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# Element Rotation Tolerance in a Low-Frequency Aperture Array Polarimeter

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**Abstract** — We present a rotation error tolerance analysis for dual-polarized dipole-like antennas commonly found in low-frequency radio astronomy. A concise Jones matrix expression for the phased array is derived which facilitates calculations of rotation error effects in polarimetry. As expected, for random rotation error and number of elements approaching infinity, the estimation error converges to that of the error-free case. However, as in practice large but finite number of antennas are involved, we present a simple analysis to estimate rotation error effects. An example calculation based on a “baseline” design for a low-frequency Square Kilometre Array (SKA) “station” is discussed.

## 1 INTRODUCTION

How tightly should (dual linearly-polarized) low-frequency aperture arrays (LFAA) antenna elements be aligned? It seems clear that misalignment should degrade array gain. It also appears on the surface that polarization properties should be at least as sensitive if not more so. How about if “large” number of antennas are involved? These are the questions that motivate our investigation.

As opposed to microwave antennas where precise machine tolerances may be maintained, LFAA antennas are inevitably human deployed in their (often harsh) environment. Demanding very tight alignment tolerance on each antenna seems impractical due to the time and cost involved. The Murchison Widefield Array (MWA) [1, 2] largely obviates this problem by their architecture as regularly spaced “bowties” are clipped onto (N-S and E-W) pre-surveyed metallic grids, forming “tiles” as seen in Fig. 1.

Although this method was found satisfactory for the MWA and could be replicated in larger SKA LFAA context, a review of this topic is warranted for a few reasons. Firstly, recently developed high-gain antennas such as the log-periodic “SKALA” candidate element [3, 4] have been designed to be directional in absence of a metallic ground plane. Hence, the possibility of having to align the antennas in the absence of a grid and the tolerance involved should be considered. Secondly, our field

experience using a simple compass for alignment yields a standard deviation of approximately  $5^\circ$ ; clearly, we need to understand if this is acceptable. Thirdly, we seek to review this problem in light of a recently introduced fundamental figure-of-merit for radio polarimeters—the intrinsic cross-polarization ratio (IXR) [5]—as it may reveal an essential insight.



Figure 1: An MWA tile consisting 16 dual-polarized bowties on a 5 m X 5 m wire mesh. The output of each antenna polarization is routed to an analog beamformer box seen on the middle right side of the photo.

## 2 JONES MATRIX DERIVATION

For dipole-like antennas, a polarimeter response with small rotation errors may be expressed as:

$$\mathbf{f}^i = \mathbf{E}^i \mathbf{J}_L \mathbf{e} \quad (1)$$

where  $\mathbf{f}^i = (f_x^i, f_y^i)^T$  indicates the measured vector,  $\mathbf{e} = (e_\theta, e_\phi)^T$  denotes the sky Jones vector, and for linear polarization

$$\mathbf{J}_L = \begin{pmatrix} J_{x\theta} & J_{x\phi} \\ J_{y\theta} & J_{y\phi} \end{pmatrix} \quad (2)$$

and the rotation error matrix is given by

$$\mathbf{E}^i = \begin{pmatrix} \cos \delta_i & \sin \delta_i \\ -\sin \delta_i & \cos \delta_i \end{pmatrix} \quad (3)$$

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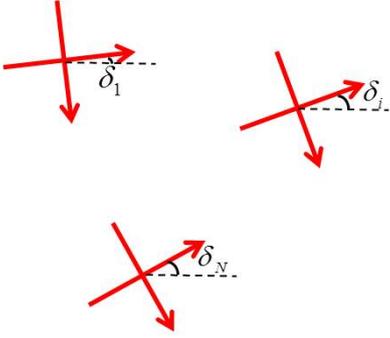


Figure 2: Rotational misalignment in an array of identical cross-dipole elements (top view).  $\delta_i$  is the rotation error of pair  $i$ .

Assuming that the Jones matrices for the antenna pairs are identical, the polarimeter response for a planar array is given by

$$\mathbf{f} = \frac{1}{N} \left( \sum_{i=1}^N e^{j(k_x x_i + k_y y_i + \varphi_i)} \mathbf{E}^i \right) \mathbf{J}_L \mathbf{e} \quad (4)$$

where  $k_x = (2\pi/\lambda) \sin \theta \cos \phi$ ,  $k_y = (2\pi/\lambda) \sin \theta \sin \phi$ ,  $x_i, y_i$  are the element positions, and  $\varphi_i$  is the phase shift for pair  $i$ . Converting the array measurement and sky bases from linear polarization (LP) to circular polarization (CP) [6], we obtain

$$\mathbf{f}_C = \frac{1}{N} \mathbf{R} \left( \sum_{i=1}^N e^{j(k_x x_i + k_y y_i + \varphi_i)} \mathbf{E}^i \right) \mathbf{J}_L \mathbf{R}^H \mathbf{e}_C \quad (5)$$

where  $\mathbf{f}_C = (f_l, f_r)^T$  (subscripts  $l, r$  refer to left hand [LH] and right hand [RH] CP, respectively),  $\mathbf{e}_C = (e_l, e_r)^T$  is the CP sky vector, and the transformation matrix is

$$\mathbf{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} -j & 1 \\ 1 & -j \end{pmatrix} \quad (6)$$

After simplification, the rotation error for the array may be expressed as a diagonal matrix that pre-multiplies the error-free Jones matrix

$$\mathbf{f}_C = \begin{pmatrix} F^- & 0 \\ 0 & F^+ \end{pmatrix} \mathbf{J}_C \mathbf{e}_C \quad (7)$$

where  $\mathbf{J}_C = \mathbf{R} \mathbf{J}_L \mathbf{R}^H$  is the CP Jones matrix and

$$F^\pm = \frac{1}{N} \sum_{i=1}^N e^{j(k_x x_i + k_y y_i + \varphi_i \pm \delta_i)} \quad (8)$$

### 3 IXR DEGRADATION

At the intended beam scanning direction  $(\theta_t, \phi_t)$

$$F_t^\pm = \frac{1}{N} \sum_{i=1}^N e^{\pm j \delta_i} \quad (9)$$

Note that  $F_t^+ = (F_t^-)^*$  and we can write

$$\begin{aligned} \mathbf{f}_{Ct} &= F_t^- \begin{pmatrix} 1 & 0 \\ 0 & 1 \angle (-2F_t^-) \end{pmatrix} \mathbf{J}_C \mathbf{e}_C \\ &= F_t^- \mathbf{M}^{err} \mathbf{J}_C \mathbf{e}_C \end{aligned} \quad (10)$$

IXR including rotation error may be calculated for  $\mathbf{J}_C^{err} = F_t^- \mathbf{M}^{err} \mathbf{J}_C$  using [5]

$$\text{IXR} = \left( \frac{\sigma_{\max}/\sigma_{\min} + 1}{\sigma_{\max}/\sigma_{\min} - 1} \right)^2 \quad (11)$$

where  $\sigma_{\max}/\sigma_{\min}$  is the (spectral norm) condition number for the Jones matrix.

Recall that IXR is unitarily invariant [5, 6] and is unaffected by scalar multiplication. Since  $\mathbf{M}^{err}$  is a unitary matrix

$$\text{IXR}(\mathbf{J}_C^{err}) = \text{IXR}(\mathbf{J}_C) \quad (12)$$

Hence given our assumptions (identical Jones matrices and small  $\delta_i$ ), IXR is unaffected by rotation error. It is important to note that no assumption regarding error distribution nor number of elements has entered the picture, therefore (12) applies to arbitrary distributions for any  $N$ .

One must *not*, however, infer from the above statement that polarimetry is unaffected by rotation error. Let us interpret (12) properly: if the total relative input error to the polarimeter is held constant despite rotation error, then the relative output error upper bound remains unchanged; begging the question: how does rotation error affect total relative input error? This question is addressed next.

### 4 UNPOLARIZED SOURCE

As radio astronomy sources are generally unpolarized or are weakly polarized [7], we now assume that there exists a single unpolarized point source at the intended beam direction  $(\theta_t, \phi_t)$ . The auto-correlation matrix may be written as [5, 8]

$$\mathbf{V} = \mathbf{J} \mathbf{B} \mathbf{J}^H \quad (13)$$

where  $\mathbf{B} = \langle \mathbf{e} \mathbf{e}^H \rangle$  (for an unpolarized source,  $\mathbf{B} = \mathbf{I}/2$  [8]) is the brightness matrix and  $\mathbf{J}$  is the Jones matrix for the entire array at the intended beam

direction. Let (13) be the error-free case. Keeping the the same source at  $(\theta_t, \phi_t)$ , let the case with rotation error be

$$\tilde{\mathbf{V}} = \tilde{\mathbf{J}}\mathbf{B}\tilde{\mathbf{J}}^H \quad (14)$$

where  $\tilde{\cdot}$  indicates the presence of error.

Assuming the errors are not corrected in the ‘‘calibration’’ procedure (which introduces an *input error* to the polarimetry), the the following brightness matrix estimate is obtained.

$$\tilde{\mathbf{B}} = \frac{1}{2}\mathbf{J}^{-1}\tilde{\mathbf{J}}\tilde{\mathbf{J}}^H(\mathbf{J}^H)^{-1} \quad (15)$$

Assuming CP bases and small rotation errors

$$\tilde{\mathbf{B}} \approx \frac{1}{2}\mathbf{J}^{-1}\mathbf{F}\mathbf{J}\mathbf{J}^H\mathbf{F}^H(\mathbf{J}^H)^{-1} \quad (16)$$

where  $\mathbf{F} = F_t^- \mathbf{M}^{err}$  as per (10). Note that the presence of the diagonal matrix  $\mathbf{F}$  introduces off-diagonal ‘‘leakage’’ (i.e., non-zero cross-polarization correlation:  $\langle e_l e_r^* \rangle$ ,  $\langle e_l e_r^* \rangle$ ) terms in  $\tilde{\mathbf{B}}$ .

Introducing statistical rotation errors to the problem, let  $\delta_i$  be a Gaussian random variable with zero mean and variance  $\sigma^2$  (independent and identically distributed for every  $i$ ). The mean and variances (real  $[\sigma_r]$  and imaginary  $[\sigma_i]$ ) for  $F_t$  is well known from array tolerance theory [9, 10].

$$\mathbb{E}[F_t] = e^{-\sigma^2/2} \quad (17)$$

$$\sigma_r^2 = \frac{1 + e^{-2\sigma^2}}{2N} - \frac{e^{-\sigma^2}}{N}$$

$$\sigma_i^2 = \frac{1 - e^{-2\sigma^2}}{2N} \quad (18)$$

For  $N \rightarrow \infty$ , the variances vanish and

$$\tilde{\mathbf{B}} \approx \frac{1}{2}\mathbb{E}[F_t]^2 \quad (19)$$

which is simply a scaled version of  $\mathbf{B} = \mathbf{I}/2$ . Assuming  $\mathbb{E}[F_t]$  introduces only small degradation in system sensitivity (which is reasonable: e.g., for  $\sigma = 10^\circ$ ,  $\mathbb{E}[F_t]^2 = 0.97$ —i.e., 3% degradation), the scaling factor may be appropriately re-scaled resulting in the correct answer (i.e., detection of a purely unpolarized source). In this particular case, polarimetry *is* unaffected by small rotation errors.

Let us transition to a more practical case where  $N$  is high but finite. Multiplying the matrices in (16), the leakage levels may be estimated as

$$\tilde{B}_{1,2} \approx 4J_\Delta \text{Im}(F_t^-) \quad (20)$$

where  $J_\Delta$  has been introduced to refer to a cross-diagonal component in the CP Jones matrix. The approximate variance of the leakage level is

$$\text{VAR}[4J_\Delta \text{Im}(F_t^-)] = (4J_\Delta)^2 \sigma_i^2 \approx \frac{(4J_\Delta \sigma)^2}{N} \quad (21)$$

For finite number of elements and leakage with a certain desired standard deviation, one may calculate the minimum number of elements required for a given  $\sigma$  and  $J_\Delta$ . This is illustrated in Tab. 1 for a baseline SKA LFAA station where  $N = 289$  [4] and a presumed tolerable leakage standard deviation of 0.5%. Note that a moderate  $J_\Delta$  of 0.2 requires rotation standard deviation of  $\approx 6.1^\circ$  whereas a ‘‘good’’  $J_\Delta$  of 0.1 may tolerate rotation error of  $\approx 12^\circ$ . We point out that the above performances are achievable with power gain degradation of less than 5% (as reported in the third column). These rotation tolerances are well within our field alignment accuracy based on a compass of approximately 5 $^\circ$  standard deviation (sample size: 16 elements).

Table 1: Rotation error standard deviation ( $\sigma$ ) required to achieve leakage with standard deviation of 0.5% for  $N = 289$ .

$J_\Delta$	$\sigma$ ( $^\circ$ )	$1 - (\mathbb{E}[F_t])^2$ (%)
0.1	12.2	4.4
0.2	6.1	1.1
0.33	3.7	0.4

## 5 Conclusion

For an array of dual-polarized dipole-like pairs where the Jones matrices are identical, the polarimeter’s Jones IXR is unaffected by (small) element rotation error. Furthermore, when the rotation error is random (Gaussian) and the number of elements approaches infinity, the array with rotation errors correctly detects a single unpolarized source in the intended beam direction without recourse to an error-correction scheme. For practical ‘‘large’’ numbers of antennas, we presented calculations for 289 elements in an SKA LFAA station. We found that a moderate antenna raw-cross polarization of 0.2 requires rotation standard deviation of  $\approx 6.1^\circ$ . This tolerance is consistent with a compass-based alignment method, which from our experience achieves approximately 5 $^\circ$  standard deviation.

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