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A note on the union-closed sets conjecture

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Abstract

A collection $\mathcal{A}$ of finite sets is closed under union if $A, B \in \mathcal{A}$ implies that $A \cup B \in \mathcal{A}$. The Union-Closed Sets Conjecture states that if $\mathcal{A}$ is a union-closed collection of sets, containing at least one non-empty set, then there is an element which belongs to at least half of the sets in $\mathcal{A}$. We show that if q is the minimum cardinality of $\cup\mathcal{A}$ taken over all counterexamples $\mathcal{A}$, then any counterexample $\mathcal{A}$ has cardinality at least $4q - 1$.

A collection $\mathcal{A}$ of finite sets is *closed under union* if $A, B \in \mathcal{A}$ implies that $A \cup B \in \mathcal{A}$. The Union-Closed Sets Conjecture (also called Frankl's Conjecture) states that if $\mathcal{A}$ is a union-closed collection of sets, containing at least one non-empty set, then there is an element which belongs to at least half of the sets in $\mathcal{A}$. The conjecture dates from 1979 and is generally attributed to Peter Frankl [2]. Some authors place the further condition on $\mathcal{A}$ that it does not contain the empty set. The results of this note apply to either version of the conjecture. A number of necessary conditions for a counterexample have been established, including that if $\mathcal{A}$ is a counterexample then $|\mathcal{A}| \geq 37$ [4] and that $|\cup\mathcal{A}| \geq 12$ [1] where $\cup\mathcal{A} = \cup_{A\in\mathcal{A}}A$. See [1] and [3] and their bibliographies for more background. In this note we show that if q is the minimum cardinality of $\cup\mathcal{A}$ taken over all counterexamples then for any counterexample $\mathcal{A}$ we have $|\mathcal{A}| \geq 4q - 1$. It is known that $q \geq 12$ [1] so this implies that a counterexample has cardinality at least 47.

We say that A is a *basis set* in $\mathcal{A}$ if it is not the union of other sets in $\mathcal{A}$. Suppose that $\mathcal{A}$ is a counterexample with $|\mathcal{A}| = 2n+2$ for some integer n. Then no element occurs in more than n of its sets. If we remove a basis set from $\mathcal{A}$, then we get a smaller collection which is also a counterexample. A counterexample of minimum size must therefore contain an odd number of sets.

Let q be the minimum cardinality of $\cup\mathcal{A}$ taken over all counterexamples $\mathcal{A}$. We will obtain a lower bound for $|\mathcal{A}|$ in terms of q. Henceforth $\mathcal{A}$ will be a minimal counterexample to the conjecture with $|\mathcal{A}| = 2n+1$. Set $S = \cup\mathcal{A}$ and note that $|S| \geq q$. We define

$$\begin{aligned}
\mathcal{A}_x &= \{A \in \mathcal{A} : x \in A\} \\
\mathcal{A}_{\overline{x}} &= \{A \in \mathcal{A} : x \notin A\} \\
C_x &= \cup\mathcal{A}_{\overline{x}} \\
\mathcal{C} &= \{C_x : x \in S\} \\
\mathcal{D} &= \mathcal{A}\backslash\{S\}\backslash\mathcal{C} \\
\mathcal{D}_x &= \{A \in \mathcal{D} : x \in A\}.
\end{aligned}$$

Clearly, C_x is an element of $\mathcal{A}$ for each element x of S. Since $\mathcal{A}$ is a minimal counterexample there will be elements which occur in exactly n of the sets in $\mathcal{A}$ (otherwise we could remove a basis set to give a smaller counterexample). We set $H = \{x : |\mathcal{A}_x| = n\}$. Let $\mathcal{B}$ be the collection of basis sets in $\mathcal{A}$. If a and b are elements of $\mathcal{A}$, then we say that a *dominates* b if whenever b occurs in a set A in $\mathcal{A}$ then a also occurs in that set. This is equivalent to saying $b \notin C_a$. We will assume that $\mathcal{A}$ contains no mutually dominating pair of elements, since if such a pair existed then we could replace it with a single element without altering the size of $\mathcal{A}$. Note that by this assumption the sets C_a are distinct.

Lemma 1 *If $a \notin H$, then $H \subseteq C_a$ and if $x \in H$, then $H\backslash\{x\} \subseteq C_x$.*

Proof: Let $y \in H$. We show that $y \in C_a$ where $a \notin H$. Since y occurs in n sets and a occurs in fewer than n sets, there exists A in $\mathcal{A}$ such that $y \in A$ and $a \notin A$ which implies $y \in C_a$. It follows that $H \subseteq C_a$.

For the second part let $y \in H$, $y \neq x$. We are assuming that x and y do not dominate each other. So there exists $A \in \mathcal{A}$ such that $y \in A$, $x \notin A$ so $A \subseteq C_x$ and therefore $y \in C_x$. $\square$

Lemma 2 *No set C_a is a basis set.*

Proof: Suppose that C_a is a basis set. If $a \notin H$ then by Lemma 1 $\mathcal{A}\backslash\{C_a\}$ is a union-closed collection in which no element occurs more than $n-1$ times, contradicting the minimality of $\mathcal{A}$. If, on the other hand, $a \in H$, then choose a basis set B which contains a. Then by Lemma 1 $\mathcal{A}\backslash\{B, C_a\}$ is a union-closed collection of $2n-1$ sets in which no element occurs more than $n-1$ times, again contradicting the minimality of $\mathcal{A}$. $\square$

Theorem 3 *If $x \in H$ and $a \notin H$, then $|\mathcal{D}_x| = n - q$ and $|\mathcal{D}_a| < n - q$.*

Proof: Let $x \in H$. By Lemma 1, x occurs in all sets C_b with $b \neq x$ as well as in S, so x occurs in $|\mathcal{A}_x| - q = n - q$ sets in $\mathcal{D}$.

Now assume, without loss of generality, that $|\mathcal{D}_a|$ is maximal for $a \in S \backslash H$. .

If there exists $b \neq a$ such that $a \notin C_b$, then b dominates a and there exists a basis set which contains b but not a. By Lemma 2 this belongs to $\mathcal{D}$ so b occurs more often in $\mathcal{D}$ than a which is a contradiction.

If, on the other hand, $a \in C_b$ for all $b \neq a$ then a occurs in q sets of $\mathcal{A} \backslash \mathcal{D}$. If a occurs in $n - q$ or more sets in $\mathcal{D}$ then it occurs in at least n sets in $\mathcal{A}$ which means $a \in H$. □

Theorem 4 *If $\mathcal{A}$ is a minimal counterexample, then $|\mathcal{A}| \geq 4q - 1$.*

Proof: Consider $\mathcal{A}_x$ and $\mathcal{A}_{\overline{x}}$ for some $x \in H$. Then $|\mathcal{A}_x| = n$, $|\mathcal{A}_{\overline{x}}| = n + 1$ and $|\mathcal{D}_x| = n - q$ by Theorem 3. Since $\mathcal{A}_{\overline{x}}$ is union-closed, there exists an element a in at least half its sets. By Theorem 3 a occurs in at most $n - q$ sets in $\mathcal{D}$. By Lemma 1, each set in $\mathcal{A} \backslash \mathcal{D}$ except C_x contains x so $\mathcal{A}_{\overline{x}} \subseteq \mathcal{D} \cup \{C_x\}$. Thus a is in at most $n - q + 1$ sets in $\mathcal{A}_{\overline{x}}$. Hence $|\mathcal{A}_{\overline{x}}| \leq 2(n - q + 1)$ which implies $n \geq 2q - 1$. Thus $|\mathcal{A}| = 2n + 1 \geq 4q - 1$. □

Corollary 5 *If $\mathcal{A}$ is a minimal counterexample, then $|\mathcal{A}| \geq 47$.*

Proof: Bošnjak and Marković [1] have recently shown that a minimal counterexample to the conjecture has $q \geq 12$. □

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