the individual PPP-RTK corrections \( \delta_r^p \), \( \delta_r^s \), and \( \delta_r^d \). But what matters, of course, is their combined effect. The comparison between the PPP-RTK model and the single-baseline model has made this clear as well. In the following we will refer to the corrections
\[ \hat{c}_\phi^{ps} = -e \hat{d}_r^{ps} - \hat{A} \hat{d}_r^{ps} \]
\[ \hat{c}_p^{ps} = -e \hat{d}_r^{ps} - \hat{E} \hat{d}_r^{ps} \]  
(29)
as the PPP-RTK corrections in combined form. With the aid of (17) it is not difficult to verify that
\[ \hat{c}_\phi^{ps} = \hat{c}_\phi^{ps} + \mu \delta_r^p \]
\[ \hat{c}_p^{ps} = \hat{c}_p^{ps} - \mu \delta_r^p \]  
(30)
This shows that the combined PPP-RTK corrections represent a biased version of the original reference station observations. This bias explains the presence of the biased ionospheric delay \( \hat{\rho}_r^{ps} \) in the PPP-RTK model (24). It can be eliminated if next to \( \hat{d}_r \), \( \hat{d}_s \), and \( \hat{d}_d \), also the ionospheric delay takes part in the corrections. The combined correction then takes the form
\[ \hat{c}_\phi^{ps} = \hat{c}_\phi^{ps} - \mu \delta_r^p = \phi_p^{ps} \]
\[ \hat{c}_p^{ps} = \hat{c}_p^{ps} + \mu \delta_r^p = p_p^{ps} \]  
(31)
in which case a complete correspondence with the single-baseline model is obtained. For a quick reference, a brief comparison between the single-station PPP-RTK setup and the single-baseline RTK setup is given in Table 1.

3.2 Individual vs combined corrections

When evaluating the generation of PPP-RTK corrections, it is usually the individual corrections that are judged on quality in the literature, instead of their combined form, see e.g. (Li and Zhang, 2012; Zhang et al., 2013; Li et al., 2013). Such an analysis of the individual corrections is useful if one wants to study the characteristics and estimation quality of the individual parameters \( \delta_r^{ps}, \delta_r^{ps} \), and \( \delta_r^{ps} \). However, from a PPP-RTK application point of view, such an individual analysis is far from sufficient. It is far better to aim at a quality analysis of their combined effect. This is the more so an analysis restricted to the individual corrections disguises important information that may result in serious pitfalls. Here we give two examples that underline the importance of this viewpoint.

Example 1 (Code-dominated corrections) If we apply the variance propagation law to (17), we obtain the variance matrices of the single-station PPP-RTK corrections as
\[ D(d_l^{ps}) = 2 (\mu T_s C_p \mu r); \quad \text{code-dominated} \]
\[ D(\phi_r^{ps}) = 2 (A^{-1} C_{\phi} A^{-1} + M_{\phi} M_{\phi}^T) \quad \text{code-dominated} \]
\[ D(d_r^{ps}) = 2 (E^{-1} C_r E^{-1}); \quad \text{code-dominated} \]
\[ D(\delta_r^{ps}) = 2 (\mu T_r C_p \mu r); \quad \text{code-dominated} \]  
(32)
in which 2\( C_p \) and 2\( C_\phi \) are the cofactor matrices for the satellite-differenced pseudorange and carrier-phase, respectively. This result clearly shows that the precision of the individual corrections is governed by the rather poor precision of the code observations. This seems to be at odds however with the quality that the PPP-RTK user-phase corrections are required to have to enable user-ambiguity resolution.

The reason for this apparent inconsistency lies in the high correlation that exists between the individual corrections. This becomes clear if we express the phase-bias solution as
\[ \delta_r^{ps} = \Lambda^{-1} (\phi_p^{ps} + e \hat{d}_r^{ps} + \mu \delta_r^p) \]  
(33)
This expression shows that the phase-bias solution is indeed highly correlated with the clock- and ionospheric
corrections. It is this high correlation that ‘repairs’ the situation when forming the combined PPP-RTK user-phase correction. Instead of the code-dominated individual corrections, the precise combined correction
\[
\hat{c}_\phi = -A \hat{\delta} - \epsilon \hat{d} + \mu \hat{t}_r,
\]
\[
\hat{c}_p = -e \hat{d} + \mu \hat{t}_r
\]
is obtained, the precision of which will be at the phase-noise level instead of at the code-noise level.

The conclusion reads therefore that the combined form of the PPP-RTK corrections should be used for performance evaluation and not the code-noise dominated time series of the individual PPP-RTK corrections. Note that the same conclusion is reached if one would follow the derivation in ionosphere-free form. Hence, the conclusion is not dependent on whether or not the ionospheric correction is provided.

\textbf{Example 2} (Perfectly known phase-biases) As PPP-RTK users may not always use corrections from one single provider, we now consider a case that can be considered as an example where corrections of two different providers are used. From the first provider, the single-station, the user uses the earlier derived single-station clock- and ionospheric solutions, \(d t\) and \(t_r\). And from a second provider the user obtains very precise phase-bias corrections, denoted as \(\hat{\delta}_{\text{ps}}\). For argument-sake the phase biases \(\hat{\delta}_{\text{ps}}\) are assumed so precise that they can be considered non-random for this example.

The corresponding combined corrections now read
\[
\hat{c}_{\phi}^{\text{ps}} = -e \hat{d} + \epsilon \hat{d} - \mu \hat{t}_r - A \hat{\delta}_{\text{ps}}
\]
\[
\hat{c}_p^{\text{ps}} = -e \hat{d} + \mu \hat{t}_r
\]
(35)

Note that while the code-correction \(\hat{c}_p\) remains unchanged, the phase-correction \(\hat{c}_{\phi}^{\text{ps}}\) differs from its single-station counterpart \(\hat{c}_{\phi}\) as,
\[
\hat{c}_{\phi}^{\text{ps}} = \hat{c}_p + A \epsilon \hat{d} = \hat{c}_p + A \epsilon \hat{d} - A \hat{\delta}_{\text{ps}}
\]
(36)
in which \(\hat{\epsilon}_{\text{ps}} = \hat{\delta} - \hat{\delta}_{\text{ps}}\) is zero mean with dispersion \(D(\hat{\delta})\). Application of the variance propagation law gives
\[
D(\hat{c}_{\phi}^{\text{ps}}) = AD(\hat{\delta}^{\text{ps}})A^T + D(\hat{c}_p) \rightarrow \text{code-dominated}
\]
(37)

This shows that the provision of the perfectly known phase-bias \(\hat{\delta}_{\text{ps}}\) has turned the previously very precise phase-correction \(\hat{c}_{\phi}^{\text{ps}}\) into a less precise code-dominated phase-correction \(\hat{c}_p\).

So what have we gained? Again one should not fall in the trap of making the judgment on the basis of an individual correction. That is, one should consider the PPP-RTK corrections in their combined form. By doing so, one will note the role played by the non-zero correlation between \(\hat{\epsilon}_{\text{ps}}\) and the code-corrections \(\hat{c}_p\).

Indeed, if \(\hat{\delta}_{\text{ru}}\) and \(\hat{\delta}_{\text{su}}\) denote the two types of float solutions of the user-ambiguities based on the two sets of corrections \(\hat{c}_p, \hat{c}_p\), respectively, then it follows from (36) that \(\hat{\epsilon}_{\text{ps}}\) gets fully absorbed in the estimator of the user DD ambiguity \(\hat{\delta}_{\text{ps}}\). Hence, the float solution of all user parameters except the ambiguities remains unchanged, while the user ambiguity solution simply changes as \(\hat{\delta}_{\text{ru}} = \hat{\delta}_{\text{su}} - \hat{\delta}_{\text{ps}}\). Since the covariance of \(\hat{\delta}_{\text{ru}}\) and \(\hat{\epsilon}_{\text{ps}}\) is given by \(D(\hat{\delta}_{\text{ps}})\), application of the variance propagation law gives
\[
D(\hat{\delta}_{\text{ru}}) = D(\hat{\delta}_{\text{su}}) - D(\hat{\delta}_{\text{ps}})
\]
(38)

thus showing that a more precise ambiguity solution is obtained. This shows that despite the poorer precision of the phase-correction \(\hat{c}_{\phi} \) vs \(\hat{c}_p\), the use of \(\hat{\delta}_{\text{ps}}\) does result in more precise user ambiguities and therefore in an improved ambiguity resolution performance. This again demonstrates that one should use the combined form of the PPP-RTK corrections, i.e. \(\hat{c}_{\phi}^{\text{ps}}, \hat{c}_p\), for performance evaluation and not the individual corrections.

\section{4 Geometry-Free Network Derived Corrections}

\subsection{4.1 Multivariate formulation}

So far we restricted ourselves to the observation equations of a single network station \(r\). We now extend the results to \(n\) network stations. We use a multivariate formulation and therefore define the undifferenced phase observation vector of station \(r\) as \(\phi_r = [\phi_{r,1}, \ldots \phi_{r,f}] \in \mathbb{R}^{lm}, \phi_{r,j} = [\Delta \phi_{r,1}^{p,j}, \ldots \Delta \phi_r^{p,m}], j = 1, \ldots, f\), with a likewise definition for the code observation vector \(p_r\).

For the \(n\) stations, the network observation matrices are defined as \(\Phi = [\phi_1, \ldots, \phi_n]\) and \(P = [p_1, \ldots, p_n]\). The compact multivariate formulation of the full-rank, multi-epoch, network observation equations becomes then
\[
\text{E}((I_f \otimes D_{\Phi}^T) \Phi(i)) = (e \otimes I_{m-1}) \hat{\Phi}(i) - (\mu \otimes I_{m-1}) \hat{i}(i) + (A \otimes I_{m-1}) \hat{A}
\]
(39)

\[
\text{E}((I_f \otimes D_{\Phi}^T) P(i)) = (e \otimes I_{m-1}) \hat{P}(i) + (\mu \otimes I_{m-1}) \hat{i}(i) - (E \otimes I_{m-1}) \hat{E} \hat{e}_n^T
\]
where \(D_{\Phi}^T\) denotes an \((m-1) \times m\) between-satellite differencing matrix. The index \(i\) refers to the epoch at which the observations are collected. This system generalizes the single-station system (12). The compact and insightful formulation of (39) is in a large part due to the application of the efficient Kronecker product \(\otimes\) (Henderson et al, 1983), which was first introduced for GNSS models in (Teunissen, 1997a).
Table 2 Network's parameters in index-, vector- and multivariate-forms (p ≠ s)

<table>
<thead>
<tr>
<th>Non-dispersive SD delays</th>
<th>Vector-form</th>
<th>Multivariate-form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\rho} = \tilde{\rho} - d_{\tilde{\rho}}$</td>
<td>$\tilde{\rho} = [\tilde{\rho}_1, \ldots, \tilde{\rho}_n]^T$</td>
<td>$\tilde{\rho} = [\tilde{\rho}_1, \ldots, \tilde{\rho}_n]$</td>
</tr>
<tr>
<td>Dispersive SD delays</td>
<td>$\tilde{d} = d_{\tilde{d}}$</td>
<td>$\tilde{d} = [\tilde{d}_1, \ldots, \tilde{d}_n]^T$</td>
</tr>
<tr>
<td>Estimable SD ambiguities</td>
<td>$\tilde{a} = a_{\tilde{a}} - M_{\tilde{a}} d_{\tilde{a}}$</td>
<td>$\tilde{a} = [\tilde{a}<em>{r,1}, \ldots, \tilde{a}</em>{r,1}]^T, [\tilde{a}<em>{r,f}, \ldots, \tilde{a}</em>{r,f}]^T$, $\tilde{A} = [\tilde{a}_1, \ldots, \tilde{a}_n]$</td>
</tr>
</tbody>
</table>

Following (ibid), the above model is referred to as *geometry-free* (GF) since no information about the relative receiver-satellite geometry is present in its design matrix. The matrices $\tilde{\rho}(i), \tilde{d}(i)$ and $\tilde{A}$ contain the network’s SD estimable non-dispersive delays, ionospheric delays and ambiguities, respectively (Table 2). The SD estimable code biases on the third frequency and beyond are collected in vector $\tilde{d}$.

As we assume the station integer ambiguities $z^p_{\tilde{a}}$ ($l = 1, \ldots, n$), the satellite phase biases $\phi^p_{\tilde{a}}$ and the satellite code biases $d_{\tilde{a}}$ of the between-satellite single differences to be time-constant, the time-constant estimable parameters of the above full-rank model are the $f(m - 1) \times n$ ambiguity matrix $\tilde{A}$ and the $(f - 2)(m - 1) \times 1$ code bias vector $\tilde{d}$. As before we take stations $r$ as the station to define our estimable satellite clock and estimable satellite phase bias, i.e. $d_{\tilde{a}}(i) = -\tilde{d}^p_{\tilde{a}}(i)$ and $\phi_{\tilde{a}} := -\phi^p_{\tilde{a}}$ (cf. 16). The goal is now to derive the estimators for the network-based PPP-RTK corrections and to analyse the improvements that can so be achieved.

The stochastic model of the network’s observables is assumed given as

$$
D = \begin{bmatrix}
(I_f \otimes D^T_m) \phi_r(i) \\
(I_f \otimes D^T_m) p_r(i)
\end{bmatrix} = \begin{bmatrix}
C_\phi & 0 \\
C_p & 0
\end{bmatrix} \otimes C_3(i)
$$

in which $C_3(i) = D^T_m C_S(i) D_m$, with $C_S(i)$ the $m \times m$ co-factor matrix that captures the satellite elevation dependency at epoch $i$. The scalar $c^2_r$ ($r = 1, \ldots, n$) is a receiver-dependent co-factor. In this study all receivers are assumed to be of the same quality and thus $c^2_r = 1$ for all $r$. The $f \times f$ positive-definite matrices $C_\phi$ and $C_p$ are the co-factor matrices of the phase and code observable types, respectively.

### 4.2 Geometry-free network redundancy

To identify the relevance of the network for PPP-RTK, we need to understand how its redundancy contributes to the best linear unbiased estimator (BLUE) of the PPP-RTK corrections. The network redundancy is defined here as the number of its observations minus the number of its estimable parameters. Would one discard this redundancy, then the network-derived corrections simply follow from the single-station solutions (31). In multivariate form they then read

$$
\begin{bmatrix}
\tilde{C}_\phi(i) \\
\tilde{C}_p(i)
\end{bmatrix} = \begin{bmatrix}
(I_f \otimes D^T_m) \phi_r(i) \\
(I_f \otimes D^T_m) p_r(i)
\end{bmatrix}
$$

They can be further improved however by exploiting the network redundancy. For the network redundancy, we discriminate between two cases:

- **Ambiguity float**: the case that the DD ambiguities are treated as real-valued parameters; and
- **Ambiguity fixed**: the case that the DD ambiguities are successfully resolved as integers.

For the geometry-free (GF) network model (39), the ambiguity-float $k$-epoch redundancy is given as

$$
\# GF-floating redundancy = \{k(f - 2)(n - 1) + (k - 1)(fn + f - 2)\}(m - 1)
$$

We now show how this redundancy is built up from the various elements of the network. For a quick reference,

Table 3 Geometry-free network redundancy brought by the ambiguity-float and fixed-scenarios, giving the total size of $2(kn - 1)(f - 1)(m - 1)$
a summary of the elements building up the network redundancy is provided in Table 3.

In the single-epoch, multi-frequency case, the network redundancy stems from the fact that all single-station solutions of the estimable code biases $\tilde{d}_{ps}$ have the same mean, that is

$$E(\vec{E}^{-1}p_{lps}^s) = -\tilde{d}_{ps}, \quad l = 1, \ldots, n$$

or

$$E(\vec{E}^{-1}p_{rlps}^s) = 0, \quad l \neq r$$

In multivariate form this reads as

$$E((\vec{E}^{-1} \otimes \vec{D}_{T,m}^T)P D_n) = 0$$

in which $D_n$ is the $n \times (n-1)$ between-station differencing matrix. Thus in the single-epoch case the redundancy is $(f-2)(m-1)(n-1)$. Hence, there is no redundancy in the single-station case $(n = 1$, see previous section) and no redundancy in case of dual-frequency data $(f = 2)$.

In case of $k$ epochs, all the additional single-station solutions of $\tilde{d}_{ps}$ (of the second epoch and beyond) have the same mean as those of (43). This is the case since the estimable code biases are assumed to be constant in time. This gives an additional redundancy of $(k-1)$ times $(f-2)(m-1)n$. Similarly, an additional redundancy of $(k-1)$ times $f(m-1)n$ is then also obtained due to the time-constancy of the ambiguity matrix $\vec{A} = [\vec{a}_1, \ldots, \vec{a}_n]$. Summing these redundancies up gives (42).

Now we consider the ambiguity-fixed network redundancy. It is given as

$$\# \text{ GF – fixed redundancy} = 2(kn-1)(f-1)(m-1)$$

Compare this to the ambiguity-floating redundancy (42) and note that it is $f(m-1)(n-1)$ larger. This increase in redundancy is due to the successfully resolved integer ambiguities. As the between-station differences of the single-station solutions of the estimable SD ambiguities $\tilde{a}_{rl}^{ps} = \tilde{d}_{rl}^{ps} - \tilde{a}_{rl}^{ps}$ have integer-valued means, we have

$$E[(\vec{A}^{-1} \otimes \vec{D}_{T,m}^T) \Phi D_n + (\vec{M} \otimes \vec{D}_{T,m}^T)P D_n] \in \mathbb{Z}^f(m-1)(n-1)$$

with $\vec{M} = \vec{A}^{-1}(\mu_\rho^T \otimes \mu_\rho^T)$. Hence, successfully resolving the integer ambiguities results in an additional $f(m-1)(n-1)$ condition equations and ditto redundancy. The total ambiguity-fixed redundancy is therefore given by (46). Note that now there already exists redundancy when $k = 1$, $f = 2$ and $n > 1$, this in contrast to the ambiguity-floating case (cf. 42). However, for $k = 1$, $f = 2$ and $n = 1$ there is still no redundancy as the single-station case does not enable the formation of integer ambiguities. For $n > 1$ such integers can be formed and the redundancy becomes then, for example, for $k = 1$, $f = 2$ and $n = 2$ equal to $2(m-1)$, which is indeed the number of dual-frequency DD ambiguities that can be formed in case of a single baseline.

### 4.3 The Ambiguity-Floating GF corrections

In this section we present our analytical analysis of the geometry-free, ambiguity-floating network-based PPP-RTK corrections. First we derived the BLUE estimators of the individual PPP-RTK corrections $d i_{ps}, \tilde{d}_{ps}$ and $\tilde{d}_{ps}$. Their precision is described by the variance-covariance matrices as given in Table 4. Note that their covariance matrices are given in the table as well. In the table, additional terms, indicated by the $\Delta$ symbol, show up themselves to characterize the contribution of the multi-frequency code data. Thus when $f = 2$, they vanish, that is, $\Delta Q[p,l] = 0$. To gain a better insight into the results, let us start with the dual-frequency case, where the (co)variance matrices corresponding to

<table>
<thead>
<tr>
<th>Table 4</th>
<th>(Co)Variance matrices of the ambiguity-floating geometry-free (GF) corrections. The ambiguities, phase and code biases are assumed constant over $k$ epochs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{Akk}$ &amp; $[\tilde{a}<em>{rl}^{ps} \otimes C_s] + [c</em>{ps}^{\theta} \otimes (C_s(i) - \tilde{\Delta}<em>p^{\theta} \otimes C_s)] - \frac{n+1}{kn} \Delta c</em>{sk}^{\theta} \otimes C_s$</td>
<td></td>
</tr>
<tr>
<td>$Q_{Akl}$ &amp; $\frac{1}{kn} \Lambda^{-1}(\Delta \tilde{\Delta}_p^{\theta} \otimes C_s)$</td>
<td></td>
</tr>
<tr>
<td>$Q_{Akl}$ &amp; $\frac{1}{kn} (\tilde{e}^T C_p E^T) \otimes C_s$</td>
<td></td>
</tr>
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<td>$Q_{Akl}$ &amp; $\frac{1}{kn} \Lambda^{-1}(\Delta \tilde{\Delta}_p^{\theta} \otimes C_s)$</td>
<td></td>
</tr>
</tbody>
</table>
the code biases \( \tilde{d}^{ps} \) are absent, as in that case \( E \) does not exist by definition. In that case, all the (co)variance matrices, except those of the satellite clocks, follow the 1-over-\( k \) rule. For not too large \( k \), almost the same rule applies to the variance matrices of the satellite clocks as well, since \( \langle c_{\rho}/c_{\rho^2} \rangle \approx 0 \). Let us now consider the multi-frequency case \( \ell > 2 \). As extra frequencies enter, the precision of the satellite clocks and phase biases improves as their (co)variance matrices decrease by a factor governed by \( \Delta Q_{[h,i]} \neq 0 \). Next to those of the satellite clocks and phase biases however, the (co)variance matrices of the code biases \( d^{ps} \) enter. They follow the 1-over-\( kn \) rule. Thus they are largely driven by both the number of epochs \( k \) and stations \( n \).

Based on the individual corrections one can construct the BLUE of the combined corrections, \( (\tilde{c}_d^T, \tilde{c}_e^T)^T \) (cf. 29) or \( \tilde{c}_d^T \tilde{c}_e^T \) (cf. 31). As these two correction types are related as

\[
\begin{bmatrix}
\tilde{c}_d \\
\tilde{c}_e
\end{bmatrix} = \begin{bmatrix}
\tilde{c}_d \\
\tilde{c}_e
\end{bmatrix} - \begin{bmatrix}
-\mu \\
+\mu
\end{bmatrix}_I
\]

both can be used for our analytical analysis of the user PPP-RTK positioning performance. The difference of the two type of corrections lies namely in the range of \( (-\mu^T, +\mu^T)^T \) and will therefore be completely absorbed by the ionospheric delays of the user. Hence, their difference will not affect the estimation of user-positioning nor that of user-ambiguity resolution. This can alternatively be understood by noting that both type of corrections have the same ionosphere-free combination. Hence, both corrections lead to an identical ionosphere-free user-correction.

In the following we work with the combined corrections \( (\tilde{c}_d^T, \tilde{c}_e^T)^T \). First we present their BLUE and then their variance-covariance matrix. The following two averaging operators will be frequently used in the remainder of this contribution

**station-averaging:** \( ()_{r} = \frac{1}{n} \sum_{r=1}^{n} ()_{r} \),

**epoch-averaging:** \( ()_{\tilde{i}} = [I_f \otimes \sum_{i=1}^{k} C_{S}^{-1}(i)]^{-1} \sum_{i=1}^{k} [I_f \otimes C_{S}^{-1}(i)] ()_{i} \)\( ()_{\tilde{i}} \)

We also make use of the notations \( ()_{r} = ()_{r} - ()_{r} \) and \( ()_{\tilde{i}} = ()_{\tilde{i}} - ()_{\tilde{i}} \). Thus in the single-station case we have \( ()_{r} = ()_{r} - ()_{r} \), therefore \( ()_{r} = 0 \). Likewise, in the single-epoch case we have \( ()_{\tilde{i}} = ()_{\tilde{i}} - ()_{\tilde{i}} \), therefore \( ()_{\tilde{i}} = 0 \).

**Theorem 1** (GF ambiguity-free corrections) The \( k \)-epoch geometry-free ambiguity-free BLUE of the network-derived corrections, at epoch \( \tilde{i} \), is given as

\[
\begin{bmatrix}
\tilde{c}_{d,GF}(\tilde{i}) \\
\tilde{c}_{e,GF}(\tilde{i})
\end{bmatrix} = I - \hat{\Phi} - \hat{M}
\]

with

\[
I = (I_f \otimes D_m^T) \begin{bmatrix}
\phi_r(i) \\
\rho_r(i)
\end{bmatrix}
\]

\[
\hat{\Phi} = (P_{[\tilde{e},\tilde{r}]}^\perp \otimes D_m^T) \begin{bmatrix}
\phi_r(\tilde{ii}) \\
\rho_r(\tilde{ii})
\end{bmatrix}
\]

\[
\hat{M} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\hat{M} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

and the projectors

\[
P_{[\tilde{e},\tilde{r}]} = C_p C_p^{-1}, \quad P_{[\tilde{e},\tilde{r}]} = I_f - P_{[\tilde{e},\tilde{r}]},
\]

\[
P_{[\tilde{e},\tilde{r}]} = I_{2f} - P_{[\tilde{e},\tilde{r}]},
\]

where \( \tilde{e} = [e^T, e^T]^T \), \( \mu = [-\mu^T, \mu^T]^T \), and

\[
C_p = [e, \mu] Q_{\tilde{r},\tilde{r}} [e, \mu]^T,
\]

\[
Q_{\tilde{r},\tilde{r}} = ([e, \mu]^T C_p^{-1} [e, \mu])^{-1},
\]

\[
C_{\tilde{e},\tilde{e}} = [\tilde{e}, \tilde{\mu}] Q_{\tilde{r},\tilde{r}} [\tilde{e}, \tilde{\mu}]^T
\]

\[
Q_{\tilde{r},\tilde{r}} = ([\tilde{e}, \tilde{\mu}]^T \text{blkdiag}(C_{\mu}^{-1}, C_p^{-1}) [\tilde{e}, \tilde{\mu}])^{-1}.
\]

**Proof** See Appendix.

So as to facilitate its interpretation, the GF-correction (51) has been written in terms of three expressions. The first expression \( I \) equals the single-station, single-epoch solution of the previous section (cf. 41), while the second expression \( \hat{\Phi} \) describes the multi-epoch contribution and the third expression \( \hat{M} \) the multi-station contribution. Thus \( \hat{\Phi} = 0 \) if \( \tilde{i} = \tilde{r} \) and \( \hat{M} = 0 \) if \( \tilde{r} = r \).

As the two terms, \( \hat{\Phi} \) and \( \hat{M} \), further adjust the single-station solution, they have a zero mean, \( \mathbb{E}(\hat{\Phi}) = \mathbb{E}(\hat{M}) = 0 \). Thus, although the three-term expression (51) provides the BLUE, any of the following combinations provides a LUE and therefore an unbiased estimator of the corrections: \( I, I - \hat{\Phi}, \) and \( I - \hat{\Phi} \).

Note that the third term not only vanishes for \( \tilde{r} = r \), but also if \( f = 2 \), since then \( P_{[\tilde{e},\tilde{r}]} = I_2 \) and thus \( P_{[\tilde{e},\tilde{r}]} = 0 \). Hence, in the dual-frequency case, not the number of stations, but only the number of epochs contribute to further improving these geometry-free float corrections.

Also note that the third term only contains code data, this in contrast to the first two terms. Hence, if there are more than one station, then only the code data contribute to a further improvement of these GF float corrections.

An important outcome of Theorem 1 is that the combined network correction can indeed be viewed as an adjusted version of the observations of a single station. Therefore once the corrections are applied to the PPP-RTK user data, the user corrected observation equations can be interpreted as if a single baseline is
formed between the user and a network-adjusted reference station.

The precision of the above corrections is obtained after applying the variance propagation law. The result is given in the following lemma.

**Lemma 1** (Precision GF ambiguity-float corrections) The variance matrix of the corrections (51) is given as

\[
D\left(\begin{bmatrix}\hat{\epsilon}_{\phi,GF}(i) \\ \hat{\epsilon}_{p,GF}(i)\end{bmatrix}\right) = D(I) - D(\hat{\Pi}) - D(\bar{\Pi})
\]

with

\[
D(I) = \begin{bmatrix} C_\phi & 0 \\ 0 & C_p \end{bmatrix} \otimes C_s(i)
\]

\[
D(\hat{\Pi}) = \frac{n-1}{n} \left[ P_{i[i,\mu]}^C \begin{bmatrix} C_\phi & 0 \\ 0 & C_p \end{bmatrix} \otimes [C_s(i) - 1/\hat{\Pi}C_s] \right]
\]

\[
D(\bar{\Pi}) = \frac{n-1}{n} \left[ P_{i[i,\mu]}^C C_p \otimes 1/\hat{\Pi}C_s \right]
\]

where \(\hat{C}_s \approx \frac{1}{n} \sum_{i=1}^{n} C_s^{-1}(i).\)

**Proof** Follows by an application of the variance propagation law to (51). \(\square\)

This result shows how the precision of the corrections is driven by the various contributing factors, like precision of observables, number of epochs, number of network stations and the frequencies. In the following we will show the extent to which these contributing factors contribute to the precision of the corrections. We will then also show the impact that these corrections are expected to have on the user’s ability to perform successful integer ambiguity resolution.

**Role of k, n and f:** As the code observables are the less precise observables, it is particularly of interest to understand how the precision of the code correction, \(D(\hat{\epsilon}_{c,p,GF}(i))\), benefits from increasing number of epochs \(k\), stations \(n\) and/or frequencies \(f\). Note that both \(D(\hat{\Pi})\) and \(D(\bar{\Pi})\) depend on \(k\), while only \(D(\hat{\Pi})\) depends on \(n\). For \(k = 1\) the second term vanishes, while for \(n = 1\) the third term vanishes. Furthermore, this last term also vanishes if \(f = 2\).

A plot of the square root of the mean variance of an individual satellite at zenith (when \(c_s(i) = 1\)), i.e. \(\text{trace}(D(\hat{\epsilon}_{c,p,GF}(i))) / f\), is given in Figure 1. It shows that the impact of the number of stations gets less the larger the number of epochs. This can be understood from the contribution of the third term \(D(\bar{\Pi})\). Thus only when not too many epochs are used, will the number of stations have a significant effect. For the case of a single station, the third term vanishes, and the variance of the code corrections behaves as \(1/k\).

...and so on...

**Fig. 1** GPS L1, L2, L5 scenario: the squared-root of the zenith-referenced mean variance of the GF ambiguity-float code corrections as a function of the number of stations \(n\) for different number of epochs \(k\) \((\sigma_{\phi} = 3\ [\text{mm}], \sigma_p = 30\ [\text{cm}], c_s = 1)\).

### 4.4 The Ambiguity-Fixed GF Corrections

We now present our results for the geometry-free ambiguity-fixed network-based PPP-RTK corrections. Again we first derived the BLUE estimators of the individual corrections \(d\hat{\rho}^p, \tilde{d}\hat{\rho}^n\) and \(\Delta Q_{[k,i]}\). Their precision is described by the variance-covariance matrices as given in Table 5. To discuss the table, let us start with the dual-frequency case, where the (co)variance matrices corresponding to the code biases \(\tilde{d}\hat{\rho}^n\) are absent and \(\Delta Q_{[k,i]}^\phi = 0\). In that case, the (co)variance matrices, corresponding to the satellite phase biases \(\tilde{d}\hat{\rho}^p\), are reduced in accordance with the 1-over-\(n\) rule. For not too large \(n\) however, almost the same rule applies to the variance matrices of the satellite clocks as well, since \((c_\phi/c_p) \approx 0\). Let us now consider the multi-frequency case \((f > 2)\). It is remarkable to see that the (co)variance matrices, corresponding to the satellite code biases \(\tilde{d}\hat{\rho}^p\), remain **unchanged** after network ambiguity resolution. This is indeed due to the fact that the satellite code biases \(\tilde{d}\hat{\rho}^p\) are **uncorrelated** with the float DD ambiguities (Teunissen and Khodabandeh, 2014). Also note, since \(\Delta Q_{[k,i]}^\phi \neq 0\) when \(f > 2\), that the (co)variance matrices of the satellite clocks and phase biases do not follow the 1-over-\(n\) improvement anymore.

Based on the individual one can construct the ambiguity-fixed combined corrections \((\hat{\epsilon}_{\phi,GF}^F, \hat{\epsilon}_{p,GF}^F)^T\).

**Theorem 2** (GF ambiguity-fixed corrections) The \(k\)-epoch geometry-free ambiguity-fixed BLUE of the network-derived corrections, at epoch \(i\), is given as

\[
\begin{bmatrix} \hat{\epsilon}_{\phi,GF}(i) \\ \hat{\epsilon}_{p,GF}(i) \end{bmatrix} = I - \hat{\Pi}
\]
with
\[ I = (I_2 \otimes D_m^T) \begin{bmatrix} \phi_r(i) \\ p_r(i) \end{bmatrix} \]
\[ \tilde{I} = (P_{[\bar{e}, \bar{p}]} \otimes D_m^T) \begin{bmatrix} \phi_r(ii) + \phi_r(i) \\ p_r(ii) + p_r(i) \end{bmatrix} \]

(58)
in which the ambiguity-fixed phase data are defined as
\[ \phi_r = \phi_r \]
\[ \phi_l = \phi_l - [A \otimes L] \tilde{z}_l, \quad l \neq r \]

(59)
with \( L \) being an \( m \times (m-1) \) matrix formed by removing the \( p^{th} \) column of \( I_m \) (given \( p \) as the pivot satellite).

Proof See Appendix.

Compare the above results with that of (51). In particular note that now in the ambiguity-fixed case the time-averaging and the station-averaging have the same contribution to the corrections. In the ambiguity-float case, the contribution from the station-averaging was confined to the code data only (cf. \( \tilde{I} \) in 51). In the ambiguity-fixed case however, also the ambiguity-fixed carrier phase data \( \phi_l \) acts as pseudorange data and will therefore contribute in a likewise manner in the station-averaging.

The precision of the above corrections is obtained after applying the variance propagation law. The result is given in the following lemma.

Lemma 2 (Precision GF ambiguity-fixed corrections)
The precision of the corrections (57) is given as
\[ D \left[ \begin{bmatrix} \tilde{c}_{\phi,GF}(i) \\ \tilde{c}_{p,GF}(i) \end{bmatrix} \right] = D(I) - D(\tilde{I}) \]

(60)
with
\[ D(I) = \begin{bmatrix} C_\phi & 0 \\ 0 & C_p \end{bmatrix} \otimes C_s(i) \]
\[ D(\tilde{I}) = P_{[\bar{e}, \bar{p}]} \begin{bmatrix} C_\phi & 0 \\ 0 & C_p \end{bmatrix} \otimes [C_s(i) - \frac{1}{k} \hat{C}_s] \]

(61)

Table 5 (Co)Variance matrices of the ambiguity-fixed geometry-free (GF) corrections. The ambiguities, phase and code biases are assumed constant over \( k \) epochs.

<table>
<thead>
<tr>
<th>Term</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_{\phi \phi}^{GF} )</td>
<td>[ \frac{1}{k} \sum_{i=1}^k Q_{\phi \phi}^{GF} ]</td>
</tr>
<tr>
<td>( Q_{p \phi}^{GF} )</td>
<td>[ \frac{1}{k} \sum_{i=1}^k Q_{p \phi}^{GF} ]</td>
</tr>
<tr>
<td>( Q_{p p}^{GF} )</td>
<td>[ \frac{1}{k} \sum_{i=1}^k Q_{p p}^{GF} ]</td>
</tr>
</tbody>
</table>

4.5 Relevance of PPP-RTK corrections for user-IAR

As the goal of PPP-RTK is in first instance not so much to improve the float solution of the user-position, but rather to enable user integer ambiguity resolution for obtaining a good ambiguity-fixed user-position, we revisit the above BLUE corrections (51) and (57), and identify the part that takes an active role in user integer ambiguity resolution. We first consider the network ambiguity-float case and then the network ambiguity-fixed case.

4.5.1 Network ambiguity-float case

Since any part of the corrections that lies in the range space of \( P_{[\bar{e}, \bar{p}]} \) gets completely absorbed by the user position and user ionospheric delay parameters, one can discard the part \( (P_{[\bar{e}, \bar{p}]} \otimes D_m^T) \begin{bmatrix} \phi_r(ii) \\ p_r(ii) \end{bmatrix} \) in (51).
as far as the user-IAR performance is concerned. Hence, for the purpose of user-IAR it then suffices to consider the following simplified corrections
\[
\begin{bmatrix}
(I_I \otimes D_m^T) \phi_r (i) \\
(I_I \otimes D_m^T)p_r (i) - (P_{[e,i]} \otimes D_m^T)p_r (i)
\end{bmatrix}
\]
(62)

instead of the complete BLUE corrections (51). This shows that only the time-averaged network data is of relevance for user-IAR. Note that this simplified correction even simplifies further in the dual-frequency case, since then \(P_{[e,i]} = 0\).

The variance matrix of the above simplified correction (62) is given as
\[
\frac{1}{k} \begin{bmatrix}
C_\phi & 0 \\
0 & (I_I - \frac{1}{n-1}P_{[e,i]}C_p)
\end{bmatrix} \otimes C_s
\]
(63)

4.5.2 Network ambiguity-fixed case

In the network ambiguity-fixed case it is the part \((P_{[e,i]} \otimes D_m^T)\) \( [(\phi_r (i) + \phi_r (i)]^T, [p_r (i) + p_r (i)]^T \) in (57) that has no role in user integer ambiguity resolution. Hence, for the purpose of user-IAR it then suffices to consider the following simplified corrections
\[
(I_I \otimes D_m^T) \begin{bmatrix}
\tilde{\phi}_r (i) \\
p_r (i)
\end{bmatrix},
\]
(64)

This shows that now, next to the time-averaging, also the station-averaging contributes to the user-IAR. The variance-matrix of (64) reads
\[
\frac{1}{kn} \begin{bmatrix}
C_\phi & 0 \\
0 & C_p
\end{bmatrix} \otimes C_s
\]
(65)

now clearly showing how \(k\) and \(n\) work in tandem for improved user-IAR.

4.6 User Ambiguity Dilution of Precision

In this subsection, we use the Ambiguity Dilution of Precision (ADOP) measure (Teunissen, 1997b) to characterize the role of the network’s contributing factors, i.e. the number of epochs \(k\) and stations \(n\), as well as the role of the number of frequencies \(f\) on the strength of user ambiguity resolution. The ADOP is defined as the square-root of the geometric mean of the ambiguity variance matrix’s eigenvalues, thus representing the average ambiguity precision. The smaller the ADOP, the larger the ambiguity success rate. As a rule of thumb, ADOP-values smaller than about 0.10 cycle correspond with success rates larger than 0.999 (Odijk and Teunissen, 2008).

The following lemma presents an analytical expressions of the user single-epoch ADOP, once the geometry-free network corrections are applied to the user observations.

**Lemma 3** (User single-epoch ADOP: geometry-free network corrections) Let \(C_\phi = \sigma_\phi^2 I_I\) and \(C_p = \sigma_p^2 I_p\), respectively, be the co-factor matrices of the network’s phase and code observable types in (40), where \(C_S (i) = \mathcal{C}_S\). With a likewise structure, let the user phase and code co-factor matrices be given as \(C_{\phi_n} = \sigma_{\phi_n}^2 I_f\) and \(C_{p_n} = \sigma_{p_n}^2 I_f\), respectively. The user single-epoch ADOP, based on the \(k\)-epoch geometry-free network corrections, reads then
\[
ADOP = c_o (\frac{\epsilon_{e,c}}{1 - \epsilon})^{\frac{1}{2}} (\frac{\epsilon_{p,c}}{1 - \epsilon})^{\frac{1}{2}} (1 + \epsilon) \frac{c_o}{\sigma_{\phi}} \frac{c_o}{\sigma_{p}} \frac{\sigma_p}{\sigma_\phi} \frac{1}{\sqrt{(1 + \epsilon)}} + \cdot \cdot \cdot (66)
\]

where
\[
\sigma_{\phi_n}^2 = \begin{cases} \sigma_{\phi_n}^2 + \frac{1}{f} \sigma_r^2, \quad \text{network ambiguity-float} \\ \sigma_{\phi_n}^2 + \frac{1}{kn} \sigma_r^2, \quad \text{network ambiguity-fixed} \end{cases}
\]
(67)

with \(\phi = \{\phi, p\}, \epsilon = (\sigma_{\phi_n}^2 / \sigma_{p_n}^2), \lambda = \prod_{j=1}^f \lambda_j^\frac{1}{f}, \mu = (1/f) \sum_{j=1}^f \mu_j, \sigma_r^2 = (1/f) \sum_{j=1}^f (\mu_j - \mu)^2\), and
\[
c_o = \left( \sum_{s=1}^m c_s^2 / \prod_{s=1}^m c_s^2 \right)^{\frac{1}{2(m-1)}}
\]
(68)

with \(c_s^2\) being the diagonal entries of the elevation weighting matrix \(C_S\).

**Proof** Follows from an application of the results of Odijk and Teunissen (2008). \(\square\)

The above lemma clearly shows that the number of stations \(n\) has no role in the user ADOP, when the ambiguity-float network corrections are applied (cf. 67). While in the network ambiguity-float case, the number of epochs \(k\) governs the user ADOP, in the network ambiguity-fixed case, both the number of epochs \(k\) and stations \(n\) work in tandem to reduce the user ADOP. This reduction is, however, bounded by the precision of the user data. For sufficiently large large number of epochs and stations, one can at most tackle the uncertainty of the network corrections, thereby leaving the precision of the user data to solely govern user ambiguity resolution.

Numerical graphs for the user ADOP, when the ambiguity-fixed network corrections are applied, are given in Figure 3 (solid lines). It shows the ADOP values decrease as both \(k\) and \(n\) increase. As one would expect, the ADOPs do not get smaller than certain values, because of the nonzero variances of the user data.
Next to the ADOP values, Figure 3 also gives the ratio of the ADOP, based on the ambiguity-fixed network, to the ADOP based on the ambiguity-fixed network (dashed lines). It is observed that these float-to-fixed ratios get larger, the larger the number of stations. As the number of epochs gets larger, the larger the number of stations (dashed lines). It is observed that these float-to-fixed ratios (dashed lines) as a function of the number of stations n. Top: GPS L1, L2 scenario; Bottom: GPS L1, L2, L5 scenario. The values follow from (66) by setting \(m = \nu + 1\) (i.e. \(m = 2\nu + 1\)), the quantity in (69) decreases from \(e^{-1/8} \approx 3.16\) \((f = 2)\) to \(e^0 = 1\) \((f = 3)\). In this case, the single-epoch ADOP decreases by almost a factor of 3.

5 Geometry-Based Network Derived Corrections

5.1 Extra redundancy by the geometry-based scenario

So far we based our network analysis on the geometry-free model (39), where information about the relative receiver-satellite geometry was absent in its design matrix. We now consider the case where the receiver-satellite geometry is incorporated into the model and study the impact such increase in redundancy brings. The underlying model is referred to as geometry-based (GB) (Teunissen, 1997a).

In the following, we show that the k-epoch redundancy of the GB network model is given by

\[
\# \text{GB redundancy} = \# \text{GF redundancy} + (k[m - 1] - \nu)(n - 1)
\]

A quick overview of the elements building up the above extra redundancy is provided in Table 6.

Recall from (9) that the \(\nu\)-vectors \(\Delta x_r\) \((r = 1, \ldots, n)\) of the stations’ position increments/ZTDs are linked, through \(\Delta p_{\nu} = g_{\nu T} \Delta x_r\), to the estimable non-dispersive delays \(\tilde{p}_{\nu}^r\). For the sake of presentation, we assume the network to be such that \(g_{\nu}^p = g_{\nu}^p\), \(r = 1, \ldots, n\). This assumption admits the inclusion of small to regional networks in our discussion. With this in mind, the DD non-dispersive delays \(\Delta \rho_{\nu}^r = \tilde{p}_{\nu}^r - \rho_{\nu}^r = \Delta p_{\nu}^r - \Delta \rho_{\nu}^r\) can be further parametrized as

\[
\Delta \rho_{\nu}^r = g_{\nu T}(\Delta x_l - \Delta x_r)
\]
Summing the extra redundancies (extra redundancy of (\(\Delta \rho\)) following Teunissen (1997a), if the time-averaged receiver-phase bias of the second epoch and beyond) are linked to the individual PPP-RTK corrections.

5.2 The Ambiguity-Float GB corrections

\[ Q_{\text{d}l}^{(i)} = Q_{\text{d}d}^{(i)} \left( \frac{1}{n-1} \sum_{i=1}^{n} (e_\phi(i) + e_p(i)) \right) \]
\[ Q_{\text{d}l}^{(i)} = Q_{\text{d}d}^{(i)} \left( \frac{1}{m} \sum_{i=1}^{m} (e_\phi(i) + e_p(i)) \right) \]
\[ Q_{\text{d}d}^{(i)} = Q_{\text{d}d}^{(i)} \]

\[ Q_{\text{d}l}^{(i)} = Q_{\text{d}d}^{(i)} + \frac{1}{n} \left( e_\phi(i) + e_p(i) \right) \]
\[ Q_{\text{d}l}^{(i)} = Q_{\text{d}d}^{(i)} \]
\[ Q_{\text{d}d}^{(i)} = Q_{\text{d}d}^{(i)} \]

\[ e_\mu = e_\mu - \frac{1}{n} \mu: \quad \hat{C}_s = D_{1}^{T} P_{1}^{-1} C s D_{m}; \quad \hat{G} = [e_{m}, G] \]

or in the multivariate form as

\[ \hat{p} D_{n} = (D_{m}^{T} G) \hat{A} X D_{n} \] (72)

with \( \hat{A} = [\Delta x_{1}, \ldots, \Delta x_{n}] \) and the geometry matrix \( G = [g_{1}^{T}, \ldots, g_{m}^{T}]^{T} \).

Equation (72) shows that \((n-1)\) times \((m-1)\) non-dispersive parameters \(p D_{n}\) are replaced by \((n-1)\) times \(n-\nu\) parameters \(\Delta X D_{n}\), when one switches from the geometry-free model to the geometry-based model. Thus in the single-epoch case, the redundancy increases by \((m-1)\) (\(n-1\)) times \(\nu\) (\(n-\nu\)) (\(n-1\)) terms of (74). Thus \(\hat{\rho} D_{n} = (D_{m}^{T} G)^{-T} \hat{p} D_{n} = 0\) (73)

where \((D_{m}^{T} G)^{-1}\) denotes an orthogonal complement basis matrix of \(D_{m} G\) (Teunissen, 2000).

In the \(k\)-epoch case, would one assume the relative position increments and ZTDs \((\Delta x_{i} - \Delta x_{r})\) to be time-invariant, all the DD non-dispersive delays \(\Delta l_{l}(i), i = 2, \ldots, k\) (of the second epoch and beyond) are linked to their first-epoch counterparts in (71). This yields an extra redundancy of \((k-1)\) times \((m-1)(n-1)\) following Teunissen (1997a), if the time-averaged receiver-satellite geometry is used as approximation, the corresponding condition equations can be written as

\[ \hat{p}(i) - \hat{p}(1) D_{n} = 0, \quad i = 2, \ldots, k \] (74)

Summing the extra redundancies \((m-1)\) \((n-1)\) and \((k-1)(m-1)(n-1)\) gives (70).

5.2 The Ambiguity-Float GB corrections

Our analytical results of the geometry-based, ambiguity-Float network-based PPP-RTK corrections are presented in this section. We first derived the BLUE estimators of the individual PPP-RTK corrections \(d\rho_{\phi}, \hat{\delta}\rho_{\phi}\) and \(d\rho_{p}\). Their precision is described by the variance-covariance matrices as given in Table 7. The results are linked to their GF counterparts. It is remarkable that the (co)variance matrices, corresponding to the satellite code biases \(d\rho_{\phi}\), remain unchanged by switching from the GF-model to the GB-model. Thus the satellite code biases \(d\rho_{\phi}\) are not only unrelated with the float DD ambiguities, but also with the relative position increments/VTDs \((\Delta x_{i} - \Delta x_{r})\). The precision improvement in the satellite clocks and phase biases is governed by matrix \(\hat{C}_{s} = D_{m}^{T} P_{1}^{-1} C s D_{m}\), where \(\hat{G} = [e_{m}, G] \) (\(e_{m}\) is the \(m\)-vector of ones). In the absence of satellite redundancy, we have \(m = n + \nu\) and therefore \(\hat{C}_{s} = 0\). In that case, the (co)variance matrices, corresponding to the satellite phase biases \(\hat{\delta}\rho_{\phi}\), remain unchanged. For the variance matrix of the satellite clocks \(d\rho_{\phi}\), there is still a slight improvement, one that can be explained by the assumed time-invariance of \((\Delta x_{i} - \Delta x_{r})\), cf. (74).

Based on the individual corrections we construct the geometry-based, ambiguity-Float combined corrections \(\hat{e}_{\phi, \rho}(\hat{c}_{\phi}, \hat{c}_{p})^{T}\). First we present their estimators and then their variance-covariance matrices.

Theorem 3 (GB ambiguity-Float corrections) The \(k\)-epoch geometry-based ambiguity-Float BLUE of the network-derived corrections, at epoch \(i\), is given as

\[
\begin{bmatrix}
\hat{\rho}_{\phi, \rho}(i) \\
\hat{\rho}_{p, \rho}(i)
\end{bmatrix}
= 
\begin{bmatrix}
\hat{\rho}_{\phi, \rho}(i) \\
\hat{\rho}_{p, \rho}(i)
\end{bmatrix}
- \hat{N} - \hat{V}
\]

(75)

with

\[ \hat{N} = ([P_{[\rho]} - P_{\rho}] \otimes D_{m}^{T} \phi_{r}(i)) \]
\[ \hat{V} = ([P_{[\rho]} - P_{\rho}] \otimes D_{m}^{T} P_{G}^{T} \phi_{r}(i)) \]

and the projectors

\[ P_{\rho} = e_{i_{l}}^{T} \mu_{l} \mu_{l}^{T} C_{l}^{T} \]
\[ P_{\rho} = e_{i_{l}}^{T} \mu_{l} \mu_{l}^{T} \text{blkdiag}(C_{l}^{T} C_{l}^{-1}) \]
\[ P_{\rho} = \hat{G}^{T} C_{s}^{-1} \hat{G}^{T} C_{s}^{-1}, \quad P_{G} = I_{m} - P_{\hat{G}} \]

where \(\hat{G} = [e_{m}, G]\), and

\[ c_{i_{l}}^{2} = \mu_{l} C_{l}^{T} \mu_{l}^{-1} \]
\[ c_{i_{l}}^{2} = \mu_{l} \text{blkdiag}(C_{l}^{T} C_{l}^{-1}) \mu_{l}^{-1} \]

(78)

\[ \text{Proof} \text{ see Appendix.} \]

The GB-correction (75) is linked to its GF counterpart (51) through the zero-mean terms \(\hat{N}\) and \(\hat{V}\). The first term \(\hat{N}\) describes the contribution of the additional multi-epoch condition equations of (74). Thus \(\hat{N} = 0\) if \(i \neq i\). On the other hand, the second term \(\hat{V}\) describes the contribution of the geometry-based condition equations of (73). Thus \(\hat{V}\) vanishes in the absence of satellite...
redundancy, i.e. $\tilde{V} = 0$ if $m = \nu + 1$. Also note that both terms are absent when a single network station is considered, i.e. $\tilde{V} = V = 0$ if $r = r$.

It should be remarked that the third term $\tilde{V}$ only contains code data. Hence, in the ambiguity-float case, only the code correction $\tilde{\phi}$ benefits from the contribution of the geometry-based condition equations of (73).

The precision of the above corrections is given in the following lemma.

**Lemma 4** (Precision GB ambiguity-float corrections)

The variance matrix of the corrections (75) is given as

$$D(\tilde{\phi}_{\phi,an}(i)) = D(\tilde{\phi}_{p,cr}(i)) = D(\tilde{\phi}) - D(\tilde{V}) \quad (79)$$

with

$$D(\tilde{\phi}) = \frac{n}{m} [P_{e,\hat{p}} - P_{\hat{p}}] \begin{bmatrix} C_{\phi} & 0 \\ 0 & C_{p} \end{bmatrix} \otimes [C_s(i) - \frac{1}{s} \tilde{C}_s]$$

$$D(\tilde{V}) = \frac{n}{m} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} P_{e,\hat{p}} - P_{\hat{p}} \end{bmatrix} \otimes \frac{1}{s} \tilde{C}_s \quad (80)$$

where $\tilde{C}_s = D_m^T P_{\hat{G}}^{-1} \tilde{C}_S D_{m}^T$.

**Proof** Follows by an application of the variance propagation law to (75). \(\square\)

To evaluate the maximum precision improvement brought by switching to the geometry-based model, we consider the extreme case, namely, the geometry-fixed case. The geometry-fixed (GFI) case refers to the situation where all the relative position increments/ZTDs ($\Delta x_i - \Delta x_r$), $i \neq r$, are assumed known. The GFI covariance matrices follow by setting $\tilde{C}_s = \tilde{C}_s$ in (79). Figure 4 gives a plot of the square root of the mean variance of an individual satellite at zenith, i.e. trace($D(\tilde{\phi}_{\phi,an}(i))$)/$f$. Compare the plot with its GF-counterpart in Figure 1.

The number of stations $n$ now has a larger impact on the precision improvement of the code correction. This is mainly due to the extra condition equations of (73) that link the non-dispersive delays $\tilde{\rho}_{\rho}$ ($r = 1, \ldots, n$) to one another. Similar to the GF case however, the stated impact gets less the larger the number of epochs $k$.

5.3 The Ambiguity-Fixed GB Corrections

We now present our analytical analysis of the geometry-based, ambiguity-fixed network-based PPP-RTK corrections. The precision of the BLUE estimators of the individual PPP-RTK corrections $\tilde{\phi}_{\rho}$, $\tilde{\rho}_{\rho}$ and $\tilde{\phi}_{\phi}$ is described by the variance-covariance matrices as given in Table 8. The ambiguity-fixed results are expressed in their ambiguity-float counterparts. In contrast to the GF-model, here the impact of ambiguity resolution is dependent on the strength of the GB-model. The stronger the model, the lower the impact of ambiguity resolution. The strongest model follows by the geometry-fixed case. In this extreme case, with $\tilde{C}_s = C_s$, no precision improvement is realized. On the other hand, the impact of ambiguity resolution gets maximum for the weakest model, i.e. when there is no satellite redundancy ($\tilde{C}_s = 0$ as $m = \nu + 1$).

We now present the geometry-based, ambiguity-fixed combined corrections ($\tilde{\phi}_{\phi}, \tilde{\phi}_{\rho}$).}

**Theorem 4** (GB ambiguity-fixed corrections) The $k$-epoch geometry-based ambiguity-fixed BLUE of the network-derived corrections, at epoch $i$, is given as

$$\begin{bmatrix} \tilde{\phi}_{\phi,an}(i) \\ \tilde{\phi}_{p,cr}(i) \end{bmatrix} = \frac{1}{k} \begin{bmatrix} \tilde{\phi}_{\phi,an}(i) \\ \tilde{\phi}_{p,cr}(i) \end{bmatrix} - \tilde{V} = \tilde{V} \quad (81)$$

**Table 8** (Co)variance matrices of the ambiguity-fixed geometry-based (GB) corrections. The ambiguities, phase and code biases are assumed constant over $k$ epochs.

<table>
<thead>
<tr>
<th>$Q_{GB_{dd}}^{(i)}$</th>
<th>$Q_{GB_{dd}}^{(i)} - \frac{k}{n} (c_p^2 - c_p^2) (\tilde{C}_s - \tilde{C}_s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{GB_{dd}}^{(i)}$</td>
<td>$Q_{GB_{dd}}^{(i)} - \frac{k}{n} (c_p^2 - c_p^2) (\tilde{C}_s - \tilde{C}_s)$</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
<td>$Q_{GB_{dd}}^{(i)}$</td>
<td>$Q_{GB_{dd}}^{(i)} - \frac{k}{n} (c_p^2 - c_p^2) (\tilde{C}_s - \tilde{C}_s)$</td>
</tr>
</tbody>
</table>

$$Q = \frac{k}{n} (c_p^2 - c_p^2) A^{-1} (\tilde{C}_s - \tilde{C}_s)$$
with
\[
\hat{\bf N} = \left( [P_{\tilde{\varepsilon},\tilde{\varepsilon}}] - P_{\bar{\mu}} \right) \otimes D_{m}^{T} \begin{bmatrix} \phi_{\nu}(ii) \\ p_{\nu}(ii) \end{bmatrix}
\]
where the ambiguity-fixed phase data are given as
\[
\hat{\phi}_{r} = \phi_{r},
\hat{\phi}_{l} = \phi_{l} - [L \otimes L] \tilde{z}_{rl},\quad l \neq r
\]
with \( L \) being an \( m \times (m-1) \) matrix formed by removing the \( p^{th} \) column of \( I_{m} \) (given \( p \) as the pivot satellite).

Proof See Appendix.

The GB ambiguity-fixed correction (81) is linked to its GF-counterpart (57) through the zero-mean terms \( \hat{\bf N} \) and \( \hat{\bf V} \). Compare the results with those of the GB ambiguity-float (75). The first term \( \hat{\bf N} \), due to the multi-epoch condition equations of (74), remains unchanged. This is what one would expect, since the epoch-differenced observations \( \phi_{\nu}(ii) \) and \( p_{\nu}(ii) \) are uncorrelated with the float ambiguities. The second term \( \hat{\bf V} \) is, however, replaced by its ambiguity-fixed counterpart \( \hat{\bf V} \), describing the contribution of the geometry-based condition equations of (73). Next to the code data, the ambiguity-fixed carrier phase data \( \hat{\phi}_{l} \) also contribute in a likewise manner to \( \hat{\bf V} \).

An application of the variance law propagation to the above corrections gives their precision as presented in the following lemma.

**Lemma 5** (Precision GB ambiguity-fixed corrections)
The variance matrix of the corrections (81) is given as
\[
\begin{align*}
\text{D}(\hat{\varepsilon}_{\phi,\nu,\alpha}(i)) &= \text{D}(\hat{\varepsilon}_{\phi,\nu,\alpha}(i)) - \text{D}(\hat{\bf N}) - \text{D}(\hat{\bf V}) \quad (84)
\intertext{with}
\text{D}(\hat{\bf N}) &= \frac{n-1}{n} [P_{\tilde{\varepsilon},\tilde{\varepsilon}}] - P_{\bar{\mu}} \begin{bmatrix} C_{\phi} & 0 \\ 0 & C_{p} \end{bmatrix} \otimes [C_{s}(i) - \frac{1}{k} \bar{C}_{s}]
\text{D}(\hat{\bf V}) &= \frac{n-1}{n} [P_{\tilde{\varepsilon},\tilde{\varepsilon}}] - P_{\bar{\mu}} \begin{bmatrix} C_{\phi} & 0 \\ 0 & C_{p} \end{bmatrix} \otimes \frac{1}{k} \bar{C}_{s} \quad (85)
\end{align*}
\]

Proof Follows by an application of the variance propagation law to (81).

We again consider the extreme case, the geometry-fixed case, to evaluate the maximum precision improvement brought by switching to the geometry-based model. The stated improvement follows by setting \( \bar{C}_{s} = C_{s} \) in (84). A plot of the square root of the mean variance of an individual satellite at zenith, i.e. \( \text{trace}(\text{D}(\hat{\varepsilon}_{p,\nu,\alpha}(i)))/f \), is given in Figure 5. Compare the plot with its GF-counterpart in Figure 2. The precision improvement is indeed insignificant. Provided that successful network ambiguity resolution is applied, one should therefore not expect a considerable improvement in the precision of the corrections, by switching to the geometry-based model. Once the network’s ambiguities are resolved, the GF and GB performances do not differ by much.

![Fig. 5](image)  
**Fig. 5** GPS L1, L2, L5 scenario: the squared-root of the zenith-referenced mean variance of the geometry-fixed, ambiguity-fixed code corrections as a function of the number of stations \( n \) for different number of epochs \( k \) (\( \sigma_{\phi} = 3 \text{ [mm]}, \sigma_{p} = 30 \text{ [cm]}, c_{s} = 1 \)).

### 5.4 User ADOP improvement by the GB corrections
To evaluate how much switching to the geometry-based model pays off, we again consider the user ADOP to characterize the improvement in the strength of user ambiguity resolution. Since adopting the geometry-based scenario further strengthens the network model, one would expect the user ADOP to get smaller upon replacing the geometry-free corrections by the geometry-based corrections. The maximum reduction in the ADOP follows when the geometry-fixed case is considered. The stated reduction is formulated in the following lemma presenting the user single-epoch ADOP GF-to-GF ratio.

**Lemma 6** (User single-epoch ADOP ratio: from geometry-free to geometry-based network corrections) Let \( C_{\phi} = \sigma_{\phi}^{2} I_{f} \) and \( C_{p} = \sigma_{p}^{2} I_{f} \), respectively, be the co-factor matrices of the network’s phase and code observable types in (40), where \( C_{S}(i) \approx \bar{C}_{s} \). With a likewise structure, let the user phase and code co-factor matrices be given as \( C_{\phi u} = \sigma_{\phi u}^{2} I_{f} \) and \( C_{pu} = \sigma_{pu}^{2} I_{f} \), respectively. The ratio of the user single-epoch ADOP, based on the
with γ

ometry-based corrections. To gain insight into the size ADOP by switching from the geometry-free to the geo-

cis considered, there is a slight reduction in the user ADOP in either case, the user ADOP remains the

after successful network ambiguity resolution, no mat-

ter whether the user is provided with the geometry-

fixed (GFi) network corrections (dashed lines) as a

function of the number of stations n. Top: GPS L1, L2 scenario; Bottom: GPS L1, L2, L5 scenario. The values follow from (86) by setting m = ν + 1 = 6, σφu = σφ = 3 [mm], σpν = σp = 30 [cm], cν = 1.

k-epoch geometry-free network corrections (ADOPGF),
to its geometry-fixed counterpart (ADOPGFi) reads then

ADOPGFi = \left\{ \begin{array}{ll}
(1 + kγn + kνγν + k(n-1)γν) σ\sigma_2^{2(\nu-1)}, & \text{net. amb.-float (86)} \\
1 & \text{net. amb.-fixed}
\end{array} \right.

with γu = (σpν/σp)^2, and

\tilde{\epsilon} = \frac{\epsilon}{(1 + \epsilon) + \frac{4p^2}{(1 + \epsilon)σp^2}}

(87)

Proof Follows from an application of the results of Odijk and Teunissen (2008).

The above lemma conveys two important messages. First, after successful network ambiguity resolution, no matter whether the user is provided with the geometry-

free corrections or with the geometry-based corrections, in either case, the user ADOP remains the same (i.e. ADOPGF = ADOPGFi).

Second, when the network ambiguity-float scenario is considered, there is a slight reduction in the user ADOP by switching from the geometry-free to the geo-

metry-based corrections. To gain insight into the size of this reduction, we make some approximation. Note that one can neglect \( \epsilon k(n-1)γν \), compared to the first term in the denominator of (86). Assuming the user’s data to be of the same quality as those of the first network receivers (i.e. γu = 1), the first expression of (86) would then takes the following form (k = 1)

\[
\frac{\text{ADOPGF}}{\text{ADOPGFi}} \approx \left[ 1 + \frac{n - 1}{2(n + 1)} \right]^{\frac{\epsilon}{\gamma u(1 + \epsilon)σp^2}}
\]

(88)

In the absence of satellite redundancy (m = ν + 1), the above ADOP GF-to-GFi ratio becomes around 1.10 (when f = 2) and 1.07 (when f = 3), would the number of network stations be n = 100. When the number of satellites increases by ν (i.e. m = 2ν + 1), the above ADOP GF-to-GFi ratio even gets smaller, around 1.05 (f = 2) and 1.03 (f = 3). In either case, the stated ratio is therefore close to one, meaning that only a slight improvement in the strength of user ambiguity resolution is realized through replacing the ambiguity-float, geometry-free corrections by their geometry-based counterparts. This analysis is consistent with the numerical results that are shown in Figure 6 (solid lines) for the user-ADOP GF-to-GFi ratios, for the dual- and triple frequency case and different number of epochs k and stations n. This is further corroborated by the user ambiguity success-rates as shown in Figure 7, based on the GFi (blue circles) and the GF ambiguity-float (red triangles) corrections. In order to consider the maximum gain in the user-IAR capacity achieved by switching to the GB model, we consider the dual-frequency single-

epoch network corrections (f = 2, k = 1) where the number of stations is assumed to be very large (i.e.
The user corrected data have been partitioned into 28 groups, each of size 100 epochs with the sampling-interval of 30 seconds. As predicted by the ADOP analysis, Figure 7 confirms that the user ambiguity success-rates based on the GF corrections do not differ too much from their GFi versions.

Although switching from the ambiguity-float, geometry-free network scenario to its geometry-based counterpart does not improve the capacity of user ambiguity resolution by much, one must, however, note that it does play a prominent role in improving the capacity of network ambiguity resolution (Teunissen and Khodabandeh, 2015). Furthermore, such a GF-to-GB network switch also improves the float solution of the user position/UTD, as the clock corrections approximately improve from a 1-over-k rule to a 1-over-kn rule (cf. Table 7 for $C_k = C_s$).

Next to the network ambiguity-float, GF-to-GB switch, we also consider the effect of network ambiguity fixing. To compare the user-ambiguity impact of the ambiguity-fixed network corrections with their ambiguity-float counterparts, the user-ADOP float-to-fixed ratios (dashed lines) are shown in Figure 6 for a GF-network. The float-to-fixed ratio (dashed lines) is around 1.27 (when $f = 2$) and 1.31 (when $f = 3$) for $k = 1$. When the $k$-epoch network corrections are applied, the stated ratio does even get smaller as the number of epochs $k$ increases. For instance, the float-to-fixed ratio drops to 1.05 ($f = 2$) and 1.07 ($f = 3$) for $k = 5$.

From the above one may conclude that user ambiguity resolution performance, when based on the PPP-RTK corrections $d^s$, $\delta^s$, and $\vec{d}^s$, will not benefit too much from ambiguity fixing in the network. This does not mean, of course, that network ambiguity resolution has no important role to play. It plays a significant role, for instance, in improving the precision of the estimated ionospheric delays in the network (Teunissen and Khodabandeh, 2014).

The reason for the rather modest impact of network ambiguity resolution on the user ambiguity resolution performance lies in the way the user’s ionospheric delays are treated. In our formulation, the user’s ionospheric delays are treated as unknown, thus resulting in a rather weak model in terms of ambiguity resolution capability. But as was already pointed out in Sect. 2.3.1, one can improve user ambiguity resolution performance significantly if the PPP-RTK corrections would be extended with an ionospheric component, thus enabling the user to make use of the stronger ionosphere-weighted model. In that case, network ambiguity resolution would improve the provided ionospheric information (Odijk, 2002; Grejner-Brzezinska et al, 2004; Mervart et al, 2013; Odijk et al, 2014).

5.5 Corrections’ precision relevant to user-IAR

As stated earlier in Sect. 4.5, not all the components of the PPP-RTK corrections contribute to user integer ambiguity resolution. Any part of the corrections that lies in the range space of $\mathcal{P}_{[\mu]}$, gets fully absorbed by the user position and user ionospheric delay parameters, thus not affecting the estimator of the user ambiguities. This in turn allows one to identify which part of the variance matrix of the corrections is relevant to user-IAR, see e.g. (63) and (65). Drawing a similar analogy to the geometry-free network corrections, the following part of the network ambiguity-fixed case: (89) and (90) are, respectively, given as,

$$\text{network ambiguity-float case:}$$
$$
\begin{bmatrix}
(I_f \otimes D_{\mu}^T)\phi^s(i) \\
(I_f \otimes D_{\mu}^T)p^s(i) - (P_{[\mu]} \otimes D_{\mu}^T)\psi^s(i) - (|P_{[\mu]} - P_{[\mu]}| \otimes D_{\mu}^T)\psi^s(i)
\end{bmatrix}
\quad(89)
$$

$$\text{network ambiguity-fixed case:}$$
$$
(I_{2f} \otimes D_{\mu}^T)\phi^s(i) - (P_{[\mu]} \otimes D_{\mu}^T)p^s(i)
\quad(90)
$$

Compare the above equations with their GF counterparts (62) and (64). This again shows that only the time-averaged network data is of relevance for user-IAR. While the GB ambiguity-fixed part (90) is identical to that of the GF ambiguity-fixed case (64), the GB ambiguity-float part (89) differs from its GF version (62). This is due to the difference in their code corrections only. The stated code-difference is formed by the projector $P_{[\mu]} - P_{[\mu]}$. For the dual-frequency case (i.e. for $P_{[\mu]} = I_f$), this projector is simplified as $P_{[\mu]} - P_{[\mu]} = P_{[\mu]}$. The projector $P_{[\mu]}$ is referred to as the ionosphere-free projector, since it nullifies the ionospheric vector $\mu$, i.e. $P_{[\mu]} \mu = 0$. Thus in the dual-frequency case, the network ambiguity-float GF-to-GB switch only leads the ionosphere-free code data to contribute to a further improvement of the relevant GF corrections.

The corresponding variance matrices of (89) and (90) are, respectively, given as,

$$\text{network ambiguity-float case:}$$
$$
\begin{bmatrix}
C_\phi & 0 \\
0 & (I_f - \frac{1}{2}P_{[\mu]}C_\phi)C_\phi - \frac{1}{2} |P_{[\mu]} - P_{[\mu]}| C_\phi
\end{bmatrix}
\quad(91)
$$

$$\text{network ambiguity-fixed case:}$$
$$
\begin{bmatrix}
C_\phi & 0 \\
0 & C_\phi
\end{bmatrix}
\quad(92)
$$

where an overview of the interactions of the above (co)variance components with their single-station and geometry-free counterparts is presented by the diagram given in Figure 8. To highlight the role of the number of
When the multi-frequency GF-float network scenario takes place (i.e., $f > 2$), the role of the number of epochs $k$ and stations $n$ is characterized by switching from the single-station scenario to the GF- and GB-network scenarios.

In Sect. 2, an analogy between the single-station PPP-RTK setup and the single-baseline RTK setup was given (cf. Table 1). It was shown that the user corrected data are nothing else but DD observations formed between the user and a single network station. This is the case as the single-station corrections stand in one-to-one correspondence with the single-station observations (cf. 17). On the other hand, through the presentation of Theorems 1–4, the PPP-RTK network corrections were shown to be an adjusted version of single-station observations. It is therefore evident that the user’s corrected observations, on the basis of the network corrections, can also be interpreted as DD observations between the user and the network stations. This notion is visualized in Figure 9 and made precise via the following theorem.

Theorem 5 (PPP-RTK in DD-form) Let $[\hat{c}_0(i), p_u^T(i)]^T$ be the user observations at epoch $i$. Also let the k-epoch network correction $[\hat{c}_0^T(i), \hat{c}_p^T(i)]^T$ be a linear unbiased estimator of $[c_0^T(i), c_p^T(i)]^T$. Then the user corrected observations can always be written as

$$\begin{align*}
&\begin{pmatrix}
(I_f \otimes D_m^T) \phi_u(i) \\
(I_f \otimes D_m^T) p_u(i)
\end{pmatrix} = \begin{pmatrix}
\hat{c}_0(i) \\
\hat{c}_p(i)
\end{pmatrix} \\
&\sum_{q=1}^{k} \sum_{l=1}^{n} W_{q,l}(i) \begin{pmatrix}
(I_f \otimes D_m^T) \phi_u(q) \\
(I_f \otimes D_m^T) p_u(q)
\end{pmatrix}
\end{align*}$$

for some weight matrices $W_{q,l}(i)$ satisfying

$$\sum_{q=1}^{k} \sum_{l=1}^{n} W_{q,l}(i) = I$$

with the DD observations

$$\begin{align*}
&\begin{pmatrix}
(I_f \otimes D_m^T) \phi_u(q) \\
(I_f \otimes D_m^T) p_u(q)
\end{pmatrix} = \begin{pmatrix}
(I_f \otimes D_m^T) [\phi_u(i) - \phi_l(q)] \\
(I_f \otimes D_m^T) [p_u(i) - p_l(q)]
\end{pmatrix}
\end{align*}$$
Proof: Follows by an application of Theorems 1–4. □

According to the above theorem, the user corrected observations can always be viewed as a weighted-average of DD observations formed between the user and the network stations. To give an example, consider the user-corrected code observables based on the single-epoch structure of the code correction we may write then

\[
\hat{c}_{p,\text{DR}} = P_{[\cdot,\mu]} p_{\mu}^p + p_{\mu}^s
\]

thus forming the user corrected code observations as

\[
p_u^p - \hat{c}_{p,\text{DR}} = P_{[\cdot,\mu]} p_{\mu}^u + P_{[\cdot,\mu]}^\perp \frac{1}{n} \sum_{l=1}^n p_{lu}^p
\]

with the DD observations \(p_{lu}^p = p_u^p - \hat{c}_{p,\text{DR}}\). Equation (96) can then be expressed as a weighted-average of \(p_{lu}^p\), \(l = 1, \ldots, n\), that is

\[
p_u^p - \hat{c}_{p,\text{DR}} = \sum_{l=1}^n W_l p_{lu}^p, \quad \text{with } \sum_{l=1}^n W_l = I
\]

in which the weight matrices \(W_l\) are defined as

\[
W_l = \begin{cases} P_{[\cdot,\mu]} + \frac{1}{n} P_{[\cdot,\mu]}^\perp, & l = r \\ \frac{1}{n} P_{[\cdot,\mu]}^\perp, & l \neq r \end{cases}
\]

It therefore follows from the DD-like structure of the PPP-RTK user corrected observations that the PPP-RTK setup can be considered equivalent to the more traditional network-RTK setup, be it that their ionospheric parametrization could be different.

6 Summary and conclusions

The contributions of this paper are summarized as follows:

- **Single-station PPP-RTK:** It was shown that one should not rely on the quality-judgment of the individual corrections. Instead, the quality of the combined version of the corrections must be evaluated. This is because of the high correlation that exists between the individual corrections. By means of some illustrative examples we demonstrated the potential pitfalls of ignoring the stated correlation.

- **Single-station PPP-RTK is single-baseline RTK:** We demonstrated the equivalence between the single-station PPP-RTK setup and the more traditional single-baseline RTK setup (cf. Table 1). It was shown that both formulations are identical except for their ionospheric delay parameters. With the PPP-RTK user model a biased ionospheric delay is obtained, whereas an unbiased DD ionospheric delay is obtained with the single-baseline model.

- **Network redundancy for PPP-RTK:** We identified the network redundancy and its impact on the precision of the PPP-RTK corrections (cf. Tables 3 and 6). This was done for both the geometry-free (GF) network model and the geometry-based (GB) network model, with and without network-IAR. The precision impact of the number of epochs \(k\), number of stations \(n\) and number of frequencies \(f\) was shown for both the individual corrections as well as for their combined form. Furthermore it was demonstrated that the estimable code biases are uncorrelated with the float DD ambiguities and the stations’ relative positions/ZTDs. Hence, their (co)variance matrices remain unchanged when switching to the geometry-based model and/or when performing integer ambiguity resolution.

- **BLUEs of PPP-RTK corrections:** We derived the best linear unbiased estimators of the PPP-RTK network corrections in analytical form. The BLUEs of the combined corrections are expressed in terms of time- and station-averaged network observations and time- and station-differenced network observations. By using the conditional least-squares approach, our result is formulated such that it clearly shows how the single-station corrections are further improved by the network information. Therefore once the corrections are applied to the user data, the user corrected observation equations can be interpreted as if a single baseline is formed between the user and a network-adjusted reference station.

- **Only time-averaged network data relevant for user-IAR:** The closed-form expressions of the BLUE corrections allow one to identify which part of the combined corrections really contributes to user integer ambiguity resolution. For all four network scenarios (i.e. the GF-float, cf. 62, the GF-fixed, cf. 64,
the GB-float, cf. 89, and the GB-fixed, cf. 90) it was shown that the network contribution to the float-estimated user-ambiguities is only through the time-averaged network data. For the two ambiguity-fixed network scenarios (i.e. GF- and GB-fixed, cf. 64 and 90), the network contribution to the float-estimated user-ambiguities becomes even confined to the station-average of the time-averaged network data.

- **Precision impact on user-float position/ZTD:** The GF-to-GB network switch can improve the float solution of the user position/ZTD. This improvement however, largely depends on the geometrical strength of the GB-model and on whether or not network-IAR is applied. The geometrically stronger the model, the larger the precision improvement becomes. In the strongest case, namely, the geometry-fixed (GFi) case, the precision of the user float position/ZTD, the strongest case, namely, the geometry-fixed (GFi) counterpart. For their network ambiguity-fixed versions, an addition that would then benefit most from using the geometry-based, ambiguity-fixed network model. Without such corrections, the user performance corresponds to that of a long baseline ionosphere float model.

- **PPP-RTK user-parameters are function of DD data:** It was shown that the PPP-RTK user corrected data can always be viewed as a weighted-average of the double-differenced (DD) observations that are formed between the user and the network stations. This shows the equivalence between the PPP-RTK formulation and the more traditional network-RTK formulation, be it that their ionospheric parametrizations could be different.

- **Network can at most overcome half the uncertainty of the reference-user data:** Recall that the user corrected model of observation equations can be interpreted as being that of a single baseline formed between the user and network-adjusted reference station. Strengthening the network model would therefore only improve the quality of the reference station’s data (i.e. the network corrections), which in the extreme case of perfectly known (i.e. non-random) corrections, would still leave the uncertainty of the user data to drive the user positioning performance.

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**Table 9** Precision impact of the network corrections on the user float ambiguities and on the float position/ZTD for the geometry-free (GF) and the geometry-fixed (GFi) network scenarios. With the approximation \( C_i(i) = C_s \), \( i = 1, \ldots, k \), the role of the number of epochs \( k \) is highlighted. The variance matrices \( D(I) \), \( Q_1 \) and \( Q_2 \) are given in Figure 8. The \( f \times (f - 1) \) matrix \( B \) is a basis matrix orthogonal-complement to the \( f \)-vector \( \mu \), thus forming ionosphere-free combinations.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Variance matrix relevant to the user-ambiguities</th>
<th>Variance matrix relevant to the user-float position/ZTD</th>
</tr>
</thead>
<tbody>
<tr>
<td>GF: ( Q_{\text{float}}^{\text{GF}} ) = ( \frac{1}{2}D(I) - \frac{n}{\rho_e}Q_2 ), ( Q_{\text{fixed}}^{\text{GF}} = \frac{1}{2}Q_{\text{float}}^{\text{GF}} )</td>
<td>GF: ( Q_{\text{float}}^{\text{GF}} = \frac{1}{2}Q_{\text{float}}^{\text{GF}} ), ( Q_{\text{fixed}}^{\text{GF}} = \frac{1}{2}Q_{\text{float}}^{\text{GF}} )</td>
<td></td>
</tr>
<tr>
<td>GFi: ( Q_{\text{float}}^{\text{GFi}} = \frac{1}{2}Q_{\text{float}}^{\text{GFi}} ), ( Q_{\text{fixed}}^{\text{GFi}} = \frac{1}{2}Q_{\text{float}}^{\text{GFi}} )</td>
<td>GFi: ( Q_{\text{float}}^{\text{GFi}} = \frac{1}{2}Q_{\text{float}}^{\text{GFi}} + \frac{1}{k}Q_{\rho_e}^{\text{GFi}} ), ( Q_{\text{fixed}}^{\text{GFi}} = \frac{1}{2}Q_{\text{float}}^{\text{GFi}} + \frac{1}{k}Q_{\rho_e}^{\text{GFi}} )</td>
<td></td>
</tr>
</tbody>
</table>

\( Q_e = (B^T C_B B \odot C_s) \), \( \Delta Q = \rho_e Q_{\rho_e}^{\text{GFi}} \), \( Q_{\rho_e}^{\text{GFi}} = (1/\rho_e)(B^T C_B E^{-T} \odot C_s) \), \( Q_{\rho_e}^{\text{GF}} = (1/\rho_e)(E^T C_B E^{-T} \odot C_s) \)

---

**Relevance of ionospheric information:** Through our user-ADOP analysis the above improvements were also quantified. It was shown that they are not as significant as would be the case when the user would be able to include ionosphere-weighting in his model. This underlines the importance of being able to include network-based ionospheric information in the corrections, an addition that would then benefit most from using the geometry-based, ambiguity-fixed network model.
Proof of Theorem 1

To prove (51), we apply the least-squares conditional adjustment (Teunissen, 2000) to the single-station correction $\hat{t}$. Given the GF ambiguity-fixing network redundancy (Table 3), the following uncoupled sets of misclosures are formed:

$$t = (I_n - \otimes E \otimes D_n^T) \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_n(t) \end{bmatrix}^T,$$

$$t_i = (I_n - \otimes A_i - M_i - E \otimes D_n^T_i) \begin{bmatrix} \phi_i(t) \\ \phi_{i1}(t) \\ \vdots \\ \phi_{in}(t) \end{bmatrix}^T \tag{99}$$

$l = 1, \ldots, n$. The first set of misclosures $t$ is due to the fact that all single-station solutions of the estimable ambiguities $\phi_i^e$ and code biases $d^e$ have the same mean. The $n$ sets of misclosures $t_i$ are due to the fact that all single-station solutions of the estimable ambiguities $\phi_i^e$ and code biases $d^e$ are assumed constant over $k$ epochs. According to the least-squares conditional adjustment, the GF ambiguity-fixing network correction (51) is obtained as

$$\begin{bmatrix} \tilde{\epsilon}_{\phi,GR}(t) \\ \tilde{\epsilon}_{\phi,GF}(t) \end{bmatrix} = 1 - Q_{t,tt}^{-1} t - \sum_{i=1}^n Q_{t,t_i}^{-1} t_i \tag{100}$$

This, together with the following equalities

$$Q_{t,tt}^{-1} t = \tilde{\mu}, \quad \text{and} \quad Q_{t,t_i}^{-1} t_i = \tilde{t}_i \quad (0, l \neq r),$$

completes the proof.

Proof of Theorem 2

The proof goes along the same lines as the proof of Theorem 1. The ambiguity-fixed network correction (57) is obtained through replacing the role of the misclosure vector $t$ in (100) by its higher-dimensional counterpart (cf. Table 3)

$$\bar{t} = (I_n - \otimes (D^T G))^{-1} t - \sum_{i=1}^n Q_{t,t_i}^{-1} t_i \tag{101}$$

together with the equality

$$Q_{t,tt}^{-1} \bar{t} = \tilde{\mu} - \tilde{\mu} \tag{102}$$

Proof of Theorem 3

We apply the least-squares conditional adjustment to the GF ambiguity-fixing network correction (51). Given the extra redundancy by the geometry-based network model (Table 6), the following sets of misclosures are formed:

$$t_g = (I_n - \otimes (D^T G))^{-1} t - \sum_{i=1}^n Q_{t,t_i}^{-1} t_i \tag{104}$$

$$t_{g1} = (I_n - \otimes D_n^T) \begin{bmatrix} \phi_1(t) \\ \phi_{i1}(t) \\ \vdots \\ \phi_{in}(t) \end{bmatrix}^T \tag{105}$$

$l = 1, \ldots, n$. The first set of misclosures $t_g$ is due to the ‘geometry-parametrization’ of (72). The $(n-1)$ sets of misclosures $t_{g1}$ are due to the fact that the positive relative positions and ZTDs $(\Delta r_i - \Delta r_j)$ are assumed constant over $k$ epochs. According to the least-squares conditional adjustment, the GB ambiguity-fixing network correction (75) is obtained as

$$\begin{bmatrix} \tilde{\epsilon}_{\phi,GR}(t) \\ \tilde{\epsilon}_{\phi,GF}(t) \end{bmatrix} = y - Q_{y,t_g} t_g - Q_{y,t_{g1}} t_{g1} \tag{105}$$

with $y = [\tilde{\phi}_{\phi,GR}(t), \tilde{\phi}_{\phi,GF}(t)]^T$ and $t_{g1} = [\tilde{t}_{g1} \otimes \tilde{t}_{g1} \otimes \tilde{t}_{g1}]^T$. Equation (75) follows then by substituting

$$Q_{y,t_g} t_g - \bar{y}, \quad \text{and} \quad Q_{y,t_{g1}} t_{g1} - \bar{y}_L = \tilde{N} \tag{106}$$

into (105).

Proof of Theorem 4

We apply the least-squares conditional adjustment to the GF ambiguity-fixed network correction (57), on the basis of the extra geometry-based misclosures given in (104). The GB ambiguity-fixed network correction (81) follows then through replacing the role of $y$ in (105) by $\tilde{y} = [\tilde{\phi}_{\phi,GR}(t), \tilde{\phi}_{\phi,GF}(t)]^T$, together with the equalities

$$Q_{y,t_g} t_g - \bar{y}, \quad \text{and} \quad Q_{y,t_{g1}} t_{g1} - \bar{y}_L = \tilde{N} \tag{107}$$

References

Baarda W (1973) S-transformations and Criterion Matrices.
Blewitt G (1989) Carrier phase ambiguity resolution for the Global Positioning System applied to geodetic baselines up to 2000 km. J Geophys Res 94(B8)