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UNIFORM STABILITY OF STOCHASTIC IMPULSIVE SYSTEMS: A NEW COMPARISON METHOD

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Abstract. This paper studies uniform stability problems of stochastic impulsive systems by using a new comparison method. We firstly establish a comparison principle between the stochastic impulsive system and its scalar comparison system. Based on the obtained comparison result, uniform stability and uniform asymptotic stability of stochastic impulsive systems are established by analyzing those of comparison systems. Finally, a numerical example of a power system with random perturbations is presented to illustrate our results.

Keywords. impulsive effects, comparison principle, stochastic stability, stochastic impulsive system.

There are a variety of impulsive factors in real world systems which cause the states' abrupt changes at certain moments of time. For example, when a ball is bouncing back from the ground, the velocity changes sharply due to the impulsive force. Such evolution process with this kind of discontinuous dynamic behaviors can be described by an impulsive system which has been used in a variety of fields including medicine, biology, economics, and engineering. Moreover, many results of analysis and control of these impulsive systems can be found in the literature, see [1]-[4] and the references therein. On the other hand, besides impulsive behaviors, stochastic effects likewise can always be found in these systems as well. These random/stochastic behaviors, which produce randomness and uncertainty in one or more parts of the evolution processes, can be modelled as a stochastic system. For any given input, the stochastic system does not always produce the same output. A few components of systems which can be stochastic in nature include stochastic inputs, random time-delays, disturbances, and even stochastic dynamic processes [5], which can be applied in many disciples such as neural networks [11]. In fact, for those dynamical systems involving both stochastic characteristics and abrupt state changes, such as stochastic failures of the components, sudden environment changes and sharp changes in stochastic subsystems, the model in form of stochastic impulsive systems provides a natural description of such systems, see [9] and [10].

In recent years, the stability problems of stochastic impulsive systems have attracted much attention. Stability and optimal control results of nonlinear impulsive stochastic systems have been presented in [7] and [9]. Exponential stability of nonlinear impulsive stochastic systems have been investigated in [8]. Moreover, stability problems of stochastic impulsive systems with time delay have been discussed in [8] and [10]. Most of these results are obtained for stochastic systems with Markovian jump parameters and impulsive effects. Recently, a framework for stochastic impulsive systems has been provided in [6] and stability of solutions for stochastic impulsive systems are established via comparison approach. Based on the above results, we will establish new comparison theorems for stochastic impulsive systems. We apply the comparison method to a general class of stochastic impulsive systems which can cover those in [6]. Then, sufficient conditions for these stochastic impulsive systems can be obtained by analyzing the given comparison system. Our stability criteria are much easier than those in [6]. These criteria are applied to study a practical problem involving power systems to illustrate their effectiveness.

The reminder of this paper is organized as follows. Section 2 introduces stochastic impulsive systems and presents necessary definitions. By the Lyapunov-like function method and Ito's formula, new comparison methods of stochastic impulsive systems are devised which are further used to derive the stability criteria of stochastic impulsive systems in Section 3. Section 4 provides a numerical example which presents the stability results of a power system with random perturbations and impulsive effects. Section 5 gives some conclusion remarks.

1 Problem Statement

Consider the following stochastic impulsive system

$$\begin{cases} dx(t) = f(t, x(t))dt + g(t, x(t))dw(t), & t \neq t_k, \\ \Delta x(t_k) = h_k(t_k, x(t_k)), & t = t_k, \\ x(t_0^+) = x_0, & k = 1, 2, \cdots, \end{cases} \tag{1}$$

where $x \in R^n$, $f : R^+ \times R^n \to R^n$, $g : R^+ \times R^n \to R^{n\times m}$, and $h_k : R^+ \times R^n \to R^n$ with $f(t, 0) = 0$, $g(t, 0) = 0$, $h_k(t, 0) = 0$. $w(t) = (w_1(t), w_2(t), \cdots, w_m(t))^T$ is an m-dimensional Brownian motion on a complete probability space $(\Omega, \tilde{f}, P)$ with a natural filtration$\{\tilde{f}_t\}_{t\geq 0}$. The state impulses trigger at the time instants t_k, $k = 1, 2, \cdots$, and $0 = t_0 < t_1 < t_2 < \cdots < t_k \cdots$, as $\lim\limits_{k\to\infty} t_k = \infty$. Impulsive increments are defined as $\Delta x(t_k) = x(t_k^+)-x(t_k)$, where $x(t_k^+) = \lim\limits_{t\downarrow t_k} x(t)$, and $x(t_k) = x(t_k^-) = \lim\limits_{t\uparrow t_k} x(t)$. Denote $x(t) = x(t, t_0, x_0)$ the equilibrium of the stochastic impulsive system (1) with the initial condition $x(t_0^+) = x_0$. It can be seen that $x(t) = 0$ is the trivial equilibrium of the stochastic impulsive system (1). Moreover, we consider a scalar impulsive system in the form of

$$\begin{cases} \dot{y}(t) = \varphi(t, y(t)), & t \neq t_k, \\ y(t_k^+) = \psi_k(t_k, y(t_k)), & t = t_k, \\ y(t_0^+) = y_0 = E(V(t_0, x_0)), & k = 1, 2, \cdots, \end{cases} \tag{2}$$

where $\varphi : R^+ \times R^+ \to R$, and $\psi_k : R^+ \times R^+ \to R^+$ satisfying $\varphi(t, 0) = 0$, and $\psi_k(t, 0) = 0$, as a comparison system.

Let v_0 be the class of Lyapunov-like functions $V : R^+ \times R^n \to R^+$, where V is continuous everywhere except t_k, $k \in N$, at which V is left continuous and the right limit $V(t_k^+)$ exists. To proceed, we need the following definitions [6].

Definition 1. *The trivial equilibrium of the stochastic impulsive system (1) is said to be stable if for any* $\varepsilon > 0$*, there exists* $\delta = \delta(t_0, \varepsilon) > 0$ *such that* $E(\|x(t)\|) \leq \varepsilon$ *holds for all* $E(\|x(t_0)\|) < \delta$*.*

Definition 2. *The trivial equilibrium of the stochastic impulsive system (1) is said to be asymptotically stable if it is stable and there exists* $\delta = \delta(t_0) > 0$ *such that* $\lim\limits_{t \to \infty} E(\|x(t)\|) = 0$ *for all* $E(\|x(t_0)\|) < \delta$*.*

Definition 3. *The right-hand upper and lower Dini derivatives of a Lyapunov-like function* $V : R^+ \times R^n \to R^+$ *is defined by*

$$D^+V(t, x) = \lim_{s \to 0^+} \sup \frac{1}{s}[V(t + s, x(t + s)) - V(t, x(t))], \tag{3}$$

and

$$D_+V(t, x) = \lim_{s \to 0^+} \inf \frac{1}{s}[V(t + s, x(t + s)) - V(t, x(t))]. \tag{4}$$

Definition 4. *For all* $V(t, x) \in v_0$*, an operator* L *is defined by*

$$\begin{aligned} LV(t, x(t)) = \ & V_t(t, x(t)) + V_x(t, x(t))f(t, x) + \\ & \tfrac{1}{2} trace[g^T(t, x)V_{xx}g(t, x)], \end{aligned} \tag{5}$$

where $V_t(t, x) = \frac{\partial V(t,x)}{\partial t}$*,* $V_x(t, x) = \left(\frac{\partial V(t,x)}{\partial x_1}, \frac{\partial V(t,x)}{\partial x_2}, \cdots, \frac{\partial V(t,x)}{\partial x_n}\right)^T$*, and* $V_{xx} = \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right)_{n \times n}$*.*

2 Main results

In this section, we will establish comparison theorems for stability analysis of the stochastic impulsive system (1). Firstly, we will present a comparison result which is useful for the further stability analysis.

Theorem 1. *Suppose that there exist a Lyapunov function $V : R^+ \times R^n \to R^+$, continuous functions $\varphi : R^+ \times R^+ \to R$, and $\psi_k : R^+ \times R^+ \to R$, where $\psi_k(t, y)$ is nondecreasing in y such that the following conditions are satisfied,*

i)

$$E(\varphi(t, V(t, x))) \leq \varphi(t, E(V(t, x))), \tag{6}$$

for $E(V(t, x)) \leq \alpha(E(\|x\|))$, where $\alpha(\cdot) \in K$,

ii)

$$E(LV(t, x)) \leq E(\varphi(t, V(t, x))), t \neq t_k, \tag{7}$$

iii)

$$E(V(t_k, x(t_k) + h_k(t_k, x(t_k)))) \leq \psi_k(t_k, E(V(t_k, x(t_k)))). \tag{8}$$

Then, $E(V(t, x)) \leq y(t)$, where $y(t)$ is the right-hand maximal solution of (2) on $[0, \infty)$ with initial condition $y(t_0^+) = y_0 = E(V(t_0, x_0))$.

Proof. When $t \in (t_k, t_{k+1}]$, by using

$$dV(t, x(t)) = LV(t, x(t))dt + V_x(t, x)g(t, x(t))dw(t) \tag{9}$$

and Ito's formula, it is followed that

$$\begin{aligned}&\frac{1}{\Delta t}(E(V(t+\Delta t, x(t+\Delta t))) - E(V(t, x(t))))\\ =&\frac{1}{\Delta t}\int_t^{t+\Delta t} E(LV(s, x(s)))ds, \Delta t > 0.\end{aligned} \tag{10}$$

As Δt approaches to zero, by (6) and (7), we have

$$\begin{aligned}&D^+(E(V(t, x(t)))) \leq E(\varphi(t, V(t, x)))\\ \leq&\varphi(t, E(V(t, x))), \quad t \in (t_k, t_{k+1}].\end{aligned} \tag{11}$$

Let $G(t) = E(V(t, x(t))) - y(t)$, then we can verify that $G(t_0) = E(V(t_0, x(t_0))) - y(t_0) = 0$. We claim that $G(t) \leq 0$ holds on $(t_k, t_{k+1}]$. Otherwise, there must exist T_1 and T_2, $t_k < T_1 < T_2 < t_{k+1}$ such that $G(t) \leq 0$ for $t \in (t_k, T_1)$, $G(T_1) = 0$ and $G(T_2) > 0$. Hence, it holds that $D^+G(T_1) > 0$. On the other hand, since (8) and (9) are satisfied for $t > t_0$ and $t \neq t_k$, by using (11), we get $D^+(E(V(T_1, x(T_1)))) \leq \varphi(T_1, E(V(T_1, x(T_1)))) = \varphi(T_1, y(T_1))$. Furthermore, since

$$\begin{aligned}D^+G(T_1) =&D^+(E(V(T_1, x(T_1))) - y(T_1))\\ \leq&D^+(E(V(T_1, x(T_1))) - D_+(y(T_1)))\end{aligned}$$

and

$$D_+y(T_1) = \lim_{s\to 0^+} \inf \frac{1}{h}[y(T_1 + s) - y(T_1)] = \varphi(T_1, y(T_1)).$$

Then, we get $D^+G(T_1) \le 0$, which results in a contradiction. Thus, $E(V(t,x)) \le y(t)$ holds on $(t_k, t_{k+1}]$. When $t = t_{k+1}$, we know

$$\begin{aligned} &E(V(t_{k+1}^+, x(t_{k+1}^+))) \\ =&E(V(t_{k+1}, x(t_{k+1})) + h_{k+1}(V(t_{k+1}, x(t_{k+1})))). \end{aligned} \tag{12}$$

By applying (9) to (12) and noting $\psi_k(t,y)$ is nondecreasing in y, it follows that

$$\begin{aligned} &E(V(t_{k+1}^+, x(t_{k+1}^+))) \le \psi_{k+1}(t_{k+1}, E(V(t_{k+1}, x(t_{k+1})))) \\ \le&\psi_{k+1}(t_{k+1}, y(t_{k+1})) = y(t_{k+1}^+). \end{aligned}$$

Therefore, $E(V(t,x)) \le y(t)$ holds for all $t \in [0,\infty)$. This completes this proof. □

Based on the above result, we can establish the following theorem which shows their relationship of stability properties between the stochastic impulsive system (1) and its comparison system (2).

Theorem 2. *Suppose that there exist a Lyapunov function $V : R^+ \times R^n \to R^+$, continuous functions $\varphi : R^+ \times R^+ \to R$, and $\psi_k : R^+ \times R^+ \to R$, $\psi_k(y)$ is nondecreasing in y such that the following conditions are satisfied,*
i)

$$E(\varphi(t, V(t,x))) \le \varphi(t, E(V(t,x))), \tag{13}$$

while

$$\beta(E(\|x\|)) \le E(V(t,x)) \le \alpha(E(\|x\|)), \tag{14}$$

where $\alpha(\cdot), \beta(\cdot) \in K$.
ii)

$$E(LV(t,x(t))) \le E(\varphi(t, V(t,x))), t \ne t_k, \tag{15}$$

iii) there exist $\rho_0 > 0$ and $\rho_1 > 0$ such that $E(\|x(t_k)\|) \in S_{\rho_0}$ implies that

$$E(\|x(t_k) + h_k(t_k, x(t_k))\|) \in S_{\rho_1}.$$

iv)

$$E(V(t_k, x(t_k) + h_k(t_k, x(t_k)))) \le \psi_k(t_k, E(V(t_k, x(t_k)))). \tag{16}$$

Then, the stability properties of the trivial equilibrium of the comparison system (2) imply those of the trivial equilibrium of the stochastic impulsive system (1).

Proof. Without loss of generality, we only consider the stability and asymptotical stability properties of the stochastic impulsive system (1) and the comparison system (2). The relationship of other stability properties of those systems can be derived similarly. First, we assume that the comparison system (2) is stable, i.e., for all $0 < \delta_2 < \rho = \min\{\rho_0, \rho_1\}$, there exists

$0 < \varepsilon_1(t_0, \delta_2) < \rho$ such that $0 < y_0 < \varepsilon_1$ implies $y(t) < \delta_1 = \beta(\delta_2)$. Let $E(\|x_0\|) \leq \varepsilon_2 = \beta^{-1}(\varepsilon_1(t_0, \delta_2))$. It is can be seen that (13)-(15) and (16) can satisfy the conditions of Theorem 1. Hence, by using Theorem 1 and (14) it follows that

$$E(\|x(t)\|) \leq \beta^{-1}(E(V(t, x(t)))) \leq \beta^{-1}(y) < \delta_2.$$

which shows that the stochastic impulsive system (1) is stable. Let us consider the asymptotical stability case. Suppose that the comparison system (2) is asymptotically stable, then the trivial equilibrium of (2) is stable and attractive. Hence, from the above proof procedure, we know that the s-tochastic impulsive system (1) must be stable. Then, we only need to verify the attractive property of the stochastic impulsive system (1). Assume that the comparison system (2) is attractive, then for every $0 < \delta_3 < \rho$, there exist $\varepsilon_3 = \varepsilon_3(\delta_3) > 0$ and $T_3 = T_3(t_0, \varepsilon_3) > 0$ such that $0 < y_0 < \varepsilon_3$ implies $y(t) < \beta(\delta_3)$, $t \geq t_0 + T_3$, From Theorem 1, it follows that

$$E(V(t, x(t))) \leq y(t), y_0 \geq \beta(E\,\|x_0\|), t > t_0.$$

Hence, for $E(\|x_0\|) < \varepsilon_4 = \beta^{-1}(\varepsilon_3(\delta_3))$, we have

$$E(\|x(t)\|) \leq \beta^{-1}(E(V(t, x(t)))) \leq \beta^{-1}(y(t)) < \delta_3, t > t_0,$$

which indicates that the trivial equilibrium of (1) is attractive. Therefore, the stochastic impulsive system (1) is asymptotically stable. □

Remark 1. *By Theorem 2, we can see that the stability properties of the stochastic impulsive system (1) can be obtained by those of its comparison system (2). Hence, if we can find an appropriate comparison system, the stability analysis of stochastic impulsive systems will be transformed to that of a scalar impulsive comparison system.*

Next, we will present sufficient conditions to ensure stability and asymptotical stability of the stochastic impulsive system (1) by using stability properties of a scalar impulsive system.

Theorem 3. *Suppose that the conditions of Theorem 2 are satisfied and let* $\varphi(t, x) = \dot{\mu}(t)x$, $\mu : R^+ \to R^+$, $\dot{\mu} > 0$, *and* $\psi_k(x) = a_k x$, $a_k \geq 0$. *If*

$$\mu(t_{k+1}) - \mu(t_k) + \ln(a_k) \leq 0, \text{ for all } k, \tag{17}$$

is satisfied, then the trivial equilibrium of the stochastic impulsive system (1) is stable. Moreover, if

$$\mu(t_{k+1}) - \mu(t_k) + \ln(\gamma a_k) \leq 0 \text{ for all } k, \text{ where } \gamma > 1, \tag{18}$$

is satisfied, then the trivial equilibrium of the stochastic impulsive system (1) is asymptotically stable.

Proof. When $\varphi(t,x) = \dot{\mu}(t)x$ and $\psi_k(x) = a_k x$, the comparison system can be rewritten as

$$\begin{cases} \dot{y}(t) = \dot{\mu}(t)y, & t \neq t_k, \\ \Delta y(t_k) = a_k y(t_k), & t = t_k, \\ y(t_0^+) = y_0 \geq 0, & k = 1, 2, \cdots . \end{cases} \tag{19}$$

Then, its solution is given by

$$y(t) = y_0 \prod_{t_0 < t_k < t} a_k e^{\mu(t) - \mu(0)}, t > t_0. \tag{20}$$

From the proof of Theorem 3.1.4 in [4], it can be seen that the comparison system is stable when (17) holds and it is asymptotically stable when (18) holds. By Theorem 2, it follows that that the stochastic impulsive system (1) is stable and asymptotically stable respectively. □

Consider the autonomous stochastic impulsive system described by

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dw(t), & t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k), & t = t_k, \\ x(t_0^+) = x_0, & k = 1, 2, \cdots , \end{cases} \tag{21}$$

where $x(t) \in R^n$, $f : R^n \to R^n$, $g : R^n \to R^{n \times m}$, B_k is an $n \times n$ constant matrix. We have the following corollary which gives the asymptotical stability criteria of the stochastic impulsive system (21).

Corollary 1. *Suppose that there exists a symmetric positive definite matrix P such that*

$$\eta(t_{k+1} - t_k) \leq -\ln(\gamma \lambda_k), \gamma > 1, \tag{22}$$

where λ_k is the largest eigenvalue of the matrix $P^{-1}(I + B_k^T)P(I + B_k)$ and η satisfies

$$f^T(x)Px + x^T Pf(x) + \frac{1}{2} trace(g^T(x)Pg(x)) \leq \eta x^T Px. \tag{23}$$

Then the trivial equilibrium of the stochastic impulsive system (21) is asymptotically stable.

Proof. It can be derived straightforwardly from Theorem 3 by letting $V(t,x) = x^T Px$ and choosing the following comparison system

$$\begin{cases} \dot{y}(t) = \eta y, & t \neq t_k, \\ \Delta y(t_k) = \lambda_k y(t_k), & t = t_k, \\ y(t_0^+) = y_0 \geq 0, & k = 1, 2, \cdots . \end{cases}$$

□

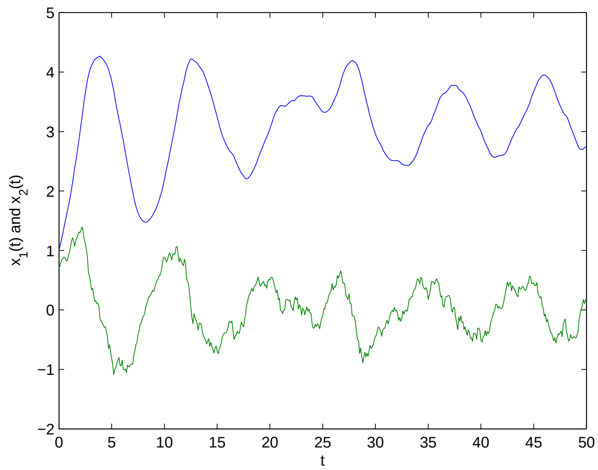

Figure 1: The state trajectories of x_1 and x_2 in (25) without impulsive effects

3 A numerical example of power systems

Consider the following simplified stochastic system model representing a power system with random perturbations,

$$\left\{ \begin{array}{lcl} dx_1 & = & x_2 dt, \\ dx_2 & = & -\frac{D}{M} x_2 dt + \frac{EVG}{M} \sin x_1 dt + \frac{x_1 E^2}{M} dw, \end{array} \right. \tag{24}$$

where D is the damping coefficient, M is the inertia constant, E, V are the voltage magnitudes, G is the system load, x_1 is the phase angle, and x_2 is the angular frequency [12]. By choosing $D = 0.5$, $M = 2$, $E = 1.2$, $V = 1$, and $G = 0.8$, the corresponding stochastic system can be given by

$$\left\{ \begin{array}{lcl} dx_1 & = & x_2 dt, \\ dx_2 & = & -0.25 x_2 dt + 0.48 \sin x_1 dt + 0.72 x_1 dw. \end{array} \right. \tag{25}$$

Fig. 1 shows for the state trajectories of x_1 and x_2 with the initial condition $[1, 0.7]^T$, which indicates that the stochastic system is not stable. Assume that system (25) evolves under impulsive effects at the time instants t_k in form of

$$\Delta x(t_k) = B_k x(t_k) = \left[\begin{array}{cc} -0.6 & 0 \\ 0 & -0.4 \end{array} \right]. \tag{26}$$

Letting $\Delta t_k = t_k - t_{k-1} = 0.2$, and $P = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$, we can verify that

$$f^T(x)Px + x^T P f(x) + \frac{1}{2}\text{trace}(g^T(x)Pg(x)) \leq 1.6 x^T P x.$$

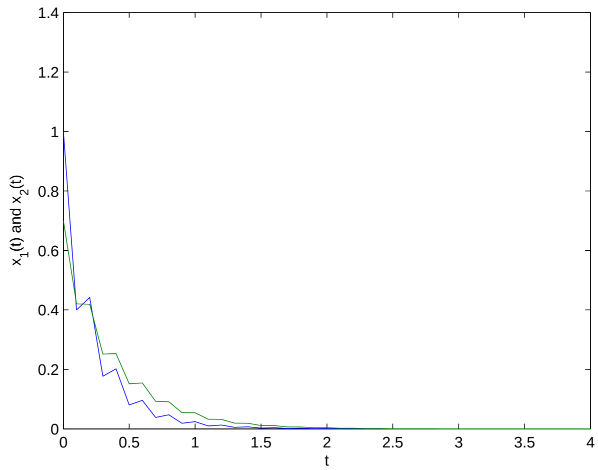

Figure 2: The state trajectories of x_1 and x_2 of the stochastic impulsive system (25)-(26), where the impulsive interval is 0.2s.

By choosing $\eta = 1.6$ and $\gamma = 2$, we can obtain that

$$\Delta t_k = t_{k+1} - t_k \leq \frac{-\ln(\gamma\lambda_k)}{\eta} = 0.2053.$$

Then, by Corollary 1, we can conclude that the trivial equilibrium of the stochastic impulsive system (25)-(26) is asymptotically stable. Numerical simulations are conducted to verify the effectiveness of the results obtained. Fig. 2 shows that the state trajectories of x_1 and x_2 of the stochastic impulsive system (25)-(26). In Fig. 2, it can been seen that the trivial solution of the stochastic impulsive system (25)-(26) is asymptotically stable in less than 3 seconds.

4 Conclusion

We have studied stability problems of a class of stochastic impulsive systems, which represent a general model of dynamic systems involving impulsive effects and stochastic disturbances. Sufficient conditions of stability and asymptotic stability of the stochastic impulsive systems are obtained by using a new comparison method. Finally, a numerical example of uncertain power systems is given to illustrate the effectiveness of our results obtained.

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