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Self-Bounded Subspaces for Nonstrictly Proper Systems and Their Application to the Disturbance Decoupling With Direct Feedthrough Matrices

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***Abstract*—In this paper, the concept of self-bounded controlled invariance is extended to nonstrictly proper systems. Moreover, its use in connection with the disturbance decoupling problem with internal stability is investigated in the case where the feedthrough matrices from the control input and from the disturbance to the output to be decoupled are possibly nonzero.**

***Index Terms*—Disturbance decoupling with internal stability, fixed poles of the closed-loop, geometric approach, nonstrictly proper systems, self-bounded subspaces.**

I. Introduction

The disturbance decoupling problem (DDP) played a key role in the development of the so-called geometric approach to control theory since, from the very beginning, its solvability conditions were expressed by means of inclusions involving certain subspaces.

The basic decoupling problem, consisting of the rejection of a disturbance from the output of the given system by means of a static state-feedback, was solved by Basile and Marro [1], and independently, by Wonham and Morse [2], via the introduction of the concept of controlled invariant subspaces, which were then found to be a powerful tool in the understanding of many system-theoretic properties of linear time-invariant (LTI) systems, and in the solution of several control problems. The DDP with the extra requirement of internal stability of the closed loop was taken into account by Wonham and Morse [2] via the introduction of (A, B) stabilizability subspaces. An alternative solution to the same problem was suggested by Basile and Marro [3], relying on the geometric concept of the self-bounded controlled invariance, thus avoiding eigenspace computation: the condition of solvability of the disturbance decoupling with stability (DDPs) was expressed in terms of a geometric structural condition, and a further condition concerning the internal stabilizability of a particular self-bounded controlled invariant subspace (denoted by $\mathcal{V}_m$ in [4], and herein denoted by $\mathcal{R}^*_{\tilde{\Sigma}}$), hence, is referred to as the *stability condition*. To this purpose, also see [5].

A fundamental contribution to the understanding of the advantages deriving from the adoption of self-bounded controlled invariant subspaces in the solution of the DDPs was given by Malabre *et al.* [6], where it was shown that, in the solution of both the DDPs and the measurable signal decoupling problem with internal stability (MSDPs), there is a number of eigenvalues of the closed-loop system that are fixed for any feedback matrix solving the decoupling problem; these nonassignable eigenvalues are usually referred to as the *fixed poles* of the decoupling problem. Taking the feedback matrix associated with the self-bounded subspace $\mathcal{R}^*_{\tilde{\Sigma}}$ as the solution to the decoupling problem is the best choice in terms of pole assignment, since it ensures that the maximum number of eigenvalues of the closed loop can be freely assigned.

In the last two decades, many papers have been written on the solution of different control problems involving the concept of self-boundedness (see [7]–[9], and the references therein). Moreover, much effort has been devoted to the generalization of these classic results to nonstrictly proper systems. For example, in [4, p. 245] and in [10, p. 99], the extension of the geometric approach to systems with a direct feedthrough matrix is carried out via a state extension due to the fictitious addition of an integral stage at the input (or at the output) of the given system. However, this contrivance can be avoided by a more direct use of output-nulling and input-containing subspaces, [11]. Within the context of the DDP, for instance, this leads to a more elegant formulation of the solvability conditions in terms of the problem data, thus not involving additional fictitious variables. For a detailed and well-organized extension of the geometric approach to control theory with the direct feedthrough term, we refer to [12]. The role of the output-nulling and input-containing subspaces in the disturbance decoupling by a dynamic output feedback has been discussed by Stoorvogel and van der Woude [13] for systems with direct feedthrough, thus generalizing the approach to the DDPs based on (A, B) stabilizability subspaces. To the author's knowledge, so far a similar extension to the DDPs (and to the MSDPs) by means of self-bounded subspaces has been neglected, notwithstanding the several advantages connected with the use of self-bounded subspaces instead of stabilizability subspaces. This paper addresses this issue. In Section II, the basic notions of the geometric approach for quadruples (A, B, C, D) are recalled. In Section III, the notion of self-boundedness is extended to nonstrictly proper systems, and in Section IV, their role in the solution of the DDPs and MSDPs is addressed. The relation between the solution proposed here and that obtainable from [13] by considering the DDPs and the MSDPs as particular cases of the disturbance decoupling by dynamic output feedback are discussed in Remark 2 and in Section IV-C.

Notation. Throughout this paper, the symbol O_n stands for the origin of the vector space $\mathbb{R}^n$. The image and the kernel of matrix A are denoted by $\operatorname{im} A$ and $\ker A$, respectively. Given a subspace $\mathcal{Y}$ of $\mathbb{R}^n$, the symbol $A^{-1}\,\mathcal{Y}$ stands for the inverse image of $\mathcal{Y}$ with respect to the linear map A. The symbol $\sigma(A)$ denotes the spectrum of A. If $A : \mathcal{X} \longrightarrow \mathcal{Y}$ and $\mathcal{J} \subseteq \mathcal{X}$, the restriction of the map A to $\mathcal{J}$ is denoted

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by $A\,|\mathcal{J}$. If $\mathcal{X}=\mathcal{Y}$ and $\mathcal{J}$ is A-invariant, the eigenvalues of A restricted to $\mathcal{J}$ are denoted by $\sigma\,(A\,|\mathcal{J})$. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are A-invariant subspaces and $\mathcal{J}_1\subseteq\mathcal{J}_2$, the mapping induced by A on the quotient space $\mathcal{J}_2/\mathcal{J}_1$ is denoted by $A\,|\frac{\mathcal{J}_2}{\mathcal{J}_1}$. Given the matrix $A\in\mathbb{R}^{n\times n}$ and the subspace $\mathcal{B}$ of the linear space $\mathbb{R}^n$, the symbol $<A,\,\mathcal{B}>$ stands for the smallest A-invariant subspace of $\mathbb{R}^n$ containing $\mathcal{B}$. The symbol $\times$ stands for the Cartesian product, while $\uplus$ denotes aggregation.

II. Geometric Preliminaries

In what follows, whether the underlying system evolves in continuous or discrete time is irrelevant, and accordingly, the time index set of any signal is denoted by $\mathbb{T}$, on the understanding that this represents either $\mathbb{R}^+$ in the continuous time or $\mathbb{N}$ in the discrete time. The symbol $\mathbb{C}_g$ denotes either the open left-half complex plane $\mathbb{C}^-$ in the continuous time or the open unit disc $\mathbb{C}^\circ$ in the discrete time. Consider an LTI system Σ modeled by

$$\begin{aligned}\rho\,x(t)&=A\,x(t)+B\,u(t)\\ y(t)&=C\,x(t)+D\,u(t)\end{aligned}\tag{1}$$

where, for all $t\in\mathbb{T}$, $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input, $y(t)\in\mathbb{R}^p$ is the output, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$, and $D\in\mathbb{R}^{p\times m}$. The operator ρ denotes either the time derivative in the continuous time, i.e., $\rho x(t)=\dot{x}(t)$ or the unit time shift in the discrete time, i.e., $\rho x(t)=x(t+1)$. Let the system Σ described by (1) be identified by the quadruple (A,B,C,D). In this paper, we are interested in the case where the feedthrough matrix D may be different from the zero matrix, i.e., when the system is possibly nonstrictly proper. In order to simplify the notation, let $\widehat{A}\stackrel{\triangle}{=}[\,A^\top\;\;C^\top\,]^\top$ and $\widehat{B}\stackrel{\triangle}{=}[\,B^\top\;\;D^\top\,]^\top$. For the readers' convenience, some fundamental definitions and results of the geometric approach that will be used in the sequel are briefly recalled (for a detailed discussion on these topics, we refer to [4], [10], [12]). First, we define an *output-nulling subspace* $\mathcal{V}_\Sigma$ of Σ as a subspace of $\mathbb{R}^n$ satisfying the inclusion $\widehat{A}\,\mathcal{V}_\Sigma\subseteq(\mathcal{V}_\Sigma\times\mathcal{O}_p)+\operatorname{im}\widehat{B}$. The set $\mathcal{V}(\Sigma)$ of output-nulling subspaces of Σ is an upper semilattice with respect to subspace addition. Thus, the sum of all the output-nulling subspaces of Σ is the largest output-nulling subspace of Σ, and is denoted by $\mathcal{V}^*_\Sigma$. The subspace $\mathcal{V}^*_\Sigma$ represents the set of all initial states of Σ for which an input function exists, such that the corresponding output function is identically zero. In the following lemma, the most important properties of output-nulling subspaces are presented.

Lemma 1: The following results hold.

1) The subspace $\mathcal{V}_\Sigma$ is output-nulling for Σ iff a matrix $F\in\mathbb{R}^{m\times n}$ exists such that

$$(\widehat{A}+\widehat{B}\,F)\,\mathcal{V}_\Sigma\subseteq\mathcal{V}_\Sigma\times\mathcal{O}_p.\tag{2}$$

2) The sequence of subspaces $(\mathcal{V}^i_\Sigma)_{i\in\mathbb{N}}$ described by the recurrence

$$\begin{cases}\mathcal{V}^0_\Sigma&=\;\mathbb{R}^n\\ \mathcal{V}^i_\Sigma&=\;\widehat{A}^{-1}\left(\left(\mathcal{V}^{i-1}_\Sigma\times\mathcal{O}_p\right)+\operatorname{im}\widehat{B}\right)\end{cases}\tag{3}$$

is monotonically nonincreasing. An integer $k\le n-1$ exists such that $\mathcal{V}^{k+1}_\Sigma=\mathcal{V}^k_\Sigma$. For such k the identity $\mathcal{V}^*_\Sigma=\mathcal{V}^k_\Sigma$ holds.

Any matrix F satisfying (2) is referred to as a *friend* of the output-nulling subspace $\mathcal{V}_\Sigma$. We denote by $\Im_\Sigma(\mathcal{V}_\Sigma)$ the set of friends of $\mathcal{V}_\Sigma$. The dual of the output-nulling subspaces are the *input-containing* subspaces, defined as the subspaces $\mathcal{S}_\Sigma$ satisfying the inclusion $[A\quad B]\,((\mathcal{S}_\Sigma\times\mathbb{R}^m)\cap\ker[C\quad D])\subseteq\mathcal{S}_\Sigma$. The set $\mathcal{S}(\Sigma)$ of input-containing subspaces of Σ is closed under subspace intersection, and its smallest element $\mathcal{S}^*_\Sigma$ can be computed as the limit of the recursion

$$\begin{cases}\mathcal{S}^0_\Sigma&=\;\mathcal{O}_n\\ \mathcal{S}^i_\Sigma&=\;[A\quad B]\left(\left(\mathcal{S}^{i-1}_\Sigma\times\mathbb{R}^m\right)\cap\ker[C\quad D]\right)\end{cases}\tag{4}$$

which converges in at most $n-1$ steps. The *output-nulling reachability subspace* on the output-nulling $\mathcal{V}_\Sigma$ is the smallest $(A+B\,F)$-invariant subspace of $\mathbb{R}^n$ containing the subspace $\mathcal{V}_\Sigma\cap B\ker D$, where $F\in\Im_\Sigma(\mathcal{V}_\Sigma)$. We denote by $\mathcal{R}^*_\Sigma$ the output-nulling reachability subspace on $\mathcal{V}^*_\Sigma$, i.e., $\mathcal{R}^*_\Sigma\stackrel{\triangle}{=}<A+B\,F,\mathcal{V}^*_\Sigma\cap B\ker D>$. The subspace $\mathcal{R}^*_\Sigma$ can be computed as $\mathcal{R}^*_\Sigma=\mathcal{V}^*_\Sigma\cap\mathcal{S}^*_\Sigma$ [12, Th. 8.22], [14]. Consider an output-nulling subspace $\mathcal{V}_\Sigma$ of Σ and define by $\mathcal{R}_\Sigma$ the reachable subspace on $\mathcal{V}_\Sigma$. For $F\in\Im_\Sigma(\mathcal{V}_\Sigma)$, the eigenvalues of $(A+B\,F)$ restricted to $\mathcal{V}_\Sigma$, i.e., in $\sigma(A+B\,F|\mathcal{V}_\Sigma)$, can be split into two sets: the eigenvalues of $(A+B\,F|\mathcal{R}_\Sigma)$ are all freely assignable by a suitable choice of F in $\Im_\Sigma(\mathcal{V}_\Sigma)$, provided that the eigenvalues to be assigned are mirrored with respect to the real axis. The eigenvalues in $\Gamma^{\rm int}_\Sigma(\mathcal{V}_\Sigma)\stackrel{\triangle}{=}\sigma\,(A+B\,F|\frac{\mathcal{V}_\Sigma}{\mathcal{R}_\Sigma})$ are fixed for all the choices of F in $\Im_\Sigma(\mathcal{V}_\Sigma)$ [11]; if $\Gamma^{\rm int}_\Sigma(\mathcal{V}_\Sigma)\subset\mathbb{C}_g$, the output-nulling $\mathcal{V}_\Sigma$ is said to be *internally stabilizable*. Similarly, by denoting with $\mathcal{R}$ the reachable subspace from the origin, i.e., $\mathcal{R}=<A,\operatorname{im}B>$, the eigenvalues $\sigma(A+B\,F|\frac{\mathbb{R}^n}{\mathcal{V}_\Sigma})$ are split into two sets: the eigenvalues of $(A+B\,F|\frac{\mathcal{V}_\Sigma+\mathcal{R}}{\mathcal{V}_\Sigma})$ are all freely assignable by a suitable choice of F in $\Im_\Sigma(\mathcal{V}_\Sigma)$, whereas the eigenvalues in $\Gamma^{\rm ext}_\Sigma(\mathcal{V}_\Sigma)\stackrel{\triangle}{=}\sigma(A+B\,F|\frac{\mathbb{R}^n}{\mathcal{V}_\Sigma+\mathcal{R}})$ are fixed. If $\Gamma^{\rm ext}_\Sigma(\mathcal{V}_\Sigma)\subset\mathbb{C}_g$, the output-nulling $\mathcal{V}_\Sigma$ is said to be *externally stabilizable*. Hence, given a friend F of $\mathcal{V}_\Sigma$, then the set $\Gamma_\Sigma(\mathcal{V}_\Sigma)\stackrel{\triangle}{=}\Gamma^{\rm int}_\Sigma(\mathcal{V}_\Sigma)\uplus\Gamma^{\rm ext}_\Sigma(\mathcal{V}_\Sigma)$ does not depend on the choice of $F\in\Im(\mathcal{V}_\Sigma)$.

III. Self-Bounded Subspaces

In this section, the concept of self-bounded controlled invariance defined in [3] is extended to systems with direct feedthrough.

Definition 1: The output-nulling subspace $\mathcal{V}_\Sigma$ of Σ is said to be *self-bounded* if, for all $x(0)\in\mathcal{V}_\Sigma$, any control yielding $y=0$ is such that $x(t)\in\mathcal{V}_\Sigma$ for all $t\in\mathbb{T}$.

Clearly, $\mathcal{V}^*_\Sigma$ is self-bounded, since, by definition, it is the locus of initial states for which an input exists such that the output function is identically zero. The following lemma is very useful in providing a geometric characterization of the concept of self-bounded output-nulling subspaces, generalizing that presented in [3].

Lemma 2: Let $\mathcal{V}_\Sigma$ and $\tilde{\mathcal{V}}_\Sigma$ be two output-nulling subspaces of Σ such that $\mathcal{V}^*_\Sigma\cap B\ker D\subseteq\mathcal{V}_\Sigma$, $\mathcal{V}^*_\Sigma\cap B\ker D\subseteq\tilde{\mathcal{V}}_\Sigma$ and $\tilde{\mathcal{V}}_\Sigma\subseteq\mathcal{V}_\Sigma$. Then $\Im_\Sigma(\mathcal{V}_\Sigma)\subseteq\Im_\Sigma(\tilde{\mathcal{V}}_\Sigma)$.

Proof: Let $F\in\Im_\Sigma(\mathcal{V}_\Sigma)$. By the addition of $\widehat{A}\,\tilde{\mathcal{V}}_\Sigma\subseteq(\tilde{\mathcal{V}}_\Sigma\times\mathcal{O}_p)+\operatorname{im}\widehat{B}$ and $\widehat{B}\,F\,\tilde{\mathcal{V}}_\Sigma\subseteq\operatorname{im}\widehat{B}$, we get $(\widehat{A}+\widehat{B}\,F)\tilde{\mathcal{V}}_\Sigma\subseteq(\tilde{\mathcal{V}}_\Sigma\times\mathcal{O}_p)+\operatorname{im}\widehat{B}$, which, once intersected with $(\widehat{A}+\widehat{B}\,F)\tilde{\mathcal{V}}_\Sigma\subseteq(\widehat{A}+\widehat{B}\,F)\mathcal{V}_\Sigma\subseteq\mathcal{V}_\Sigma\times\mathcal{O}_p$, yields $(\widehat{A}+\widehat{B}\,F)\tilde{\mathcal{V}}_\Sigma\subseteq(\mathcal{V}_\Sigma\times\mathcal{O}_p)\cap(\tilde{\mathcal{V}}_\Sigma\times\mathcal{O}_p+\operatorname{im}\widehat{B})=\tilde{\mathcal{V}}_\Sigma\times\mathcal{O}_p+(\mathcal{V}_\Sigma\cap B\,\ker\,D\times\mathcal{O}_p)$. Since $\tilde{\mathcal{V}}_\Sigma\supseteq\mathcal{V}^*_\Sigma\cap B\,\ker\,D\supseteq\mathcal{V}_\Sigma\cap B\,\ker\,D$, we find $(\widehat{A}+\widehat{B}\,F)\,\tilde{\mathcal{V}}_\Sigma\subseteq\tilde{\mathcal{V}}_\Sigma\times\mathcal{O}_p$. Hence, $F\in\Im_\Sigma(\tilde{\mathcal{V}}_\Sigma)$. ■

The following theorem provides a necessary and sufficient geometric condition for self-boundedness, which is the extension to nonstrictly proper systems of the one given in [3].

Theorem 1: The output-nulling subspace $\mathcal{V}_\Sigma$ of Σ is self-bounded iff $\mathcal{V}^*_\Sigma\cap B\ker D\subseteq\mathcal{V}_\Sigma$.

Proof: (If): Let $x(0)\in\mathcal{V}_\Sigma$, and let u be an input such that $y=0$. Then, $x\in\mathcal{V}^*_\Sigma$, which implies that u can be expressed as $u(t)=F\,x(t)+v(t)$, where $F\in\Im_\Sigma(\mathcal{V}^*_\Sigma)$ and $v(t)\in B^{-1}\,\mathcal{V}^*_\Sigma\cap\ker D$ for all $t\in\mathbb{T}$, as shown in [12, Th. 7.11]. Since $\mathcal{V}^*_\Sigma\cap B\ker D\subseteq\mathcal{V}_\Sigma\subseteq$

$\mathcal{V}_{\Sigma}^{*}$, by Lemma 2, it follows that $F \in \Im_{\Sigma}(\mathcal{V}_{\Sigma})$. The state and output equations can, thus, be written as

$$\begin{bmatrix} \rho x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A + B\,F \\ C + D\,F \end{bmatrix} x(t) + \begin{bmatrix} B \\ D \end{bmatrix} v(t). \tag{5}$$

Hence, for all $t \in \mathbb{T}$, there holds $\widehat{B}\,v(t) \in \widehat{B}\,(B^{-1}\,\mathcal{V}_{\Sigma}^{*} \cap \ker D) = (\mathcal{V}_{\Sigma}^{*} \cap B \ker D) \times O_p$ [12, p. 164], so that, since $F \in \Im_{\Sigma}(\mathcal{V}_{\Sigma})$ and $\mathcal{V}_{\Sigma} \supseteq \mathcal{V}_{\Sigma}^{*} \cap B \ker D$, it is found that $x(t) \in \mathcal{V}_{\Sigma}$ for all $t \in \mathbb{T}$.
(Only if): Let $\mathcal{V}_{\Sigma}$ be self-bounded, and let $x(0) \in \mathcal{V}_{\Sigma}$. The set of control functions ensuring that the output function is zero is parameterized by $u(t) = F\,x(t) + v(t)$, where $F \in \Im_{\Sigma}(\mathcal{V}_{\Sigma}^{*})$ and $v(t) \in B^{-1}\,\mathcal{V}_{\Sigma}^{*} \cap \ker D$ for all $t \in \mathbb{T}$, by [12, Th. 7.11]. From (5), it follows that the state trajectory lies on $\mathcal{V}_{\Sigma}$ only if $\widehat{B}\,v(t) \in \mathcal{V}_{\Sigma} \times O_p$ for all $t \in \mathbb{T}$, which implies that $\mathcal{V}_{\Sigma}^{*} \cap B \ker D \subseteq \mathcal{V}_{\Sigma}$. ■

The set $\Phi(\Sigma)$ of self-bounded subspaces of Σ, unlike $\mathcal{V}(\Sigma)$, admits both a maximal and a minimal element. In fact, $\Phi(\Sigma)$ is closed under subspace addition and intersection. To see this, let $\mathcal{V}_1, \mathcal{V}_2 \in \Phi(\Sigma)$, and let $F \in \Im_{\Sigma}(\mathcal{V}_{\Sigma}^{*})$; by Lemma 2, it follows that $F \in \Im_{\Sigma}(\mathcal{V}_1) \cap \Im_{\Sigma}(\mathcal{V}_2)$, so that $(\widehat{A} + \widehat{B}\,F)\,\mathcal{V}_i \subseteq \mathcal{V}_i \times O_p$ for $i \in \{1, 2\}$. Finally, since $\mathcal{V}_i \supseteq \mathcal{V}_{\Sigma}^{*} \cap B \ker D$, $i \in \{1, 2\}$, it follows that $\mathcal{V}_1 + \mathcal{V}_2 \supseteq \mathcal{V}_{\Sigma}^{*} \cap B \ker D$ and $\mathcal{V}_1 \cap \mathcal{V}_2 \supseteq \mathcal{V}_{\Sigma}^{*} \cap B \ker D$, so that $\mathcal{V}_1 + \mathcal{V}_2 \in \Phi(\Sigma)$ and $\mathcal{V}_1 \cap \mathcal{V}_2 \in \Phi(\Sigma)$. Now, given $\mathcal{V}_1, \mathcal{V}_2 \in \Phi(\Sigma)$, it is easily seen that their sum $\mathcal{V}_1 + \mathcal{V}_2$ is the smallest element of $\Phi(\Sigma)$ containing both $\mathcal{V}_1$ and $\mathcal{V}_2$, and $\mathcal{V}_1 \cap \mathcal{V}_2$ is the largest element of $\Phi(\Sigma)$ contained in both $\mathcal{V}_1$ and $\mathcal{V}_2$. Hence, $(\Phi(\Sigma), +, \cap; \subseteq)$ is a lattice. As such, it admits a maximum element, which is $\mathcal{V}_{\Sigma}^{*}$, and a minimum element, which is $\mathcal{R}_{\Sigma}^{*}$.

IV. DDPS With Internal Stability

Consider an LTI system described by

$$\begin{aligned} \rho x(t) &= A\,x(t) + B\,u(t) + H\,w(t) \\ y(t) &= C\,x(t) + D\,u(t) + G\,w(t) \end{aligned} \tag{6}$$

where, for all $t \in \mathbb{T}$, $w(t) \in \mathbb{R}^{r}$ is the disturbance input to be decoupled from the output $y(t) \in \mathbb{R}^{p}$, $H \in \mathbb{R}^{n \times r}$, and $G \in \mathbb{R}^{p \times r}$. From now on, we identify the *undisturbed system* characterized by the quadruple (A, B, C, D) with the symbol Σ and the *disturbed system* characterized by the quadruple $(A, [\,B \;\; H\,], C, [\,D \;\; G\,])$ with the symbol $\tilde{\Sigma}$. The classic formulation of the *DDPs*, in which the disturbance input $w(t)$ is considered to be nonavailable for measurement, can be stated as follows.

Problem 1: Find (if possible) a static state-feedback control $u(t) = F\,x(t)$, $t \in \mathbb{T}$, such that the following hold.

1) The output y of the closed-loop system is not affected by the disturbance w, i.e., $(C + D\,F)\,[\,\zeta\,I_n - (A + B\,F)\,]^{-1}\,H + G = 0$ for all $\zeta \in \mathbb{C} \setminus \sigma(A + B\,F)$.
2) The closed loop is internally stable, i.e., $\sigma(A + B\,F) \subset \mathbb{C}_g$.

Besides the classic formulation of the disturbance decoupling problem, a case of interest is that in which the disturbance input w is measurable; hence, we formulate the so-called *MSDPs*.

Problem 2: Find (if possible) a static control function $u(t) = F\,x(t) + S\,w(t)$, $t \in \mathbb{T}$ such that the following are satisfied.

1) The output y of the closed-loop system is not affected by the signal w, i.e., $(C + D\,F)\,[\,\zeta\,I_n - (A + B\,F)\,]^{-1}\,(H + B\,S) + (G + D\,S) = 0$ for all $\zeta \in \mathbb{C} \setminus \sigma(A + B\,F)$.
2) The closed loop is internally stable, i.e., $\sigma(A + B\,F) \subset \mathbb{C}_g$.

The basic requirement for Problems 1 and 2 to admit solutions is that the pair (A, B) be stabilizable. The complete solutions of these problems is given in Section IV-B.

A. Geometric Preliminaries

Before presenting the solution of Problems 1 and 2, some useful results on self-bounded subspaces are introduced. These results are the extension of the properties in [4, pp. 225–226] to nonstrictly proper systems. Again, to simplify notation, let $\widehat{A} \stackrel{\triangle}{=} [\,A^{\top} \;\; C^{\top}\,]^{\top}$, $\widehat{B} \stackrel{\triangle}{=} [\,B^{\top} \;\; D^{\top}\,]^{\top}$, and $\widehat{H} \stackrel{\triangle}{=} [\,H^{\top} \;\; G^{\top}\,]^{\top}$.

Lemma 3: If $\operatorname{im} \widehat{H} \subseteq (\mathcal{V}_{\Sigma}^{*} \times O_p) + \operatorname{im} \widehat{B}$. The following facts hold.

1) $\mathcal{V}_{\Sigma}^{*} = \mathcal{V}_{\tilde{\Sigma}}^{*}$.
2) $\Phi(\tilde{\Sigma}) \subseteq \Phi(\Sigma)$.
3) For all $\mathcal{V}_{\tilde{\Sigma}} \in \Phi(\tilde{\Sigma})$, $\operatorname{im} \widehat{H} \subseteq (\mathcal{V}_{\tilde{\Sigma}} \times O_p) + \operatorname{im} \widehat{B}$.

Proof: Consider 1). The two sequences $(\mathcal{V}_{\Sigma}^{i})_{i \in \mathbb{N}}$ and $(\mathcal{V}_{\tilde{\Sigma}}^{i})_{i \in \mathbb{N}}$ converge to $\mathcal{V}_{\Sigma}^{*}$ and $\mathcal{V}_{\tilde{\Sigma}}^{*}$, respectively. Since it is assumed that $\operatorname{im} \widehat{H} \subseteq (\mathcal{V}_{\Sigma}^{*} \times O_p) + \operatorname{im} \widehat{B}$, the image of $\widehat{H}$ can be written as the sum of two subspaces $\mathcal{J}_1$ and $\mathcal{J}_2$, which are, respectively, contained in $\mathcal{V}_{\Sigma}^{*} \times O_p$ and in the image of $\widehat{B}$. Since the sequences $(\mathcal{V}_{\Sigma}^{i})_{i \in \mathbb{N}}$ and $(\mathcal{V}_{\tilde{\Sigma}}{}^{i})_{i \in \mathbb{N}}$ are nonincreasing, $\mathcal{J}_1 \subseteq \mathcal{V}_{\Sigma}^{*} \times O_p$ implies $\mathcal{J}_1 \subseteq \mathcal{V}_{\Sigma}^{i} \times O_p$ for all $i \in \mathbb{N}$. We verify by induction that $\mathcal{V}_{\Sigma}^{i} = \mathcal{V}_{\tilde{\Sigma}}^{i}$ for all $i \in \mathbb{N}$. This fact is clearly true when $i = 0$. Let us now suppose that it holds for a given $i - 1$, i.e., $\mathcal{V}_{\Sigma}^{i-1} = \sqcup V^{\,i-1}$, and let us prove the same fact for i. In particular, we show that $(\mathcal{V}_{\Sigma}^{i-1} \times O_p) + \operatorname{im} \widehat{B} = (\mathcal{V}_{\Sigma}^{i-1} \times O_p) + \operatorname{im} [\widehat{B} \;\; \widehat{H}]$. Since $\mathcal{J}_1 \subseteq \mathcal{V}_{\Sigma}^{i-1} \times O_p$ and $\mathcal{J}_2 \subseteq \operatorname{im} \widehat{B}$, we get $(\mathcal{V}_{\Sigma}^{i-1} \times O_p) + \operatorname{im} [\widehat{B} \;\; \widehat{H}] = ((\mathcal{V}_{\Sigma}^{i-1} \times O_p) + \mathcal{J}_1) + \operatorname{im} \widehat{B} + \mathcal{J}_2 = (\mathcal{V}_{\Sigma}^{i-1} \times O_p) + \operatorname{im} \widehat{B}$. It follows that $\mathcal{V}_{\Sigma}^{i} = \mathcal{V}_{\tilde{\Sigma}}^{i}$ for all $i \in \mathbb{N}$, leading to $\mathcal{V}_{\Sigma}^{*} = \mathcal{V}_{\tilde{\Sigma}}^{*}$. Now, consider 2). By 1), $\mathcal{V}_{\Sigma}^{*} = \mathcal{V}_{\tilde{\Sigma}}^{*}$. Let $F \in \Im_{\Sigma}(\mathcal{V}_{\Sigma}^{*})$. It follows that $\tilde{F} \stackrel{\triangle}{=} [F^{\top} \;\; 0]^{\top} \in \Im_{\tilde{\Sigma}}(\mathcal{V}_{\tilde{\Sigma}}^{*})$. In fact, by defining $\tilde{B} \stackrel{\triangle}{=} [B \;\; H]$ and $\tilde{D} \stackrel{\triangle}{=} [D \;\; G]$, we find $A + \tilde{B}\,\tilde{F} = A + B\,F$ and $C + \tilde{D}\,\tilde{F} = C + D\,F$. In view of Lemma 2, $\tilde{F} \in \Im_{\tilde{\Sigma}}(\mathcal{V}_{\tilde{\Sigma}})$ for any $\mathcal{V}_{\tilde{\Sigma}} \in \Phi(\tilde{\Sigma})$. Hence, $(\widehat{A} + \widehat{B}\,F)\,\mathcal{V}_{\tilde{\Sigma}} = [(A + \tilde{B}\,\tilde{F})^{\top} \;\; (C + \tilde{D}\,\tilde{F})^{\top}]^{\top}\,\mathcal{V}_{\tilde{\Sigma}} \subseteq \mathcal{V}_{\tilde{\Sigma}} \times O_p$, so that $\mathcal{V}_{\tilde{\Sigma}} \in \mathcal{V}(\Sigma)$. Moreover, since $\mathcal{V}_{\tilde{\Sigma}}^{*} \cap \tilde{B} \ker \tilde{D} \supseteq \mathcal{V}_{\Sigma}^{*} \cap B \ker D$, it follows that $\mathcal{V}_{\tilde{\Sigma}} \in \Phi(\Sigma)$. Consider 3). Let $\mathcal{V}_{\tilde{\Sigma}} \in \Phi(\tilde{\Sigma})$. As in the proof of 1), let the image of $\widehat{H}$ be written as the sum of two subspaces $\mathcal{J}_1$ and $\mathcal{J}_2$, which are, respectively, contained in $(\mathcal{V}_{\Sigma}^{*} \times O_p)$ and in the image of $\widehat{B}$. By the definition of self-boundedness, we find

$$\begin{aligned} \mathcal{V}_{\tilde{\Sigma}} \times O_p &\supseteq (\mathcal{V}_{\tilde{\Sigma}}^{*} \cap [B \;\; H] \ker [D \;\; G]) \times O_p \\ &\supseteq (\mathcal{V}_{\tilde{\Sigma}}^{*} \times O_p) \cap \operatorname{im} [\widehat{B} \;\; \widehat{H}] \\ &= (\mathcal{V}_{\tilde{\Sigma}}^{*} \times O_p) \cap (\operatorname{im} \widehat{B} + \mathcal{J}_1 + \mathcal{J}_2) \\ &= (\mathcal{V}_{\tilde{\Sigma}}^{*} \times O_p \cap \operatorname{im} \widehat{B}) + \mathcal{J}_1 \end{aligned}$$

which proves that $\mathcal{J}_1 \subseteq \mathcal{V}_{\tilde{\Sigma}} \times O_p$, which, in turn, yields 3). ■

The following theorem is the extension of a well-known property that was first presented as a conjecture by Basile and Marro [3], and then, proved by Schumacher [5] in the case when both D and G are zero. Here, we prove it in the nonstrictly proper case.

Theorem 2: Let $\operatorname{im} \widehat{H} \subseteq (\mathcal{V}_{\Sigma}^{*} \times O_p) + \operatorname{im} \widehat{B}$. If an internally stabilizable output-nulling subspace $\mathcal{V}_{\Sigma} \in \mathcal{V}(\Sigma)$ exists such that $\operatorname{im} \widehat{H} \subseteq (\mathcal{V}_{\Sigma} \times O_p) + \operatorname{im} \widehat{B}$, then the subspace $\mathcal{R}_{\tilde{\Sigma}}^{*} = \min \Phi(\tilde{\Sigma})$ is internally stabilizable.

Proof: Let $\bar{\mathcal{V}}_{\Sigma} \stackrel{\triangle}{=} \mathcal{V}_{\Sigma} + \mathcal{R}_{\tilde{\Sigma}}^{*}$. The subspace $\bar{\mathcal{V}}_{\Sigma}$ is output-nulling for Σ since such are $\mathcal{V}_{\Sigma}$ and $\mathcal{R}_{\tilde{\Sigma}}^{*}$. We first show that $\bar{\mathcal{V}}_{\Sigma} \in \Phi(\tilde{\Sigma})$. Since $\bar{\mathcal{V}}_{\Sigma} \supseteq \mathcal{V}_{\Sigma}$ and $\operatorname{im} \widehat{H} \subseteq (\mathcal{V}_{\Sigma} \times O_p) + \operatorname{im} \widehat{B}$, we find

$$\operatorname{im} \widehat{H} \subseteq (\bar{\mathcal{V}}_{\Sigma} \times O_p) + \operatorname{im} \widehat{B}.$$

Moreover, since $\bar{\mathcal{V}}_{\Sigma} \supseteq \mathcal{R}_{\tilde{\Sigma}}^{*} \supseteq \mathcal{V}_{\Sigma}^{*} \cap B \ker D$, we get

$$\bar{\mathcal{V}}_{\Sigma} \times O_p \supseteq (\mathcal{V}_{\Sigma}^{*} \times O_p) \cap \operatorname{im} \widehat{B}.$$

By adding $\operatorname{im}\widehat{B}$ to both sides of (7), we get $\operatorname{im}\widehat{H} + \operatorname{im}\widehat{B} \subseteq (\bar{\mathcal{V}}_\Sigma \times O_p) + \operatorname{im}\widehat{B}$; by intersecting $\mathcal{V}^*_\Sigma \times O_p$ to both sides of the latter and by applying the modular rule to the relation thus obtained, we find $(\mathcal{V}^*_\Sigma \times O_p) \cap (\operatorname{im}\widehat{H} + \operatorname{im}\widehat{B}) \subseteq (\bar{\mathcal{V}}_\Sigma \times O_p) + ((\mathcal{V}^*_\Sigma \times O_p) \cap \operatorname{im}\widehat{B}) = (\bar{\mathcal{V}}_\Sigma \times O_p)$, in view of (8). Since $\mathcal{V}^*_\Sigma = \mathcal{V}^*_{\tilde{\Sigma}}$, the latter can be written as $(\bar{\mathcal{V}}_\Sigma \times O_p) \supseteq (\mathcal{V}^*_{\tilde{\Sigma}} \times O_p) \cap \operatorname{im}[\widehat{B} \quad \widehat{H}] = (\mathcal{V}^*_{\tilde{\Sigma}} \cap [B \quad H] \ker [D \quad G]) \times O_p$, which proves that $\bar{\mathcal{V}}_\Sigma \in \Phi(\tilde{\Sigma})$. Since $\bar{\mathcal{V}}_\Sigma$ is stabilizable with respect to the input u, a matrix $F \in \mathbb{R}^{m\times n}$ exists such that $(\widehat{A} + \widehat{B}\,F)\,\bar{\mathcal{V}}_\Sigma \subseteq \bar{\mathcal{V}}_\Sigma \times O_p$ and $\sigma(A + B\,F|\bar{\mathcal{V}}_\Sigma) \subset \mathbb{C}_g$. This fact also implies that $\bar{\mathcal{V}}_\Sigma$ is stabilizable as an element of $\Phi(\tilde{\Sigma})$ as well, i.e., with respect to inputs u and w; indeed, a feedback matrix meeting this requirement is $\tilde{F} \stackrel{\triangle}{=} \begin{bmatrix} F^\top & 0 \end{bmatrix}^\top$. By virtue of Lemma 2, it follows that $\tilde{F} \in \Im_{\tilde{\Sigma}}(\mathcal{R}^*_{\tilde{\Sigma}})$ and $\sigma(A + \tilde{B}\,\tilde{F}|\mathcal{R}^*_{\tilde{\Sigma}}) \subset \mathbb{C}_g$. In view of Lemma 2, the latter implies that $\sigma(A + B\,F|\mathcal{R}^*_{\tilde{\Sigma}}) \subset \mathbb{C}_g$. Hence, $\mathcal{R}^*_{\tilde{\Sigma}}$ is internally stabilizable with respect to the sole input u as well. ■

If, instead of the condition $\operatorname{im}\widehat{H} \subseteq (\mathcal{V}^*_\Sigma \times O_p) + \operatorname{im}\widehat{B}$, the stronger condition $\operatorname{im}\widehat{H} \subseteq \mathcal{V}^*_\Sigma \times O_p$ holds, all the results presented in this section still apply, as the following corollary points out.

Corollary 1: Let $\operatorname{im}\widehat{H} \subseteq \mathcal{V}^*_\Sigma \times O_p$. Then, $\mathcal{V}_\Sigma{}^* = \mathcal{V}^*_{\tilde{\Sigma}}$, $\Phi(\tilde{\Sigma}) \subseteq \Phi(\Sigma)$, and for all $\mathcal{V}_{\tilde{\Sigma}} \in \Phi(\tilde{\Sigma})$, $\operatorname{im}\widehat{H} \subseteq (\mathcal{V}_{\tilde{\Sigma}} \times O_p)$ holds. Moreover, if an internally stabilizable output-nulling subspace $\mathcal{V}_\Sigma \in \mathcal{V}(\Sigma)$ exists such that $\operatorname{im}\widehat{H} \subseteq \mathcal{V}_\Sigma \times O_p$, then the subspace $\mathcal{R}^*_{\tilde{\Sigma}}$ is internally stabilizable.

B. Solutions of Problems 1 and 2

In Theorems 3 and 4, we present a constructive solution to Problems 1 and 2, respectively, based on the results of the previous section. In both cases, the solvability conditions are two: the first consists of a pure geometric relation, and is sometimes referred to as *structural*. The second is referred to as the *stability condition*, and is the same for both problems, and is expressed in terms of the self-bounded subspace $\mathcal{R}^*_{\tilde{\Sigma}}$.

Theorem 3: Let the pair (A, B) be stabilizable. The DDPs is solvable iff

1) $\operatorname{im}\widehat{H} \subseteq \mathcal{V}^*_\Sigma \times O_p$, or equivalently, $\operatorname{im} H \subseteq \mathcal{V}^*_\Sigma$ and $G = 0$;
2) $\mathcal{R}^*_{\tilde{\Sigma}}$ is internally stabilizable.

The proof of this theorem is omitted as it can be derived straightforwardly as a particular case from the proof of Theorem 4. Observe that the structural condition 1) is a generalization of the well-known condition $\operatorname{im} H \subseteq \mathcal{V}^*_\Sigma$ [1], which was obtained in the case where both D and G are zero. The presence of a nonzero G is more interesting in the solution of Problem 2, which is presented in the following theorem.

Theorem 4: Let the pair (A, B) be stabilizable. The MSDP with internal stability is solvable iff

1) $\operatorname{im}\widehat{H} \subseteq (\mathcal{V}^*_\Sigma \times O_p) + \operatorname{im}\widehat{B}$;
2) $\mathcal{R}^*_{\tilde{\Sigma}}$ is internally stabilizable.

Proof: It is easily seen that the MSDPs is equivalent to finding a feedback matrix $F \in \mathbb{R}^{m\times n}$ and a matrix $S \in \mathbb{R}^{m\times r}$ such that $(C + D\,F)\ <A + B\,F, \operatorname{im}(H + B\,S)> \ = \ O_p, G = -D\,S$, and such that $\sigma(A + B\,F) \subset \mathbb{C}_g$. Suppose that 1) and 2) are true. By Lemma 2, from 1), it follows that $\operatorname{im}\widehat{H} \subseteq (\mathcal{R}^*_{\tilde{\Sigma}} \times O_p) + \operatorname{im}\widehat{B}$. The subspace $\mathcal{R}^*_{\tilde{\Sigma}}$ is internally stabilizable, and in view of the stabilizability of the pair (A, B), it is externally stabilizable as well, so that a matrix $F \in \Im_\Sigma(\mathcal{R}^*_{\tilde{\Sigma}})$ exists such that $A + BF$ is stable. Since $\operatorname{im}\widehat{H} \subseteq (\mathcal{R}^*_{\tilde{\Sigma}} \times O_p) + \operatorname{im}\widehat{B}$, it follows that, given a basis matrix R of $\mathcal{R}^*_{\tilde{\Sigma}}$, two matrices Φ and Ψ exist such that the identities $H = R\,\Phi + B\,\Psi$ and $G = D\,\Psi$ hold. By taking $S = -\Psi$, on the one hand, we get $H + B\,S = R\,\Phi$, which implies that $\operatorname{im}(H + B\,S) \subseteq \mathcal{R}^*_{\tilde{\Sigma}}$. On the other hand, $G + D\,S = 0$. As a result, $<A + B\,F, \operatorname{im}(H + B\,S)> \ \subseteq \ <A + B\,F, \mathcal{R}^*_{\tilde{\Sigma}}> \ = \mathcal{R}^*_{\tilde{\Sigma}} \ \subseteq \ \ker(C + D\,F)$, and $G = -D\,S$, since $F \in \Im_\Sigma(\mathcal{R}^*_{\tilde{\Sigma}})$, so that MSDPs admits solutions. Conversely, let $F \in \mathbb{R}^{m\times n}$ and $S \in \mathbb{R}^{m\times r}$ be the solution of the MSDPs. Let $\mathcal{V}_\Sigma \stackrel{\triangle}{=} <A + B\,F, \operatorname{im}(H + B\,S)>$. The subspace $\mathcal{V}_\Sigma$ is such that $(C + D\,F)\,\mathcal{V}_\Sigma = O_p$. Moreover, since $\sigma(A + B\,F) \subset \mathbb{C}_g$, it follows that $\mathcal{V}_\Sigma$ is an internally and externally stabilizable output-nulling subspace, and $\mathcal{V}_\Sigma \subseteq \mathcal{V}^*_\Sigma$; since $\mathcal{V}_\Sigma \supseteq \operatorname{im}(H + B\,S)$ and $G + D\,S = 0$, it follows that $\mathcal{V}_\Sigma \times O_p \supseteq \operatorname{im}(\widehat{H} + \widehat{B}\,S)$. Hence, $\operatorname{im}\widehat{H} \subseteq (\mathcal{V}^*_\Sigma \times O_p) + \operatorname{im}\widehat{B}$. Moreover, by Theorem 2 it follows that $\mathcal{R}^*_{\tilde{\Sigma}}$ is internally stabilizable. ■

When matrix G is zero, the structural condition 1) can be written as $\operatorname{im} H \subseteq \mathcal{V}^*_\Sigma + B \ker D$. Moreover, in the particular case when both the feedthrough matrices D and G are zero, condition 1) reduces to the well-known inclusion $\operatorname{im} H \subseteq \mathcal{V}^*_\Sigma + \operatorname{im} B$ [4], [10].

Remark 1: As observed in the proofs of Theorems 3 and 4, the solutions of DDPs and MSDPs given here are constructive, in the sense that if the solvability conditions are met, $\mathcal{R}^*_{\tilde{\Sigma}}$ is itself a solution to the decoupling problem considered by taking $F \in \Im_\Sigma(\mathcal{R}^*_{\tilde{\Sigma}})$ such that $\sigma(A + B\,F) \subset \mathbb{C}_g$ (and in the case of Problem 2, a matrix S such that $G + D\,S = 0$).

Remark 2: By using the approach presented in [13], i.e., by considering the DDPs and the MSDPs as special cases of the dynamic output feedback case, the solvability conditions for Problems 1 and 2 become, respectively,

$$\operatorname{im}\widehat{H} \subseteq (\mathcal{V}^*_g \times O_p) \quad \text{and} \quad \operatorname{im}\widehat{H} \subseteq (\mathcal{V}^*_g \times O_p) + \operatorname{im}\widehat{B} \qquad (9)$$

where the subspace $\mathcal{V}^*_g$ is the so-called "good" part of $\mathcal{V}^*_\Sigma$, and is defined as the largest subspace $\mathcal{V}_\Sigma$ of Σ such that there exists a matrix F satisfying $(\widehat{A} + \widehat{B}\,F)\,\mathcal{V}_\Sigma \subseteq \mathcal{V}_\Sigma \times O_p$ and $\sigma(A + B\,F\,|\,\mathcal{V}_\Sigma) \subset \mathbb{C}_g$. These conditions are more compact than those given in Theorems 3 and 4. On the other hand, the latter have important advantages over those expressed by (9). First, the conditions in Theorems 3 and 4 are split into a so-called *structural condition* (which is the solvability condition of the same decoupling problem considered without stability requirements) and a *stability condition* (which is independent of the signal to be decoupled being either unaccessible, measurable or even known in advance with finite preview [8]). As such, when more information about the disturbance is available, what changes is only the structural condition—which becomes weaker—whereas the stability condition remains valid in all cases. Hence, the independent role of each condition in the solution of different decoupling problems is made more explicit when these are expressed as shown in Theorems 3 and 4. Second, from a computational point of view, checking the set of conditions of Theorems 3 and 4 is much simpler than checking (9). In fact, $\mathcal{R}^*_{\tilde{\Sigma}}$ can be simply computed as the intersection $\mathcal{V}^*_{\tilde{\Sigma}} \cap \mathcal{S}^*_{\tilde{\Sigma}} = \mathcal{V}^*_\Sigma \cap \mathcal{S}^*_{\tilde{\Sigma}}$,[1] whereas the determination of $\mathcal{V}^*_g$ requires eigenspace computations, which is rather critical for high-order systems [4], [12], [15].

Moreover, it is easily seen that the inclusion $\mathcal{R}^*_{\tilde{\Sigma}} \subseteq \mathcal{V}^*_g$ holds. As a result of this, when feedforward schemes are adopted for the implementation of the solution of the MSDP, or when employing such a solution to solve the exact model matching problem along the lines of [16] (see Th. 3.2), the advantage of using $\mathcal{R}^*_{\tilde{\Sigma}}$ is even more evident. Indeed, the order of the compensator solving these problems is, in general, equal to the dimension of the output-nulling where the state trajectory lies. Hence, since the solvability conditions given in Theorems 3 and 4 are

[1]This operation can be implemented by resorting to the standard routines of the geometric approach. To this end, the MATLAB routine `ints.m` may be utilized, which is included in the geometric approach toolbox *ga*. This toolbox is downloadable at www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm

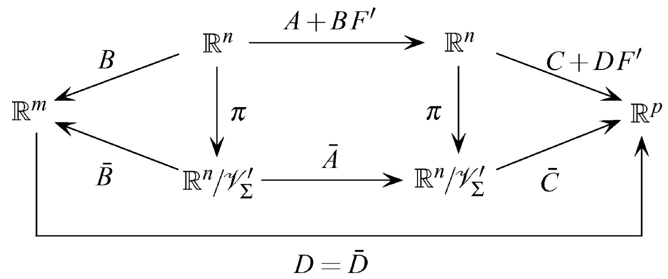

Fig. 1. Commutative diagram.

constructive, the solution of the MSDPs based on this set of conditions leads to compensators whose order is smaller than that of compensators designed by resorting to conditions (9). As a result, the use of $\mathcal{R}^*_{\hat{\Sigma}}$ allows us to reduce the state dimension of the decoupling filter. Furthermore, the set of fixed poles associated with $\mathcal{V}^*_g$ is larger than that associated with $\mathcal{R}^*_{\hat{\Sigma}}$, the latter being minimal as in the strictly proper case. This issue is discussed in the next section.

C. Fixed Poles of the DDP

As shown in [5, p. 375], in general, $\mathcal{R}^*_{\hat{\Sigma}}$ is not the smallest internally stabilizable output-nulling of Σ such that $\mathcal{R}^*_{\hat{\Sigma}} \times O_p$ contains im $\widehat{H}$ (for instance, in the case of Problem 1). However, it can be shown that taking $\mathcal{R}^*_{\hat{\Sigma}}$ as the solving output-nulling subspace of the decoupling problem is the best solution in terms of the pole assignment of the closed-loop eigenvalues. Indeed, as already noticed, when either the DDP or the MSDP is solvable, a maximal set of eigenvalues of the closed-loop exists that is present for any feedback solution (these eigenvalues are usually referred to as the fixed poles of the decoupling problem), and at least one state-feedback solution F exists, such that all the remaining eigenvalues can be assigned arbitrarily. In 1997, Malabre *et al.* [6] showed that the fixed poles of both the DDP and the MSDP are exactly $\Gamma_\Sigma(\mathcal{R}^*_{\hat{\Sigma}})$. The extension of this result to nonstrictly proper systems is straightforward, provided that [17, Lemma A.1], which is exploited in the proof in [6, Th. 1], is generalized to quadruples. This issue is addressed in the following lemma.

Lemma 4: Let $\mathcal{V}_\Sigma, \mathcal{V}'_\Sigma \in \mathcal{V}(\Sigma)$ be such that $\mathcal{V}_\Sigma = \mathcal{V}'_\Sigma + \mathcal{R}_\Sigma$, where $\mathcal{R}_\Sigma$ is the reachability subspace on $\mathcal{V}_\Sigma$. Then, $\Gamma^{\rm int}_\Sigma(\mathcal{V}_\Sigma) \subseteq \Gamma^{\rm int}_\Sigma(\mathcal{V}'_\Sigma)$.

Proof: Let $F' \in \Im_\Sigma(\mathcal{V}'_\Sigma)$, and denote with $\pi: \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathcal{V}'_\Sigma$ the canonical projection on the quotient space $\mathbb{R}^n/\mathcal{V}'_\Sigma$ [11]. Let $\bar{A} \triangleq (A + B\,F'|\frac{\mathbb{R}^n}{\mathcal{V}'_\Sigma})$, so that $\bar{A}\,\pi = \pi\,(A + B\,F')$. Moreover, define $\bar{B} \triangleq \pi\,B$, $\bar{D} \triangleq D$, and let $\bar{C}$ be such that $\bar{C}\,\pi = C + D\,F'$ (see Fig. 1). Let $\bar{\Sigma} \triangleq (\bar{A}, \bar{B}, \bar{C}, \bar{D})$. It is first shown that $\pi\,\mathcal{V}_\Sigma \in \mathcal{V}(\bar{\Sigma})$. By definition of $\bar{A}$, it is found that

$$\begin{bmatrix} \pi\,\bar{A} \\ C \end{bmatrix} \mathcal{V}_\Sigma \subseteq (\pi\,\mathcal{V}_\Sigma \times O_p) + \mathrm{im} \begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}.$$

Now, since $\mathcal{V}_\Sigma$ is controlled invariant in Σ, $\pi\,A\,\mathcal{V}_\Sigma \subseteq \pi\,\mathcal{V}_\Sigma + \mathrm{im}\,\bar{B}$, so that

$$\begin{bmatrix} \bar{A} \\ \bar{C} \end{bmatrix} \pi\mathcal{V}_\Sigma \subseteq (\pi\,\mathcal{V}_\Sigma \times O_p) + \mathrm{im} \begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}$$

which implies

$$[\,\bar{A}^\top \quad \bar{C}^\top\,]^\top\,\pi\mathcal{V}_\Sigma \subseteq (\pi\,\mathcal{V}_\Sigma \times O_p) + \mathrm{im}\,[\,\bar{B}^\top \quad \bar{D}^\top\,]^\top.$$

Hence, $\pi\,\mathcal{V}_\Sigma \in \mathcal{V}(\bar{\Sigma})$. Let $\bar{F} \in \Im_{\bar{\Sigma}}(\pi\,\mathcal{V}_\Sigma)$; we want to show that $F \triangleq F' + \bar{F}\,\pi \in \Im_\Sigma(\mathcal{V}_\Sigma)$. Indeed,

$$\begin{aligned} \begin{bmatrix} \pi\,(A + B\,F) \\ C + D\,F \end{bmatrix} \mathcal{V}_\Sigma &\subseteq \begin{bmatrix} \pi\,(A + B\,F') \\ C + D\,F' \end{bmatrix} \mathcal{V}_\Sigma + \begin{bmatrix} \pi\,B \\ D \end{bmatrix} \bar{F}\,\pi\,\mathcal{V}_\Sigma \\ &= \begin{bmatrix} \bar{A} + \bar{B}\,\bar{F} \\ \bar{C} + \bar{D}\,\bar{F} \end{bmatrix} \pi\,\mathcal{V}_\Sigma \subseteq \pi\,\mathcal{V}_\Sigma \times O_p \end{aligned}$$

which leads to $[(A + B\,F)^\top \quad (C + D\,F)^\top]^\top \mathcal{V}_\Sigma \subseteq (\mathcal{V}_\Sigma + \ker\pi) \times O_p = (\mathcal{V}_\Sigma + \mathcal{V}'_\Sigma) \times O_p = \mathcal{V}_\Sigma \times O_p$, which proves that $F \in \Im_\Sigma(\mathcal{V}_\Sigma)$. Since $F = F' + \bar{F}\,\pi$ and $\mathcal{V}'_\Sigma = \ker\pi$, it follows that $A + B\,F|\mathcal{V}'_\Sigma = A + B\,F'|\mathcal{V}'_\Sigma$. We want to show that $\pi\,\mathcal{V}_\Sigma = \pi(\mathcal{V}'_\Sigma + \mathcal{R}_\Sigma) = \pi\,\mathcal{R}_\Sigma$ is the smallest $(\bar{A} + \bar{B}\,\bar{F})$-invariant subspace containing $\pi\,\mathcal{R}_\Sigma \cap \bar{B}\ker\bar{D}$. We recall that $\mathcal{R}_\Sigma$ is the smallest subspace such that $(A + B\,F)\,\mathcal{R}_\Sigma \subseteq \mathcal{R}_\Sigma$ and $\mathcal{R}_\Sigma \supseteq \mathcal{V}_\Sigma \cap B\ker D$. By applying the linear operator π, it is found that $\pi\,(A + B\,F)\,\mathcal{R}_\Sigma \subseteq \pi\,(\,(A + B\,F') + B\,\bar{F}\,\pi)\,\mathcal{R}_\Sigma = (\bar{A} + \bar{B}\,\bar{F})\,\pi\,\mathcal{R}_\Sigma \subseteq \pi\,\mathcal{R}_\Sigma$ and $\pi\,\mathcal{R}_\Sigma \supseteq \pi\,\mathcal{R}_\Sigma \cap \pi\,B\ker D$, so that $\pi\,\mathcal{R}_\Sigma$ is $(\bar{A} + \bar{B}\,\bar{F})$-invariant and it contains $\pi\,\mathcal{R}_\Sigma \cap \bar{B}\ker\bar{D}$. The subspace $\pi\,\mathcal{R}_\Sigma$ of $\mathbb{R}^n/\mathcal{V}'_\Sigma$ is the smallest subspace enjoying this property. In fact, let $\mathcal{W}_{\bar{\Sigma}}$ be the smallest subspace of $\mathbb{R}^n/\mathcal{V}'_\Sigma$ such that $(\bar{A} + \bar{B}\,\bar{F})\,\mathcal{W}_{\bar{\Sigma}} \subseteq \mathcal{W}_{\bar{\Sigma}}$ and $\mathcal{W}_{\bar{\Sigma}} \supseteq \pi\,\mathcal{R}_\Sigma \cap \bar{B}\ker\bar{D}$. Since the map π is epic, there exists a subspace of Σ, say $\mathcal{W}_\Sigma$, such that $\mathcal{W}_{\bar{\Sigma}} = \pi\,\mathcal{W}_\Sigma$. The subspace $\mathcal{W}_\Sigma$ is $(A + B\,F)$-invariant and contains $\mathcal{V}_\Sigma \cap B\ \ker\ D$, so that $\mathcal{W}_\Sigma \supseteq \mathcal{R}_\Sigma$. It follows that $\mathcal{W}_{\bar{\Sigma}} = \pi\,\mathcal{R}_\Sigma$. We may conclude that the controllability subspace on $\pi\,\mathcal{V}_\Sigma$ is $\pi\,\mathcal{V}_\Sigma$ itself, so that by means of a suitable choice of $\bar{F}$, we can assign all the eigenvalues of $\bar{A} + \bar{B}\,\bar{F}$ that are internal to $\pi\,\mathcal{V}_\Sigma$. As a result, $\bar{F}$ can be chosen so as to place all the eigenvalues of $(\bar{A} + \bar{B}\,\bar{F})$ restricted to $\pi\,\mathcal{V}_\Sigma$ arbitrarily. In other words, if $\Lambda \subset \mathbb{C}$ is a symmetric set of $\dim(\mathcal{V}_\Sigma/\mathcal{V}'_\Sigma)$ elements, F can be chosen so that $\sigma(A + B\,F|\frac{\mathcal{V}_\Sigma}{\mathcal{V}'_\Sigma}) = \Lambda$. The last part of the proof presents no differences with respect to that in [17] for nonstrictly proper systems, and therefore is omitted. ∎

We present the main result concerning the fixed poles of the DDP and MSDP [6].

Corollary 2: Let the conditions of Theorem 3 (respectively, Theorem 4) be satisfied. Then, the set of fixed poles of the DDP (respectively, MSDP) is given by $\Gamma_\Sigma(\mathcal{R}^*_{\hat{\Sigma}})$.

V. Concluding Remarks

In this paper, the extension of the concept of self-boundedness to nonstrictly proper systems has been carried out via the exploitation of output-nulling subspaces. Hence, several classical results obtained by assuming that the matrix D is zero have been extended. In the second part of this paper, it has been shown how the DDPs can be reformulated and solved when both the feedthrough matrices from the control and the disturbance to the output to be decoupled are possibly nonzero. With respect to other existing techniques based on the introduction of fictitious variables, this approach has the advantage of leading to a generalized version of the geometric solvability conditions that are written in terms of the matrices of the given system. It is also worth noticing that the problem of the decoupling of previewed input signals and the regulator problem with output stability can be tackled similarly by exploiting the definition of self-boundedness and the techniques herein presented, so that with the same philosophy, the material in [8] and [7] can be generalized as well.

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Fixed-Order Controller Design for Polytopic Systems Using LMIs

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***Abstract*—Convex parameterization of fixed-order robust stabilizing controllers for systems with polytopic uncertainty is represented as a linear matrix inequality (LMI) using the Kalman–Yakubovich–Popov (KYP) lemma. This parameterization is a convex inner approximation of the whole nonconvex set of stabilizing controllers, and depends on the choice of a central polynomial. It is shown that, with an appropriate choice of the central polynomial, the set of all stabilizing fixed-order controllers that place the closed-loop poles of a polytopic system in a disk centered on the real axis can be outbounded with some LMIs. These LMIs can be used for robust pole placement of polytopic systems.**

I. INTRODUCTION

Nowadays, many control design problems are formulated as convex optimization problems, and are solved efficiently using recently developed numerical algorithms. Yet, a challenging problem is the design of restricted order controllers by convex optimization methods. The main problem stems from the fundamental algebraic property that the stability domain in the space of a polynomial's parameters is nonconvex for polynomials with order higher than 2 [1]. To overcome the nonconvexity, there are different strategies, which are explained in [2]. One possibility is to consider an approximation of the nonconvex domain with an outer-or-inner convex set. Although an inner approximation introduces some conservatism in the design method, it is preferred because the stability is ensured. Several convex inner approximations of the stability domain around a central polynomial have been proposed in the literature. However, the linear matrix inequality (LMI) approximations are more flexible since they can represent the other convex sets like polytopes, spheres, and ellipsoids.

The problem becomes more complicated when a fixed-order controller should stabilize a model with structured polytopic uncertainty. This problem is usually studied in the state space representation of the system for the full-order controllers using Lyapunov equation. A conservative solution is to find one Lyapunov function to stabilize all models. The other solution that is less conservative is to design a parameter-dependent Lyapunov function. However, it is not easy to find this Lyapunov function for polytopic systems. The stabilization problem can be converted to regional pole placement by using the concept of D-stability. It is to define a subregion of the stability domain and to modify accordingly the structure of the Lyapunov equation, and then, design a stabilizing controller [3]. The desired regions are restricted to strips, circles, sectors, and hyperbolas. In [4], a unified robust pole placement design method for both continuous and discrete-time systems is introduced. The controller meets the H_2 and/or H_∞ specifications for a nominal plant model, and assigns the closed-loop poles in an LMI region that is introduced in [5], and covers many desired regions, using LMI constraints. This problem is extended to the case of systems with a specific type of unstructured uncertainty in [6]. Recently, the design of a state feedback controller for a polytopic uncertain system that assigns the closed-loop poles in the same

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