A \textsc{$\alpha$}-\textsc{cut} approximate algorithm for goal-based bilevel risk management systems

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Bilevel programming techniques are developed for decentralized decision problems with decision makers located in two levels. Both upper and lower decision makers, termed as leader and follower, try to optimize their own objectives in solution procedure but are affected by those of the other levels. When a bilevel decision model is built with fuzzy coefficients and the leader and/or follower have goals for their objectives, we call it fuzzy goal bilevel (FGBL) decision problem. This paper first proposes a \textsc{$\alpha$}-\textsc{cut} set based FGBL model. A programmable \textsc{$\alpha$}-\textsc{cut} approximate algorithm is then presented in detail. Based on this algorithm, a FGBL software system is developed to reach solutions for FGBL decision problems. Finally, two examples are given to illustrate the application of the proposed algorithm.

Keywords: Bilevel decision making; goal programming; fuzzy sets; optimization; risk management.

1. Introduction

Bilevel programming techniques, initiated by Von Stackelberg,\textsuperscript{25} are mainly developed for solving decentralized management problems with decision makers in a two-level hierarchy. The upper decision maker is termed leader and the lower the follower.\textsuperscript{2} Fuzzy bilevel programming techniques, which handle bilevel decision

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problems when coefficients are described by fuzzy sets.\textsuperscript{30} are recognized effective on analyzing potential risks and generating warnings in risk management.

The investigation of bilevel decision problems is strongly motivated by real world applications, and bilevel programming techniques have been applied with remarkable success in different domains such as decentralized resource planning,\textsuperscript{28} electronic power market,\textsuperscript{29} logistics,\textsuperscript{29} civil engineering,\textsuperscript{1} and road network management.\textsuperscript{28, 31} For risk management, which aims to measure and assess any risk and develop strategies to manage it,\textsuperscript{30} bilevel programming techniques play significant roles as well. Decision makers face the challenge of allocating supply resources, transportation ability, rescue aid and whatever to minimize the effect of threat. These decision makers may be located at different levels within a management network and thus have inconsistent concerns. For example, when a severe earthquake occurs,\textsuperscript{4} the roadway systems usually get different degrees of damage, which reduces the through capacity and causes traffic congestion. The commander of an Emergency-Response Center, in the upper level, aims at allowing traffic to go through the disaster areas as much as possible within the roadway capacity, while the road users, located at the lower level, always choose the shortest route to actualize emergency rescues. The decision from the commander and the road users will inevitably influence the choice from each other. In this situation, bilevel programming should be a suitable technique to solve this decision problem.

Fuzzy numbers, which are useful for representing numerical quantities in a vague environment,\textsuperscript{27} have been applied in subsequent research on bilevel decision problems. Shih et al.\textsuperscript{21} and Lai\textsuperscript{22} first applied fuzzy set approach to bilevel decision problems. Their method, however, sometimes might cause a final undesirable solution due to the inconsistency between fuzzy goals of the objective functions and the decision variables.\textsuperscript{22} To overcome this problem, Sakawa et al.\textsuperscript{22} developed an interactive fuzzy set approach by deriving a satisfactory solution and updating the satisfactory degrees of decision makers with considerations of overall satisfactory balance among all levels. In our research lab, an approximation approach has been developed\textsuperscript{28, 30} based on framework building and models formatting.\textsuperscript{16, 17} Solutions can be reached by solving associated multiple objectives bilevel decision problem under different cut sets.

Goal programming was originally proposed by Charnes and Cooper\textsuperscript{4} in 1961 for a linear model. It has been further developed by Lee,\textsuperscript{14} Ignizio,\textsuperscript{11, 12} Charnes and Cooper.\textsuperscript{3} Recent research on goal programming can be found from Refs. 10, 15, 18–20. Goal programming requests a decision maker to set a goal for the objective that he/she wishes to attain. A preferred solution is then defined to minimize the deviation from the goal. Therefore, goal programming seems to yield a satisfactory solution rather than an optimal one. In fuzzy bilevel decision problems, when both a leader and follower set goals for their objectives respectively, the problem becomes a FGBL decision problem, which is addressed by this study.

This paper is organized as follows. After the introduction, Sec. 2 reviews related definitions and theorems of FGBL programming. In Sec. 3, a λ-cut set based FGBL
model and a λ-cut approximate algorithm to solve FGBL problems are presented. Meanwhile, a FGBL software system which implemented the proposed algorithm is described. A numerical example and a case-based example on traffic management in a disaster area are shown in Sec. 4. Conclusions and further study are discussed in Sec. 5.

2. Preliminaries

In this section, some definitions and formulations used in subsequent sections are presented.

Throughout this paper, \( \mathbb{R} \) represents the set of all real numbers, \( \mathbb{R}^n \) is \( n \)-dimensional Euclidean space, \( F^*(\mathbb{R}) \) and \( F^*(\mathbb{R}^n) \) are the set of all finite fuzzy numbers and the set of all \( n \)-dimensional finite fuzzy numbers on \( \mathbb{R}^n \), respectively.

**Definition 2.1.** (Ref. 21) The λ-cut set of a fuzzy set \( A \) is defined as an ordinary set \( A_\lambda \) for which the degree of its membership function exceeds the level \( \lambda \):

\[
A_\lambda = \{ x | \mu_A(x) \geq \lambda \}, \quad \lambda \in [0, 1].
\]

\( A_\lambda \) is a nonempty bounded closed interval and it can be denoted by

\[
A_\lambda = [A_\lambda^L, A_\lambda^U],
\]

where \( A_\lambda^L \) and \( A_\lambda^U \) are the lower and upper bounds of the interval, respectively.

**Definition 2.2.** (Ref. 30) For any \( n \)-dimensional fuzzy numbers \( \tilde{a}, \tilde{b} \in F(\mathbb{R}^n) \), under a certain satisfactory degree \( \alpha \), we define

\[
\tilde{a} \leq \alpha \tilde{b} \iff a_i^L \leq b_i^L \quad \text{and} \quad a_i^U \leq b_i^U, \quad i = 1, 2, \ldots, n, \quad \forall \lambda \in [0, 1],
\]

where \( \alpha \) is the adjustable satisfactory degree, which means, when comparing two fuzzy numbers all values with membership grades smaller than \( \alpha \) are neglected.

**Definition 2.3.** A fuzzy linear bilevel (FLBL) decision problem is defined as:

For \( x \in X \subset \mathbb{R}^n, y \in Y \subset \mathbb{R}^m, F : X \times Y \rightarrow F^*(\mathbb{R}), \) and \( f : X \times Y \rightarrow F^*(\mathbb{R}) \),

\[
\min_{x \in X} F(x, y) = c_1 x + d_1 y \quad (1a)
\]

subject to \( \tilde{A}_1 x + \tilde{B}_1 y \preceq_A \tilde{b}_1 \) \quad (1b)

\[
\min_{y \in Y} f(x, y) = c_2 x + d_2 y \quad (1c)
\]

subject to \( \tilde{A}_2 x + \tilde{B}_2 y \preceq_A \tilde{b}_2 \) \quad (1d)

where \( \tilde{c}_1, \tilde{c}_2 \in F^*(\mathbb{R}^n), \tilde{d}_1, \tilde{d}_2 \in F^*(\mathbb{R}^m), \tilde{b}_1 \in F^*(\mathbb{R}), \tilde{b}_2 \in F^*(\mathbb{R}) \). \( \tilde{A}_1 = (a_{ij})_{n \times n}, \)

\( \tilde{A}_2 = (a_{ij})_{n \times m}, \tilde{b}_1 \in F^*(\mathbb{R}), \tilde{B}_1, \tilde{B}_2 \in F^*(\mathbb{R}) \). \( \tilde{c}_1, \tilde{c}_2 \in F^*(\mathbb{R}) \), \( \tilde{d}_1, \tilde{d}_2 \in F^*(\mathbb{R}) \). \( F^*(\mathbb{R}) \) is the set of all finite fuzzy numbers.
Theorem 2.1. (Ref. 30) For $x \in X \subseteq \mathbb{R}^n$, $y \in Y \subseteq \mathbb{R}^m$, if all the fuzzy coefficients $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_j, \tilde{d}_i, \tilde{c}_i$, and $\tilde{d}_i$, have membership functions in FLBL problem (1):

\[
\mu_i(t) = \begin{cases} 
0 & t < \alpha_{i0}^L \\
\frac{\lambda_i - \alpha_{i0}^L}{\alpha_{i1}^L - \alpha_{i0}^L} (t - \alpha_{i0}^L) + \lambda_0 & \alpha_{i0}^L \leq t < \alpha_{i1}^L \\
\frac{\lambda_i - \alpha_{i1}^L}{\alpha_{i2}^L - \alpha_{i1}^L} (t - \alpha_{i1}^L) + \lambda_1 & \alpha_{i1}^L \leq t < \alpha_{i2}^L \\
\vdots & \vdots \\
\frac{\lambda_i - \alpha_{i(n-1)}^L}{\alpha_{i(n-1)}^L - \alpha_{i(n-2)}^L} (t - \alpha_{i(n-2)}^L) + \lambda_{n-1} & \alpha_{i(n-2)}^L \leq t < \alpha_{i(n-1)}^L \\
\frac{\lambda_i - \alpha_{i(n-1)}^L}{\alpha_{i0}^L - \alpha_{i(n-1)}^L} (t - \alpha_{i(n-1)}^L) + \lambda_0 & \alpha_{i(n-1)}^L \leq t \leq \alpha_{i0}^L \\
0 & \alpha_{i0}^L < t
\end{cases}
\]  

(2)

where $\tilde{x}$ denotes $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_j, \tilde{d}_i$, respectively, then, it is the solution of problem (1) that $(\mathbf{x}^*, \mathbf{y}^*) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying

\[
\min_{x \in X} (F(x, y))^L_{\mathbf{x}_0} = c_1^L x + d_1^L y
\]

\[
\vdots
\]

\[
\min_{x \in X} (F(x, y))^R_{\mathbf{x}_0} = c_1^R x + d_1^R y
\]

subject to $A_1^L x + B_1^L y \leq b_1^L$,

\[
\vdots
\]

\[
A_1^R x + B_1^R y \leq b_1^R
\]

\[
\min_{y \in Y} (f(x, y))^L_{\mathbf{y}_0} = c_2^L x + d_2^L y
\]

\[
\vdots
\]

\[
\min_{y \in Y} (f(x, y))^R_{\mathbf{y}_0} = c_2^R x + d_2^R y
\]  

(3a)

(3b)

(3c)
subject to $A_{2i,x}x + B_{2i,y}y \leq b_{2i,n}$.
\vdots
$A_{2i,x}x + B_{2i,y}y \leq b_{2i,n}$,
$A_{2i,x}x + B_{2i,y}y \leq b_{2i,n}$,
\vdots
$A_{2i,x}x + B_{2i,y}y \leq b_{2i,n}$.

(3d)

3. A $\lambda$-Cut Approximate Algorithm for Fuzzy Goal Bilevel Decision Problems

First, we give the definition of a multiple objective bilevel (MOBL) decision problem:

**Definition 3.1.** For $x \in X \subset \mathbb{R}^n$, $y \in Y \subset \mathbb{R}^m$, a MOBL decision problem is defined as

\[
\begin{align*}
\min_{x \in X} & \quad F(x, y) \\
\text{subject to} & \quad G(x, y) \leq 0, \\
\min_{y \in Y} & \quad f(x, y) \\
\text{subject to} & \quad g(x, y) \leq 0,
\end{align*}
\]

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.

Associated with MOBL problem (4), some definitions are listed below:

**Definition 3.2.**

1. **Constraint region of MOBL problem (4)**

   $$S \triangleq \{(x, y) : x \in X, y \in Y, G(x, y) \leq 0, g(x, y) \leq 0\}.$$  

   It refers to all possible combinations of choices that the leader and follower may make.

2. **Projection of $S$ onto the leader’s decision space**

   $$S(X) \triangleq \{x \in X : \exists y \in Y, G(x, y) \leq 0, g(x, y) \leq 0\}.$$  

3. **Feasible set for the follower $\forall x \in S(X)$**

   $$S(x) \triangleq \{y \in Y : (x, y) \in S\}.$$  

4. **Follower’s rational reaction set for $x \in S(x)$**

   $$P(x) \triangleq \{y \in Y : y \in \arg\min\{f(x, y) : y \in S(x)\}\}.$$
where \( \text{argmin}\{f(x, y) : y \in S(x)\} = \{y \in S(x) : f(x, y) \leq f(x, \hat{y}), \hat{y} \in S(x)\} \),

which means, the follower observes the leader’s action and reacts by selecting

\( y \) from his/her possible set to minimize his/her objective function.

(5) Inducible region

\[
IR \triangleq \{(x, y) : (x, y) \in S, y \in P(x)\},
\]

which represents the set over which the leader may optimize his or her objective.

To ensure that (4) is well posed, it is assumed that \( S \) is nonempty and compact,

and that for all decisions taken by a leader, the follower has some room to respond:

i.e. \( P(x) \neq \emptyset \). Thus, in terms of the above notation, a MOBL problem can be

written as

\[
\min \{P(x, y) : (x, y) \in IR\}.
\]

Goals given for objectives of a leader and follower in (1) are denoted by fuzzy

numbers \( \hat{g}_L \) and \( \hat{g}_F \) with membership functions \( \mu_{g_L} \) and \( \mu_{g_F} \), respectively, and our

concern is to make the objectives of both the leader and the follower as near to

their goals as possible. The differences between \( P(x, y) \) and \( \hat{g}_L \), \( f(x, y) \) and \( \hat{g}_F \)

are usually defined as deviation functions. Initiated by the idea of Theorem 2.1, we use

\( \lambda \)-cut set of fuzzy number to format a FGBL model as in Definition 3.3.

**Definition 3.3.** The \( \lambda \)-cut set based FGBL model is defined as

\[
\begin{align*}
\min_{x \in X} c_{i,1}^L x + d_{i,1}^L y - g_{i,1}^L, \\
\min_{x \in X} c_{i,2}^L x + d_{i,2}^L y - g_{i,2}^L, \\
\text{subject to } A_1^L x + B_1^L y \leq b_1^L, \\
A_2^L x + B_2^L y \leq b_2^L, \\
\min_{y \in Y} c_{i,1}^R x + d_{i,1}^R y - g_{i,1}^R, \\
\min_{y \in Y} c_{i,2}^R x + d_{i,2}^R y - g_{i,2}^R, \\
\text{subject to } A_1^R x + B_1^R y \leq b_1^R, \\
A_2^R x + B_2^R y \leq b_2^R, \\
\end{align*}
\]

(7a) \( \lambda \)-cut maximization

\[
\begin{align*}
\min_{x \in X} c_{i,1}^L x + d_{i,1}^L y - g_{i,1}^L, \\
\min_{x \in X} c_{i,2}^L x + d_{i,2}^L y - g_{i,2}^L, \\
\text{subject to } A_1^L x + B_1^L y \leq b_1^L, \\
A_2^L x + B_2^L y \leq b_2^L, \\
\min_{y \in Y} c_{i,1}^R x + d_{i,1}^R y - g_{i,1}^R, \\
\min_{y \in Y} c_{i,2}^R x + d_{i,2}^R y - g_{i,2}^R, \\
\text{subject to } A_1^R x + B_1^R y \leq b_1^R, \\
A_2^R x + B_2^R y \leq b_2^R, \\
\end{align*}
\]

(7b) \( \lambda \)-cut minimization

where \( \epsilon_1, \epsilon_2 \in F^*(\mathbb{R}^n) \), \( \delta_1, \delta_2 \in F^*(\mathbb{R}^n) \), \( \tilde{b}_1 \in F^*(\mathbb{R}^n) \), \( \tilde{b}_2 \in F^*(\mathbb{R}^n) \), \( \tilde{A}_1 = (\tilde{a}_{ij})_{m \times n} \), \( \tilde{B}_1 = (\tilde{b}_{ij})_{m \times n} \), \( \tilde{A}_2 = (\tilde{a}_{ij})_{m \times n} \), \( \tilde{B}_2 = (\tilde{b}_{ij})_{m \times n} \), \( \tilde{a}_{ij} \), \( \tilde{b}_{ij} \), \( \tilde{a}_{ij} \), \( \tilde{b}_{ij} \), \( \tilde{a}_{ij} \), \( \tilde{b}_{ij} \in F^*(\mathbb{R}) \).

For a clearer understanding of the idea adopted, we define

\[
\epsilon_{i,1}^L = \frac{1}{2}|c_{i,1}^L x + d_{i,1}^L y - g_{i,1}^L| - (c_{i,1}^L x + d_{i,1}^L y - g_{i,1}^L)|.
\]
\[ v^L_{x|\lambda} = \frac{1}{2} \| c^L_{x|\lambda} x + d^L_{x|\lambda} y - g^L_{x|\lambda} \| + (c^L_{x|\lambda} x + d^L_{x|\lambda} y - g^L_{x|\lambda}) \]
\[ v^R_{y|\lambda} = \frac{1}{2} \| c^R_{y|\lambda} x + d^R_{y|\lambda} y - g^R_{y|\lambda} \| - (c^R_{y|\lambda} x + d^R_{y|\lambda} y - g^R_{y|\lambda}) \]
\[ v^R_{x|\lambda} = \frac{1}{2} \| c^R_{x|\lambda} x + d^R_{x|\lambda} y - g^R_{x|\lambda} \| + (c^R_{x|\lambda} x + d^R_{x|\lambda} y - g^R_{x|\lambda}) \]
\[ v^L_{y|\lambda} = \frac{1}{2} \| c^L_{y|\lambda} x + d^L_{y|\lambda} y - g^L_{y|\lambda} \| - (c^L_{y|\lambda} x + d^L_{y|\lambda} y - g^L_{y|\lambda}) \]
\[ v^L_{x|\lambda} = \frac{1}{2} \| c^L_{x|\lambda} x + d^L_{x|\lambda} y - g^L_{x|\lambda} \| + (c^L_{x|\lambda} x + d^L_{x|\lambda} y - g^L_{x|\lambda}) \]  \hspace{1cm} (8)

Associated with the FGL problem defined by (7), we now consider the following bilevel decision problem:

For \((v^L_{x|\lambda}, v^R_{x|\lambda}, v^L_{y|\lambda}, v^R_{y|\lambda}) \in \mathbb{R}^4, X' \subseteq X \times \mathbb{R}^3, (v^L_{x'}, v^R_{x'}, v^L_{y'}, v^R_{y'}) \in \mathbb{R}^4, Y' \subseteq Y \times \mathbb{R}^3, \) let \(x = (x_1, \ldots, x_n) \in X, x' = (x_1, \ldots, x_n, v^L_{y|\lambda}, v^R_{y|\lambda}, v^L_{x|\lambda}, v^R_{x|\lambda}), y' = (y_1, \ldots, y_m) \in Y, y'' = (y_1, \ldots, y_m, v^L_{y|\lambda}, v^R_{y|\lambda}, v^L_{x|\lambda}, v^R_{x|\lambda}) \in Y', \) and \(v^L_{x|\lambda}, v^R_{x|\lambda}, v^L_{y|\lambda}, v^R_{y|\lambda} : X' \times Y' \to F^\prime (\mathbb{R}).\)

\[
\begin{align*}
\min_{x' \in X} v^L_{x'|x'} = v^L_{x'|x'} + v^R_{x'|x'}, \\
\min_{x' \in X} v^L_{x'|x'} = v^R_{x'|x'} + v^L_{x'|x'}. \\
\end{align*}
\hspace{1cm} (9a)

subject to \(c^L_{x|\lambda} x + d^L_{x|\lambda} y + v^L_{x|\lambda} - v^L_{x|\lambda} = g^L_{x|\lambda}, \)
\(c^R_{x|\lambda} x + d^R_{x|\lambda} y + v^R_{x|\lambda} - v^R_{x|\lambda} = g^R_{x|\lambda}, \)
\(c^L_{y|\lambda} x + d^L_{y|\lambda} y + v^L_{y|\lambda} - v^L_{y|\lambda} = g^L_{y|\lambda}, \)
\(c^R_{y|\lambda} x + d^R_{y|\lambda} y + v^R_{y|\lambda} - v^R_{y|\lambda} = g^R_{y|\lambda}, \)
\(c^L_{x|\lambda} x + d^L_{x|\lambda} y + v^L_{x|\lambda} - v^L_{x|\lambda} = 0, \)
\(c^R_{x|\lambda} x + d^R_{x|\lambda} y + v^R_{x|\lambda} - v^R_{x|\lambda} = 0, \)
\(v^L_{x|\lambda} - v^L_{x|\lambda} = 0, \)
\(v^R_{x|\lambda} - v^R_{x|\lambda} = 0, \)
\(A^L_{x|\lambda} x + B^L_{y|\lambda} y \leq b^L_{x|\lambda}, \)
\(A^R_{x|\lambda} x + B^R_{y|\lambda} y \leq b^R_{x|\lambda}, \)
\(\min_{y' \in Y'} v^L_{x'|y'} = v^L_{x'|y'} + v^R_{x'|y'}, \)
\(\min_{y' \in Y'} v^L_{x'|y'} = v^R_{x'|y'} + v^L_{x'|y'}. \)
\hspace{1cm} (9b)

\[
\begin{align*}
\min_{y' \in Y'} v^L_{x'|y'} = v^L_{x'|y'} + v^R_{x'|y'}, \\
\min_{y' \in Y'} v^L_{x'|y'} = v^R_{x'|y'} + v^L_{x'|y'}. \\
\end{align*}
\hspace{1cm} (9c)
subject to \( e_{2}^{T}A_{j}x + d_{2j}^{T}y + e_{2}^{R} - v_{2j}^{T}A_{j}y = g_{2j}^{T} \),
\( e_{2}^{R}x + d_{2}^{T}y + e_{2}^{R} - v_{2}^{T}A_{j}y = g_{2}^{T} \),
\( e_{2j}^{T}y - v_{2j}^{R}A_{j}y = v_{2j}^{R}A_{j}y \),
\( e_{2j}^{R}y - v_{2j}^{R}A_{j}y = v_{2j}^{R}A_{j}y \),
where \( j = 1, 2, \ldots, l \).

(9d)

**Theorem 3.1.** Let \((x^{*}, y^{*}) = (x^{*}, v_{1j}^{T}A_{j}x + d_{1j}^{T}y + g_{1j}^{T})\) be the optimal solution to the bilevel decision problem (9); then, \((x^{*}, y^{*})\) is the optimal solution to the bilevel decision problem defined by (7).

**Proof.** By Definition 3.2, let the notations associated with problem (7) be denoted by

\[ S = \{(x, y) : A_{i}^{T}x + B_{i}^{T}y \leq b_{i}^{T}, A_{i}^{R}x + B_{i}^{R}y \leq b_{i}^{R}, 1 \leq i \leq k, 0 \leq j \leq m \}, \]  

(10a)

\[ S(X) = \{x \in X : \exists y \in Y, A_{i}^{T}x + B_{i}^{T}y \leq b_{i}^{T}, A_{i}^{R}x + B_{i}^{R}y \leq b_{i}^{R}, 1 \leq i \leq k, 0 \leq j \leq m \}, \]  

(10b)

\[ S(x) = \{y \in Y : (x, y) \in S\}. \]  

(10c)

\[ P(x) = \{y \in Y : y \in \arg \min \{e_{2j}^{T}y + d_{2j}^{R}y - g_{2j}^{T} \} \}. \]  

(10d)

Problem (7) can be written as

\[ \min \{e_{2j}^{T}x + d_{2j}^{T}y - g_{2j}^{T} \} : (x, y) \in IR \]  

(11)

and those of problem (9) are denoted by

\[ S' = \{(x', y') : A_{i}^{T}x + B_{i}^{T}y \leq b_{i}^{T}, A_{i}^{R}x + B_{i}^{R}y \leq b_{i}^{R}, 1 \leq i \leq k, 0 \leq j \leq m \}, \]  

(12a)

\[ v_{1j}^{T} - v_{1j}^{R} = 0, v_{2j}^{T} - v_{2j}^{R} = 0, i = 1, 2, \]  

(12b)

\[ e_{2j}^{T}x + d_{2j}^{T}y + v_{2j}^{T} - v_{2j}^{R}A_{j}y = g_{2j}^{T}, \]  

(12c)

\[ e_{2j}^{R}x + d_{2j}^{R}y + v_{2j}^{R}A_{j}y = g_{2j}^{T}, \]  

(12d)

\[ j = 0, 1, 2, \ldots, l \].
\[ S(x') = \{ x' \in X' : \exists y' \in Y', A_{i}^{R} x + B_{i}^{R} y \leq b_{i}^{R}, A_{i}^{L} x + B_{i}^{L} y \leq b_{i}^{L}, \]
\[ e_{i}^{L} y = 0, e_{i}^{R} y = 0, i = 1, 2, \]
\[ c_{i}^{u} x + d_{i}^{u} y + e_{i}^{u} y - e_{i}^{l} y = g_{i}^{u}, c_{i}^{l} x + e_{i}^{l} y - e_{i}^{u} y = g_{i}^{l}, \]
\[ f_{i}^{u} x + d_{i}^{u} y + e_{i}^{u} y - e_{i}^{l} y = g_{i}^{u}, f_{i}^{l} x + d_{i}^{l} y + e_{i}^{l} y - e_{i}^{u} y = g_{i}^{l}, \]
\[ j = 0, 1, \ldots, l \}. \]

(12b)

\[ S(x') = \{ y' \in Y' : (x', y') \in S' \}, \]
\[ P(x') = \{ y' \in Y' : y' \in \arg\min \{ d_{i}^{u} + e_{i}^{u} y - e_{i}^{l} y : y' \in S(x') \} \}. \]

(12c)

(12d)

(12e)

Problem (9) can be written as
\[ \min \{ r_{i}^{L} + r_{i}^{L}, v_{i}^{R} - v_{i}^{R} : (x', y') \in I'R' \}. \]

(13)

As \((x'^{*}, y'^{*})\) is the optimal solution to the problem (9), from (13), it can be obtained that \(\forall (x', y') \in I'R'\), we have: \(r_{i}^{L} + r_{i}^{L} \geq v_{i}^{R} - v_{i}^{R} \) and \(v_{i}^{R} - v_{i}^{R} \geq r_{i}^{L} + r_{i}^{L} \).

As
\[ c_{i}^{u} x + d_{i}^{u} y + v_{i}^{R} - v_{i}^{R} = g_{i}^{u}, \]
we have
\[ r_{i}^{L} + r_{i}^{L} = |c_{i}^{u} x + d_{i}^{u} y - g_{i}^{u}|, \]
\[ v_{i}^{R} - v_{i}^{R} = |c_{i}^{u} x + d_{i}^{u} y - g_{i}^{u}|. \]

So,
\[ |c_{i}^{u} x + d_{i}^{u} y - g_{i}^{u}| \geq |c_{i}^{u} x^{*} + d_{i}^{u} y^{*} - g_{i}^{u}|. \]

(14a)

Similarly, we can get that
\[ |v_{i}^{R} - v_{i}^{R}| \geq |v_{i}^{R} - v_{i}^{R}|. \]

(14b)

Now we prove that the projection of \(S'\) onto the \(X \times Y\) space, denoted by \(S'_{X,Y}\), is equal to \(S\).

On the one hand, \(\forall (x, y) \in S'_{X,Y}\), from constraints: \(A_{i}^{R} x + B_{i}^{R} y \leq b_{i}^{R}, A_{i}^{L} x + B_{i}^{L} y \leq b_{i}^{L}, i = 1, 2\) in \(S'\), we have: (\(x, y\) \(\in S\), so \(S'_{X,Y} \subseteq S\).

On the other hand, \(\forall (x, y) \in S\), by (8), we can always find such \(r_{i}^{L}, v_{i}^{R}, e_{i}^{L}, e_{i}^{R}, i = 1, 2\), which make constraints: \(v_{i}^{R} - v_{i}^{R} = 0, i = 1, 2, \)
\[ c_{i}^{u} x + d_{i}^{u} y + v_{i}^{R} - v_{i}^{R} = g_{i}^{u}, c_{i}^{l} x + v_{i}^{R} - v_{i}^{R} = g_{i}^{l}, \]
\[ f_{i}^{u} x + d_{i}^{u} y + v_{i}^{R} - v_{i}^{R} = g_{i}^{u}, f_{i}^{l} x + v_{i}^{R} - v_{i}^{R} = g_{i}^{l}, \]
\[ c_{i}^{u} x + d_{i}^{u} y + v_{i}^{R} - v_{i}^{R} = g_{i}^{u}, c_{i}^{l} x + v_{i}^{R} - v_{i}^{R} = g_{i}^{l}, \]
satisfied. Together
with the inequations of $A_{2i}^L x + B_{2i}^L y \leq b_{2i}^L$ and $A_{2i}^R x + B_{2i}^R y \leq b_{2i}^R$, $i = 1, 2$ requested by $S$, we have $(x, v_{1i}^L, v_{1i}^R, y, v_{2i}^L, v_{2i}^R) \in S'$; thus, $(x, y) \in S'_{|X,Y}$, $S \subseteq S'_{|X,Y}$.

So, we can prove that
\[ S'_{|X,Y} = S. \]  
(15)

Similarly, we have
\[ S(x)'_{|X,Y} = S(x), \]  
(16a)
\[ S(Y)'_{|X,Y} = S(Y). \]  
(16b)

Also, from
\[ c_{2i}^L x + d_{2i}^L y + v_{2i}^L - v_{2i}^L = g_{2i}^L, \quad \text{and} \quad v_{2i}^L - c_{2i}^L = 0, \]
we have
\[ v_{2i}^L + v_{2i}^R = |c_{2i}^L x + d_{2i}^L y - g_{2i}^L|. \]  
(17a)

Similarly, we have
\[ v_{2i}^R + v_{2i}^R = |c_{2i}^R x + d_{2i}^R y - g_{2i}^R|. \]  
(17b)

Thus,
\[ P(x') = \{ y' \in Y' : \exists \ y' \in \text{argmin}\{v_{2i}^L + v_{2i}^R : v_{2i}^L = g_{2i}^L\} \} \]
\[ = |c_{2i}^L x + d_{2i}^L y - g_{2i}^L| : y' \in S(x') \}. \]  
(18)

From (15) and (18), we get
\[ P(x')_{|X,Y} = P(x). \]  
(19)

From (10c), (12c), (15), and (19), we get
\[ IR'_{|X,Y} = IR, \]  
(20)
which means, the leaders of (7) and (9) share the same optimizing space in $X \times Y$ space.

Thus, from (14) and (20) and the discussions above, we have
\[ \forall (x, y) \in IR, \exists (x', y') \in \text{argmin}\{v_{1i}^L + v_{1i}^R + v_{1i}^L + v_{1i}^R : v_{1i}^L, \}
\[ = |c_{2i}^L x + d_{2i}^L y - g_{2i}^L, \} \]  
\[ = |c_{2i}^R x + d_{2i}^R y - g_{2i}^R| : \}
\[ = S(x'). \}
(21a)

By adopting weighting method, (9) can be further transferred into (21):
\[ \min_{x \in X, y \in Y} c_{1i}^L x + v_{1i}^L + v_{1i}^R + v_{1i}^R. \]  
(21a)
subject to \( c_{L_i}^f x + d_{L_i}^f y + v_i^{L_i} - v_i^{R_i} = g_{L_i}^f \),
\[ \begin{align*}
    c_{R_i}^f x + d_{R_i}^f y + v_i^{R_i} - v_i^{L_i} = g_{R_i}^f, \\
    v_i^{L_i} - v_i^{R_i} = 0, \\
    v_i^{R_i} - v_i^{L_i} = 0, \\
    A_1 y + B_1 y \leq b_1^L, \\
    A_2 x + B_2 y \leq b_2^L, \\
    \min_{y \in \mathbb{F}} v_i^{L_i} + v_i^{R_i} + v_i^{R_i} + v_i^{L_i} = 0, \quad (21b)
\end{align*} \]

subject to \( c_{2A_i}^f x + d_{2A_i}^f y + v_i^{L_i} - v_i^{R_i} = g_{2A_i}^f \),
\[ \begin{align*}
    c_{2A_i}^R x + d_{2A_i}^R y + v_i^{R_i} - v_i^{L_i} = g_{2A_i}^R, \\
    v_i^{L_i} - v_i^{R_i} = 0, \\
    v_i^{R_i} - v_i^{L_i} = 0, \\
    A_2 x + B_2 y \leq b_2^L, \\
    A_2 x + B_2 y \leq b_2^L, \\
    j = 0, 1, \ldots, l. \quad (21c)
\end{align*} \]

The nonlinear conditions of \( v_i^{L_i} - v_i^{R_i} = 0 \) and \( v_i^{R_i} - v_i^{L_i} = 0, i = 1, 2 \) need not be maintained if the Kuhn–Tucker approach together with Simplex algorithm are adopted, since only equivalence at an optimum is wanted. Further explanation can be found from Ref. 4. Thus, problem (21) is further transformed as follows:

For \((v_i^{L_i}, \ v_i^{R_i}) \in \mathbb{R}^2, X' \subseteq X \times X^2; (v_i^{L_i}, \ v_i^{R_i}) \in \mathbb{R}^2, Y' \subseteq Y \times Y^2, \) let \( x = (x_1, \ldots, x_n) \in X, y' = (y_1, \ldots, y_m) \in X', y = (y_1, \ldots, y_m) \in Y, \)
\( g' = (g_1, \ldots, g_n, v_{2A_i}, v_{2A_i}) \in Y', \) and \( A_2 x + B_2 y \leq b_2^L, \)
\[ \begin{align*}
    \min_{x \in A, y \in B} v_i = v_i^L + v_i^R, \\
    \text{subject to } (c_{L_i}^f + d_{L_i}^f)x + (d_{L_i}^f + d_{L_i}^f)y + v_i^L - v_i^R = g_{L_i}^f, \quad (22a)
\end{align*} \]
\[ \begin{align*}
    A_1 y + B_1 y \leq b_1^L, \\
    A_2 x + B_2 y \leq b_2^L, \\
    \min_{y \in \mathbb{F}} v_i = v_i^L + v_i^R, \quad (22b)
\end{align*} \]
subject to 

\[
\begin{align*}
& (d_{1j}^L + c_{2j}^R) x + (d_{1j}^U + c_{2j}^R) y + e_{2j}^L - e_{2j}^R = g_{1j}^L + g_{1j}^R, \\
& A_{2xj}^L x + B_{2yj}^L y \leq b_{2j}^L, \\
& A_{2xj}^R x + B_{2yj}^R y \leq b_{2j}^R,
\end{align*}
\]

\[j = 0, 1, \ldots, l,
\]

where \(e_{2j}^L = v_{2j}^L + v_{2j}^{R-}, \ e_{2j}^R = v_{2j}^U + v_{2j}^{R+}, \ i = 1, 2.

Problem (22) is a standard linear bilevel decision problem, which can be solved by Kuhn-Tucker approach.

Based on the discussions above, the \(\lambda\)-cut approximate algorithm for solving the FGBIL problems is detailed as follows:

**Step 1 (Input)** Get relevant coefficients of a FGBIL problem which include:

1. Coefficients of (1)
2. Coefficients of \(\hat{g}_L\) and \(\hat{g}_R\)
3. Satisfactory degree: \(\alpha\)
4. \(\varepsilon > 0\)

**Step 2 (Initializing)** Let \(k = 1\), which is the counter to record current loop.

In (7), where \(\lambda_j \in [a, 1]\), let \(\lambda_0 = a\) and \(\lambda_1 = 1\), respectively; then, each objective will be transferred into four nonfuzzy objectives, and each fuzzy constraint is converted into four nonfuzzy constraints.

**Step 3 (Computing)** By introducing auxiliary variables \(v_{ij}^L\) and \(v_{ij}^R, i = 1, 2,\) we get the format of (22).

The solution \((x, v_{i1j}^L, v_{i1j}^R, v_{i2j}^L, v_{i2j}^R, v_{i2j}^+ )\) of (22) is obtained by Kuhn-Tucker approach.

**Step 4 (Comparison)**

If \((k = 1)\) Then

\((x, v_{i1j}^L, v_{i1j}^R, y, v_{i2j}^L, v_{i2j}^R)\) \(= (x, v_{i1j}^L, v_{i1j}^R, y, v_{i2j}^L, v_{i2j}^R, v_{i2j}^+ )\); go to Step 5;

Else If \(((x, v_{i1j}^L, v_{i1j}^R, y, v_{i2j}^L, v_{i2j}^R, v_{i2j}^+ )\) \(= (x, v_{i1j}^L, v_{i1j}^R, y, v_{i2j}^L, v_{i2j}^R)\) \(< \varepsilon \) \) Then

go to Step 7;

**EndIf**

**Step 5 (Splitting)** Suppose there are \((L + 1)\) nodes \(\lambda_j (j = 0, 2, 4, \ldots, 2L)\) in the interval \([a, 1]\), insert \(L\) new nodes \(\lambda_j (j = 1, 3, \ldots, 2L - 1)\), which satisfy

\(\lambda_{2j+1} = (\lambda_{2j} + \lambda_{2j+2})/2, \quad (j = 0, 1, 2, \ldots, L - 1)\).
Step 6 (Loop)
\[ k = k + 1; \]
go to Step 3;

Step 7 (Output) \((x, y)\) is obtained as a final solution.

To realize this algorithm proposed above, a FGBL software system is developed using Visual Basic 6.0. This FGBL software system provides computerized assistance to decision makers in a decentralized organization to gather knowledge about a FGBL problem and controls the decision-making process for a better-informed decision.

The structure of the software system is depicted in Fig. 1. Within this architecture, five modules are involved, i.e. “user interface,” “model management,” “algorithm engine,” “updating system,” and “visualization.” Data are collected through user interface and formatted as a FGBL model by model management module. The core calculations are carried in algorithm engine over a FGBL model, and the solution is output through visualization module to an end user by user interface.

4. Examples

This section employs a numerical example and a case-based example to show the running procedure and the application of the proposed algorithm.

4.1. A numerical example

We first use the proposed algorithm to solve a numerical FGBL problem.

Step 1 (Input relevant coefficients)

1) Coefficients of (1):

\[
\begin{align*}
\max_{x \in X} F(x, y) &= \bar{c}_1 x + \bar{d}_1 y, \\
\text{subject to } &\bar{A}_1 x + \bar{B}_1 y \leq \bar{b}_1, \\
\min_{y \in Y} f(x, y) &= \bar{c}_2 x + \bar{d}_2 y, \\
\text{subject to } &\bar{A}_2 x + \bar{B}_2 y \leq \bar{b}_2,
\end{align*}
\]

where \(x \in \mathbb{R}, y \in \mathbb{R}\), and \(X = x \geq 0, Y = y \geq 0\).
The membership functions of the coefficients of the objective functions and the constraints of both the leader and the follower are as follows:

\[
\mu_{\beta_1}(x) = \begin{cases} 
0, & x < 5, \\
\frac{(x^2 - 25)}{11}, & 5 \leq x < 8, \\
1, & x = 6, \\
\frac{(64 - x^2)}{28}, & 6 < x \leq 8, \\
0, & x > 8,
\end{cases}
\mu_{\bar{\beta}_1}(x) = \begin{cases} 
0, & x < 2, \\
\frac{(x^2 - 4)}{5}, & 2 \leq x < 3, \\
1, & x = 3, \\
\frac{(25 - x^2)}{16}, & 3 < x \leq 5, \\
0, & x > 5.
\end{cases}
\]

\[
\mu_{\alpha_2}(x) = \begin{cases} 
0, & x < -4, \\
\frac{(16 - x^2)}{7}, & -4 \leq x < -3, \\
1, & x = -3, \\
\frac{(x^2 - 1)}{8}, & -3 < x \leq -1, \\
0, & x > -1.
\end{cases}
\mu_{\bar{\alpha}_2}(x) = \begin{cases} 
0, & x < 5, \\
\frac{(x^2 - 25)}{11}, & 5 \leq x < 6, \\
1, & x = 6, \\
\frac{(64 - x^2)}{28}, & 6 < x \leq 8, \\
0, & x > 8.
\end{cases}
\]

\[
\mu_{\beta_j}(x) = \begin{cases} 
0, & x < -2, \\
\frac{(4 - x^2)}{3}, & -2 \leq x < -1, \\
1, & x = -1, \\
\frac{(x^2 - 0.25)}{0.75}, & -1 < x \leq -0.5, \\
0, & x > -0.5.
\end{cases}
\mu_{\bar{\beta}_j}(x) = \begin{cases} 
0, & x < 2, \\
\frac{(x^2 - 4)}{5}, & 2 \leq x < 3, \\
1, & x = 3, \\
\frac{(25 - x^2)}{16}, & 3 < x \leq 5, \\
0, & x > 5.
\end{cases}
\]

\[
\mu_{\alpha_j}(x) = \begin{cases} 
0, & x < 0.5, \\
\frac{(x^2 - 0.25)}{0.75}, & 0.5 \leq x < 1, \\
1, & x = 1, \\
\frac{(4 - x^2)}{3}, & 1 < x \leq 2, \\
0, & x > 2.
\end{cases}
\mu_{\bar{\alpha}_j}(x) = \begin{cases} 
0, & x < 2, \\
\frac{(x^2 - 4)}{5}, & 2 \leq x < 3, \\
1, & x = 3, \\
\frac{(25 - x^2)}{16}, & 3 < x \leq 5, \\
0, & x > 5.
\end{cases}
\]

\[
\mu_{\beta_f}(x) = \begin{cases} 
0, & x < 2, \\
\frac{(x^2 - 4)}{5}, & 2 \leq x < 3, \\
1, & x = 3, \\
\frac{(25 - x^2)}{16}, & 3 < x \leq 5, \\
0, & x > 5.
\end{cases}
\mu_{\bar{\beta}_f}(x) = \begin{cases} 
0, & x < 2, \\
\frac{(x^2 - 4)}{5}, & 2 \leq x < 3, \\
1, & x = 3, \\
\frac{(25 - x^2)}{16}, & 3 < x \leq 5, \\
0, & x > 5.
\end{cases}
\]

\[
\mu_{\alpha_f}(x) = \begin{cases} 
0, & x < 0.5, \\
\frac{(x^2 - 0.25)}{0.75}, & 0.5 \leq x < 1, \\
1, & x = 1, \\
\frac{(4 - x^2)}{3}, & 1 < x \leq 2, \\
0, & x > 2.
\end{cases}
\mu_{\bar{\alpha}_f}(x) = \begin{cases} 
0, & x < 2, \\
\frac{(x^2 - 4)}{5}, & 2 \leq x < 3, \\
1, & x = 3, \\
\frac{(25 - x^2)}{16}, & 3 < x \leq 5, \\
0, & x > 5.
\end{cases}
\]
(2) The membership functions for the fuzzy goals of $\tilde{g}_1$ and $\tilde{g}_2$ are

$$
\mu_{\tilde{g}_1}(x) = \begin{cases} 
0, & x < 19, \\
(x^2 - 361)/80, & 19 \leq x < 21, \\
1, & x = 21, \\
(625 - x^2)/184, & 21 \leq x \leq 25, \\
0, & x > 25.
\end{cases}
$$

$$
\mu_{\tilde{g}_2}(x) = \begin{cases} 
0, & x < 25, \\
(x^2 - 625)/104, & 25 \leq x < 27, \\
1, & x = 27, \\
(961 - x^2)/232, & 27 \leq x \leq 31, \\
0, & x > 31.
\end{cases}
$$

(3) Satisfactory degree: $\alpha = 0.2$.

(4) $\varepsilon = 0.01$.

**Step 2 (Initializing)** Let $k = 1$. Associated with this example, the corresponding $\lambda$-cut set based FGL problem is

$$
\min_{x \in \lambda} \left( \sqrt{11\lambda + 25x} + \sqrt{5\lambda + 1y} - \sqrt{175\lambda + 225} \right)
$$

$$
\min_{x \in \lambda} \left( \sqrt{64 - 28\lambda x + \sqrt{29 - 16\lambda y} - \sqrt{900 - 500\lambda}} \right)
$$

subject to $-\sqrt{4 - 3\lambda x + \sqrt{5\lambda + 4} y} \leq \sqrt{80\lambda + 36}$
$-\sqrt{0.75\lambda + 0.25\lambda x + \sqrt{25 - 16\lambda y}} \leq \sqrt{625 - 184\lambda}$
\[
\min_{y \in Y} | - \sqrt{16 - 7\lambda x + \sqrt{11\lambda + 25} y} - \sqrt{48\lambda + 16} |
\]
\[
\min_{y \in Y} | - \sqrt{8\lambda + 1x + \sqrt{61 - 28\lambda y}} - \sqrt{223 - 161\lambda} |
\]
subject to $\sqrt{0.75\lambda + 0.25\lambda x + \sqrt{5\lambda + 4} y} \leq \sqrt{104\lambda + 625}$
$\sqrt{4 - 3\lambda x + \sqrt{5\lambda + 4} y} \leq \sqrt{861 - 232\lambda}$,

where $\lambda \in [0.2, 1]$.

Referring to the algorithm, only $\lambda_0 = 0.2$ and $\lambda_1 = 1$ are considered initially.

Thus, four nonfuzzy objective functions and four nonfuzzy constraints for the leader and follower are generated, respectively:

\[
\begin{align*}
\min_{x \in X} & \quad (5.2x + 2.2y - 16.1) \\
\min_{x \in X} & \quad (6x + 3y - 20) \\
\min_{x \in X} & \quad (7.6x + 4.7y - 28.3) \\
\min_{x \in X} & \quad (6x + 3y - 20) \\
\text{subject to} & \quad -1.8x + 2.2y \leq 19.1 \\
& \quad -0.6x + 4.7y \leq 24.3 \\
& \quad -x + 3y \leq 21 \\
& \quad -0.6x + 4.7y \leq 24.3 \\
& \quad -x + 3y \leq 21 \\
\min_{y \in Y} & \quad -3.8x + 5.2y - 5.1 \\
\min_{y \in Y} & \quad -3x + 6y - 8 \\
\min_{y \in Y} & \quad (1.6x + 7.6y - 13.9) \\
\min_{y \in Y} & \quad -3x + 6y - 8 \\
\text{subject to} & \quad 0.6x + 2.2y \leq 25.4 \\
& \quad x + 3y \leq 27 \\
& \quad 1.8x + 4.7y \leq 30.2 \\
& \quad x + 3y \leq 27.
\end{align*}
\]

**Step 3 (Computing)** By introducing auxiliary variables $v_i^*, v_i^+, i = 1, 2$, we get

\[
\min_{(x, v_1^*, v_2^*) \in X} v_i^* + v_i^+.
\]
subject to \(24.8x + 12.9y + v_1^+ - v_1^- = 84.4\),
\(-1.8x + 2.2y \leq 19.4\),
\(-x + 3y \leq 21\),
\(-0.6x + 4.7y \leq 24.3\),
\(-x + 3y \leq 21\),
\[
\min_{(v_1^+, v_1^-) \in \mathbb{V}^+} v_1^+ + v_1^-.
\]
subject to \(-11.4x + 24.8y + v_2^- - v_2^+ = 35\),
\(0.6x + 2.2y \leq 25.4\),
\(x + 3y \leq 7\),
\(1.8x + 4.7y \leq 30.2\),
\(x + 3y \leq 27\).

Using Branch-and-bound approach, the current solution is \((2.15366,0.0, 2.39243,0.0)\).

**Step 4 (Comparison)** Because \(k = 1\), go to **Step 5**.

**Step 5 (Splitting)** By inserting a new node \(\lambda_1 = (0.2 + 1)/2 = 0.6\), there are in total three nodes of \(\lambda_1 = 0.2, \lambda_1 = 0.6\), and \(\lambda_2 = 1\). Then, a total of 12 nonfuzzy objective functions for the leader and follower together with 12 nonfuzzy constraints for the leader and follower, respectively are generated.

**Step 6 (Loop)** \(k = 1+1 = 2\), go to **Step 3**, and the current solution of \((2.17093,0.0, 2.41756,0.0)\) is obtained. As \(|2.15366 - 2.17093 + 2.39243 - 2.41756| = 0.04 > \varepsilon = 0.01\), the algorithm keeps going until the solution of \((2.13535,0.0, 2.42797,0.0)\) is obtained. The computing results are listed in Table 1.

**Step 7 (Output)** As \(|2.12393 - 2.13535 + 2.43436 - 2.42797| = 0.0178 < \varepsilon = 0.02\), \((x^*, y^*) = (2.1354, 2.4280)\) is the final solution of this FGBL decision problem. The

<table>
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<th>(k)</th>
<th>(x)</th>
<th>(y)</th>
<th>(v_1^+)</th>
<th>(v_1^-)</th>
<th>(v_2^+)</th>
<th>(v_2^-)</th>
</tr>
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<td>2.39243</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>2.41756</td>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
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<td>2.43436</td>
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</tr>
<tr>
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<td>2.42797</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
4.2. A case-based example on traffic management in a disaster area

This section develops a case-based example on the traffic management in a disaster area by the FGBL model. When a disaster occurs, the blockage on roads and streets will cause severe problems for the missions of evacuation, restoration, and rescue. It is necessary to balance the travel demand and service supply in order to relieve traffic congestion. This study addresses this problem from a two-level aspect to present the interactive decision process between the roadway control decision makers and the road user. We treat the commander of the Emergency-Response Center for the disaster-raid area as the leader, whose objective ($F$) is to allow traffic to go through the disaster area as much as possible under the condition of not exceeding the available roadway capacity. The road users, as the followers, will reasonably choose the shortest routes with regard to travel time, which are the objectives ($f_1$ and $f_2$) for the followers. In the FGBL model, the decision maker for the Emergency-Response Center, the leader, controls the number of vehicles ($x$) to enter the earthquake-raid area, while the road users, the followers, decide their objectives obtained for the leader and the follower under $(x^*, y^*) = (2.1354, 2.4280)$ are

\[
\begin{aligned}
F(x^*, y^*) &= F(2.1354, 2.4280) = 2.1354d_1^L + 2.4280d_2^L, \\
f(x^*, y^*) &= F(2.1354, 2.4280) = 2.1354d_1^R + 2.4280d_2^R,
\end{aligned}
\]

and their membership functions are shown in Figs. 2(a) and 2(b).

The above example illustrated the detailed working process of the proposed algorithm.
specific route \((y_1, y_2)\). The leader may have certain goal of traffic throughout \((\hat{g}_k)\) for his or her objective, and the followers wish to meet the emergency rescue needs \((\hat{g}_{\hat{k}})\) of the objective as well.

When modeling this problem, the main difficulty is to set up coefficients for the objectives and constraints of both the leader and the follower. We can only estimate these values according to our experience and previous data. Thus, by using fuzzy numbers to describe these uncertain values, a FCGM model is established below:

\[
\max_{x \in X} F(x, y_1, y_2) = 6x + 3y_1 + 4y_2,
\]

subject to \(Lx + \hat{3}y_1 + \hat{6}y_2 \leq \bar{L}_1\),

\[
\min_{y_1 \in Y_1} f(x, y_1) = 1x + 3y_1,
\]

\[
\min_{y_2 \in Y_2} f(x, y_2) = \hat{3}x + \hat{6}y_2,
\]

subject to \(Lx + \hat{3}y_1 + \hat{1}y_2 \leq \bar{L}_1\).

where \(x \in \mathbb{R}, y_1, y_2 \in \mathbb{R}\), and \(X = x > 0, Y_1 = y_1 \geq 0, Y_2 = y_2 \geq 0\).

The membership functions of the coefficients of the objectives functions and the constraints of both the leader and the followers are as follows:

\[
\mu_0(x) = \begin{cases} 
0 & x < 5, \\
(x^2 - 25)/11 & 5 \leq x < 8, \\
x = 6, & \mu_3(x) = \begin{cases} 
0 & x < 2, \\
(x^2 - 4)/5 & 2 \leq x < 3, \\
x = 3, & (25 - x^2)/16 & 3 \leq x \leq 5, \\
x > 5, & 0, \\
\end{cases}
\end{cases}
\]

\[
\mu_2(x) = \begin{cases} 
0 & x < 3, \\
(x^2 - 9)/7 & 3 \leq x \leq 4, \\
x = 4, & \mu_1(x) = \begin{cases} 
0 & x < 0.5, \\
(x^2 - 0.25)/0.75 & 0.5 \leq x \leq 1, \\
x = 1, & (4 - x^2)/3 & 1 \leq x \leq 2, \\
x > 2, & 0, \\
\end{cases}
\end{cases}
\]

\[
\mu_0(x) = \begin{cases} 
0 & x < 19, \\
(x^2 - 361)/80 & 19 \leq x < 21, \\
x = 21, & \mu_1(x) = \begin{cases} 
0 & x < 0.5, \\
(x^2 - 0.25)/0.75 & 0.5 \leq x \leq 1, \\
x = 1, & (4 - x^2)/3 & 1 \leq x \leq 2, \\
x > 2, & 0, \\
\end{cases}
\end{cases}
\]

\[
\mu_2(x) = \begin{cases} 
0 & x < 19, \\
(x^2 - 225)/175 & 19 \leq x < 20, \\
x = 20, & \mu_1(x) = \begin{cases} 
0 & x < 15, \\
(x^2 - 225)/175 & 15 \leq x < 20, \\
x = 20, & (200 - x^2)/500 & 20 \leq x \leq 30, \\
x > 30, & 0, \\
\end{cases}
\end{cases}
\]

The membership function of the fuzzy goal given to the leader is
The membership functions of the fuzzy goals set for the followers are:

$$
\mu_{g_{x1}}(x) = \begin{cases} 
0 & x < 10, \\
(x^2 - 100)/225 & 10 \leq x < 15, \\
1 & x = 15, \\
(400 - x^2)/175 & 15 \leq x \leq 20, \\
0 & x > 20, \\
(x^2 - 49)/32 & 7 \leq x < 9, \\
1 & x = 9, \\
(121 - x^2)/40 & 9 \leq x \leq 11, \\
0 & x > 11. 
\end{cases}
$$

Fig. 3. Objectives for the leader and followers.
Following all the steps of the proposed \( \lambda \)-cut approximate algorithm, the solution to this problem is: \((x^*, y_1^*, y_2^*) = (1.0, 11.82, 0.02)\). The objectives for the leader and followers under this solution are shown in Figs. 3(a)–3(c).

5. Conclusion and Future Study

Many organizational decision problems can be formulated by bilevel decision models. In a bilevel decision model, the leader and/or the follower may wish that their objectives attain some goals, which are different from simple optimization problems. This kind of bilevel decision problems are studied by goal programming in this paper. Meanwhile, we take into consideration of the situation where coefficients which formulate a bilevel decision model are not precisely known to us. Fuzzy set method is thus applied to handle these coefficients.

This paper has proposed an approximate algorithm to solve FGBL decision problems, demonstrated the software system, and presented two examples to further explain this algorithm. In the future, we will develop a method to handle the situation where the leader and the follower in a FGBL problem have multiple objectives, respectively.

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References