Science and Mathematics Education Centre

Students' Difficulties, Conceptions and Attitudes Towards Learning Algebra: An Intervention Study to Improve Teaching and Learning

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DECLARATION

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgement has been made.

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ABSTRACT

The skills necessary to identify and analyse errors and misconceptions made by students are needed by teachers of all levels especially at the lower secondary school level in Malaysia. If students are to be successful in tackling mathematical problems later in their schooling, the one prerequisite is the mastery of basic concepts in algebra. Despite the best efforts of the teachers, students still develop algebra misconceptions. Is it possible to reduce or eliminate these misconceptions? The research involved a survey of 14 year-old students in Form 2 (Grade 8) in the Penampang district of Sabah, East Malaysia. The focus of this study lies in students' difficulties, conceptions and attitudes towards learning algebra in the framework of conceptual change. A possible way to help students overcome their learning difficulties and misconceptions is by implementing diagnostic teaching involving conflict to foster conceptual change. The study involved evaluating the efficacy of a conceptual change instructional programme involving cognitive conflict in (1) facilitating Form 2 students' understanding of algebra concepts, and (2) assessing changes in students' attitudes towards learning mathematics, in a mixed quantitative-qualitative research design.

A 24-item Algebra Diagnostic Test and a 20-item Test of Mathematics-Related Attitudes (TOMRA) questionnaire were administered as a pretest and a posttest to 39 students in each of a heterogeneous high-achieving class and a below-average achieving class. In addition 9 students were purposefully selected to participate in the interview.

The results of the study indicated that students' difficulties and misconceptions from both classes fell into five broad areas: (1) basic understanding of letters and their place in mathematics, (2) manipulation of these letters or variables, (3) use of rules of manipulation to solve equations, (4) use of knowledge of algebraic structure and syntax to form equations, and (5) generalisation of rule for repetitive patterns or sequences of shapes.

The results also showed that there was significant improvement in students' achievement in mathematics. Further, students' attitude towards inquiry of

mathematics lessons showed significant positive improvement. Enjoyment remained high even though enjoyment of mathematics lesson showed no change. Also, changes in students' understanding (from unintelligible to intelligible, intelligible to plausible, plausible to fruitful) illustrated the extent of changes in their conceptions.

Different pedagogies can affect how conceptual change and challenge of misconceptions occurs. Therefore, knowledge of the origin of different types of misconceptions can be useful in selecting more effective pedagogical techniques for challenging particular misconceptions. Also, for teachers to create an effective learning experience they should be aware of and acknowledge students' prior knowledge acquired from academic settings and from everyday previous personal experiences. Since all learning involves transfer from prior knowledge and previous experiences, an awareness and understanding of a student's initial conceptual framework and/or topic can be used to formulate more effective teaching strategies. If this idea is taken a step further, it could be said that, because misconceptions comprise part of a conceptual framework, then understanding origins of misconceptions would further facilitate development of effective teaching strategies.

Further research is needed to help teachers to understand how students experience conflict, how students feel when they experience conflict, and how these experiences are related to their final responses because cognitive conflict has both constructive and destructive potential. Thus, by being able to interpret, recognise and manage cognitive conflict, a teacher can then successfully interpret his/her students' cognitive conflict and be able to make conceptual change more likely or help students to have meaningful learning experiences in secondary school algebra.

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Rationale of the Study

1.0   Introduction

Mathematics arises from the attempt to organise and explain the phenomena of our
environment and experience (Bell, 1993). It has been expressed thus: "Mathematics
is ... an activity of organising fields of experience" (Freudenthal, 1973, p. 123); and
"Mathematics concerns the properties of the operations by which individual orders,
organises and controls his environment" (Peel, 1971, p. 157). These descriptions are
somewhat non-specific, though Peel is clearly referring to basic mental operations
such as classifying, comparing, combining, and representing. A more specific
characterisation of mathematics is given by Gattegno (1963):

> *To do mathematics is to adopt a particular attitude of mind in which*
> *what we term relationships per se are of interest. One can be*
> *considered a mathematician when one can isolate relationships from*
> *real and complex situations and later on when relationships can be*
> *used to create new situations in order to discover further*
> *relationships. Teaching mathematics means helping one's pupils to*
> *become aware of their relational thought, of the freedom of mind in*
> *its creation of relationship; it means encouraging them to develop a*
> *liking for such an attitude and to consider it as a human richness*
> *increasing the power of intellect in its dialogue with the universe"* (p.
> 55).

Other authors have offered more descriptions which emphasise the logical aspects.
Kaput and Roschelle (1998) regard "mathematics as a culturally shared study of
patterns and languages that is applied and extended through systematic forms of
reasoning and argument" (p. 155). Accordingly, it is both an object of understanding
and a means of understanding. They reiterate that these patterns and languages are an
essential way of understanding the worlds we experience – physical, social, and even

mathematical. While our universe of experience can be comprehended and organised in many ways – through the arts, humanities, the physical and social sciences (Kaput & Rochelle, 1998) – important aspects include those that are subject to measure and quantification, that embody quantifiable change and variation, that the world we inhabit and construct, and that involve algorithms and more abstract structures (Kaput & Rochelle, 1998). In addition, "mathematics embodies languages for expressing, communicating, reasoning, computing, abstracting, generalising and formalising" (p. 155) – all extending the limited powers of the human mind. The authors conclude that mathematics embodies systematic forms of reasoning and argument to help establish the certainty, generality, and the reliability of our mathematical assertion.

Battista (1999), however, opines that "mathematics is a form of reasoning". Thinking mathematically consists of thinking in a logical manner, formulating and testing conjectures, making sense of things, and forming and justifying judgments, inferences, and conclusions. Research has shown that mathematical behaviour is demonstrated by being able to recognise and describe patterns, construct physical and conceptual models of phenomena, create symbol systems to help in representing, manipulating, and reflecting on ideas, and inventing procedures to solve problems (Battista, 1999). Mathematics is also known as a language of science (Jaworski, 2006), an ideal tool for modelling scientific theories, deriving qualitative consequences from them and forecasting events. Mathematics is used in areas as diverse as space research, weather forecasting, geological exploration, and high finance. Mathematics is also a language (Jaworski, 2006) for everyday life, a central part of human communication, and a means of articulating patterns, relationships, rationality and aesthetic.

Meanwhile Bramall and White (2000) add that mathematics is a science related to life that has been used and valued by people ever since the emergence of civilisation or even before then, in many known and unknown ways. Ernest (2000) suggests that we understand our lives through the conceptual meshes of the clock, calendar, working timetables, travel planning and timetables, finance and currencies, insurances, pensions, tax, measurements of weight, length, area and volume, graphical and geometric representations, etc. To Skemp (1985) "mathematics is one

of the most powerful and adaptable mental tools which the intelligence of man has constructed for his own use cooperatively over the centuries. Hence, its importance in today's world of rapidly advancing science, technology, manufacturing, and commerce" (p. 447). These tools include logical reasoning, problem solving skills, and the ability to think in abstract ways. Mathematics therefore, is important in everyday life and the subject transcends cultural boundaries where its importance is universally recognised.

For all these reasons, mathematics is a central subject in the school curriculum for students at all levels and lends itself as a tool and a way of thinking to many other subjects. The government of Malaysia for the past decades has put great emphasis on the learning of mathematics, science and technology in education (MOE, 2006) encouraging that students should learn not only to use and apply mathematics rules, processes and formulae, but also develop principled understandings of mathematics, ways of thinking mathematically and ways of tackling a wide range of problems using mathematics (Mason, Burton, & Stacey, 1982).

Yet despite great effort or emphasis given to the teaching and learning of mathematics in the classrooms, the level of achievement of mathematics is still not promising. Mathematics achievement in Malaysia is not as strong as educators or society in general would like to see. This view is reflected in both international and national studies such as Trends in International Mathematics and Science Study (TIMSS) and Sijil Pelajaran Malaysia (SPM - Form Five public examination) evaluation. Looking at the performance of Malaysian students in comparisons to students from 44 countries participating in the TIMSS assessment (Mullis, Martin, Gonzalez, & Chrostowski, 2004) in 2003, Malaysian Form Two students scored 504, on the average, in mathematics. Although this average score exceeded the international average, the Malaysian students were out-performed by students from five Asian countries (Singapore, Republic of Korea, Hong Kong, Chinese-Taipei, and Japan), four European countries (Belgium-Flemish, Netherlands, Estonia and Hungary) and Australia. Malaysia was placed 18th overall and had rankings of 10 and 21 in mathematics and science respectively, in the Trends in International Mathematics and Science Study (TIMSS, 2003). Students in schools are achieving just above the international averages. National studies show many areas of

mathematics to be problematic for students (MOE, 2006) and evidence abounds that students' performance and achievement was (and is still) low, and has consequently affected students' attitudes and perceptions towards mathematics.

Students' attitudes towards mathematics as a school subject, including both the subject itself and the learning of the subject, and their implications for mathematics instructions have long received considerable attention from the mathematics education community. In particular, the relationship between attitudes towards mathematics and achievement in mathematics had traditionally been a major concern in mathematics education research (Ma & Kishor, 1997). For example, Neale (1969) described the relationship between the two as one of a reciprocal influence, that is, attitudes affect performance and performance in turn affects attitudes. In addition, there is research reporting that the relationship is not statistically significant (e.g., Ng-Gan, 1987; Papanastasiou, 2002). There is also research evidence showing that students' high performance in mathematics is not necessarily positively associated with their attitudes about mathematics and mathematics learning. For instance, the results of the Third International Mathematics and Science Study (TIMSS) revealed that while Japanese students outperformed students from many other countries in mathematics, they displayed relatively negative attitudes towards mathematics (Mullis et al., 2000).

Although the above evidence indicates that there are no consistent results in the relationship between students' attitudes towards mathematics and their mathematics achievement, fostering positive attitudes in students towards mathematics is still highly recognised as one crucial component in developing students' mathematical ability and understanding. In fact, many countries have set it as one of the aims of mathematics education in schools. In the case of Malaysia, the national mathematics curriculum states that "mathematics education aims to enable pupils to develop positive attitudes towards mathematics including confidence, enjoyment and perseverance" (MOE, 2003, p. 9). The challenge in education today is to effectively teach students of diverse ability and differing rates of learning in a way that enables them to learn mathematics concepts with understanding and developing positive attitudes towards mathematics learning.

Student attitudes to mathematics [algebra specifically] are central to this process because early failure in algebra is likely to result in passive withdrawal from further study in the area (MacGregor, 2004). Research conducted over the past two-to-three decades has shown that positive attitudes can impact on students' inclination for further studies and careers in mathematics-related fields (Trusty, 2002). For example, a recent study using the Third International Mathematics and Science Study (TIMSS) data from Canada, Norway and the United States found that attitudes towards mathematics as the strongest predictor of student participation in advanced mathematics courses (Ercikan, McCreith, & Lapointe, 2005). Hence, algebra study may act as a filter for further study in mathematics (MacGregor, 2004; Stacey & Chick, 2004), therefore, the development of positive attitudes to the subject is essential to increase student enrolments in advanced mathematics subjects.

Researchers have noted that early educational and socialisation processes are critical to children's learning and perceptions and subsequent participation in education (Khoon & Ainley, 2005). Student perceptions, which include their expectations of success and the value they attribute to particular tasks, have been found to correlate strongly with later participation in the study (Ethington, 1992; Wigfield & Eccles, 2000). The analysis of this relationship in the TIMSS data for Australian school students by Thompson and Fleming (2003) supports theorised connections between perceptions, participation and performance. Perceptions shape the information individuals attend to and how it is interpreted (De Bono, 2004). MacGregor (2004), Stacey and Chick (2004) attribute the decreasing participation of students in mathematics to be related to the interaction of three perceptions held by an increasing proportion of students that algebra is perceived as uninteresting; algebra is based upon symbolic manipulation with limited meaning and little relevance to everyday life; and algebra is perceived as difficult.

1.1 Background of the Study

Algebra is considered a course appropriately taken by students in the 8th grade (equivalent of Form 1) in many countries (although algebra is often not a separate course in other countries, but is integrated into the mathematics coursework). In the

Malaysian educational system, mathematics is a compulsory core subject in which all secondary school students must enrol, whereas, Additional Mathematics (a combination of algebra, trigonometry, geometry and pre-calculus) is an elective subject which is included in the pure science stream and the technical package. A list of algebra topics taught in Malaysian secondary schools mathematics syllabus shows much depth and breadth required in this area. Topics for Form 1 (Year 7 equivalent) are algebraic expressions 1, and patterns and sequences; topics for Form 2 (Year 8 equivalent) include algebraic expressions 2 (with two or more unknowns), linear equations 1 (with one variable), and patterns and sequences; topics for Form 3 cover algebraic formulae, linear equations 2, linear inequalities and graphs of functions; topics for Form 4 are quadratic expressions and equations, and the straight line (linear function); and topics for Form 5 include graphs of function 2, variations, and gradient and area under a graph (Ministry of Education, 2003, 2006; National Education Blue Print, 2006).

Algebra has traditionally been introduced when it was considered that students have acquired the necessary arithmetic skills. Besides, algebra has usually been developed separately from arithmetic without taking advantage of their strong link. "Usually in arithmetic (mathematics) we apply operations to numbers and obtain results after each operation; but in algebra, we usually do not start solving a problem using the given numbers, doing calculations with them, and obtaining a numeric result. In algebra, students have to identify the unknowns, variables and relations among them, and express them symbolically in order to solve the problem" (Martinez, 2002, p. 8). In its recent final report, the National Mathematics Advisory Panel (2008) in the United States outlined three fundamental elements in mathematics that students need to master in order to be fully prepared for exploring algebra. Those three critical elements are: (1) fluency with numbers, (2) fluency with fractions, and (3) particular aspects of geometry and measurement. Fluency, they said, means conceptual understanding and problem solving skills as well as computational fluency.

A fundamental requirement of algebra is an understanding that the equal sign indicates equivalence and that information can be processed in either direction (Kieran, 1981; Linchevski, 1995). It has been noted previously that many students' understanding of equals is action indication (e.g., "makes or gives" – Stacey &

MacGregor, 1997, p. 113) or syntactic (showing the place where the answer should be written – Filloy & Rojano, 1989). Misconceptions relating to equal concept make it difficult for students to transform and solve equations (Kieran, 1992; Linchevski & Herscovics, 1996).

Understanding the arithmetic operational laws (commutative, associative and distributive) and the order of convention for the operations are also important in algebra (Bell, 1995; Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 1998). For example, Kieran (1992) reported that some students read algebraic expressions from the left to right and ignored bracketing, a behaviour that indicates inability to apply the order of convention. The difficulties students who study algebra face without adequate arithmetic prerequisite knowledge can be easily seen in the following Year 9 task: "Solve for x: $2(x – 1) + 2 = − (4 – 3x)$". Completing this algebra task requires understanding of the equal concept, order of convention, operational laws and directed numbers.

Traditionally, many students have found the study of algebra difficult. Algebra is an abstract system in which interactions reflect structure of arithmetic (Cooper, Williams & Baturo, 1999). Its processes are abstract schemas (Ohlsson, 1993) or structure conceptions (Sfard, 1991) of the arithmetic operations, equals, and operational laws, combined with the algebraic notion of variable (Cooper, Boulton, Lewis, Atweh, Wilss & Mutch, 1997). Arithmetic does not operate at the same level of abstraction as algebra for, although they both involve written symbols and an understanding of operations (e.g., order of operations and inverse operations – Linchevski & Herscovics, 1994), arithmetic is limited to numbers and numeral computations (Sfard & Linchevski, 1994). Arithmetic and algebra differ fundamentally in that arithmetic computational procedures are separated from the object obtained (Linchevski & Herscovics, 1996). That is, students in arithmetic are not expected to conceive of groups of numbers and symbols as objects, whereas in algebra this is necessary.

In recent years, many research projects on mathematics education have focused on learning difficulties of students related to algebra. Different research studies as well as the experience in teaching mathematics show multiple difficulties students

encounter in learning algebra (Molina, Ambrose, & Castro, 2004), when the complexity and cognitive demand of school mathematics seem to increase, causing students to memorise rules without meaning or sense and to lose interest in mathematics. Research suggests that solutions to the problem of students' inability to be successful in algebra are many and frequently interconnected (Norton & Irwin, 2007). In order to overcome these difficulties and to ease transition to algebra many researchers propose an earlier introduction of algebra. They suggested some solutions which include: making explicit algebraic thinking inherent to arithmetic in children's earlier learning (Lin & Kaput, 2004; Warren & Cooper, 2006); explicit teaching of nuances and processes of algebra in an algebraic and symbolic setting (Kirshner & Awtry, 2004; Sleeman, 1986; Stacey & MacGregor, 1997, 1999) especially in ransformational activities (Kieran & Yerushalmy, 2004; Stacey & Chick, 2004) using multiple representations including the use of technology (Kieran & Yerushalmy, 2004; Van de Walle, 2006), and recognising the importance of embedding algebra into contextual themes (Stacey & Chick, 2004). Other research has shown that students' errors in algebra can be ascribed to fundamental differences between arithmetic and algebra. For instance, if students want to adopt an algebraic way of reasoning, they have to break away from certain arithmetical conventions and need to learn to deal with algebraic symbolism.

Understanding of algebra in school mathematics is one of the most important goals for mathematics education. On the other hand, algebra has been an obstacle and a challenge for many students. Yet we know that many students fail algebra in eighth grade, and many fail it in the ninth grade. Not only are these students unprepared for algebra, they also come to fear and dislike it (have a negative attitude towards mathematics) and to think of themselves as mathematically weak. Algebra is frequently taught as though it is unrelated to any prior mathematics that students have experienced (Carraher & Schliemann, 2007). Armstrong and Kahl (1979) concluded that one of the major reasons for students either continuing in mathematics or avoiding mathematics was their own perception about how good they are at mathematics. I have noticed that the attitudes of my students have changed as a result of this feeling of inadequacy and it is a major concern for me as an educator, researcher and mathematics teacher in particular.

"Despite massive effort, relatively little is accomplished by remediation programs. No one - not educators, mathematicians, or researchers – knows how to reverse a consistent early pattern of low achievement and failure" (Ball, 2003, p. 13). Thus, it is understandable why so many students give up learning mathematics if they keep on making mistakes and keep on experiencing learning difficulty. Why are students' misconceptions of such a "simple" concept such as the use of the "equal sign" which is a basic topic in elementary mathematics so robust, or resistant to change? Researchers found that even college students had trouble understanding and using the "equal sign" (Barcellos, 2005). Is it because of the characteristics of mathematics or something else?

1.2   Rationale of the Study

Learning of mathematics starts from when we begin to learn how to count. Then, we use mathematics in our everyday lives, sometimes without even realising. In these situations, what is needed to be learned – the 'basic numerical concepts' (Ansari, 2004) – was nothing but a way of expressing ourselves (in a language), in order to communicate with and relate to others. Mathematical skills develop as we grow and become involved in more and more activities, for example, measuring flour while making cakes or maybe rushing to "the sale" calculating (in our head) how much money could be saved. What we need in these occasions is mostly common sense, a practical approach towards obtaining a solution and some prior experience. Everyday mathematics does not require the 'brains' of an academic mathematician (Ansari, 2004). Sometimes it is surprising how people even with barely any formal education can deal with calculations so quickly.

Many children have well developed informal and intuitive mathematical competence before they start formal education (Clements & Sarama, 2004; Ginsburg, 2002; Ginsburg, Inoue, & Seo, 1999; Kilpatrick, Swafford, & Findell, 2001; Pepper & Hunting, 1998; Urbanska, 1993; Young-Loveridge, 1989). Children engage in all kinds of everyday activities that involve mathematics (Anderson, 1997), and consequently develop a wide range of informal knowledge (Baroody & Wilkins, 1999; Perry & Dockett, 2004). From infancy to preschool, children develop a base of

skills, concepts and understanding about numbers and mathematics. Perry and Dockett (2002) noted that "much of this learning has been accomplished without the assistance of formal lessons and with the interest and excitement of the children intact. This is a result that teachers would do well to emulate in our children's school mathematics learning" (p. 96). However, as children learn mathematics in a formal setting (the school curriculum), the sense they make of what they are presented with can differ from what the teachers might expect. The concepts can be counterintuitive and they do not understand the fundamental ideas or basic concepts covered in the mathematics class. Poincaré's statement below captures eloquently both the inextricable relation between mathematics and understanding, and the difficulty that learning mathematics entails:

> *How is it that there are so many minds that are incapable of understanding mathematics? Is there not something paradoxical about this? Here is a science which appeals only to fundamental principles of logic, to the principle of contradiction, for instance, to what forms, so to speak, the skeleton of our understanding, to what we could not be deprived of without ceasing to think, and yet there are people who find it obscure, and actually they are the majority.*
>
> (Poincaré, 1914, pp. 117-118)

While learning mathematics with understanding has increasingly received attention from mathematics educators and psychologists and has progressively been elevated to one of the most important goals of the mathematical education of all students, the realisation of this goal has long been problematic (Stylianides, & Stylianides, 2007). For too many people, mathematics stopped making sense somewhere along the way. Either slowly or dramatically, they gave up on the field as hopelessly baffling and difficult (McNeill, 1988), and they grew up to be adults who – confident that others share their experience – nonchalantly announce, "Math was just not for me" or "I was never good at it" (p. 45). Usually the process is gradual, but for McNeill (1988), the turning point was clearly defined as she described how it happened to her in an article in the Journal of Mathematical Behaviour:

What did me in was the idea that a negative number times a negative number comes out to a positive number. This seemed (and still seems) inherently unlikely – counterintuitive, as mathematicians say. I wrestled with the idea, for what I imagine for several weeks, trying to get a sensible explanation from my teacher, my classmates, my parents, anybody. Whatever explanations they offered could not overcome my strong sense that multiplying intensifies something, and thus two negative numbers multiplied together should properly produce a very negative result... Meanwhile, the curriculum kept rolling on, ...the book and the teacher and the general consensus of the algebra survivors of society were clearly more powerful than I was. I capitulated.

(McNeill, 1988, p. 45)

Imagine a person who has difficulty in understanding and a dislike of mathematics as described above. Somewhere in the past is a negative experience with the subject – it may have been a single event, or it may have been a series of negative events. The nature of the experience is not clear, but what is clear is the impact that it had on him as a learner of mathematics. He feels that he 'was never good at mathematics' or that he 'can't do the mathematics'. Perhaps he is even afraid of the subject, suffering anxiety at the thought of having to endure the mathematics course (Liljedahl, 2005).

Such a person is not difficult to imagine. I encounter such individuals on a regular basis. They are the students of my mathematics class over the span of my more than 20 years of teaching mathematics in secondary schools. I have noticed that many students at the school have problems with the understanding of mathematical concepts and symbols. They are able to operate with them but they cannot tell what they are doing, why they are doing certain procedures and what is the meaning with mathematical symbolism (Sierpinska, 1994; Attorps, 2003, 2005). Often students would comment to me in the mathematics lesson that algebra "is just letters isn't it (and what those letters mean causes students confusion when they do not realise that the letters represent numbers)? They would describe themselves as either being incapable of doing mathematics [especially algebra], or having a phobia about the learning of mathematics, or both. For many of them mathematics has long been a

subject devoid of wonder, surprise, and discovery (Liljedahl, 2005). As a researcher I became interested in what such a population would describe themselves as incapable of doing mathematics, giving up on mathematics as hopelessly baffling and difficult and having a negative perception on mathematics. Many of the low achieving students made essentially no improvement, in spite of the fact that I paid special attention to their learning. They were still not good at solving even simple problems or understanding certain fundamental concepts. I found this a common phenomenon and with my colleagues believed that some students were born with certain innate ability in mathematics while others lacked such innate ability in mathematics and we thought that the abstractness of mathematics was the problem that causes the learning difficulties. It was when I was doing my research on difficulties and misconceptions in mathematics education that I discovered that there are other explanations for this malaise. That is, the low achieving students only memorised a few facts, formulas, and algorithms, without any deep understanding of them. The lack of understanding prevented them from applying mathematics knowledge to new contexts in a flexible way.

But, what is "understanding and how to improve mathematics understanding of the students?" Hiebert and Carpenter (1992) define understanding as making connections. To understand a new concept means to construct a relationship between the new concept and the old conceptual network and ways to achieve understanding are through reflecting and communicating, working on proper problems, or communicating with partners. Hiebert and Carpenter (1992), however, highlighted the importance and difficulties of promoting students' understanding as "one of the widely accepted ideas within the mathematics education community is the idea that students should understand mathematics. The goal ... to promote learning with understanding. ... but designing school learning environments that successfully promote understanding has been difficult" (p. 65).

The aim of mathematics education is surely to promote understanding and the success of all students, yet it seems to be a fact of life that whilst a few students are successful in mathematics, a much greater number find mathematics difficult. However hard teachers may try, there are students who begin to struggle and who will need appropriate help to be able to pursue mathematics further. The objective of

any mathematics curriculum includes fostering favourable feelings towards mathematics as well as imparting cognitive knowledge. Yet, the general belief and common saying among secondary school students that only the exceptionally brilliant students can successfully learn mathematics leaves much to be desired. This is because such belief has not only affected students' attitude towards mathematics but also their cognitive achievement or understanding in the subject. Many researchers are unanimous in the submission that secondary school students often show negative attitudes towards mathematics (Mullis et al., 2000), and that such negative attitudes often result in lack of interest in the subject, which consequently results in poor cognitive outcomes in mathematics assessment (Ma & Kishor, 1997). As a result, researchers in mathematics education have emphasised and recommended the use of diverse methods (Ansari, 2004) of teaching mathematics to promote learning with understanding and as a means of promoting positive attitudes towards the subject.

In Malaysia, there are few studies available about students' difficulties and misconceptions in algebra concepts and attitudes towards mathematics. There is a need for scientific studies of students' knowledge, understanding, difficulties, misconceptions and attitudes in algebra in the Malaysian context. The present research project is focussed on finding out ways in which learning can be improved, on how better learning outcomes can be achieved in algebra in secondary schools in Malaysia and on helping teachers develop relevant teaching practices.

It will be interesting to determine if the different kinds of difficulties and misconceptions (errors) are relevant to Malaysian students. Learners make mistakes (errors) for many reasons: 1) due to lapses in concentration; 2) hasty reasoning; 3) failure to notice important features of a problem; and 4) others are symptoms of more profound difficulties. According to Liu (2010), misconceptions are systematic patterns in errors in interpreting, understanding or applying mathematical concepts. These conceptual errors are responsible for students' mistakes. Misconceptions are a result of consistent and alternative interpretation of mathematical ideas. Errors are mistakes made due to lack lapses of concentration, hasty reasoning or failure to notice important features of the problem. It is expected that the analysis used in this study would help to identify the kinds of wrong strategies that lead to incorrect

responses. Hopefully, such identification could provide teachers with some definite guidance for remedial education of the students when such errors occur.

The difficulties and misconceptions students have (as mentioned above) on different aspects (concepts) of algebra are widespread among my students and because of these difficulties and misconceptions I would like to investigate what they know and subsequently teach in a way that may improve their understanding of algebra. To this end, I have been developing cognitive conflict problems as tools to elicit and probe students' understanding of algebra in an intervention strategy called diagnostic teaching. Knowledge of the common algebra errors and misconceptions of children can provide teachers with an insight into student thinking and a focus for teaching and learning (Bell, Swan, Onslow, Pratt, & Purdy, 1985; Black & William, 1988; Hart, 1981; Schmidt et al, 1996; Stigler & Hiebert, 1999). A social constructivist view of learning suggests that errors are ripe for classroom consideration; via discussion, justification, persuasion and finally even change of mind, so that it is the student who reorganises his or her own conception (Cobb, Yackel & McClain, 2000; Ryan & Williams, 2003; Tsamir & Tirosh, 2003). As Chan, Burtis, and Bereiter (1997) pointed out, "a common approach fostering conceptual change is based on a conceptual conflict strategy". The usual cognitive paradigm involves: a) identifying students' current state of knowledge; b) confronting students with contradictory information which is usually presented through texts and interviewers, making explicit the contradictions or only guiding the debate with the students or among peers (small groups or the whole classroom (i.e. Dreyfus, Jungwirth, & Eliovitch, 1990; Treagust, 2007), or by the teacher and new technologies; and c) evaluating the degree of change between students' prior ideas or beliefs and a post-test measure after the instructional intervention. Often, conflict is induced by presenting information that clearly – for the experimenter or for the teacher – contradicts children's or students' ideas beliefs or theories. This present study intends to find out whether diagnostic teaching in the framework of conceptual change in a cooperative instructional methodology could enhance conceptual understanding, achievement and positive attitudes of lower secondary school students towards mathematics [algebra].

1.3   Research Questions

The focus of this study lies in students' difficulties, conceptions and attitudes towards learning algebra in the framework of conceptual change. In order to gain information on the relevance of these areas, the research questions that will guide this study are:

1.   Are there learning gains in understanding algebra concepts evident after the six weeks intervention?
2.   What conceptual difficulties and misconceptions do Form 2 Malaysian students have with algebra?
3.   Is there any evidence of students' conceptual change in algebra concepts following the teaching intervention (diagnostic teaching)?
4.   Are students' attitudes towards algebra enhanced after the six weeks intervention?

1.4   Significance

The important role that algebra plays is acknowledged by most educators and educationists, but it is a fact that the inherent nature of algebra concepts is quite complex. As already stated, most students find it difficult to understand these concepts. For many years teachers have been seeking for answers why students have misconceptions and what they can do to help them to develop mathematically correct conceptions.

In order to provide learning experiences that students can use to develop understanding of important algebra concepts, teachers must learn what students understand, and they must be sensitive to possible misconceptions held by the students. The learning experiences should allow students to confront their misconceptions and build upon knowledge they already understand. If students are unfamiliar with vocabulary, symbols, representations, or materials, the meaning students gain might differ significantly from what the teacher intends (Hiebert et al., 1997). Simply lecturing to students on a particular topic will not help most students

give up their misconceptions. Since students actively construct knowledge, teachers must actively help them dismantle their misconceptions. Teachers must also help students reconstruct conceptions capable of guiding their learning in the future.

The identification of students' difficulties and misconceptions in algebra ought to be of significance for it might lead to a better understanding of the students' thought processes and the quality of learning that takes place. By knowing students' misconceptions or alternative conceptions of algebra, teachers can gain a greater insight into the subject matter of the topic, their teaching, and the learning process of the students. They are likely to be more receptive and willing to try or develop alternative teaching strategies if they find their present methods are inadequate in addressing students' difficulties. The identification of algebra misconceptions can create worthwhile opportunities to enhance learning and steps can be taken by teachers to help students to remove these misconceptions and providing better understanding of algebra concepts. Knowledge of misconceptions can be employed in planning more effective teaching strategies and methods. It can also be used to present richer learning experiences to students and the researcher can assist other teachers in teaching algebra in a more effective way.

1.5   Summary

This thesis contains five chapters – Rationale of the Study, Literature Review, Methodology, Findings and Discussion, and Conclusions.

Chapter 1 has presented the different descriptions of mathematics, mathematics learning and its achievement and difficulty, and students' attitudes to mathematics in general. Also contained in Chapter 1 is the background of this thesis, a rationale for the thesis, the statement of research questions, the significance of the study and the limitations of the study.

Chapter 2 will review literature relevant to this study. This literature will focus on previous studies that are related to the current study, and also on literature relevant to each research question proposed in Section 1.3. The literature will attempt to show

the relevance of the current study and to substantiate or negate information gained in previous studies.

Chapter 3 will describe the research methodology incorporated in this study. This will be determined by using both quantitative and qualitative data instruments that were adapted (developed) as a means of eliciting succinct answers to each of the research questions. As these instruments were either new or amended from previous instruments, a description of their make-up, and the criteria used in their formation will be described. Additionally, a description of the student population used in this study will be found in this chapter. Care will be given to maintaining student and school anonymity through the use of pseudonyms for students and school. Finally, this chapter will provide details of methods used to gain data and a procedural description of the data analysis.

Chapter 4 which contained the research findings will analyse the data collected and report salient findings determined from the study. The researcher will use data tables, diagrams, charts and statements to effectively present these findings.

The conclusion of the thesis addressing the four research questions is contained in Chapter 5. The first section of this chapter will give a brief overview of the background, objectives, and methodology of this study. The second section of this chapter will relate the major findings of this study by addressing each of the original research questions. Section three will address the implications of this study and will attempt to show the relevance that this study has on the improvement of mathematics teaching and learning, and specifically on teaching algebra. Section four will describe the limitations and implications of this study. Finally, section five will give suggestions for further research.

Chapter 2

Literature Review

In their research, mathematicians generally faced problems nobody knows how to solve (on the other hand, pupils face problems which they believe their teachers can solve). To solve their problems, mathematicians are led to create concepts as tools.

(Douady, 1985, p. 35)

2.0   Introduction

The literature review serves to provide a theoretical and methodological framework for this study on Form Two students' difficulties, conceptions and attitudes of algebra: An intervention study to improve the teaching and learning of algebra. There are five main sections in this literature review. The first section describes students' conceptions and the nature of alternative conceptions; the second section discusses previous research in students' difficulties and misconceptions of algebra concepts; the third section outlines research relating to attitudes in mathematical learning; the fourth section introduces and describes the conceptual change approach to mathematics teaching and learning; and the final section discusses diagnostic (conflict) teaching using a cooperative learning methodology as an intervention strategy for conceptual change.

2.1   Students' Conceptions and Nature of Alternative Conceptions

Ausubel, in the preface to his book, Educational Psychology: A Cognitive View, says that "If [he] had to reduce all of educational psychology to one principle, [he] would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel, 1968, p. vi). This widely shared principle in educational psychology stated by Ausubel (1968) posits that the most important factor to influence learning is the student's previous

knowledge. The acceptance of this principle resulted in greater interest within the mathematics education research in exploring students' conceptions (Confrey, 1990).

Since the time of Piaget, researchers have been keenly interested in how students view concepts of science and mathematics (Confrey, 1990). Interest in student conceptions surged with the emergence of constructivism (Osborne, 1996; Solomon, 1994), which brought along with it "a language with new descriptive power" (Solomon. 1994, p. 6). This language, Solomon argues, transmuted the study of common student mistakes, previously of little appeal to anyone, to something exciting and of great interest.

Essential to the constructivist view is the recognition that learning is viewed as a process of active construction which is shaped by students' prior knowledge and conceptions. Many researchers agree that the most significant things that students bring to class are their conceptions (Ausubel, 1968, 2000; Driver & Oldham, 1986). This current cognitive structure of the students composing a set of abstract ideas, concepts and generalisation builds upon facts is the organisation of the students' present knowledge in a subject (Confrey, 1990). The students' previously stored knowledge or ideas play an important role when teaching new concepts (Dochy & Bouwens, 1990; Goss, 1999). During instruction students generate their own meaning based upon their experience and abilities (Nakhleh, 1992). For meaningful learning to occur, new knowledge must be related by the student to the relevant existing concepts in the students' cognitive structure (Ausubel, 1968, 2000). Teachers can be astonished to learn that despite their best efforts, students do not understand fundamental ideas or basic concepts covered in mathematics class. Some of the students give the right answers but these are only from correctly memorised algorithms. Students are often able to use algorithms to solve numerical problems without completely understanding the mathematical concepts (West & Fensham, 1974).

Concept means in everyday life 'a term' or 'a word' (Bolton, 1972, p. 23; Nelson, 1985). Concepts are described both in older and newer sources in a similar way. For instance, it is said, that "concepts are perceived regularities or relationships within a group of objects or events and are designated by some sign or symbol" (Heinze-Fry

& Novak, 1990, p. 461). Concepts can be considered as ideas, objects or events that help us understand the world around us (Eggen & Kauchak, 2004). Concepts that are embedded in mental structures are defined by Cohen, Manion and Morrison as follows:

> *Concepts enable us to impose some sort of meaning on the world; through them reality is given sense, order and coherence. They are the means by which we are able to come to terms with our experience.*
>
> (Cohen, Manion & Morrison, 2000, p. 13)

Each student constructs and reconstructs a wide range of complex, integrated, idiosyncratic, and epistemologically legitimate conceptions on an ongoing basis as s/he negotiates his/her classroom experience (Confrey, 1990). Therefore, students' real world conceptions play a critical role in their view of the world (Novick & Nussbaum, 1982). So, what is a conception? The term 'conception' itself must be explained since it is widely used in science and mathematics education but with very different meanings (Kaldrimidou & Tzekaki, 2006).

Conceptions are systems of explanation (White, 1994). Glynn and Duit (1995) viewed conceptions as learner's mental models of an object or an event. Duit and Treagust (1995) however define conceptions as "the individual's idiosyncratic mental representations", while concepts are "something firmly defined or widely accepted" (p. 47). Consequently, Kattmann, Duit, Gropengießer and Komorek (1996) define conceptions "as all cognitive constructs which students use in order to interpret their experience" (p. 182). These constructs are located on different epistemological levels of complexity, comprising for example concepts, intuitive rules, thinking forms, and local theories (Gropengießer, 2001; Prediger, 2008).

Osborne and Wittrock (1983) summarised student conceptions succinctly in their statement that "children develop ideas about their world, develop meanings for words used in science [mathematics], and develop strategies to obtain explanations for how and why things behave as they do" (p. 491). Children develop these ideas and beliefs about the natural world through their everyday experiences. These include sensual experiences, language experiences, cultural background, peer groups,

mass media as well as formal instruction (Duit & Treagust, 1995). Some of these ideas and beliefs such as those about "the failure to accept the possibility of dividing a smaller by a larger number, and the assumption that multiplication always makes bigger and division smaller" (Bell, 1986, p. 26) may be similar across cultures as children have very similar personal experience with phenomena. These categories of children's beliefs, theories, meanings, and explanations will form the basis of the use of the term student conceptions. Simply stated, "conceptions can be regarded as the learner's internal representations constructed from the external representations of entities constructed by other people such as teachers, textbook authors or software designers" (Treagust & Duit, 2008, p. 298).

Students' conceptions are critical to subsequent learning in formal lessons because there is interaction between the new knowledge that the students encounter in class and their existing knowledge. Research with infants has shown that the process of constructing naïve physics starts soon after birth. By the time children go to school they are deeply committed to an ontology and causality that distinguishes physical from psychological objects and which forms the basis for the knowledge acquisition process (Carey, 1985; Vosniadou, 1994; 2001). Naïve physics facilitates further learning when the new, to be acquired information is consistent with existing conceptual structures. In the learning of mathematics, students also develop a "naïve mathematics" (Vosniadou, 2001) on the basis of everyday experience, which appears to consist of certain core principles or presuppositions (such as the presupposition of discreteness in the number concept) that facilitate some kinds of mathematical learning but may inhibit others (Gelman, 2000).

Johnstone (2000) states that when a person tries to store material in long term memory and cannot find existing knowledge with which to link it, s/he may try to 'bend' the knowledge to fit somewhere, and this gives rise to erroneous ideas (Gilbert, Osborne & Fensham, 1982). When people place instances in sets, or concepts, different from those determined by the community of scientists, the term misconceptions (Nesher, 1987; Perkins & Simmons, 1988) is used. Other researchers use other terms for these constellations of beliefs (Confrey, 1990), including the following: alternative frameworks (Driver, 1981; Driver & Easley, 1978), student conceptions (Duit & Treagust, 1995), preconceptions (Ausubel, Novak, & Hanesian,

1978), alternative conceptions (Abimbola, 1988; Hewson, 1981), intuitive beliefs (McCloskey, 1983), children's science (Osborne & Freyberg, 1985), children's arithmetic (Ginsburg, 1977), mathematics of children (Steffe, 1988), naïve theories (Resnick, 1983), conceptual primitives (Clement, 1982) and private concepts (Sutton, 1980). Mathematics education researchers have conceptualised these conceptions as a 'belief system' (Frank, 1985), as a 'network of beliefs' (Schoenfeld, 1983), as a 'mathematical world view' (Silver, 1982) and as 'conceptions of mathematics and mathematical learning' (Confrey, 1984). Although different authors may have good reasons for preferring particular terms in the context of particular studies, there is no generally agreed usage across the literature (Taber, 2009). There is currently a large body of research literature which has been summarised in articles by Duit (2009).

2.1.1 Nature and Significance of Students' Conceptions

Students' existing ideas are often strongly held, resistant to traditional teaching and form coherent though mistaken conceptual structures (Driver & Easley, 1978). Rather than being momentary conjectures that are quickly discarded, misconceptions consistently appear before and after instruction (Smith, diSessa, & Roschelle, 1993). Students may undergo instruction in a particular topic, do reasonably well in a test on a topic, and yet, do not change their original ideas pertaining to the topic even if these ideas are in conflict with the concepts they were taught (Fetherstonhaugh & Treagust, 1992). Duit and Treagust (1995) attribute this to students being satisfied with their own conceptions and therefore seeing little value in the new concepts. Therefore, it is difficult for students to change their thinking. Another reason the authors proposed was that students look at the new learning material "through the lenses of their preinstructional conceptions" (p. 47) and may find it incomprehensible. Osborne and Wittrock (1983) state that students often misinterpret, modify or reject scientific viewpoints based upon the way they really think about how and why things behave, so it is not surprising that research shows that students may persist almost totally with their existing views (Treagust, Duit, & Fraser, 1996). When the students' existing knowledge prevails, the scientific concepts are rejected or they may be misinterpretation of the concepts to fit or even support their existing knowledge.

As Clement (1982a) has shown in elementary algebra where college students make the same reversal error in translating multiplicative reasoning relationships into equations (e.g., translating "there are four people ordering cheesecake for every five people ordering strudel" into "4C = 5S"), whether the initial relations were stated in sentences, pictures, or data tables. In domains of multiplication (Fischbein, Deri, Nello, & Marino, 1985), probability (Shaughnessy, 1977), and algebra (Clement, 1982a; Rosnick, 1981), misconceptions continue to appear even after the correct approach has been taught. Sometimes misconceptions coexist alongside the correct approach (Clement, 1982a). Such results are compatible with the conceptual theoretical framework, which predict difficulties in learning when the new knowledge to be acquired comes in conflict with what is already known (Vosniadou, 1994). If the concepts are accepted, it may be that they are accepted as special cases, exceptions to the rule (Hashweh, 1986), or in isolation from the students' existing knowledge, only to be used in the classroom (de Posada, 1997; Osborne & Wittrock, 1985) and regurgitated during examinations. Additional years of study can result in students acquiring more technical language but still leave the alternative conceptions unchanged (de Posada, 1997).

In the mathematics education research, there has been much evidence to show that prior knowledge about natural numbers stand in the way of understanding rational numbers. Students make use of their knowledge of whole numbers, to interpret new information about rational numbers (Moskal & Magone, 2000; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989; Vamvakoussi, & Vosniadou, 2004). This gives rise to numerous misconceptions pertaining to both conceptual and operation aspects of numbers. For example, properties of natural numbers such as "the more digits a number has the bigger it is" are used in the case of decimals (Stafylidou & Vosniadou, 2004). And in the context of mathematical operations, the well known misconception such as "multiplication always makes bigger" reflects the effects of prior (existing) knowledge about multiplication with natural numbers (Fischbein, Deri, Nello, & Marino, 1985).

These preconcepts are tenacious and resistant to extinction (Ausubel, 1968, 2000); deep seated and resistant to change (Clement, 1987); and consistent across individuals, very strongly held and not easily changed by classroom instruction

(McCloskey, 1983). Smith et al. (1993) opine that students can "doggedly hold onto mistaken ideas even after receiving instruction designed to dislodge them" (p. 121). This persistence does not necessary mean that instruction has failed completely. However, it is vital to acknowledge that misconceptions because of their strength and flawed content can interfere with learning expert concepts. Research has demonstrated the persistence of student misconceptions and the tendency for regressing to these preconceptions even after instruction to dislodge them. Mestre (1989) reports "students who overcome a misconception after ordinary instruction often return to it only a short time later" (p. 2).

2.1.2 Genesis of Students' Alternative Conceptions

Arguably the most general source of conceptions is limited experience or exposure to limited examples - both in learning outside of school and during formal instruction (Greer, 2004). Alternative conceptions may arise when students are presented with concepts in too few contexts or when concepts presented are beyond their developmental level (Gabel, 1989). It may be that, when first introduced to a new exposition of a scientific idea, students have not yet attained as high a level of abstract thinking as the instructors assume. Perhaps instructors provide part of the concept or a more simplified concept in the belief this will lead students to better understanding, as sometimes happens when the concept is difficult or known to be troublesome. Such limited explanation, however, may prevent some students from crossing a cognitive threshold and entering through a door to a higher level of understanding (Liljedahl, 2005). Other possible explanations could be both the pace at which the algebra concepts are covered and also the formal approach often used in its presentation. It seems that many teachers and textbook authors are unaware of the serious cognitive difficulties involved in the learning of algebra. As a result, many students do not have the time to construct a good intuitive basis for the ideas of algebra or to connect these with the pre-algebraic ideas that they have developed. They fail to construct meaning for the new symbolism and are reduced to performing meaningless operations on symbols they do not understand (Herscovics & Linchevski, 1994).

Within the domain of algebra, Kieran (1992) contended that "one of the requirements for generating and adequately interpreting structural representations such as equations is a conception of the symmetric and transitive character of equality – sometimes referred to as the 'left-right equivalence' of the equal sign" (p. 398). Yet, there is abundant literature that suggests students do not view the equal sign as a symbol of equivalence (i.e., a symbol that denotes a relationship between two quantities), but rather as an announcement of the result or answer of an arithmetic operation (e.g., Falkner, Levi, & Carpenter, 1999; Molina & Ambrose, 2008). It has been suggested that this well documented (mis)conception might be due to students' elementary school experiences (Carpenter, Franke, & Levi, 2003; Seo & Ginsburg, 2003), where the concept of equality and its symbolic instantiation are traditionally introduced during students' early elementary school years, with little instructional time explicitly spent on the concept in the later grades and the equal sign was nearly always presented in the operations equals answer context (e.g., $2 + 5 = \_$).

Another source of confusion is the different meaning of common words in different subjects and in everyday use. This applies not only to words, but also symbols. Harrison and Treagust (1996) reported that students were confused between the nucleus of an atom and the nucleus of a cell, and that one student actually drew a cell for an atom. They also found students having alternative conceptions about electron clouds, and cautioned that teachers need to qualify the sense in which they transfer the attributes of the analog to the target. Many scientifically associated words are used differently in the vernacular (Gilbert & Watts, 1983), for example, energy has a cluster of 'life-world' associations that do not match its technical use (Solomon, 1992). Likewise, Tall and Thomas (1991) indicate that, in the natural language 'and' and 'plus' have similar meanings. Thus, the symbol 'ab' is read as 'a and b' and interpreted as '$a + b$'.

Schmidt has pointed out how linguistic cues in scientific terminology may lead to inappropriate inferences being drawn (Schmidt, 1991). In the same way, verbal cues – certain words such as 'more', 'times', reduction' have strong associations with particular operations, and these sometimes "displace the true meanings of the phrases in which they appear" (Bell, Swan & Taylor, 1981). For example, "the milkman brought 11 bottles of milk on Sunday. That was four more than he brought on

Monday. How many bottles did he bring on Monday?" Here, the word 'more' was shown to greatly increase the likelihood of addition being chosen as the correct operation to perform (Anderson, Reder & Simon, 2000; Nesher & Teubal, 1975).

The problem raised by Fischbein (1987, p. 198) is that: A certain interpretation of a concept or operation may be initially very useful in the teaching process as a result of its intuitive qualities (concreteness, behavioral meaning, etc.). But as a result of the primacy effect that interpretation may become so rigidly attached to the respective concept that it may become impossible to discard it later on. The initial interpretation of a concept may become an obstacle which can hinder the passage to a higher-order interpretation - the more general and more abstract – of the same concept. For example, an early understanding of natural number and its properties supports children's understanding of notions such as potential infinity (Hartnett & Gelman, 1998), while at the same time it stands in the way of students' understanding of the properties and operations of rational numbers (Moskal & Magone, 2000; Yujing & Yong-Di, 2005). There is a great deal of evidence showing that students at various levels of instruction make use of their knowledge of natural number to conceptualise rational numbers and make sense of decimal and fraction notation, often resulting in making systematic errors in ordering, operations, and notation of rational numbers. Many researchers attribute these difficulties to the constraints students' prior knowledge about natural numbers imposes on the development of rational number concept.

McDermott (1988) suggests that some alternative conceptions may arise from failure to integrate knowledge from different topics and from concept interference that comprises "situations where the correct application of a conception by students is hindered by their misuse of another concept that they have learned" (p. 539). This occurs when students do not have an adequate conceptual framework to know which concept to apply in a situation. Concept interference may also be due to set effects (Hashweh, 1986) where certain knowledge or conceptions are brought to mind due to strong ties with certain features of a given situation through previous experience. For example, students may have as part of their concept image for subtraction that when things are subtracted the numbers become smaller. This could become a conflict when they are later introduced to negative numbers because when they are subtracted

the numbers become larger (Gallardo, 1995, 2002; Vlassis, 2001)). Additionally, student conceptions of the associative and commutative properties of numbers think that subtraction like addition, is commutative, that order does not affect the answer (Bell, Greer, Grimison, & Mangan, 1989; Brown, 1981), so 7 – 3 and 3 – 7 are the same, or rather they have the same answer. According to Olivier (1984), the main contributory factor for seeing subtraction as commutative is probably that students have extensive experience of the commutativity of addition and multiplication when learning tables. They are over-generalising over operations.

Booth (1984), Chalouh and Herscovics (1988) refer to the students' frame of reference or the context of a problem on the answer that the student chooses and also the uncertainty about what is required as the source of the misconception. For example, students may often hear teachers say that "you can't add unlike terms" but then in a test are instructed: "Add 4 onto 3n" (Kuchemann, 1981, p. 108). Since these are unlike terms, how can they be added? Similarly, when told "x is any number: "Write the number which is 3 more than x" (Bell, Costello & Kuchemann, 1983, p. 138), the large number of students who gave *x* an arbitrary value and added 3 to it may have thought that since *x* is "any number" that is what they are supposed to do. Other wrong responses reflect students' assumption, developed from arithmetic, about what an acceptable "answer" should look like (e.g., that it should not contain an operation sign).

Teachers need to be aware that they also can be the sources of alternative conceptions. When teachers have the same alternative conceptions as their students (Wandersee, Mintzes, & Novak, 1994), they think that there is nothing wrong with their students' conceptions. Consequently, teachers can unwittingly pass their own alternative conceptions to their students, and the way they teach, for instance, using imprecise terminology, can also cause confusion (Chang, 1999; Lin, Cheng, & Lawrenz, 2000). The incorrect direct transliteration of verbal statements into algebra, where $3a + 4b$ could be derived from 3 apples and 4 bananas (irreverently known as "fruit-salad algebra"), has been shown to be a frequent source of error (MacGregor & Stacey, 1997b; Kuchemman, 1981), and is encouraged by some algebra texts.

In research conducted by MacGregor and Stacey (1997a), they found that a new misinterpretation of algebraic letters may be caused by students misinterpreting teachers' explanations. Students sometimes misinterpret $'x'$ is any number. It appears that when teachers stress that "x without the coefficient is $1x$", they may also be misunderstood, since some students interpret this to mean "x by itself is one". Another possible source of confusion is the fact that in the context of indices the power of x is 1 if no index is written (i.e., $x = x^1$). When interviewed, several Year 10 students explained, "By itself x is 1, it's because it hasn't got a number".

Thus, teachers must be aware that they "cannot assume that what is taught is what is learned" (Driver & Scott, 1996, p. 106). Gray and Tall (1994) underlie the fact that the same notation may be viewed as signifying a process or an object, so that, for example, a teacher may offer a representation of the function $y = 2x + 3$ as an example of a linear function, but the learner may see it as an example of a procedure (for drawing a graph from an equation) (Bills, Dreyfus, Tsamir, Watson, & Zaslavsky, 2006).

Given the number of students taught over a teaching career, the generation of alternative conceptions can be quite significant. Teachers should realize that textbooks also can contain errors and misleading or conflicting illustrations and statements which can give rise to alternative conceptions (Boo, 1998; de Posada, 1999); hence, textbook should not be regarded as infallible.

Teachers need to know their students' alternative conceptions in order to help them lower the status of these conceptions in favour of the accepted algebra concepts. Unfortunately teachers are often unaware of their students' alternative conceptions (Treagust et. al., 1996), which is why Posner, Strike, Hewson and Gertzog (1982) maintain that teachers "should spend a substantial portion of their time diagnosing errors in thinking and identifying moves used by students to resist accommodation" (p. 226).

The above literature suggests that students' conceptions play an important role in their ongoing understanding and learning in formal lessons. That some of these conceptions can be stable, widespread among students, can be strongly held, resistant

to change, and can interfere with learning expert concepts causing students difficulties in their learning of certain algebra concepts. The purpose of the following section is to examine not only the inhibiting interference from these conceptions but also the other inhibiting interferences causing students difficulties in the learning of algebra. Relevant literature related to typical students' difficulties and misconceptions is reviewed and discussed in greater detail so as to have a better understanding of the problems faced by the students.

2.2 Students' Difficulties and Misconceptions of Algebra Concepts

2.2.1 Typical Students' Learning Difficulties

Teachers, mathematics educators, and mathematicians consider algebra to be one of the most important areas of school mathematics. Despite the importance placed on algebra in school mathematics curricula, many students find it abstract and difficult to comprehend (Witzel, Mercer, & Miller, 2003). Algebra concepts include unknowns and variables, expressions and equations, and the expansion of the meaning given to the equal and minus signs (Kieran, 2007). Algebra deals with expressions with symbols and the extended numbers beyond whole numbers in order to solve equations, to analyse functional relations, and to determine the structure of the representational system which consist of expressions and relations (Lew, 2004).

Centeno (1988) points out that "a difficulty is something that inhibits the student in accomplishing correctly or in understanding quickly a given item. Difficulties may be due to several causes: related to the concept that is being learned, to the teaching method used by the teacher, to the student's previous knowledge, or to his ability" (p. 142). Other possible explanations for the difficulties that students experience with algebra involve the lack of the basic knowledge needed for a correct understanding of a given concept or procedure; the pace at which the algebra concepts is covered and also the formal approach often used in its presentation (Herscovics & Linchevski, 1994). According to Kieran (1992, 2003), students' learning difficulties are centered on the meaning of letters, the change from arithmetical to algebraic

conventions, and the recognition and use of structure. Wagner and Parker (1993) agree with Kieran that most obstacles inherent to algebra stem from notational conventions or the complexity of concepts that arise with the use of letters as variables. Some of these problems are amplified by teaching approaches. The teaching methods used to convey content often exacerbate these algebra learning barriers, possibly becoming a unique barrier themselves (Leitzel, 1989; Rakes, Valentine, McGatha, & Ronau, 2010). Teaching methods that focus on skill or procedural levels rather than relational understanding of abstract mathematical ideas (often requires a lengthy, iterative process) are often insufficient for helping students understand the abstract, structural concepts necessary for supporting the demonstrated procedural activities in algebra (Kieran, 1992; Rakes et al., 2010). As a result, many students do not have the time to construct a good intuitive basis for the ideas of algebra or to connect these with the pre-algebraic ideas they have developed in primary school; they fail to construct meaning for the new symbolism and are reduced to performing meaningless operations on symbols they do not understand (Herscovics & Linchevski, 1994; Drijvers, Goddijn, & Kindt, 2011).

Various studies (e.g. Lithner, 2000; Maharaj, 2005; Mason, 2000) have focused on the teaching and learning of school mathematics. These studies have indicated some important sources of students' difficulties in mathematics. Particular studies have focused on the equal sign (e.g., Carpenter et al, 2003; McNeil & Alibali, 2005), literal symbols or variables (e.g., Drijvers et al., 2011; Schoenfeld & Arcavi, 1988), graphing (e.g., Nemirovsky, 1994; Romberg, Fennema, & Carpenter, 1993), and equations and inequalities (Bazzini & Tsamir, 2001; Boeno, Bazzini, & Garuti, 2001). These studies identify the "complexities that learners experience but that the curriculum – with its focus on algorithms or solution methods – does not address explicitly" (Chazan & Yerushalmy, 2003, p. 125). The purpose of this section was to review relevant literature related to some of the important sources and types of difficulties and misconceptions students encounter in the teaching and learning of algebra.

2.2.1.1   Abstract reasoning and different approach to problem solving

Students beginning the study of algebra face learning challenges that form a general foundational set of understandings necessary to negotiate through several complex topics with multiple sources of difficulty (Rakes et al., 2010). Algebra is often the first course in which students are asked to engage in abstract reasoning (that is, the ability to abstract common elements from situations, to conjecture, and to generalise) and problem solving (Vogel, 2008). For the students, all the previous mathematical problems had nothing that varied – they were presented with some numbers and operator (a plus, minus, multiplication, or division sign) and had to come up with the answer. With algebra, the situation is much more subtle. Instead of simply mathematical problems, students are presented with expressions and equations. Now the students must not only calculate variables, but they must also determine which operators to use (Sinitsky & Ilany, 2009) and determine the right order of computation to a written expression such as $2 + 3 \times 5 \, or \, p + m \times a$ because there is a need for a convention about the order of operation in algebra. How to overcome this paradigm shift? How to overcome the natural difficulties encountered when transitioning from concrete arithmetic to more abstract mathematics? It seems to the students as "a switch from four arithmetic operations with numeric operands into terra incognita of some quantities that are both unknown and tend to change" (Sinitsky & Ilany, 2009, p. 640).

Bednarz, Radford, Janvier, and Lepage (1992) note that "students have difficulty acquiring and developing algebraic procedures in problem-solving" (p. 65); while Geary (1994) concludes, "Most students find the solving of algebraic-word problems a cumbersome task" (p. 127). The obstacles are considered two-fold. Firstly "students encounter major difficulties in representing word problems by equations" (Herscovics, 1989, p. 63). "Generating equations to represent the relationships found in typical word problems is well known to be an area of difficulty" (Kieran, 2007, p. 721) where students are required to engage in abstract algebraic reasoning (Vogel, 2008). Similarly, Lochhead and Mestre (1988) write: "It is well known that word problems have traditionally been the nemesis of most mathematics students. The translation process from words to algebra [equation] is perhaps the most difficult step in solving word problems" (p. 134).

Studies have shown that many students are unable to write, simplify and solve algebraic equations using substitution, the order of operations, the properties of operations and the properties of equality (Bottoms, 2003). Numerous studies have also shown that when working on word problems, many students do not analyse the situation modeled in the problem when determining which operation to apply. Instead they select that operation by guessing, by trying all operations and choosing one that gives what seems to them a reasonable answer, or by studying such properties as the size of the numbers involved (Graeber & Tanenhaus, 1993).

There is considerable evidence that students use informal problem solving methods (Carpenter & Moser, 1981; van Amerom, 2003) instead of the formal (algebraic) approach to solving algebraic problems. Research in this area continues to provide evidence of students' preferences for arithmetic reasoning and their difficulties with the use of equations to solve word problems (e.g., Bednarz & Janvier, 1996; Cortes, 1998; Swafford & Langrall, 2000). For example, Stacey and MacGregor (1999) found that, at every stage of the process of solving problems by algebra, students were deflected from the algebraic path by reverting to thinking grounded in arithmetic problem-solving methods.

Many studies report that solving equations is found difficult (Carry, Lewis, & Bernard, 1980), and this is corroborated by international assessment. When 15-year-olds were asked to solve $5x + 4 = 4x - 31$ (an equation on the curriculum of the countries involved: US, England, Scotland, France, and Germany), the success rate for countries ranged from 9% to 58% (Foxman, 1992). In a study involving approximately 200 16-year-old students representing the top 20 percent of the ability range in a sample of Swedish school students, Ekenstam and Nilsson (1979) found that although 82 percent of the students solved the equation $\frac{30}{x} = 6$ correctly, only 48 percent were successful with the structurally similar example $\frac{4}{x} = 3$. In the first example, students were able to solve the equation by inspection, a procedure that could not be so readily applied in the second example. The use of informal methods can have implications for students' ability to produce (or understand) general statements in algebra (Booth, 1988).

A study dealing with the setting up of equations within the activity of word-problem solving has reported that reasoning and symbolising appear to develop as independent capabilities (van Amerom, 2003). Although some students could write equations to represent problems, they did not use these equations to find the solution, preferring instead to use more informal methods. Drijvers (2004) found further evidence of students' difficulties with the substitution method for solving systems of equations. 14- and 15-year-olds were asked to solve parametric equations, for example, "Solve for $ax + b = 5$ for x." Students experienced difficulty in accepting the expression $(5 - \dfrac{b}{a})$ as a solution. According to Drijvers (2004), this required that the students conceptualise an expression as an object (Sfard, 1991).

Da Rocha Falcao (1995), however, suggests that the difficulty is contained in the difference in approach to problem solving. Arithmetical problems can be solved directly, if necessary with intermediate answers (van Amerom, 2003). On the other hand, algebraic problems need to be translated and written in formal representations first, after which they can be solved. In algebra "the focus is on the derivation of procedures and relationships and the expression of these in general, simplified form" (Booth, 1988, p. 20). Booth (1988) explains the reason for deriving such general statements is to use them as "procedural rules" for solving appropriate problems and hence finding numerical answers, but the immediate focus is on the derivation, expression, and manipulation of the general statement itself. However, many students do not realise that they need to define the unknown(s), describe the relation(s) between the quantities, and solve the problem with algebraic means to discover and express generality of method, looking beyond specifics (van Amerom, 2002), preferring the direct approach instead to solving the problem.

To further illustrate, Usiskin (1988) has emphasised that:

> *in solving problems such as "when 3 is added to 5 times a certain number, the sum is 40," many students having difficulty moving from arithmetic to algebra. Whereas the arithmetic solution involves subtracting 3 and dividing by 5 [i.e., using the undoing operations], the algebraic form $5x + 3 = 40$ involves multiplication by 5 and*

addition of 3 [i.e., using forward operations]. That is, to set up the equation, you must think precisely the opposite of the way you would solve it using arithmetic (p. 13).

This shift from thinking about undoing or solving operations to focusing on the forward operations required in setting up an equation is one of the crucial steps in the transition from arithmetic to algebra in which students encounter difficulties. "The algebraic sentence that most naturally describes … a problem situation does not immediately fit an arithmetic procedure. This possibility of 'first describing and then calculating, is one of the key features that makes algebra different from arithmetic" (Lesh, Post, & Behr, 1987, p. 657) and one that students find complex and confusing. To the mathematics teacher (or textbook author), this approach seems superior and appropriate because each step in the solution process can be verified by arithmetical counterpart (the inverse operation), but in the eyes of the student it is not a logical or natural approach; because, the arithmetical approach is easier and works just as well (van Amerom, 2002).

Therefore, the abstract reasoning required and these different approaches to problem solving – straight-to-the-point inversion in arithmetic versus the round-about way of constructing an equation first in algebra - as shown by research (e.g., Da Rocha Falcao, 1995; Usiskin, 1988; van Amerom, 2003) – can cause students great difficulty. It has been found that students have trouble recognising the structure of the problem as they try to represent the problem symbolically (Herscovics, 1989; Kieran, 2007). They can recognise the solutions procedure (for example, inverse calculation, back-tracking, trial and error) but they cannot reason with the unknown themselves. Moreover, the informal arithmetical approaches do not go hand in hand with algebraic methods.

2.2.1.2   Difficulty in understanding algebra language, letters and meanings given to symbols

Symbolism is mentioned in a substantial portion of the research addressing algebraic understanding and misconceptions (Christou & Vosniadou, 2005; Kuchemann, 1981). Similarly, letter usage is cited in a great deal of algebra research (Knuth et al., 2005; Usiskin, 1988).

Algebra deals with letters or symbols. Letters or symbols are abstract objects in the sense that they are de-contextualised (Lew, 2004). Algebra also deals with a lot of objects, including numbers, variables, expressions, equations, functions and relations, and each of these can play many different roles. When students begin their study of algebra, they need to learn and acquire a new language, an efficient way of representing properties of operations and relationships among them in the conventional system of notation (Schifter, 1999).

The learning of algebra requires students to learn the language of mathematical symbols that is also completely foreign to their previous experiences (Kilpatrick, Swafford, & Findell, 2001). The multiple ways in which this language is described and used during instruction often prevent students from connecting algebraic symbols to their intended meaning (Blanco & Garrote, 2007). In some cases, students are completely unaware that any meaning was intended for the symbols (Fujii, 2003; Kuchemann, 1978). In other cases, they may know that meaning exists, but limited understanding prevents them from ascribing meaning to the symbols, or they may assign erroneous meaning to symbols (Fujii, 2003; Kuchemann, 1978). Hence, giving rise to various misconceptions and difficulties in the process of learning algebra.

Symbols are used in many different contexts in mathematics. This can be to represent: technical concepts (e.g., unknown, coefficient, variable); operations (e.g., $+, -, x, \div, \sqrt{\ }$) and expressions (e.g. $3x + 1, 2a - 3b, 3(a + 2)$) or equations (e.g. $ax^2 + bx + c = 0, 3y - 5 = 2y + 7$) (Maharaj, 2008). The many uses of literal symbols in algebra have been documented by Philipp (1992). Some of

the different uses of symbols are: "as labels (m , cm in 1m = 100cm); generalised number ($3a + 2a = 5a$); constants (c in $3x + c$); varying quantities (x, y in $y = 3x + 1$); unknowns (x in $2x + 1 = 0$); parameters (m, b in $y = mx + b$); and in geometry and trigonometry: vertices (A, B and C in triangle ABC); sides (BC or a in triangle ABC); angles ($\hat{A}_1$, $\hat{A}_2$ or B$\hat{A}$C) or short-forms (PQ ‖ST)" (Maharaj, 2008, p. 401).

Letters are mostly used to represent the concept of variable - the most important concept in algebra. The variable is an object to encapsulate changing objects (Lew, 2004) and it is essential for understanding a function in algebra. In other words, a variable is a mathematical entity that can be used to represent any number within a range of numbers and can stand on its own right in the algebraic formal system (Christou, Vosniadou, & Vamvakoussi, 2007). Or "In the symbolic language, variables are simply signs or symbols that can be manipulated with well-established rules, and that do not refer to a specific, context-bound meaning" (Drijvers, Goddijn, & Kindt, 2011, p. 17). Students have difficulty assigning meaning to variables and fail to recognize the systemic consistency in the multiple uses of variables. Therefore the "different notions of letters in the context of algebraic symbolism could imply different levels of difficulties for students" (Maharaj, 2008, p. 402). This view is supported by Kieran (1992) who argued that discriminating "… the various ways in which letters can be used in algebra can present difficulties to students" (p. 396).

Symbols can represent various concepts and also take on varying roles and the context is important in determining the role of the literal symbols (Philipp, 1992). Therefore the correct reading of the context could pose a problem to students of algebra. According to Linchevski and Herscovics (1996) tasks such as grouping algebraic terms and using algebraic expressions demand "… quite an advanced perception of literal symbols" (p. 43) and "once meanings are established for individual symbols, it is necessary to think about creating meanings for rules and procedure that govern actions on these symbols" (Hiebert & Carpenter, 1992, p. 72). Often, students have great difficulty in deciphering the meanings assigned/attached to individual symbols, and applying the correct rules and procedures that govern the actions on these symbols.

To illustrate the psychological complexity that mathematics education researchers see and the difficulties students encounter in school algebra, Usiskin (1988) presents five equations (in the sense of having an equal sign in them) involving literal symbols. Each has a different feel.

1. $A = LW$
2. $40 = 5x$
3. $\sin x = \cos x \cdot \tan x$
4. $1 = n \cdot 1/n$
5. $y = kx$

As Usiskin observes, "We usually call (1) a formula, (2) an equation (or open sentence) to solve, (3) an identity, (4) a property, and (5) an equation of a function of direct variation (not to be solved)" (p. 9). Each of these symbol strings reads differently. Furthermore, Usiskin (1988) argues that each of these equations has a different feel because in each case the concept/idea of variable is put to a different use:

> *In (1), A, L, and W stand for the quantities area, length, and width and have the feel of knowns. In (2), we tend to think of x as unknown. In (3), x is an argument of a function. Equation (4), unlike others, generalizes an arithmetic pattern. In (5), x is again an argument of a function, y the value, and k a constant (or parameter, depending on how it is used). Only with (5) is there a feel of "variability" from which the term variable arose.*
>
> (Usiskin, 1988, p. 9)

The illustration above shows the important differences exist between strings of symbols that are labeled as equations and the complexities that exist at the heart of school algebra. The discussion above illustrates that, despite frequent assertions to the contrary, mathematical language, notation, and representations are inherently ambiguous. This ambiguity is an advantage to those who are familiar with it, can effortlessly disambiguate through contextual cues, and understand the underlying

conceptual linkages, but for the novice it can be a major source of confusion (White & Mitchelmore, 2010).

The distinction developed in this aspect of the literature review raise important curricular questions on how to assist students in developing an understanding of these various meanings of the symbol strings of algebra and in overcoming the learning difficulty associated with the language of mathematical symbols that is foreign to their previous experiences (Kilpatrick, Swafford, & Findell, 2001). On top of that, students need to learn and acquire an efficient way of representing properties of operations and relationships among them in the algebraic conventional system of notation (Schifter, 1999) which is different from the conventions used in arithmetic.

2.2.1.3   *Change from arithmetic to algebraic conventions*

According to Kieran (1990), most of the problems which students have in their introduction to algebra arise because of the shift to a set of conventions different from those used in arithmetic. Much research conducted over the past few decades suggests that arithmetic instruction encourages students to think about mathematical operations as a series of givens which must be transformed through a series of one-way operations into output or "answers" (McNeil & Alibali, 2005; Schliemann, Carraher, Brizuela, & Jones, 1998). However, when students are introduced to algebra, "the meaning of equivalence, operations, and equations undergoes a paradigm shift. Operations are meant to describe logical relations among elements (quantities or variables) instead of events or actions. In an expression such as $a^2\text{-}b^2$, the minus sign indicates a subtraction and yet one may be expected to factor [$a^2\text{-} b^2 = (a + b)(a - b)$] and equals no longer simply means yields or gives" (Schliemann et al., 1998, p. 3).

Breiteig and Grevholm (2006) attribute the complexity of algebra "to syntactic inconsistencies with arithmetic, such as the following: a variable may simultaneously represent many numbers, the letter may be chosen freely, the absence of positional value, equality as an equivalence relation, the invisible multiple sign, the priority of rules and the use of parentheses" (p. 226). The inconsistency of conventions or rules

in the expression and interpretation of literal symbols between arithmetic and algebra have been found to give rise to difficulties and misconceptions.

One of the important ideas regarding symbolic representation is that while it is a language, its rules or conventions are not the same as those of ordinary verbal languages. As a result, those rules can be confusing to students. Smith (2003) notes that in order to succeed in algebra, one needs to learn the mathematical and logical concepts, but at the same time learn the correct way to express those concepts. As Stacey and MacGregor (1999) note:

> *[Algebra] often cannot say what we want it to say. For example, we can represent "y is more than x as y > x, but we cannot represent the statement "y is 4 more than x" in a parallel way. We must make inferences from the unequal situation just described to write such equalities as $y = x + 4$ or $y - 4 = x$. As a second example, the natural way to describe a number pattern like 2, 5, 8, 11, 14, … is to focus on the repeated addition perhaps saying, "Start at 2 and keep on adding 3." However, the algebra that most students are first taught cannot express this easy idea in any simple way. To construct a required formula, $y = 3n - 1$, students have to look at the relationship between each number and its position in the sequence. Algebra is a special language with its own conventions. Mathematical ideas often need to be reformulated before they can be represented as algebraic statements* (p. 110).

Many beginning algebra students are found "to have difficulty expanding their understanding of arithmetic sign systems to include the new sign systems attached to algebra" (Wood, 1998, p. 239). This can cause difficulties for students. According to Tall and Thomas (1991), there is considerable cognitive conflict between the deeply ingrained implicit understanding of natural language and the symbolism of algebra. The researchers explain that in most western civilizations, both algebra and natural language are spoken, written and read sequentially from left to right. And, this is nothing compared with the subtle rules of precedence which occur in algebra. For

instance, the expression $3x + 2$ is both read and processed from left to right, however, $2 + 3x$ is read from left to right as 'two plus three x", but computed from right to left, with the product of 3 and x calculated before the sum. This difficulty of unraveling the sequence in which the algebra must be processed is conflicting with the sequence of natural language. It manifests itself in various ways, for example the student may consider that ab means the same as $a = b$, because they read the symbol ab as a and b, and interpret it as $a + b$. Or the student may read the expression $2 + 3a$ from left to right as $2 + 3$ giving 5, and consider the full expression to be the same as $5a$ (Tall & Thomas, 1991).

In another context, the algebraic expression, $3x + 2$, is understood by convention to mean the quantity that is two more than the product of 3 times the variable x, rather than the product of 3 times the number which is two larger than x. If x were 8, then $3x + 2$ is 26 in the way the convention or language of algebra is used because it is 3 times 8, which is 24 and then 2 is added to give 26. But if there were no convention, someone might take it to be equivalent to 30, because they though it meant 3 times the sum of eight and two, which would be 3 times 10. In algebraic notation, that is to say that $3x + 2$ just is designated to mean the same thing as $3(x) + 2$ rather than $3(x + 2)$. This confuses students who try to understand why something is the way it is because it is simply a convention (Kieran, 1992; van Amerom, 2002).

Another typical sign whose use in arithmetic is inconsistent with its meaning in algebra is the equal sign. Beginning students tend to see the equal sign as a procedural marking telling them "to do something," or as a symbol that separates a problem from its answer, rather than a symbol of equivalence (Molina & Ambrose, 2008). Equality is commonly misunderstood by beginning algebra students (Carpenter et al., 2003; Seo & Ginsburg, 2003). Several researchers have noted that such a limited view of the equal sign exists among some students in secondary school (Herscovics & Linchevski, 1994) and also at college level (Bell, 1995). In fact the equal sign has several meanings. Using Nickerson's (1985) illustration, in each of the three following expressions, the symbol means something distinctly different from what it means in the others:

$$x^2 + x - 12 = (x + 4)(x - 3) \quad \text{- tautology}$$

$$y = 2x + 3 \quad \text{- functional relationship}$$

$$x^2 + 3x - 10 = 0 \quad \text{- constraint equations}$$

One wonders to what extent these distinctions are made clear when students learn to use the sign in these various ways. These examples do not exhaust the many uses of the equals sign but perhaps do represent the three most common situations encountered in mathematics through algebra. Another usage, widely seen in the context of computer programming, is this of an assignment operator: in this case, it means "replace the value of the variable on the left with the value of the expression on the right," and it permits such mathematically bizarre expression as $X = X + 1$ (Nickerson, 1985, p. 219–220).

As a result, students may not understand the processes needed to solve non-arithmetical equations such as $3y - 5 = 2y + 7$ because they do not accept the beginning premise that the two quantities are equivalent. Herscovics and Linchevski (1994) describe the student's difficulty at this phase as a cognitive gap. These researchers (Filloy & Rojano, 1985; Herscovics & Linchevski, 1994) discovered that whereas the solution of an equation of the form $ax + b = c$ is intuitively accessible to most students, the equation with an unknown appearing on both sides, such as $ax + b = cx + d$ poses a lot of problems for the students. Since in the equation $ax + b = c$ the equals sign still functions like in arithmetic – operations on one side and the result on the other – they called it "arithmetical".

Unfortunately, the equal sign is not only the symbol whose use in arithmetic is inconsistent with its meaning in algebra (Kieran, 1992; Kuchemann, 1981). The convention of the plus sign $(+)$ and minus sign $(-)$ can give rise to errors of interpretation: in algebraic language they do not stand for operations only (addition and subtraction); they can be part of the number itself, the relative number; the minus sign $(-)$ can be used as a unary operator to indicate the inverse. For some students the positive $(+)$ and negative $(-)$ signs have only operational definition (addition and subtraction). Thus, computing with integers and/or symbols often results in conflict with these preconceived ideas. Some commonly heard frustrations described

by Kieran and Chalouh (1993) are: "There are two 'minus' signs and you want me to ADD!?", "Aren't we doing addition $(-7 + 3)$, but you say I must subtract!?" She added that "These difficulties are further aggravated by the seemingly contradictory conventions or rules associated with integers [e.g., $(-) \times (-) = (+)$; $(-) + (-) = (-)$] and are compounded even more when the rules for operating integers are applied to letters" (pp. 195-196).

Freudenthal has pointed out decades ago that inconsistencies between arithmetic and algebra can cause great difficulties in early algebra learning. He observes that the difficulty of algebraic language is often underestimated and certainly not self-explanatory: "Its syntax consists of a large number of rules based on principles which, partially, contradict those of everyday language and the language of arithmetic, and which are even mutually contradictory" (Freudenthal, 1962, p. 35). He continues:

> *The most striking divergence of algebra from arithmetic in linguistic habits is a semantical one with far-reaching syntactic implications. In arithmetic $3 + 4$ means a problem. It has to be interpreted as a command: add 4 to 3. In algebra $3 + 4$ means a number, viz. 7. This is a switch which proves essential as letters occur in formulae. $a + b$ cannot easily be interpreted as a problem.*

> (Freudenthal, 1962, p. 35)

2.2.1.4   Nature of algebra concepts – abstract, algebraic structure and process-object duality

Many mathematics courses include abstract and complex concepts. Characteristic of all advanced mathematics (algebra included) is the demand for abstract entities. Unlike material objects, advanced mathematical notions are totally inaccessible to our senses. This means that they can by themselves be neither seen nor touched but "they can only be seen with our mind's eyes" (Sfard, 1991, p. 3). Algebra concepts and expressions are not only abstract; they also have a dual nature (Sfard, 1991).

White and Mitchelmore (2010) explain that "because mathematics is perceived as a self-contained system separated from the physical and social world, the term 'abstract' is regularly used to describe mathematical concepts. Mathematics uses everyday words, but their meaning is defined precisely in relation to other mathematical terms and not by their everyday meaning. Even the syntax of mathematical argument is different from the syntax of everyday language and is again quite precisely defined. The issue that frequently arises is that this abstract, separated mathematics is not linked to the real world and made relevant to the learner" (p. 205 – 206) resulting in many students to believe that mathematics consists of a large set of rules with little or no connection to each other and hardly any relevance to their everyday lives (Graeber, 1993; Schoenfeld, 1989). Mathematics has some connections to the real world, but for many students the link is not obvious or is at best tenuous (White & Mitchelmore, 2010). The authors believe that when given a situation that relates to real-life situations, the students tended to be confused and have difficulty in responding. This inexperience with abstractness for the construction of meaning directly affects the ability of students to manage multiple representations of algebraic objects (Kieran, 1992; Vogel, 2008). The above descriptions highlight the difficulties students face when confronting abstract concepts.

Skemp (1976) also attributes the difficulty students encountered to the abstract nature of algebra concepts. Researchers have demonstrated that the abstract nature of algebra increases its difficulty over arithmetic (Carraher & Schliemann, 2007; Kieran, 1989, 2004, 2006). Students who have experienced mathematics only at a concrete or procedural level must negotiate the difficult gap from concrete to abstract reasoning with no preparation (Freudenthal, 1983). Some researchers argue that many students at the elementary level face "difficulties in moving from an arithmetic world to an algebraic world" mainly because they cannot understand the structure and patterns of arithmetic (Warren & Cooper, 2003, p. 10). In algebra, students have to manipulate letters of different natures such as unknown numbers (Tahta, 1972), parameters, and variables. Required skills include specific rules for manipulating expressions and an ability to construct and analyze patterns. These components form the basis for the structure of school algebra, which appears to students to be abstract and rather artificial.

Merriam-Webster Dictionary defines a structure as "something arranged in a definite pattern of organisation" (Merriam-Webster, 2010) and the study of algebra begins by learning the basic rules and building blocks necessary to operate within the confines of the discipline that is the structure upon which all algebraic thinking is developed. Knowledge of mathematical structure is knowledge about mathematical objects and the relationship between the objects and the properties of these objects (Morris, 1999). According to Warren (2003), mathematical structure is "concerned with the (i) relationships between quantities (for example, are the quantities equivalent, is one less than or greater than the other); (ii) group properties of operations (for example, is the operation associative and/or commutative, do inverses and identities exist); (iii) relationships between the operations (for example, does one operation distribute over the other); and (iv) relationships across the quantities (for example, transitivity of equality and inequality)" (pp. 123 - 124).

In algebra, expressions are structured explicitly by the use of parentheses, and implicitly by assuming conventions for the order in which we perform arithmetic operations. An algebraic expression is a description of some operation involving variables, such as $x + 1, or\ 2x - y,$ where two distinct symbol systems (letters and numerals) are used together in algebra. Viewing an expression structurally really depends on what we choose to be the main focus in the structure (Liebenberg, Linchevski, Olivier, & Sasman, 1998). The focus may sometimes be only the numbers or operation in the expressions as in the expressions $x + y$ and $y + x$ where the operation is fixed while the numbers change; or in another situation in the expressions with the same structure, $x * y$ and $y * x$, where the numbers and the operations are changed to see if the commutative property is generalised over all the operations. Therefore, viewing an expression structurally implies having the ability to "see" both its "surface" structure as well as its "hidden" structure (Liebenberg et al., p. 3). For example, when looking at the following expression $a \times b + c \times d + e \times f,$ many students see six numbers (its "surface" structure), while some students see the expression as having three numbers (its "hidden" structure). Those students who are able to view the structure $a \times b + c \times d + e \times f$ as having three numbers are able to see $a \times b$ as a single object (x) and to see the structure as $x + y + z$ and the addition as the dominant operation. The ability to see the hidden

structure in complex algebraic structures and to relate the structure to its equivalent "simplified" form is one of the major obstacles that confront students when having to deal with those structures.

In another context, according to Kieran (1989), the surface structure of an algebraic expression refers to "the given form or arrangement of the terms and operations, subject … to the constraints of the order of operations" (p. 34). That is, the surface structure of the expression $6 + 2(x + 5)$ consists of taking 6 and adding to that the multiplication of 2 by $x + 5$. On the other hand, the systemic structure of an algebraic expression refers to the operational properties of commutativity, associativity, and distributivity which would permit one to express $6 + 2(x + 5)$ as $2(x + 5) + 6$.

Kieran (1989, 1992) attributed many errors committed by students of algebra appear to be a result of their long-term inattention to structure of expressions and equations. Kieran (1992) documented the results of various studies that provided evidence of the inability and the difficulties of students to distinguish structural features of algebraic expressions and equations. For example, students often fail to recognise the differences between expressions and equations. They also have difficulty conceptualising an equation as a single object rather than a collection of objects. The meaning of equality is often confused within the algebra contexts as well. These structural challenges often prevent students from recognising the utility of algebra for generalising numerical relationships. In her studies, Kieran (1992) showed that beginning students do not regard $x + 4 = 7$ and $x = 7 - 4$ as equivalent equations. In another study, Wagner, Rachlin and Jensen (1984) found that some high school students do not regard $7w + 22 = 109$ and $7n + 22 = 109$ as equivalent equations and do not perceive the structure of, for example, $4(2r + 1) + 7 = 35$ is the same as $4x + 7 = 35$. The findings of this study show that most algebra students have trouble and difficulty dealing with multi-term expressions as a single unit (including ones in which the unknown occurs on both sides.

Another difficulty of algebra is that students have to be able to view an algebraic expression as both a process and an object, and must acquire a sense of which view is

most suitable at any one time. Initially for students, a formula often has the character of a process description, a calculation procedure or a step by step plan. For example, in a formula $y = 3a + 5$ can only be found when the value of a is given. The symbols $+, \times$ and $=$ acquire an action character and appear to invite a calculation.

However, in algebra there often is nothing to be calculated at all. An identity such as $(a + b)^2 = a^2 + 2ab + b^2$ does not have a process character; it indicates that the expression on the left hand side is equivalent to the expression on the right hand side. In this case, the equal sign does not stand for 'and the result is …', but stands for 'is equivalent to'. The plus symbol also has a different character: instead of 'take a and add b' the $+$ sign in $(a + b)^2$ stands 'for the sum of a and b'. The expression $(a + b)^2$ is not a process description, but an algebraic object (Freudenthal, 1983). Several authors have pointed out students' difficulties with this dual nature of algebraic expressions, and with the object view in particular (Sfard, 1991; Tall & Thomas, 1991). For example, students find it difficult to accept an expression such as $3a - 5$, as a solution to an equation, because 'then you still don't know how much it is' (Booth, 1984). And to a student who thinks only in terms of process, the symbols $3(a + b)$ and $3a + 3b$ (even if they are understood) are quite different, because the first requires the addition of a and b before multiplication of the result by 3, but the second requires each of a and b to be multiplied by 3 and then the results added. Yet such a student is asked to understand that the two expressions are essentially the same, because they always give the same product (Tall & Thomas, 1991).

According to Kieran (1992) the overall picture that emerges from an examination of the findings of algebra research is that the majority of students do not acquire any real sense of the structural aspects of algebra. Sfard (1987), and Kieran and Sfard (1999) argue that the majority of students cannot "see" the mathematical object that two processes (structures/expressions) are equivalent to, and so cannot tell whether or not the processes are equivalent. Tall et al. (2001) considered research ranging from arithmetic through algebra to calculus felt that the key to a structural understanding is when a "symbol can act as a pivot, switching from a focus on process to compute or manipulate, to a concept that may be thought about as a manipulative entity" (p. 84).

Studies on the process-product dilemma initially began with Matz and Davis (Kieran, 1989, 1990). The researchers conducted a study in the 1970's on students' interpretation of the expression $x + 3$. In their findings, students see it as a procedure of adding 3 to x, whereas in algebra it represents both the procedure of adding 3 to x and the object $x + 3$. In other words, in algebra the distinction between the process and the object is often not clear. Matz and Davis call this difficulty the "process-product dilemma". Freudenthal (1983) illustrated the difference between a procedural (operational) and a relational (structural) outlook by comparing language use and meaning:

> *A powerful device – this formal substitution. It is a pity that it is not as formal as one is inclined to believe, and this is one of the difficulties, perhaps the main difficulty, in learning the language of algebra. On the one hand the learner is made to believe that algebraic transformations take place purely formally, on the other hand if he has to perform them, he is expected to understand their meaning. …the learner is expected to read formulae with understanding. He is allowed to pronounce:*
>
> $$a + b, \qquad a - b, \qquad ab, \qquad a^2$$
> *as* $\quad$ a plus b $\qquad$ a minus b $\qquad$ a times b $\qquad$ a square
> *yet he has to understand it as*
> sum of a and b, difference of a and b, product of a and b, square of a.
> *The action suggested a plus, minus, times, square and the linear reading order must be disregarded. The algebraic expressions are to be interpreted statically if the formal substitution is to function formally indeed* (p. 483-484).

The interaction of these three foundational understandings, abstract reasoning (Section 2.2.1.1), symbolic language acquisition (Section 2.2.1.2), and mathematical structure (Section 2.2.1.4), together with the students' lack of basic knowledge (Section 2.2.1.3) required for a correct understanding of a given concept or procedure forms a very formidable impediment to mastering algebra for many students. As a result, these students have a poor initial experience with algebra and therefore fail to

gain an adequate foundation for future learning. On top of these, another source of difficulty which students encounter in the classroom is the result of the teaching method used by the teacher (Centeno, 1988; Rakes et al., 2010; Sfard & Linchevski, 1994).

2.2.1.5   Teaching methods – formal representation

The teaching methods used to convey content often exacerbate these algebra learning barriers, possibly becoming a unique barrier themselves (Leitzel, 1989; Thorpe, 1989). Sfard (1991) found that both students and teachers often expect immediate rewards for teaching and learning efforts. Instead, Sfard noted that relational understanding of abstract mathematical ideas often requires a lengthy, iterative process. Teaching methods that focus on skill or procedural levels on cognitive demand fail to address these foundational understandings and therefore fall short of providing students with the tools necessary to find their way once they waver from a scripted path (Rakes et al., 2010). Kieran (1992) extended Sfard's findings to algebra. She suggested that a great deal of time must be spent connecting algebra to arithmetic before proceeding to the structural ideas of algebra. Instead, teachers often spend a short period of time reviewing arithmetic and then proceed directly into a textbook sequence of instruction, which are often insufficient for helping students understand the abstract, structural concepts necessary for supporting the demonstrated procedural activities in algebra (Kieran, 1992).

Some of these problems are amplified by teaching approaches. Most often the structural character of school algebra is emphasised, whilst procedural interpretations would be more accessible for children (Kieran, 1990, 1992; Sfard & Linchevski, 1994), or in what Skemp (1976) ascribes to as a mis-match between the teacher's desire for relational understanding and the student's preference for instrumental understanding:

> *The pupils just 'won't want to know' all the careful ground-work given for whatever is to be learnt next. … All they want is some kind of rule for getting the answer. As soon as this is reached, they latch*

on to it and ignore the rest. ... But many, probably a majority, attempts to convince them that being able to use the rule is not enough will not be well received. 'Well is the enemy of better', and if pupils can get the right answers by the kind of thinking they are used to, they will not take kindly to suggestions that they should try something beyond this (p. 21-22).

The above literature suggests that the difficulties of achieving competence in abstract reasoning, language acquisition, and mathematical structure within the learning of algebra require teaching strategies that purposefully target the needs of learners. For example, in recognition of the difficulties some students have learning algebra in isolation, diagnostic (conflict) teaching within a cooperative and collaborative learning environment (e.g., Slavin & Karweit, 1982) offers a relevant pedagogical option. The purpose of the following section is to review literature on students' misconceptions as a result of the learning difficulties faced by students as they progress to enhance their ability to abstract common elements from situations, to conjecture, and to generalise, in short, to do mathematics.

2.2.2 Students' Misconceptions of Algebra Concepts

Alternative conceptions research has documented that students:

enter instruction with conceptual configurations that are culturally embedded; are tied into the use of language; and connected to other concepts; have historical precursors; and are embedded in a cycle of expectation, prediction, and confirmation or rejection. For students ... it appears that the course of learning is not a simple process of accretion, but involves progressive consideration of alternative perspectives and the resolution of anomalies.

(Confrey, 1990a, p. 32)

Recent studies focusing on errors and misconceptions in school mathematics are difficult to find (Barcellos, 2005), although some past studies were carried out to

focus on students' errors caused by correctly using buggy algorithms or incorrectly selecting algorithms in elementary arithmetic (Brown & Burton, 1978; Brown & VanLehn, 1982) and in elementary algebra (Matz, 1982; Sleeman, 1982). In the efforts to explore the conceptual basis of students' procedural errors or buggy algorithms, existing research tended to attribute students' learning difficulties to their underdevelopment of logical thinking (e.g., Piaget, 1970), lack of understanding of mathematical principles underlying procedures (e.g., Resnick et. al., 1989), lack of proficiency or knowledge (Anderson, 2002; Haverty, 1999), or poor understanding of mathematical symbols (e.g., Booth, 1984; Fujii, 2003; Stacey & MacGregor, 1997; Usiskin, 1988). In contrast, rather than characterising students' difficulties or misconceptions in terms of a deficiency model, McNeil and Alibali (2005) claimed that "earlier learning constrains later learning" (p. 8), that is, students' misconceptions may be caused by their previous learning experiences.

Researchers agree that students enter classrooms with different conceptions due to different life experiences or prior learning. An important task for teachers is to identify students' preconceptions and misconceptions in order to help students learn mathematics effectively and efficiently. Ignoring students' misconceptions may have negative effects on students' new learning and may also result in the original misconceptions being reinforced. In the following sub-section, literature on some misconceptions and their genesis is reviewed briefly.

2.2.2.1   *Misconceptions in relation to the use of literal symbols*

Algebra uses its own standardized set of signs, symbols and rules about how something can be written (Drijvers et al., 2011). Algebra seems to have its own grammar and syntax and this makes it possible to formulate algebraic ideas unequivocally and compactly. In this symbolic language, "variables are simply signs or symbols that can be manipulated with well-established rules, and that do not refer to a specific, context-bound meaning" (Drijvers et al., 2011, p. 17).

There is a great deal of empirical evidence (Vosniadou, Vamvakoussi, & Skopeliti, 2008) showing that rational number reasoning (learning rational numbers) is very

difficult for students at all levels of instruction and in particular when new information about rational numbers comes in contrast with prior natural number knowledge (Ni & Zhou, 2005) in their previous learning experiences. Students have difficulties interpreting and dealing with rational number notation, in particular when it comes to fractions (Gelman, 1991; Stafylidou & Vosniadou, 2004). They do not realise that it is possible for different symbols (e.g., decimals and fractions) to represent the same number and thus they treat different symbolic representations as if they were different numbers (Khoury & Zazkis, 1994; O'Connor, 2001; Vamvakoussi & Vosniadou, 2007).

These difficulties are exacerbated to the study of algebra in which students are introduced to the principled ways in which letters are used to represent numbers and numerical relationships – in expression of generality and as unknowns – and to the corresponding activities involved with these uses of letters … justifying, proving, predicting, and solving, resulting in a series of misconceptions with the use of notation – a tool to represent numbers and quantities with literal symbols but also to calculate with these symbols (Kieran, 2007).

With regard to students' possible misconceptions of the meaning of variable, often contexts call for multiple usages and interpretations of variables. Students must "switch from one interpretation to another in the course of solving a problem which makes it difficult for an observer and for the individual himself to disentangle the real meaning being used" (Kuchemann, 1981, p. 110). In conclusion, "the meaning of variable is variable; using the term differently in different contexts can make it hard for students to understand" (Schoenfeld & Arcavi, 1988, p. 425), hence giving rise to many misconceptions students have in understanding the different uses of literal symbols in different contexts.

Previous studies have demonstrated a series of misconceptions which students have in relation to the use of literal symbols in algebra. A common naïve conception about variables is that different letters have different values. Alternatively, when students think of literal symbols as numbers they usually believe that they stand for specific numbers only (Booth, 1984; Collis, 1975; Kuchemann, 1978, 1981; Knuth et al., 2008; Stacey & MacGregor, 1997). This misconception is illustrated by students'

responses of "never" to the following question: "When is the following true –
$L + M + N = L + P + N$ - always, never or sometime?" Kuchemann (1981)
reported in the CSMS project that 51 percent of students answered "never" and
Booth (1984) reported in SESM project that 14 out of 35, that is, about 41 percent of
13 – 15 year-old students responded likewise. Olivier (1989) reported that 74 percent
of 13 year-old students also answered "never". Fujii (2003) have likewise found a
high proportion of students in his sample doing the same. Kieran (1988) in her study
found out that students do not understand that multiple occurrences of the same letter
represent the same number. Even students who have been told and are quick to say
that any letter can be used as an unknown may, nonetheless, believe that changing
the unknown can change the solution to an equation.

One of the best known of the misconceptions is the "letter as object" misconception,
described by Kuchemann (1981), in which the letter, rather than clearly being a
placeholder for a number, is regarded as being an object. For example, students often
view literal symbols as labels for objects, that is, they think that 'D' stands for David,
'*h*' for height, or they believe that '*y*' - in the task "add 3 to 5*y*"- refers to anything
with a '*y*' like a yacht. The term "fruit-salad algebra" (MacGregor & Stacey, 1997b)
is sometimes used for this misconception, infamously presented in examples such as
"a for apples and b for bananas, and so $3a + 2b$ is like 3 apples and 2 bananas, and
since you can't add apples and bananas we just write it as $3a + 2b$." One difficulty
is that 3a in algebra does not represent 3 apples, but three times an unknown number.
The second difficulty here concerns the mathematical idea of closure: in saying we
cannot add apples and bananas we contradict the fact that $3a + 2b$ is the sum.
According to Chick (2009), the letter as object misconception may be reinforced by
formulas like $A = l \times b$, where A = area.

Another typical sign whose use in arithmetic is inconsistent with its meaning in
algebra is the equal sign. Beginning students tend to see the equal sign as a
procedural marking telling them "to do something", or as a symbol that separates a
problem from its answer, rather than a symbol of equivalence (Behr, Erlwanger, &
Nichols, 1976). Equality is commonly misunderstood by beginning algebra students
(Falkner, Levi, & Carpenter, 1999; Knuth et al., 2008). Several researchers have
noted that such a limited view of the equal sign exists among some students in

secondary schools (Herscovics & Linchevski, 1994) and also at college level (Bell, 1995).

Steinberg, Sleeman, and Ktorza (1990) showed that eighth- and ninth-grade students have a weak understanding of equivalent equations. Even college calculus students have misconceptions about the true meaning of the equal sign. Carpenter, Levi, and Farnsworth (2000) posit correct interpretation of the equal sign is essential to the learning of algebra because algebraic reasoning is based on students' ability to fully understand equality and appropriately use the equal sign for expressing generalisations. For example, the ability to manipulate and solve equations requires students to understand that the two sides of the equation are equivalent expressions and that every equation can be replaced by an equivalent equation (Kieran, 1981). Consider the equation $2x - 3 = 11$. Some students see the expression on the left-side as a process and the expression on the right-side as the result (Linchevski & Herscovics, 1996; Knuth et al., 2008).

For example, parentheses (brackets) in the language of arithmetic are used only to indicate the priority of operation over the others when this priority contrasts with the convention. In the algebraic language the symbol ( ) can play the same role, but can also be used as a mere barrier between two signs that you may not write one beside the other (Malara, & Iaderosa, 1999). This new function of parentheses often brings the students to the mistake of using parentheses in the first function. A typical example of the multiplicity of meanings of signs is given by $(+\frac{1}{2} - 3 + \frac{1}{5}) \times (-3)$ where a parentheses is only a separator the other one has a double role (binary function), a negative sign (-) represents a unary operator, and the other one is part of a number. Some students apparently ignore or overlook bracketing symbols, as in $4(n + 5) = 4n + 5$ (Booth, 1984).

2.2.2.2 *Misconceptions of over-generalisation – rules and/or operations*

Studying, describing and using generalisations is one of the core concepts of algebra (Drijvers, Goddijn, & Kindt, 2011). Generalising about situations is often linked with

abstraction, because it frequently requires a more detached view of the topic of study. One of the difficulties here is the problem of over-generalisation. For example, consider the following simplification that students are sometimes tempted to apply in an illustration given by Drijvers et al. (2011, p. 20):

$$x^2 + y^2 = 25 \ so \ x + y = 5$$

This simplification indeed is visually attractive (Kirschner & Awtry, 2004). *The reasoning appears to be that you can take square root 'piece-wise':*

$$\sqrt{x^2 + y^2} = \sqrt{x^2} + \sqrt{y^2}$$

This idea isn't totally absurd because with x (multiplication) instead of + (addition) it does work:

$$\sqrt{x^2 \ x \ y^2} = \sqrt{x^2} \times \sqrt{y^2}$$

And with the operation 'multiply by 5' instead of 'take the square' it is also correct:

$$5(x^2 + y^2) = 5x^2 + 5y^2$$

But with the lens formula, it doesn't work:

$$\text{If } \frac{1}{f} = \frac{1}{v} + \frac{1}{b} \text{ then } f = v + b$$

These researchers agree that students who make such mistakes are guilty of over-generalising the distributivity of algebraic operations calling this an illusion of distributivity. Generally speaking, students have difficulty identifying generalisations and the limits of generalisations.

Matz (1982) provided an explanation of why students tend to make over-generalisations in high school algebra. He said, "Errors are the results of reasonable, though unsuccessful, attempts to adapt previously acquired knowledge to new situations' (pp. 25–26). Such as $\sqrt{x^2 + y^2} = \sqrt{x^2} + \sqrt{y^2}$ described above. He

maintained that one of the causes was "linearity" which was "a way of working with a decomposable object by treating each of its parts independently" (p. 29). He thought it was human nature to treat most mathematics operations as "linearity" because their past experiences were compatible with the above hypothesis of "linearity".

Such response patterns have been widely observed, documented, and classified for more than 30 years (Payne & Squibb, 1990; Sleeman, 1982, 1984, 1986). What is so confounding about these errors is their superficial character. Rather than reflecting misunderstanding of the meaning of correct algebra rules, they seem to indicate nothing more substantial than misperception of the forms of the correct rules. For instance, Thompson (1989) spoke of algebra students as "prone to pushing symbols without engaging their brains" (p. 138). Similarly, in his landmark study, Erlwanger (1973) observed:

> *One may be tempted to treat this kind of talk as evidence of an algebraic concept of commutativity. But, in view of the whole picture of Benny's concept of rules, it appears more likely that it involves less awareness of algebraic operations than it does awareness of patterns on the printed page* (p. 19).

Misconceptions regarding the commutativity of the subtraction and (particularly) the division operations, and their notations, have been widely documented (Bell, Fischbein, & Greer, 1984; Bell et al., 1989; Brown, 1981). Available research suggests that one guiding principle is students' erroneous conception that order does not matter, so $8 - 3$ and $3 - 8$ are the same, or rather they have the same answer is the outcome of their experience influenced by correct previous learning. This is further influenced by word problems containing phrases in an illustration by Olivier (1989, p. 8) such as "the difference between Bill's age and Mary's age is 5 years", without specifying who is older, so presumably $8 - 3$ and $3 - 8$ both produce 5 as a result. This is reinforced with the common phrase that students often hear teachers say, "Take the smaller from the larger". Although students know $8 - 3$ and $3 - 8$ have different meanings, they may reason that the method to get the answer of $3 - 8$ to calculate $8 - 3$. Olivier (1989) attributes the main contributory influence for seeing

subtraction as commutative to the extensive experience students have of the commutativity of addition and multiplication when learning their tables, and in lieu of any contradictory evidence, students have no reason to believe that subtraction and division will behave otherwise.

2.2.2.3 *Misconceptions of translation*

Drijvers et al. (2011) regard an important part of algebraic activity is the translation of a problem or situation into 'algebra'. These researchers are of the opinion that this involves more than translation; it concerns building a structure that algebraically represents the problem variables and their mutual relationships in the situation which is essentially described as modeling.

Among students' greatest difficulties is modeling equations from problem situations. Translating from verbal relational statements to symbolic equations or from English to "maths" causes students of all ages a great deal of confusion. Lodholz (1990) observed that writing equation from word problem is often a skill taught in contrived situations or in isolation. The researcher attributes this to be one of the causes for another misconception related to direct translation of verbal statement in English. He reiterates that mechanical word problems that require students to write an expression that represents "5 more than 3 times a number," when taught apart from opportunities for application, can cause students difficulty when interpreting meaningful sentences later. Students may translate English sentences to algebraic expression, simply moving from left to right. For example, "Three less than a number" is interpreted by many students as "3 − x" since the words "less than" (which means to subtract) follow the 3. Teachers must be aware of these misconceptions and address them in instruction (Lodholz, 1990).

Another incorrect direct transliteration of verbal statements into algebra, where $3a + 4b$ could be derived from 3 apples and 4 bananas (irreverently known as "fruit-salad algebra"), has been shown to be a frequent source of error (Kuchemann, 1981), and is of course encouraged by some algebra texts.

The so-called "students and professors problem" has received much attention in the research literature over the last 20 years. In the original study, several groups of university engineering students were presented with the problem:

> *Write an equation using the variables S and P to represent the following statement: At this university there are six times as many students as professors at this university. Use S for the number of students and P for the number of professors.*

<div align="right">(Rosnick & Clement, 1980, p. 4)</div>

As a solution to this task, it was found that 37% of the students answered incorrectly, and of these, 68% represented the problem as $6S = P$ instead of $6P = S$. In a similar study, Oliver (1984) reported that 58% of standard 8 students erroneously responded with $P = 6S$ referred to as "the reverse equation" (Clement, Lochhead, & Monk, 1981).This reveals a limited understanding of the concept of variable that is seen as a label referring to an object or the use of algebraic letters as abbreviated words, in this case the letter S is used as an abbreviation for 'Student', and not as a variable that refers to the number of students.

The second explanation offered by these researchers (Clement, 1982; Rosnick & Clement, 1980) for this error is that students tend to transliterate the verbal form: six students to every professor $\rightarrow 6S = P$ and the error appears to relate to an associative link perceived in the statement $6S = P$, the larger number S being associated with 6. The equation is used to represent an association of related groups, rather than equal numbers. Thus when students incorrectly translate "There are six times as many students as professors" as $6S = P$, the equation is perceived not as symbolizing a sequence of words ("six," ["times"], "students," "professors") but instead as representing a group of six students associated with one professor. The resultant error that students made was to write the "reversed" equation, $6S = P$, is illustrated as (Clement, 1982):

$$6S = P$$

Six times as many

Students as   Professors

<div align="center">57</div>
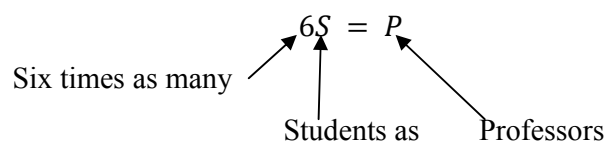
Under this interpretation, the equal sign denotes correspondence or association rather than equality, and the letters S and P are labels for students and professors (MacGregor & Stacey, 1993, p. 219). The resultant error that students made was to write the "reversed" equation, 6S = P, is illustrated as (Clement, 1982):

Similarly, MacGregor and Stacey (1993) explored this well-documented hypothesis and data from their study suggested deeper cognitive reasons for students reversing variables and putting terms in the wrong order. The students in their study actually did make an attempt to understand the situation being described in the problem but were unable to represent their cognitive model symbolically. Their inclination to translate directly from English sentences to algebraic expressions may be augmented by the procedural method many teachers use when addressing this topic in class. It is not uncommon for teachers to encourage students to look for "key words" in word problems that signify a particular operation. Wagner and Parker (1993) stated, "Though looking for key words can be a useful problem-solving heuristic, it may encourage over reliance on direct, rather than analytical mode for translating word problems to equation" (p. 128). However, it is currently accepted (Herscovics, 1989) that a major cause of the reversal error mentioned above is the attempt to translate directly from words to symbols, from left to right, without concern for meaning.

Existing research on students' errors and their conceptual sources have laid a foundation for developing possible interventions on some concepts, which is not only critical to learning (McNeil & Alibali, 2005) but also to teaching mathematics and science effectively. Resnick (1982) attributed students' learning difficulties to conceptual learning: "difficulties in learning are often a result of failure to understand the concepts on which procedures are based." (p. 136) Further research on students' misconceptions in school mathematics and science should empower teachers to use proper strategies to help students, because "one of the greatest talent of teachers is their ability to synthesise an accurate 'picture,' or model, of a student's misconceptions from the meager evidence inherent in his errors." (Brown & Burton, 1978, pp. 155-156)

The above literature suggests that the difficulties of achieving competence in abstract reasoning, language acquisition, and mathematical structure within the learning of

algebra require teaching strategies that purposefully target the needs of learners. For example, in recognition of the difficulties and misconceptions some students have learning algebra in isolation, cooperative and collaborative learning (e.g., Slavin & Karweit, 1982) offers a relevant pedagogical option. As a result, this study was interested in using diagnostic teaching and cooperative learning methodology to assist students in overcoming the learning difficulties and misconceptions they have in algebra.

The purpose of the following section is to explore literature relevant to student attitudes towards mathematics, what possible causes for these attitudes, what instruments are available for measuring these attitudes, and how attitudes were measured in this study. The initial section examines definitions of what constitutes attitudes.

2.3   Research Relating to Attitudes in Mathematics Learning

Attitudes towards mathematics curriculum and teaching have been widely researched (McLeod, 1992; Zan, Brown, Evans, & Hannula, 2006). Studies of attitudes towards mathematics have developed significantly in recent decades. Beginning from the first studies focusing on the possible relationships between positive attitude and achievement (Neale, 1969) to studies highlighting several problems linked to measuring attitude (Kulm, 1980). Next, to a meta-analysis (Ma & Kishor, 1997) and recent studies which question the very nature of attitude (Ruffell, Mason, & Allen, 1998). Finally, to a search for 'good' definitions (Daskalogianni & Simpson, 2000; Di Martino & Zan, 2001, 2003), and/or explore observation instruments very different from those traditionally used, such as questionnaires (Hannula, 2002).

Research on attitude has a long history in mathematics education. There are many different definitions of attitude in educational research. The term attitude as used today evolved in the early 1900's. Originally the term attitude was referred to "aspects of posture (as in to strike a posture) which expressed emotion" (Ruffell et al., 1998, p. 2), was meant as a physical orientation.

In the late 1910s, Thomas and Znaniecki (1918) opined that attitude is "a process of individual consciousness which determines real or possible activities of the individual in the social world" (p. 22). While, Thurstone (1928, p. 531) defined attitude as "the sum total of a man's inclinations and feelings, prejudice and bias, preconceived notions, ideas, fears, threats, and convictions about any specified topic." In 1935, Allport defined attitude as "a mental or neutral state of readiness" (Allport, 1935, p. 810). Later, Aiken (1970) referred to attitude as "a learned predisposition or tendency on the part of the individual to respond positively or negatively to some object, situation, concept, or another person" (p. 551). McLeod (1992) noted that attitudes develop with time and experience and are reasonably stable, so that hardened changes in students' attitudes may have a long-lasting effect.

It is common in the vernacular use, and in mathematics education, to describe a "person's attitude as either positive or negative" (Ruffell et al., 1998, p. 3) thus supporting Ajzen's (1988) definition of attitude as "a disposition to respond favourably or unfavourably to an object, person, institution or event" (p. 4). Currently the everyday notion of attitude refers to "someone's basic liking or disliking of a familiar target" (Hannula, 2002, p. 25) which is a synthesis of Neale's (1969) definition of attitude toward mathematics as "an aggregated measure of a liking or disliking of mathematics, a tendency to engage or avoid mathematical activities, a belief that one is good or bad at mathematics, and a belief that mathematics is useful or useless" (p. 632). In these studies, attitude is generally described as a predisposition to respond to a certain object either in a positive or in a negative way (Di Martino & Zan, 2010). In other words, attitude toward mathematics is a general emotional disposition toward the school subject of mathematics (Haladyna, Shaughnessy, & Shaughnessy, 1983).

The results of extensive research have shown that the field of educational research lacks one clear definition for attitude (DiMartino & Zan, 2001). Many different definitions of attitude can be found (Krech, Crutchfield & Ballachey, 1978). Researchers such as Ruffell et al. (1998) and Di Martino and Zan (2001) have pointed out that attitude is still an ambiguous construct. Kulm (1980) suggested that "it is probably not possible to offer a definition of attitude towards mathematics that would be suitable for all situations, and even if one were agreed upon, it would be

too general to be useful" (p. 358). Nonetheless, there seems to be an agreement on at least two aspects: that attitudes are created and modified by events and the way these are perceived; and that attitudes have affective, cognitive, and behavioural components. Regardless of what definition we give to attitude, "the most obvious problem with attitudes is the discrepancy between espoused and enacted attitudes" (Hannula, 2002, p. 26) and the difficulty of defining the term arises as a result of attitudes being non-observable, and being based on behaviour.

2.3.1 Attitudes towards Mathematics

How students feel towards mathematics is important. Students' liking or interest in mathematics has a pronounced effect upon "the amount of work attempted, the effort expended, and the learning that is acquired" (Dutton, 1956, p. 18). The attitude of students towards what they learn has been of continuing interest to teachers. Given the relatively negative view that many people have of mathematics (Lazarus, 1975) it seems reasonable to assume that mathematics teachers, in particular, are very much concerned with such attitudes.

This concern with student attitudes towards mathematics has led educators to develop instructional techniques and creating positive learning environment leading to greater motivation to learn and more positive attitudes towards the subject (Waters, Martelli, Zakrajsek, & Popovich, 1988; Graham & Fennel, 2001).

Motivations are reasons individuals have for behaving in a given manner in a given situation (Middleton, & Spanias, 1999) and they can either facilitate or inhibit achievement. Motivating students in the classroom is a particular concern for mathematics educators because of society's acceptance of poor attitudes about the potential for success in mathematics (Kloosterman & Gormen, 1990). It is a minority of adults who will remember with fondness their own childhood experiences in mathematics classes. More so, people remember difficulties and challenges associated with mathematics. The widespread dislike of mathematics is easily communicated from non-teacher adults to children (Bragg, 2007, p. 28-29).

When children commence school their attitude toward learning derives from their home environment (Lumsden, 1994). However, success or failure in the classroom impacts on these initial attitudes, and attitudes shaped by early school experience, in turn, impact on subsequent classroom situations (Bragg, 2007; Lumsden, 1994; Reynolds & Walberg, 1992). In a review article on attitudes towards mathematics, Aiken (1970) proposed that attitudes towards mathematics begin developing as soon as children are exposed to the subject. That is, the initial experiences that young children have with mathematics begin to formulate their attitude towards mathematics. However, the eventual formulation of either positive or negative attitudes towards mathematics depends on the experiences that children are offered during this early exposure to mathematics (Mandler, 1989).

There are a number of factors suggested by researchers in previous studies that impact on or influence the formulation of these attitudes. Students' personalities and self-concepts (Moore, 1973; Brassell, Petry, & Brooks, 1980), beliefs and conceptions (Andrews & Hatch, 2000), and achievement (Hannula, 2002; Lopez, Lent, Brown, & Gore, 1997; Tapia & Marsh, 2004) have been found to be important factors in determining students' attitudes towards mathematics. Variables such as motivation and self confidence (Aiken, 1976; Middleton & Spanias, 1999), cognitive variables (Schiefele & Csikzentmihalyi, 1995), feelings of inadequacy (Callahan, 1971), anxiety (Aiken, 1970; Brassell, Petry, & Brooks, 1980), and emotions (Hannula, 2002; McLeod, 1992) have been found to be equally important in determining students' attitudes towards mathematics. Other variables affecting attitudes [and learning] include the quality of instruction, time-on-task, and classroom conversations (Hammond & Vincent, 1998; Reynolds & Walberg, 1992). Students' attitudes [and learning] are also affected by interactions with peers (Fischbein & Ajzen, 1975), teacher quality and the learning environment (Haladyna, Shaughnessy, & Shaughnessy, 1983, Lewis, 2007).

Research conducted over the past two-to-three decades has shown that positive attitudes can impact on students' inclination for further studies and careers in mathematics-related fields (Haladyna et al., 1983; Maple & Stage, 1991; Trusty, 2002). For example, a recent study using the Third International Mathematics and Science Study (TIMSS) data from Canada, Norway and the United States found that

attitudes towards mathematics as the strongest predictor of student participation in advanced mathematics courses (Ercikan, McCreith, & Lapointe, 2005).

In addition, research has also indicated that positive and negative experiences of school activities can produce learned responses which may then impact on students' attitudes as they get older, when positive attitudes towards mathematics appear to weaken (Dossey, Mullis, Lindquist, & Chambers, 1988). This view is supported by Aiken (1970) and McLeod (1994) who asserted that attitudes tend to become more negative as students move from elementary to secondary school. Whether the increase in negative attitudes at this stage of development is due to greater abstractions of the mathematical material to be learned, to social/sex preoccupations, or to some other factor is not clear (Cheung, 1988). One explanation for the decline in attitudes with age is the possible relationship between the increase in curriculum difficulty and a decline in perceived mathematical ability. Specifically, Ma (2003) found that mathematical attitudes tend to improve as mathematical ability grows.

As a result, when studying the learning process attitudes have been considered a key element to be taken into account (Fennema & Sherman, 1976). Gomez-Chacon (2000) suggests that attitudes might act as cognitive guides that augment or inhibit learning. As such, literature suggests that the formulation of positive attitudes should be viewed as a goal of mathematics education. As illustrated by Suydam and Weaver (1975):

> *Teachers and other mathematics educators generally believe that children learn more effectively when they are interested in what they learn and that they will achieve better in mathematics if they like mathematics. Therefore, continual attention should be directed towards creating, developing, maintaining and reinforcing positive attitude* (p. 45).

Therefore, generating positive attitudes towards mathematics among students is an important goal of mathematics education. Haladyna et al. (1983) argued that generally a positive attitude toward mathematics is valued for the following reasons: "a positive attitude is an important school outcome in and of itself; attitude is often

positively, although slightly, related to achievement; and a positive attitude toward mathematics may increase one's tendency to elect mathematics courses in high school and college and possibly one's tendency to elect careers in mathematics or mathematics-related fields" (p. 20).

2.3.2 Relationship of Attitude to Achievement in Mathematics

The idea that attitudes towards mathematics are relevant in the teaching and learning process is shared in the mathematics education research community. Early studies about attitude in mathematics education are motivated by the belief that attitude plays a crucial role in learning mathematics (Neale, 1969) and these studies focus on the relationship between attitude towards mathematics and school mathematics achievement, trying to highlight a causal relationship. As Neale (1969, p. 631) underlines:

> *Implicit … is a belief that something called 'attitude' plays a crucial role in learning mathematics. … positive attitude toward mathematics is thought to play an important role in causing student to learn mathematics.*

Mathematics educators have traditionally taken the relationship between attitude toward mathematics and achievement in mathematics as their major concern (Ma & Kishor, 1997). Aiken (1970) argued that "the assessment of attitude toward mathematics would be of less concern if attitudes were not thought to affect performance in some way" (p. 558). This dynamic interaction between attitudes and behavior has received a great deal of attention in the social-psychological literature (Festinger, 1964).

Many reports have now been published on studies addressing the relationship between students' attitudes towards and achievement in mathematics, gender differences between boys and girls in their persistence with further study in the area (Frost, Hyde & Fennema, 1994; Ma & Kishor, 1997). While the results of these studies have been somewhat mixed, several have indicated significant relationships

between attitudes and achievement in mathematics during early and middle school years.

As summarised in Ma and Kishor (1997), there has been a popular conception among researchers that assumes the causal predominance of attitude towards mathematics over achievement in mathematics. According to this conception, if students do not have positive attitudes, they cannot achieve well (Ma & Xu, 2004, p. 257); a more positive attitude towards mathematics contributes to a higher level of achievement in mathematics (Suydam & Weaver, 1975). This view is supported by Papanastasiou (2000) who explains that generally, students who have a positive attitude towards mathematics will excel at it. His study shows that students with positive attitudes towards mathematics tend to give outstanding performances in education as a whole. Students were found to have a high level of perseverance; they will not stop trying until they manage to get the answer, and they will continue to work on the problem until they succeed in solving it. Students with high self-confidence in their mathematical ability were also found to be able to solve more difficult problems. Additionally, students seem to enjoy solving challenging questions, and so are quite willing to try. In his study, Ma (2003) also found that mathematical attitudes of students tend to improve as mathematical ability grows.

In contrast to the above view of causal predominance of attitude towards mathematics over achievement in mathematics, the prospect of reciprocal relationship of attitude and achievement has been proposed by others (Kulm, 1980). Neale (1969) and Cheung (1988) proposed that the relationship between attitudes and performance is certainly the consequence of a reciprocal influence, in that attitudes affect achievement and achievement in turn affects attitudes. McLeod (1992) suggested that neither attitude nor achievement is dependent on the other, but they "interact with each other in complex and unpredictable ways" (p. 582). However, research does not give a clear picture of the direction of causal relationships (Hart, 1989; Schoenfeld, 1989), with some, as above, arguing that positive attitudes will improve the ability to learn ((Fennema & Sherman, 1976; Meyer, 1985), while others argue that the best way to foster positive attitudes is to provide success (Tall & Razali, 1993).

But the goal of highlighting a connection between a positive attitude and achievement has not been reached. Enemark and Wise (1981) showed that "the attitudinal variables are significant indicators of mathematics achievement" (p. 22) while Steinkamp (1982) concluded that primary among the variables that determine achievement in mathematics is attitude towards mathematics. These conclusions represent the view of strong relationship between attitude towards mathematics and achievement in mathematics, with correlations above 0.40, as supported by a number of other researchers (Kloosterman, 1991; Minato, 1983; Minato & Yanase, 1984; Randhawa & Beamer, 1992). There are other findings that show a statistically significant relationship between attitude towards mathematics and achievement in mathematics with correlations ranging from 0.20 to 0.40 in absolute value (Aiken, 1970; Anttonen, 1968; Beattie, Diechmann, & Lewis, 1973; Jacobs, 1974; Quinn, 1978).

Furthermore, a number of researchers have demonstrated that the attitude towards mathematics and achievement in mathematics correlation is quite low, ranging from zero to 0.25 in absolute value, and they have concluded that the attitude towards mathematics and achievement in mathematics relationship is weak and cannot be considered to be of practical significance (Abrego, 1966; Deighan, 1971; Vachon, 1984; Wolf & Blixt, 1981). Robinson (1975) concluded that attitude towards mathematics accounts for, at best, 15% of the variance in achievement in mathematics, indicating that the relationship has no useful implications for educational practice.

On the other hand, a meta-analysis on existing literature carried out by Ma and Kishor (1997) shows that the correlation between attitude and achievement is statistically not significant. There is also research evidence showing that students' high performance in mathematics is not positively associated with mathematics and mathematics learning. For instance, the results of the Third International Mathematics and Science Study (TIMSS) revealed that while Japanese and Korean students outperformed students from many countries in mathematics, they displayed relatively negative attitudes towards mathematics (Mullis et al., 2000).

The research literature, however, failed to provide consistent findings regarding the relationship between attitude towards mathematics and achievement in mathematics. Despite the inconsistencies of research findings, mathematics educators have traditionally taken the relationship between attitude towards mathematics and achievement in mathematics as their major concern because the attitudes we possess towards mathematics affect how we approach, persist, and succeed at the subject (Thorndike-Christ, 1991). Students who come to enjoy and value mathematics increase their achievement, persistence, and confidence with the subject (Gottfried, 1985; Meece, Wigfield, & Eccles, 1990; Pokay & Blumenfeld, 1990). Students also engage in and enjoy mathematics more if they expect to be successful (Dickenson & Butt, 1989), and generally avoid the subject if possible when they perceive their ability to do mathematics as poor (Hilton, 1981; Otten & Kuyper, 1988). Attitudes were also found to be shaped in great part by the learning environments one experience (Graham & Fennel, 2001), and teachers who understand their students' attitudes are better able to create learning environments conducive to positive attitudes and better achievement (Middleton, 1995).

Awareness of these complex interacting factors informed the research in relation to the potential impact of diagnostic teaching and cooperative learning strategy on students' attitudes. As a result, this study was interested in determining whether the intervention strategy of diagnostic (conflict) teaching within a cooperative learning environment could produce a positive attitudinal change in students and if these changes could in turn affect mathematics achievement.

This section has discussed relevant literature related to the relationship between student attitudes and achievement in mathematics and the inconsistent findings regarding the relationship between attitude towards mathematics and achievement in mathematics. The following section discusses various means of assessing student attitudes, literature related to these assessments, and specifically literature related to the attitude assessment used in this study.

2.3.3   Instruments and Scales for Evaluation of Student Attitudes

Inglis (1918) in his classic work on secondary education wrote:

> *Much more important, however, than acquired knowledge(s) in the field of history are the less tangible but none-the-less real attitudes developed through the study of the subject. ... The attitudes (ideals, ambition, tendencies to act) developed through the study of history are probably more important than the specific knowledge(s) acquired through such study.*
>
> (Inglis, 1918, p. 548)

Achievement tests have shown clearly that the facts learned in any school subject are soon forgotten, educators have consistently echoed Inglis in stating that the important outcomes are not the facts but the attitudes-or at least that the attitudes are of equal importance (Longstreet, 1935, p. 202). Thus Kelly (1932) writes: "We must surely assert that one of the important outcomes of education is the establishment of what we think of as correct attitudes. …All will grant that development of and changes in attitude take place as the school years roll by" (p. 105).

The idea that the establishment of correct attitude is relevant in the teaching and learning process is shared in the mathematics education research community. Since then, there has been a revival of interest in the relationship between cognitive and affective factors such as attitudes in mathematics education (e.g., Fennema, 1996). Researchers have devoted a great deal of effort to understanding attitudes because attitudes have been shown to be useful in predicting human behavior. Because of this interest, researchers have developed theories to explain where attitudes come from and tools with which to measure attitudes.

Because attitudes are so important and assumed to typically predict behaviour, researchers have developed a variety of tools with which to measure them. Several of these techniques described by Corcoran and Gibbs (1961) were: observational methods; interviews; and self report methods such as questionnaires (open-ended questions or closed-item questionnaire), attitude scales (preference ranking

techniques), sentence completion, projective techniques, and content analysis of essays to determine students' attitudes.

Along with these techniques, several different attitude-scaling procedures such as the Thurston scales based on the works of Allport and Hartman (1925), summated rating scales exemplified by (the most common) Likert-type scales (Likert, 1932), the Guttman Scales developed by Guttman (1944), and the Semantic Differential scales of Osgood, Suci and Tannenbaum (1957), have been used to measure attitudes. The majority of these techniques are examples of self-report (paper-and-pencil) measures that require students to response to a set of items or questions. Other measurement techniques that involve drawing inferences (e.g., observation rating scales and interviews) do not appear to be widely used (Leder, 1985).

The Likert scales have been most widely used in assessing student attitudes as they pose the fewest problems to the establishment of validity. Respondents to whom the scale is administered are asked to indicate, typically on a five point scale, whether he/she strongly agrees, agrees, is undecided, disagrees, or strongly disagrees with statements expressing positive or negative attitudes towards something. Strong agreement and disagreement with favourable items are scored as 5 and 1 respectively. Appropriate ratings are given to the intermediate responses. Scoring is reversed for unfavourable items. The respondent's attitude is defined as the sum of the item scores. The sum of the scores obtained on the item denotes the respondent's attitude to mathematics. Instrument such as The Test of Science Related Attitudes (TOSRA) developed by Fraser (1981) and the Revised Science Attitude Scale developed by Thompson and Shrigley (1986) use the Likert Scaling Technique for assessing student attitudes.

In reviewing the various instruments that have incorporated these techniques and scales, research indicates several surveys have been used in the past to assess students' attitudes towards mathematics.

As a result, three instruments that this research considered relevant were the Aiken's Mathematics Attitudes Scales (Aiken, 1974; Aiken & Dreger, 1961), Fennema-Sherman Attitude Scales (Fennema & Sherman, 1976) and the Sandman's (1980)

Mathematics Attitude Inventory. However, neither of these instruments appeared appropriate for the sample population used in this study.

Aiken's Mathematics Attitudes Scales comprise two subscales of 10 items each which assess students' enjoyment of mathematics (e.g., 'Mathematics is enjoyable and stimulating to me') and their perceptions of its value as a subject area (e.g., 'Mathematics is a very worthwhile and necessary subject') – the E and V scales respectively.

The Fennema-Sherman Mathematics Attitude Scales (Fennema & Sherman, 1976) were developed in 1976, and it has become one of the most popular instruments used in research over three decades. The Fennema-Sherman Attitude Scale (1976) has nine dimensions – attitude towards mathematics success, mathematics as a male domain, mathematics anxiety, mother, father, teacher scales, confidence in learning mathematics, effective motivation in mathematics, and perceived usefulness of mathematics – for investigating gender differences. These scales were employed in an attempt to paint a wide picture of possible factors involved in a student's attitude. The Fennema-Sherman Attitude scale was somewhat long and a great many of the items were deemed to be too difficult to answer for the target population. Use of this scale would have required a great deal of modification in order for it to be useful in this study.

The Sandman's (1980) Mathematics Attitude Inventory measures attitude through the use of six scales: the value of mathematics, self-concept in mathematics, anxiety towards mathematics, enjoyment of mathematics, motivation in mathematics, and perceptions of mathematics teachers. The major difference between this and the Fennema-Sherman instrument is that it incorporates the notion of enjoyment as part of the attitude besides motivation, success and personal interaction.

At the same time, the question of attitudes of students towards science topics and classrooms was beginning to take shape and was quantified through the construction of the Test of Science-Related Attitude (TOSRA) (Fraser, 1981). This instrument is designed to measure attitudes towards science among secondary school students. Originally, seven items were developed for each scale based on a classification

scheme designed by Klopfer (1971). These items attempted to delineate meanings associated with the term 'attitude towards science'. These included areas of favourable attitudes towards science and scientists, acceptance of scientific inquiry as a way of thought, adoption of science attitudes, enjoyment of science lessons, and development of interests in science, science-related activities, or possible science-related careers. Each of these scales is defined with an item from that scale in Table 1.

Table 1   Sample Item and Scale Description for Each Scale of the original Test of Science-Related Attitudes (TOSRA)

Scale	Sample Item	Description of Scale
Social Implications of Science	Public money spent on science in the last few years has been used wisely.	Extent to which students understand the implications of science on public life.
Normality of Scientists	Scientists can have a normal life.	Extent to which students manifest favorable attitudes towards scientists
Attitude to Scientific Inquiry	I would prefer to find out why something happens by doing an experiment rather than being told.	Extent to which students accept inquiry as a way of thought
Adoption of Scientific Attitudes	I am curious about the world in which we live.	Extent to which students adopt 'scientific attitudes'
Enjoyment of Science Lessons	Science lessons are fun.	Extent to which students enjoy their science lessons
Leisure Interest in Science	I would like to be given science book or a piece of scientific equipment as a present.	Extent to which students develop interest in science and science-related activities
Career interest in Science	When I leave school, I would like to work with people who make discoveries in science.	Extent to which students develop an interest in pursuing a career in science

Based on Fraser (1981)

The TOSRA has been used to measure a variety of areas of attitudes towards science (Adolphe, Fraser, & Aldridge, 2003; Aldridge, Fraser, & Huang, 1999; Waldrip & Fisher, 2001). Some of the uses have been tied to the relationship between student attitudes and student achievement (White & Richardson, 1993), connections between gender and attitudes (Joyce & Farenga, 1999), classroom environment and attitudes (Wong & Fraser, 1996), and the training of in-service science teachers (Lott, 2002).

The TOSRA has been shown to have good internal consistency reliability and discriminant validity. The TOSRA was culturally cross-validated in junior and senior high schools in Australia and the United States (Fraser, 1981; Khalili, 1987), and can be used to monitor progress in achieving attitudinal aims in distinct areas, or to compare groups of students on particular attitudinal dimensions (Blessing, 2004; Fraser, 1981; Lewis, 2007).

Because of the close connection between mathematics and science, the TOSRA has been adapted to a mathematics context as the Test of Mathematics-Related Attitudes (TOMRA). It has been used to investigate the effectiveness of innovative mathematics programs (Spinner & Fraser, 2005) and associations between environment and attitudes (Ogbuehi & Fraser, 2007). Spinner and Fraser (2005) found no significant differences in mathematics attitudes between the group experiencing the innovative program and the one that did not. Conversely, Ogbuehi and Fraser (2007) found associations between the use of innovative teaching strategies in inner city classrooms and attitudes, achievement and conceptual development. Therefore the TOMRA was deemed useful in this study for investigating the attitudes of students towards mathematics, and for exploring possible associations between attitudes towards mathematics and achievement in mathematics. Since the achievement of favourable attitudes is an important outcome of this study, the TOMRA could be used to monitor student progress toward achieving attitudinal aims. Besides, TOMRAcould be used for measuring the status of individual students or for providing information about the changes in student attitudes after the teaching intervention.

In order to assess students' attitudes towards mathematics in this study, two scales from the TOSRA (Fraser, 1981) were used. The scales chosen were Enjoyment of Mathematics Lessons and Attitude to Mathematical Inquiry. Each scale contains 10 questions; half of the items (that is, Enjoyment of Mathematics Lessons as odd-numbered statements) are designated as positive, and another half of the items (that is, Attitude to mathematics Inquiry as even-numbered statements) are designated negative. Participants responded using a five-point Likert scale ranging from 'strongly agree' to 'strongly disagree'. Positive items are scored by allotting 5 for

'strongly agree' and 1 for 'strongly disagree' responses. Negative items are scored by allotting 1 for 'strongly agree' and 5 for 'strongly disagree' responses.

A major focus of this section is the literature on students' attitudes towards mathematics and instruments to measure these attitudes. The literature reviewed points to the fact that attitudes are an important aspect of learning and that attitudes can be directly related to achievement. The following section reviews relevant literature on efforts to change mathematics teaching practices in a way that may enhance the learning experience for the students to improve not only their disposition (Royster, Harris, & Schoeps, 1999) but also to assist them in overcoming the difficulties and misconceptions they may encounter in learning algebra. As a result, the following section reviews relevant literature related to conceptual change approach to mathematics teaching and learning.

2.4 Conceptual Change Approach to Mathematics Teaching and Learning

For decades, researchers studying knowledge acquisition have recognised that students come to the learning situation with preconceived notions about the way the world works (e.g., Anderson, Reynolds, Schallert, & Goetz, 1977; Piaget, 1952). This pre-existing knowledge serves as a platform from which learners interpret their world. Often this wide range of ideas or knowledge, developed through everyday experiences with the psychological, social, and physical world are not entirely consistent with target knowledge (Duit, 2009; Taber, 2009) or conflicts with information taught in school. Strong evidence exists that these prior ideas constrain learning in many areas. As an example, Vosniadou (2008b) argued that many young children believe that "multiplication makes bigger", as their experience with natural numbers suggests, and thus have considerably difficulty developing the conception of multiplication may result either in bigger, or smaller outcomes as in the case of $3x\frac{1}{4}$. These conceptions and ideas are firmly held and are often resistant to change (Duit & Treagust, 1998, 2003).

The focus on preconceptions represented a basic rejection of a tabula rasa approach to learning (Confrey, 1990). The assumption was that students connect new ideas to existing ideas, and that existing knowledge thus serves as both a filter and a catalyst to the acquisition of new ideas. To understand what students will learn, one must first determine what they currently believe. Researchers in science were often motivated to examine students' conceptions because it was believed that an understanding of a student's prior knowledge determined the appropriate starting point for instruction (Ausubel, 1968, 2000; Bruner, 1960; Novak, 1977a, 1977b). As Hawkins, Appelman, Colton and Flexner (1982) wrote:

> *In some contrast with studies which have the … aim of paying attention to what students don't know … our purpose is always, at least in principle, to find out conjecturally, and more firmly where possible, what students do know, and then how this knowledge can be raised by them to the level of consciousness – retrieved for their own use in further learning* (pp. C – 3).

Research on teaching in ways that are informed by the work on what students do know (conceptions) is increasingly evident. Within the community of those researching student conceptions, there are increasingly more studies of how one might use a rich database of research to improve instruction. Treagust and Duit (2008) stress the importance of student conceptions when they wrote: "Any discussion of conceptual change needs to consider the nature of conceptions" (p. 297). The nature of these concepts significantly determines what is learned and how it is learned (Strike & Posner, 1982, p. 232). Scott, Asoko, Driver, and Emberton (1994), Wittrock (1994) and Ebenezer and Erickson (1996) believe that identifying and understanding student conceptions will advance the design of [mathematics] teaching. The information obtained could be used to develop strategies to induce students' dissatisfaction with their alternative conceptions, and give them access to newer and better ideas which are intelligible, plausible and fruitful in offering new interpretations (Hewson, 1981; Posner, Strike, Hewson, & Gertzog, 1982). Learning under these circumstances often involves not only the integration of new information into memory but also the restructuring of existing knowledge representations. This restructuring is known as conceptual change or conceptual change learning

(Vosniadou, 1999). The purpose of this section is to discuss teaching for conceptual change, its brief history and development, and conceptual change and mathematics learning.

2.4.1 Teaching for Conceptual Change

Within the learning sciences, conceptual change is probably best defined by its relevance to instruction. In the broad educational experience, some topics seem systematically to be extremely difficult for students. Learning and teaching in these areas are problematic and present persistent failures of conventional methods of instruction (diSessa, 2006). Many areas in science and mathematics, from elementary school to university level have this characteristic. Many of the most important ideas or concepts in science and mathematics seem to be affected by the challenges of problematic learning. There is strong evidence that prior ideas constrain learning in many areas (diSessa, 2006; Strike & Posner, 1982) and how preconceptions influence future learning (Novak, 1977b). Before conceptual change research began, instructors who noticed student difficulties with problems would have attributed their difficulties to the abstractness of the concept, or to its complexity. Instructional interventions might include simplifying exposition or repeating basic instruction. These are "blank slate" (Confrey, 1990) reactions to student difficulties. In conceptual change, students must first build new ideas in the context of the old ones, hence the emphasis on change rather than on simple accumulation or (tabula rasa or blank slate) acquisition (diSessa, 2006).

The term conceptual change is used to describe "the kind of learning required when the new information to be learned comes into conflict with learners' prior knowledge usually acquired on the basis of everyday experiences" (Vosniadou & Liven, 2004, p. 445). It describes a learning process that requires a significant reorganization of existing knowledge base or structures (Mason, 2000; Vosniadou, 2003). Pre-instructional conceptual sets must be fundamentally restructured to allow for understanding of new knowledge (Duit & Treagust, 2003) – a conceptual change. Llinares and Krainer (2006) described conceptual change as learning that changes existing beliefs and knowledge. Dhindsa and Anderson (2004) provide a working definition of conceptual change: "Conceptual change is interpreted as a context-

appropriate change in the breadth and composition of conceptual knowledge occasioned by challenging experiences that require learners to rethink their understandings based on evidence from experience" (p. 64). Vosniadou and Verschaffel (2004), who distinguish conceptual change from most other learning, state that whereas most learning involves adding to and enrichment of existing knowledge, conceptual change deals with changing existing knowledge.

The mechanism of conceptual change can be described in the Piagetian terms of assimilation and accommodation. Assimilation is the fitting of new ideas into the existing cognitive structure or schema. If the existing schema has to be modified to assimilate an idea, accommodation takes place. Posner et al. (1982) use accommodation to describe the times when a student may need to replace or reorganise his or her exisiting conceptions and argue for the conditions under which this is likely to occur. They require that a student be dissatisfied with an exisiting conception and find the new conception intelligible, plausible and fruitful. They further indicate that accommodation is facilated when anomalies exist within the current belief system; when analogies and metaphors assist the student in accepting a new conception and make it more intelligible; and when their epistemological, metaphysical, and other beliefs (i.e., their existing knowledge or conceptual ecology) support such a change. Hewson (1992) elaborates on this position by discussing how conceptions can be in competition with each other and how in such cases of conflict, a student will raise or lower the status of one conception relative to another. This disturbance in mental balance when new ideas do not fit existing schema is called cognitive conflict, or cognitive dissonance (Posner et al., 1982).

In essence, teaching for conceptual change as introduced by Posner et al (1982) focuses on conditions under which students will choose to modify, reject, or extend their conceptions by creation of cognitive conflict through the presentation of anomalies. It involves "the teacher making students' alternative frameworks explicit prior to designing a teaching approach consisting of ideas that do not fit students' existing conceptions and thereby promoting dissatisfaction. A new framework is then presented based on formal science that may explain the anomaly" (Duit, Treagust, & Widodo, 2008, p. 630). It is assumed that all learners possess a conceptual context or ecology (Strike & Posner, 1982) by which they understand and appraise concepts in

the context of other concepts that they already possess. These ecologies include such artifacts as anomalies, analogies, metaphors, epistemological beliefs, metaphysical beliefs, knowledge from other areas of inquiry, and knowledge from other areas of competing conceptions. Learners use their existing knowledge (a product of all experiences and social interactions a person has had), to determine which conditions are met, that is whether a new conception is intelligible (knowing what it means or comprehensible), plausible (believing it to be true), and fruitful (finding it useful for explaining other conceptions or related phenomena) (Hewson, 1992).

Hewson and Thorley (1989) use the term conceptual status to classify a conception as being intelligible, plausible or fruitful and for assessing changes in students' conceptions during learning. When a competing conception does not generate dissatisfaction, the new conception may be assimilated alongside the old. However, when the new conception conflicts with existing conceptions or when dissatisfaction between competing conceptions reveals their incompatability, two conceptual events may happen. If the new conception achieves higher status than the prior conception and an available replacement conception was intelligible, plausible and/or fruitful, accommodation, which Hewson (1982) calls conceptual exchange, may occur. If the old conception retains higher status, conceptual exchange will not proceed for the time being. It should be noted that a replaced conception is not forgotten and the learner may wholly or partly reinstate it at a later date. Duit et al. (2008) proposed that the resultant conceptual changes may be permanent, temporary or too tenuous to detect.

Both Posner et al. (1982) and Hewson (1982) stress that it is the student, not the teacher, who makes the decisions about conceptual status and conceptual changes. The teacher's role is basically to facilitate conceptual change so that the student starts to consider the [wide range of] ideas they come to class with that are not entirely consistent with the target knowledge (Duit, 2009; Taber, 2009), understand, appraise and then emerges with a significant change in their exisiting ideas and beliefs consistent with the target knowledge. The purpose of conceptual teaching is "not to force students to surrender their alternative concepts to the teachers' conceptions but rather, to help students both form the habit of challenging one idea with another, and develop appropriate strategies for having alternative conceptions compete with one

another for acceptance" (Hewson, 1992, pp. 9 – 10). However, in the process there may be students who are not aware of the quality of their previous acquired knowledge and its contradiction with scientific knowledge. Hence, they do not see or understand any reasons for change. Thus, they tend to ignore or to enrich their prior representations rather than revise them (Duit, Roth, Komorek, & Wilbers, 2001; Guzzetti, Synder, Glass, & Gamas, 1993; Vosniadou, 1999). Recent empirical research has suggested that conceptual change is a very complex process that proceeds through the gradual replacement of prior beliefs and presumptions or presuppositions (e.g., Vosniadou, 2003) to the "possibility that what is already known can be radically restructured and that new, qualitative different structures emerge" (Vosniadou, 2007a, p. 55).

2.4.2   Conceptual Change and Mathematics Learning

The beginnings of the conceptual change approach can be traced in the attempts of science educators to use Thomas Kuhn's (Kuhn, 1962) explanations of theory change in science as a major source of hypotheses concerning the learning of science (Vosniadou, 2007b). Since the 1970s researchers like Novak (1977a), Driver and Easley (1978), Viennot (1979), and McCloskey (1983) realised that students bring to the science learning task alternative frameworks, preconceptions or misconceptions that are robust and difficult to extinguish. Hodzi (1990) added that if [student's] intuitions and misconceptions are ignored or dismissed out of hand, their original beliefs are likely to win in the long-run; even though they may give the test answers the teachers want. Even the brightest students in the class [have] false ideas based on enduring misconceptions that traditional instructional methods cannot overcome. Educators saw vivid examples of students responding to apparently simple, core conceptual questions in non-normative ways. Poor performance in response to such basic questions, often years into the instructional process, could not be dismissed (diSessa, 2006). They saw an analogy between theory change in science and the need for students to change their alternative frameworks and replace them with the scientific concepts instructed in school (Posner et al., 1982). This change can be brought about through a process of conceptual change that they hope will lead to cognitive restructuring or conceptual restructuring (Greer, 2004).

Conceptual change theory has been widely used to explain students' understanding in a series of developmental studies referring to science (e.g., Posner et al., 1982; Carey, 1985; Hashweh, 1986). Initially the idea of conceptual change was used in education as a way of thinking about the learning of disciplinary content such as physics (Posner et al., 1982) and biology (Carey, 1985). While much of conceptual change research has been geared to scientific concepts, recent work is increasingly emerging in other domains. Conceptual change has later been considered in other domains of disciplinary content such as chemistry, earth science, mathematics, writing, reading, and teacher education (Hewson, 1992). Conceptual change is now being examined in mathematics (Vosniadou & Verschaffel, 2004) and has been applied to explain phenomena in mathematics teaching and learning in general (Tirosh & Tsamir, 2006).

Although some historians of mathematics find that the Kuhnian conceptual change approach particularly fruitful in the case of mathematics (e.g., Corry, 1993; Dauben, 1992; Kitcher, 1992), there is a general reluctance in philosophy and history of science circles to apply conceptual change approach to mathematics. The reasons being, unlike science, mathematics is based on deductive proof and not on experiment. Mathematics is also proven to be extremely tolerant of anomalies and it does not display the radical incommensurability of theory before and after revolution (Vosniadou & Verschaffel, 2004). Moreover, according Dauben (1992) and Corry (1993), the formulation of a new theory in mathematics usually carries mathematics to a more general level of analysis and enables a wider perspective that makes possible solutions that have been impossible to formulate before, are some of the reasons why mathematics was exempted from the pattern of scientific development and change.

However, a number of researchers have pointed out that students are confronted with similar situations when they learn mathematics and science. Vosniadou and Verschaffel (2004) argued that as it is the case that students develop a naïve physics on the basis of everyday experiences, they also develop a "naïve mathematics", which appears to be neurologically based (developed through a long process of evolution), and that consists of certain core principles or presuppositions (such as the discreetness in the number concept) that facilitate some kinds of learning but inhibit

others (Dehaene, 1998; Gelman, 2000; Lipton & Spelke, 2003). Such similarities support the argument that the conceptual change approach can be fruitfully applied in the case of learning mathematics.

Large empirical literatures exist on everyday alternative conceptions or misconceptions in mathematics. A review of literature in section 2.2.2 has shown a number of widespread conceptions which clearly stand in the way of understanding basic algebraic concepts. Because of the abstract nature of the algebraic concepts, most of these concepts cannot be taught merely by showing an example and understanding of these concepts is difficult for most students. Students need to understand these concepts before meaningful learning can occur. Learning new knowledge will be meaningful if students can relate it to previous concepts that they have already understood (Ausubel, 1968, 2000). BonJaoude (1992) demonstrated that learners who relate new knowledge to relevant concepts and propositions they already know were able to use information they acquire in mathematics classes to correct misunderstandings. Therefore, learning algebra can be seen as restructuring of existing ideas rather than merely adding information to existing knowledge (Hackling & Garnett, 1985). Learning under these circumstances often involves not only the integration of new information into memory but also the restructuring of existing knowledge representations. Vosniadou (1999) calls this restructuring that occurs [in mathematics] as conceptual change or conceptual change learning.

Tirosh and Tsamir (2004) argued that conceptual change approach was applicable to mathematics education since it had explanatory and predictive power needed from a theory. When studying mathematics, in the course of accumulating mathematical knowledge, the students go through successive processes of generalisation, while also experiencing the extension of various mathematical systems (Tirosh & Tsamir, 2006); the most typical case of such kind of generalisation or extension is the number concept (see, Merenluoto & Lehtinen, 2004). Tirosh and Tsamir (2006) argued that during the early years, children develop and entrench a rich concept of counting numbers, positive integers as represented by the integer list and participate in operations of addition and subtraction. For each positive integer there is an answer to the question "Which is the next one/number?" It is the next word in the count list. Later, they are exposed to new entities-fractions and decimals. To develop an

understanding that fractions and decimals are numbers, children must restructure their concept of number from the count number to rational number and make a host of changes, including developing a mathematical understanding of division. They must also rethink many core assumptions about what numbers are. Various reflections of conceptual understanding of fractions develop in parallel across samples of children (Gelman, 1991). Researchers have also found evidence of resistance to change, as children initially try to assimilate fractions to their concept of counting number. Finally, understanding rational number was strongly related to understanding weight as a continuous property of matter (Smith, Solomon & Carey, 2005).

The conceptual change approach has the potential to enrich a social constructivist perspective and provides the needed framework to systematise the above-mentioned widespread findings and utilise them for a theory of mathematics learning and instruction (Verschaffel & Vosniadou, 2004). Some of the more obvious advantages of exploring the instructional implications of the conceptual change approach are the following: It can be used as a guide to identify concepts in mathematics that are going to cause students great difficulty, to predict and explain students' systematic errors and misconceptions, to provide student-centered explanations of counter-intuitive mathematical concepts, to alert students against the use of additive mechanisms in these cases, to find appropriate bridging analogies, etc. In a more general fashion, instructional implications of the conceptual change approach highlights the importance of developing students who are intentional learners and have developed the metacognitive skills required to overcome the barriers imposed by their prior knowledge (Schoenfeld, 1987; Vosniadou, 2003).

Large empirical evidence exists that the conceptual change approach is now being examined in mathematics (Vosniadou & Verschaffel, 2004) and has been applied to explain phenomena in mathematics teaching and learning in general (Ni & Zhou, 2005; Tirosh & Tsamir, 2006; Vamvakoussi & Vosniadou, 2007). Hence, the conceptual change approach is applied to the learning and teaching of concepts in mathematics [algebra] that seem to be affected by the challenges of problematic learning, and the existence of false ideas based on enduring misconceptions that traditional instructional methods cannot overcome (diSessa, 2006).

Use of conceptual change learning model is one way of closing the gap between children's science and scientists' science (Hewson, 1981; Posner et al., 1982) especially in bringing about the reorganisation of students' pre-existing knowledge to be consistent with target knowledge by creation of cognitive conflict. A teaching model designed to diagnose and modify students' conceptions to bring about conceptual change exploiting cognitive conflict is diagnostic (conflict) teaching. In the next section, literature relevant to diagnostic (conflict) teaching as a teaching and learning strategy aimed at conceptual change is discussed.

2.5 Diagnostic (Conflict) Teaching as a Teaching and Learning Strategy Aimed at Conceptual Change

Too often, when teachers find errors in a child's work, they mark the example wrong, assume that the child did not master the basic facts, and prescribe further drill. Careful analysis of errors through observation and interviews with the individual child is essential.

(Pincus, 1975, p. 581)

According to the American Association for the Advancement of Science (AAAS, 1990), effective learning often requires more than just multiple connections of new ideas to old ones; it sometimes requires that people restructure their thinking radically. That is, to incorporate some new ideas, learners must change the connections among the things they already know, or even discard some long-held beliefs about the world. The alternatives to the necessary restructuring are to distort the new information to fit their old ideas or to reject the new information entirely. Students come to school with their own ideas, some correct and some not, about almost every topic they are likely to encounter. "If [student's] intuitions and misconceptions are ignored or dismissed out of hand, their original beliefs are likely to win in the long run, even though they may give the test answers the teachers want" (Hodzi, 1990, pp. 52-63). Mere contradiction is not sufficient; students must be encouraged to develop new views by seeing how such views can help them make better sense of the world.

Research shows (Swan, 1983) that students may appear successful in following the normal curriculum yet still have a number of serious misconceptions which cause persistent errors. For example, Swan (1983) noted that students acquire from their elementary school experience the awareness that multiplying a number makes it bigger. This may or may not be made explicit. Later, when decimal numbers come into use, the fact that this principle does not apply when the multiplier is less than 1 may again not be made explicit, "they will continue to suggest that 0.2 x 0.3 = 0.6 despite being taught methods which give the result 0.06" (Kennewell, 1994, p. 3) and this also leads to errors in choosing correct operations. Swan (2001) believes that misconceptions are a natural stage of conceptual development.

But, the challenge is how can students' errors and misconceptions be used to produce more effective teaching? Studies carried out by Onslow (1986) and Swan (1983) indicated that merely being aware of student misconceptions was not sufficient, since even when teachers explained fully correct methods based on the knowledge of children's misconceptions, the misconceptions in most cases remained unchanged. Confrey (1990, p. 109) states that "before children will change (their) … beliefs, they must be persuaded that the(ir) ideas are no longer effective or that another alternative is preferable." This is undoubtedly why misconceptions are so resistant to change. An analysis of conceptual change by Hewson (1981) indicated that "… change is relatively difficult to obtain unless one is dissatisfied with his or her present belief and sees an alternative framework as intelligible, plausible, and fruitful" (p. 43). Bell (1993) suggests that if students were to overcome these misconceptions, the teaching must expose and discuss them and assist the student to achieve a resolution which is consistent with the claim made by Confrey (1990) when he wrote: "… learning … involves progressive consideration of alternative perspectives and the resolution of anomalies" (p. 32). This presents a need for 'conflict' in teaching. Research shows that teaching is more effective when it assesses [or diagnoses] and uses prior learning to adapt to the needs of students (Black & William, 1998). This section describes the diagnostic teaching model designed to diagnose and modify students' conceptions through exposure and rejection.

The diagnostic teaching approach is of interest because it is designed primarily to bring about conceptual change rather than teach new facts or procedural skills. The

awareness of a need for change in one's cognitive framework is brought about by a realization that something important doesn't fit in (Stacey, Sonenberg, Nicholson, Boneh, & Steinle, 2003). For this reason, Bell's (1980) research was undertaken in an attempt to find a method of teaching which would eliminate misconceptions and establish correct notions which could be transferrable to new situations and which could contribute to long term learning (Bell, Swan, Onslow, Pratt, & Purdy, 1985). Bell and his associates based at the Shell Centre for Mathematics Education have seen Piaget's views as providing theoretical justification for their view that the best way to overcome a misconception is by engineering a cognitive conflict. The notion of provoking cognitive conflict is not alien to a student's mathematical learning experiences: consider the conflict caused by noticing that a small object appears to be heavier than a big object, a tall container holds less water than a short container or that the digit 4 can be worth different amounts depending on the position of that digit in, for example, two-or three-digit numbers. These researchers have successfully devised activities for stimulating and confronting cognitive conflict under the label of diagnostic teaching or conflict teaching (Underhill, 1991).

Diagnostic teaching, also known as conflict teaching (Underhill, 1991), lays great importance on students' wrong answers and the role of cognitive conflict in the process of teaching and learning. This type of teaching is based largely on the theories of Piaget in that the method involves initially ascertaining the existing cognitive structures (as related to the topic or task being taught) possessed by the students, and on the basis of these structures attempting to both expand and refine them by presenting tasks which lead to cognitive conflict (Perso, 1991). Creating tension and cognitive conflict may then be resolved through discussion (Bell, 1993b; Brousseau, 1997). The notion that a pre-existing misconception cannot be replaced or corrected unless it is first recognised as being incorrect and cognitive conflict facilitates exposure of misconceptions was succintly described by Bell et al. (1985) when they wrote: "… persistent, well retained bodies of knowledge and skill are those which are richly inter-connected and that fresh ideas are often rejected until they become so strong that they force a reorganisation of the existing material into a new system, holding together the new idea and the transformed old ones" (p. 3).

The diagnostic teaching approach is "based on identifying key conceptual points and misconceptions. Teaching is then designed to focus on these points, giving students

substantial open challenges, provoking cognitive conflict by exposing misconceptions, and resolving through intensive discussion" (Bell, 1993b, p. 115). This approach involves "appropriate exploratory work in the topic first followed by conflict/discussion lessons where students are given carefully designed tasks together with encouragement to voice their ideas and question each other's assumptions with the teacher's support" (Kennewell, 1994, p. 3). A key feature of the method is that students are placed in situations in which they are required to make their ideas and beliefs explicit, exposing them to challenge in a supportive situation. Their ideas may be carefully challenged by the teacher through prompts and questions which will help them to adapt their thinking, or through argument with their peers who may be equally, but differently, wrong (Kennewell, 1994).

When students come to realise that there is something wrong with their existing interpretation of the situation, cognitive conflict occurs, and creating such conflict in the student is one strategy that has been recommended for situations where students need to move from one way of thinking to another (Bell, 1993a; Light & Glachan, 1985). New ideas are constructed through reflective discussion. Opportunities are provided for students to consolidate what has been learned through the application of newly constructed concepts. The role of the teacher in contrast to traditional roles of being an 'expositor' or a desk-to-desk imparter of knowledge is required to be a facilitator of the diagnostic teaching process (Perso, 1991). In essence, the teaching methodology involved carefully chosen task or problem to be resolved through a process of discussion with peers, shared methods, articulation of conflicting points of view and whole-class discussion. Through such an approach the conflict is resolved and new learning is consolidated.

The aim of diagnostic teaching is to help students to adopt more active approaches towards learning (Swan, 2006). Diagnostic teaching lesson places emphasis on students' interaction and participation, particularly in the whole class elements of each lesson, and encourages discussion and co-operation between students in group/paired work (Bell, 1980, Swan, 1983, 2006). This method allows students to engage in discussing and explaining ideas, challenging and teaching one another, creating and solving each other's questions and working collaboratively to share methods and results (Swan, 2006). The model of teaching emphasises the

interconnected nature of the subject and it confronts common conceptual difficulties through discussion and also allowing students opportunities to tackle problems before offering them guidance and support. This encourages the students to reapply pre-existing knowledge and allows the teacher to assess and then help them build the knowledge. This approach has a thorough empirically tested research base (Swan, 2006). Implementing a cognitive conflict approach has been reported in studies on a variety of topics, such as division (Tirosh & Graeber, 1991), sampling and chance statistics (Watson, 2002), decimal domain (Bell, 1993b; Liu, Huang, & Chang, 2007; Swan, 1983), fractions (Tanner & Jones, 2000), literal symbols (Fujii, 2003) or directed numbers (Shiu & Bell, 1981; Gallardo, 2002). However, Swan (1983, 2006) warns that this approach is challenging but research shows that it develops connected, long-term learning (Swan, 1983).

Earlier research into provoking cognitive conflict (Bell, 1993a, 1993b) suggests that the benefit to long-term learning is greater when students encountered misconceptions through their own work than when teachers choose to draw attention to potential errors/misconceptions in their introduction to topics. Using a teaching methodology called diagnostic teaching, the Diagnostic Teaching Project based at Nottingham University Shell Centre reported long-term retention of mathematical skills and improvement in achievement when using teaching packages that were designed to elicit and address students' misconceptions.

Developing the work of the Diagnostic Teaching Project, Swan (2001) believes that "mistakes and misconceptions should be welcomed, made explicit, discussed and modified if long-term learning is to take place" (p. 150). Askew and William (1995) commented that "we have to accept that pupils will make some generalisations that are not correct and many of these misconceptions remain hidden unless the teacher makes specific efforts to uncover them" (p. 12). Swan (2005) argued that if students' errors and misconceptions are more effectively addressed through being encouraged to examine their own ideas and confront inconsistencies and compare their own interpretations with those of other students and with accepted conventions, then adequate time would have to be given for reflection and discussion. He reiterated that when students feel less pressurised to give quick response which is correct, creating more time for such reflection and dialogue would clearly lead to less mathematical

content being taught but, perhaps, more long-term mathematical learning taking place.

Despite its promise, this teaching method has not been adopted as normal teaching practice, at least partly because of the difficulty of reliably generating usefully wrong answers (Stacey et al., 2003). As mentioned earlier in this review, diagnostic teaching is challenging and is not easy to implement (Swan, 1983, 2005). It is more time consuming than traditional or rote teaching methods. It needs an environment – a community of enquiry - which is missing from many classrooms at present (Kennewell, 1994; Swan, 2006). Furthermore, it requires that teacher possesses well-developed facilitation skills and a thorough understanding of the topic or phenomenon in question in terms of the planning of the tasks and intervention during learning activities to provide scaffolding and challenge (Hoyles & Sutherland, 1989; Noss, 1986). Or else, peer discussion in potential conflict situations may merely reinforce naïve conceptions, and the teacher may be too ready to provide the 'right answer' in a way which will not influence the students' intuitive thinking (Kennewell, 1994).

The research dealing with cognitive conflict in science and mathematics education is divided, with some research finding cognitive effective at advancing students' cognitive structure, and other research finding that cognitive conflict did not lead to conceptual change. The findings demonstrate that cognitive conflict can have constructive, destructive, or meaningless potential (Fraser, 2007). A considerable number of researchers (Kang, Scharmann, Kang, & Noh, 2010; Lee & Kwon, 2001; Liu, Huang, & Chang, 2007; Treagust & Duit, 2008) found cognitive conflict an effective method of changing students' existing conceptions (if incorrect, misconceptions). Other studies (Dreyfus et al., 1990; Tirosh, Stavey, & Cohen, 1998) found cognitive conflict was not consistently effective at modifying students' existing conceptions. A possible disagreement among the researchers as to the effectiveness of cognitive conflict is that there are different types of cognitive conflict and there may be a significant difference in the level of conflict required to properly resolve these different types of conflict (Fraser, 2007). Lee and Kwon (2001) describe sub-types of cognitive conflict researchers have categorised, for example: conflict between two internal concepts; conflict between two external

sources of information; or conflict between and internal concept and an external source of information. Kwon, Lee and Beeth (2000) and Lee and Kwon (2001) found that the different types of cognitive conflict demand different levels of conflict to resolve. There were however, key details in the methods of the two groups that attest that the strategy used to manage cognitive conflict is extremely important. The common features of these methods were: 1) an introduction of the relationships and the context of the concept; 2) presentation of a problem that will induce cognitive conflict; and 3) after having generated a conflict it is essential to provide an environment that will facilitate the proper resolution of the conflict.

The literature discussed in the above leads to a conclusion that teaching to avoid children developing misconceptions appears to be unhelpful and could result in misconceptions being hidden from the teacher (and from the students themselves). This implies that a shift in the mindset is needed for the teachers to movefrom planning mathematical lessons to avoid errors/misconceptions from occurring, to actively planning lessons which will confront students with carefully chosen examples that will allow for accommodation/ conceptual change. It appears that effective teaching of mathematics involves planning to expose and discuss erros and misconceptions in such a way that students are challenged to think, encouraged to ask questions and listen to explanations, and helped to reflect upon these experiences. Diagnostic teaching methodology provides for such an opportunity for students to go through the process. This suggests that the more aware the teachers are of the common errors and possible misconceptions associated with a topic, the more effective will be the planning to address and deals with children's potential difficulties. The role of questioning, dialogue and discussion is significant (Swan, 2006; Watson & Mason, 1998) if students are to shift their perspectives on only contributing if they think they have a correct answer they believe is wanted by their teacher.

The preceding section has reviewed literature relevant to diagnostic teaching. During this study it became apparent that in order to arrive at valid conclusions it would be necessary to obtain sound quantitative data as well as qualitative data, the historical perspective of qualitative and quantitative data, and a justification of why this research chose to combine these methods.

2.6   Combining Qualitative and Quantitative Research

Section 2.3 of this chapter gave a detailed description of a quantitative instrument used to measure student's attitudes. Whereas, another quantitative instrument used to measure students' achievement is discussed in Section 3.3. While quantitative research can be used in a myriad of studies effectively, this study chose to use both qualitative and quantitative measures to elicit a broader perspective of the research questions proposed. The purpose of the following section is to give a brief historical perspective of the combination of qualitative and quantitative research, and to support the use of multiple research methods in this study.

The challenge to the historical use of quantitative data as the main measure of social research began in the 1960s. Unfortunately, this challenge caused a split between quantitative and qualitative researchers (Punch, 1998). Many researchers seemed to be of the opinion that either one or the other type of research is superior. On the one side are those who support quantitative research, or research based on objective methods of evaluation. Some of these researchers believe that when both quantitative and qualitative data are used, the quality of the study is diminished (Bernstein & Freeman, 1975). On the other side are those supporting qualitative research which not only allows for subjectivity but encourages it (Briedenhann & Wickens, 2002). Recently, however, there have been more acceptances condoning the use of multiple means of conducting research as a manner of improving research outcomes (Bryman, 1988; 1992; Hammersley, 1992; Howe, 1988). Ultimately, the major factor that dictates which method or methods are used in research resides in the questions that we want answered. The focus of the questions determines the method of data collection and dictates whether qualitative and/or quantitative data can be best used to answer the questions.

For the purpose of this study, both quantitative and qualitative data were deemed necessary to gain a clearer perspective of the questions being asked. First, in order to determine how well students performed on the pre/post-tests on algebra diagnostic test and attitudes toward mathematics, quantitative data were necessary in order to obtain responses to the variables and the relationships that existed among them. A

second portion of the present study attempted to gain more insight into students' attitudes about mathematics and algebra difficulties and misconceptions in general. Interviews designed by the researcher, revolving around the thoughts and perceptions of mathematics, and in particular about algebra, were used for this purpose (whether after the intervention the (mis)conceptions held by the students still persist.

The interview questions were structured in such a way as to attempt to gain consistency in answers. However, conducting the interview often led the interviewer in some divergent paths that ultimately produced a much more well defined view how students perceived mathematics and evaluate whether beginning misconceptions still persisted or not (*there is a conceptual change*). Perhaps the best description for the format used in the present study emanates from Creswell (1998), when he attempts to explain qualitative research as a holistic picture gained by the interviewer through analysing words, reporting on detailed views of the informants, and conducting this research in a natural setting.

The final portion of this research study dealt with the incorporation of qualitative and quantitative data as a means of obtaining a complete spectrum of data. In this area, the literature is somewhat conflicting. Literature in this area seems to indicate various groups of purists who see only qualitative or quantitative data as being used in isolation. These purists are of the opinion that combining the two types of data diminishes the quality of the study. Recent literature, however, reports that the use of both qualitative and quantitative data is gaining acceptance, mainly because both methods allows the researcher to gain more insight into the problem being studied.

2.7   Summary

This chapter reviewed pertinent literature regarding students' conceptions and nature of alternative conceptions, students' difficulties and misconceptions of algebra concepts, students' attitudes towards mathematics, and diagnostic teaching methodology that incorporates cognitive conflict focusing on its correlation to student achievement and conceptual change.

The critical role that students' existing knowledge (conceptions) plays in learning is central to the theory of conceptual change and is embedded within a broad scope of constructivism (Hewson, 1996). The underlying theme in conceptual change literature is that it is difficult to assess. As a result, Posner et al. (1982) developed a conceptual change model in which learning is described as a process in which an individual changes his/her conceptions by capturing new conceptions, restructuring existing conceptions or exchanging existing conceptions for new conceptions. This change to the central conception occurs rationally as a result of perceived conflict with existing concepts in reference to perceived intelligibility, plausibility, and fruitfulness of the new concept.

Fostering academic achievement and conceptual change, as the research indicates, requires a good teaching methodology in an optimal learning environment. A review of the literature provided information about associations between student attitudes, achievement (outcomes) and instruction for using cognitive conflict to assess the effectiveness of diagnostic teaching methodology.

Investigations in mathematics education focus more on students than teachers. With this shift to the learner, it is noted that learning is an active process occurring within and influenced by the learner (Yager, 1991). This emphasis on the learner has resulted in a shift toward student-centered instruction (containing a rich exploratory situation) assoiated with engaging students in discussing and explaining ideas, challenging and teaching one another, creating and solving each other's questions and working collaboratively to share methods and results (Swan, 2006) in mathematics education. A common claim, when mathematics learning is discussed, is that students can master mathematical tasks more easily in groups than individually, and that their understanding of the mathematical content can be enhanced through the contributions of others (Ernest, 1991). The claim has strong theoretical support, e.g. from Vygotsky's inspired research and the notion of "proximal zone of development" or the concept of "scaffolding" pupils' cognitive endeavors (Wood, Bruner, & Ross, 1976) as well as Piagetian research and the notion of conflicting points of view (Doise & Mugny, 1984). In recent years close observations of peer interaction have been used to describe the micro-social processes involved in peer interaction (Brown & Palinscar, 1989; Wistedt, 1994) and

it has been stressed that peer interaction can enlarge and enrich students' reasoning. Studies show that when new ideas are constructed through reflective discussion to facilitate mathematical understanding, students are able to make connections to the real world and remember the new information (Pauls, Young, & Lapitkova, 1999).

Finally, this research uses a constructivist paradigm to examine ways student built on their cognitive structures through diagnostic teaching Units. The units were selected to generate cognitive disequilibrium with students' existing conceptual structures, requiring them to accommodate new conceptual understandings and potentially attaining cognitive equilibrium. Constructivism represents a way of conceptualising our understanding of how learning takes place. From this perspective, constructivism describes knowledge as an entity built upon previous knowledge; we construct and reconstruct knowledge to accommodate new understanding (Ernest, 1996; Spivey, 1995; von Glaserfeld, 1984). Learners come to know their world through their experiences, social-negotiations and reflections. Upon engaging in a new experience, learners reorganise their conceptual framework to assimilate or accommodate this experience (von Glaserfeld, 1984). This is when learning takes (Lerman, 1996), a process captured by Piaget's notion of cognitive disequilibrium, where previous knowledge is challenged by unfamiliar or misunderstood experiences and moves towards cognitive equilibrium, as these experiences are reconceptualised and the knowledge has been constructed and accommodated (Simon, 1995). Following on from Piaget (1957), constructivism is an established theoretical framework for understanding these phenomena and indeed proved fruitful in this study.

The review of the literature in this chapter guides the research and provides the foundation to subsequent chapters. The objective is to provide a theoretical framework that interconnects students' difficulties, conceptions, attitudes, achievement, conceptual change, and the diagnostic teaching intervention to improve teaching and learning mathematics.

Chapter 3

Methodology

3.0   Introduction

This study attempted to determine the difficulties and misconceptions Form 2 (Year 8) students have with algebra and investigated whether diagnostic teaching could have a positive effect on their attitudes and achievement in mathematics. Furthermore, an attempt was also made to determine whether or not there is any evidence of students' conceptual change following a teaching intervention. As stated in Chapter 1, four research questions formed the focus of this study:

1.   Are there learning gains in understanding algebra concepts evident after the six weeks intervention?

2.   What conceptual difficulties and misconceptions do Form 2 Malaysian students have with algebra?

3.   Is there any evidence of students' conceptual change in algebra concepts following the teaching intervention (diagnostic teaching)?

4.   Are students' attitudes towards algebra enhanced after the six weeks intervention?

In order to determine answers to the four research questions, both qualitative and quantitative data were used. The purpose for the use of both types of data arose out of concern by the researcher that the use of quantitative data only would not give a clear enough picture of students' conceptual change and attitudes about mathematics. Furthermore, the researcher felt that face-to-face interviews would be the most beneficial method for gathering these data. Quantitative data can provide some information about student conceptual understanding, likes and dislikes, but it cannot report on the extent to which student conceptions meet the three conditions of intelligibility, plausibility and fruitfulness as described by Hewson and Hewson (1992) as being the status of a person's conception (Treagust, Harrison, & Venville, 1996) and why one has formed these views (likes and dislikes). In addition, using

both quantitative and qualitative data allowed the researcher to use the strengths of each method while compensating for the weaknesses of each (Bryman, 1988, 1992). Therefore, both quantitative and qualitative data were deemed necessary to gain a clearer perspective of the questions being asked. First, in order to determine how well students performed on the pre/post-tests on algebra diagnostic test and attitudes toward mathematics, quantitative data were necessary in order to obtain responses to the variables and the relationships that existed among them. A second portion of the present study attempted to gain more insight into students' difficulties and misconceptions and set out to examine whether or not conceptual change took place (Treagust et al., 1996). Interviews designed by the researcher, revolving around the thoughts and understandings of mathematics, and in particular about algebra, were used for this purpose (whether after the intervention the (mis)conceptions held by the students still persist).

The intial section of this chapter, section 3.1, describes the student population used in the study. A description of diagnostic teaching intervention strategy provides the focus of Section 3.2. This section is covered at length, as some of the literature cited in Chapter 2 described diagnostic teaching methodology that incorporates cognitive conflict is effective and has a positive effect on conceptual change is followed up with the description of the process of the teaching methodology.

A description of data collection techniques, the instruments used to collect these data (both quantitative and qualitative), and data analysis procedures are found in Section 3.3 and 3.4. These sections are structured in a manner that matches each research question with the instrument used to collect data related to the question.

3.1   Students/Subjects

The study was conducted at a low middle-class suburban state secondary school in the Penampang District, located in Sabah, East Malaysia. Seventy-eight Form 2 students from two heterogeneous classes, consisting of 39 students each of high achieving and of below-average achieving, respectively, participated in the study.

The researcher was teaching mathematics for five teaching periods (lessons) a week (40-minute per lesson) in those classes during the 2009 school year. These participants were given the Algebra Diagnostic Test consisting of 24 items and the Test of Mathematics-Related Attitudes (TOMRA) comprising two scales (Attitude to Mathematics Inquiry and Enjoyment of Mathematics Lessons) of 10 items each. Following a teaching intervention, that is, diagnostic teaching that incorporates cognitive conflict in a cooperative learning environment, focusing on its correlation to student attitudes, achievement and conceptual change, all the participants were given the similar same instruments (Algebra Diagnostic Test and Test of Mathematics-Related Attitudes) as posttests.

Nine students were purposefully selected based on their responses to the Algebra Diagnostic Test participate in the qualitative portion of this study. These students were interviewed at the conclusion of the study in an attempt to correlate their responses on the interview with the quantitative data collected throughout the study.

3.2   Diagnostic Teaching Methodology

Effective teaching necessitates working publicly with students' beliefs and difficulties and that the public exposure of public conceptions or misconceptions allows for all learners greater insights into their mathematics (Andrews, Dickinson, Eade & Harrington, 1999). Research has shown that teaching becomes more effective when common mistakes and misconceptions are systematically exposed, challenged and discussed (Askew & William, 1995).

The key aspects of diagnostic teaching as described by Bell, Swan, Pratt, and Purdy (1986) are: the identification and exposure of students' misconceptions, and their resolution through collaborative conflict discussion. Several studies (Fujii, 2003; Perso, 1991; Naiz, 1995; Watson, 2002) have developed methods of managing cognitive conflict that brought about cognitive development. The common features proposed by these researchers used in this study were:

1.	An introduction of the relationships and the context of the concept.
2.	Presentation of a problem that will induce cognitive conflict.
3.	After having generated a conflict it is essential to provide an environment that will facilitate the proper resolution of the conflict.

The introduction to the concept came in the form of asking the students to solve a problem and discussing how they solve it, as there is a "necessity for students to make their conceptions explicit so they are available for change" (Watson, 2002, p. 228). The cognitive conflict was generated by giving a counter example, or two conflicting examples, where the student's familiar method of solving the problem fails. The environment after the conflict varied, but it was necessary to provide the students with an alternative conception that they could understand. This was usually done with a sharing of ideas about why a method failed or defending why one method is superior to another.

The goal of the diagnostic teaching experiment is to break students' deep-rooted tendency to give wrong responses to certain identified algebraic concepts revealed in the literature. To achieve this goal, a series of diagnostic teaching unit lessons were trialed in order to maximize the chance of obtaining conceptual change in the students in accordance to the four principles suggested by Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2004). They are: (a) the necessity of being informed about students' prior knowledge; (b) the need to explicitly address students' preconceptions during instruction; (c) the importance of facilitating students' meta-conceptual awareness (awareness of their beliefs and presuppositions and possible inconsistencies in them) and meta- cognition (monitoring their learning and problem solving processes) in order to bring conceptual change in the conscious control of the learner; and (d) the motivation source for obtaining conceptual change. These researchers reiterate that if teachers want students to invest substantial effort to change their original conceptual structures and presuppositions, they need to have meaningful learning experiences and be actively involved in learning activities.

With regard to the suggestions and discussion in the literature reviewed in Section 2.5 the researcher embarked on a teaching experiment to establish if this intervention strategy does have a positive impact on student conceptual understanding. However,

before the teaching experiment, the researcher needs to be familiar with the likely misconceptions for the chosen area of mathematics to be worked on. Then, there was a need to choose which misconceptions to focus on and select activities that would bring out a limited number of misconceptions making sure that any single answer should not be immediately obvious. The lesson series of the diagnostic teaching unit was interspersed with a rich variety of meaningful, realistic and attractive problem situations, challenging particular mathematical (mis)conceptions, beliefs and habits observed in the students, and actively involving them in the activities.

In the following section, a typical diagnostic teaching lesson implemented throughout the six weeks duration of the teaching experiment as recommended by Perso (1991, pp. 40-41) is explained followed by a sample of a typical conflict teaching lesson:

1. Introductory task – students are initially confronted with a problem containing a rich exploratory situation which contains a conceptual obstacle (misconception) identified earlier through an analysis of the common errors on algebra diagnostic test given prior to the teaching experiment and from literature reviewed in Section 2.2. Students individually come up with an answer and write down their individual responses. Students use their most relevant mental model to approach the problem; their answer reflects that model.

2. Students each set out their mental models by explaining how they arrived at their particular answers. Public differences begin to arouse dissonance; heightened dissonance and some resolution as students try to persuade each other and reach group agreement or working towards consensus as this provokes greater involvement in a less threatening manner and then maintains a greater involvement when discussion involves the whole class. Results are again recorded. At the same time, teacher monitors discussion – picks up aired misconceptions.

3. Class discussion – students are organised in groups of two or four, each group leader or spokesperson presents the group's opinions to the rest of the class, one at a time.This helps ensure that if a whole group has accepted an erroneous conclusion it can be exposed and countered. Wrong responses can

be challenged by other groups or the teacher. The teacher acts in a way as to make the situation unthreatening, while at the same time not providing any positive or negative feedback. The teacher facilitates discussion not indicating who is correct initially but offering checking methods if necessary that might allow students to work out who is right while at the same time providing further provocation or conflicting ideas where necessary in order to ensure the exposure of all misconceptions. Students reach agreement on the correct model, with varying degrees of teacher support: the conflict is resolved and dissonance abates. Need to make sure students at least know which answer was right.

4. Reflective class discussion – students can discuss how errors are made and which misconceptions they are likely to be based on. The teacher can 'sum up' the ideas presented although this is not necessary (the teacher should be continuously aware that the aim is that students resolve conflict on their own based on their own perceptions).

5. Consolidation – a new problem explored. Students are given an opportunity to consolidate new mental model. Students are presented with further questions in the form of written work, which can now be attempted in order to consolidate the newly acquired understanding. Exercises contain built-in feedback of correctness whenever possible, so that students can know immediately if they have made an error.

A typical conflict teaching lesson began with an introductory task where students were confronted with a problem containing a conceptual obstacle by asking them to solve a problem and discussing how they solve it. The cognitive conflict was generated by giving a counter example, or two conflicting examples, where the student's familiar method of solving the problems fails. The environment after the conflict varied, but it was necessary to provide students with an alternative conception that they could understand. This was usually done with a sharing of ideas about why a method failed or defending why one method is superior to another.

Here is a sample lesson (Think of a letter) (Perso, 1981, p. 380) to address the misconceptions that order of operations is unimportant, working in mathematics is always from left-to-right and neglecting the use of brackets when needed in which

students had to write down what the teacher was doing to the letter using signs, symbols and brackets:

Teacher	Carey
"Think of a letter,	y
Add four	$y + 4$
Multiply by 2	$y + 4 \times 2$
Add five	$y + 4 \times 2 + 5$
The answer is 19	$y + 4 \times 2 + 5 = 19$
The letter you thought of must be 3."	

When Carey checked this by putting '3' back into the equation in place of 'y' she got 16 as her answer. Was her equation wrong? If you think that it was, write down what you think it should have looked like and give a reason.

Because of the emphasis of the methodology on group work, it was necessary for either the class to be used to doing group work in mathematics prior to the use of the Diagnostic Teaching material, or for the teacher to explain thoroughly the need for each student to respect the opinions of other students in his/her group and in the class as a whole. The researcher as a class teacher took liberty to explain the dynamics of group work, discussing the importance of each member of the group contributing as well as listening carefully to what is said. The teacher researcher took on a role of a facilitator in contrast to traditional roles of being an 'expositor' or a desk-to-desk imparter of knowledge (Perso, 1991) throughout this teaching experiment.

First, each participant worked on a mathematical problem individually, solving the problem in writing on his/her own. Once all the students had solved the problem alone, they worked with at least one other student on the same problem. Some were grouped in pairs and others in groups of four. The students were asked to discuss their individual solutions with the other person(s) in their group and to reach agreement on a solution, whose conclusion contradicted that of the group. The students were then asked to resolve the contradiction as a group. Thus, the groups followed a two-step decision making process: the first they had to reach an

agreement on the solution; the second, they had to reach an agreement after being exposed to a solution that contradicted the group's solution.

In essence, as well as forcing students to explain their reasoning, and thus exposing misconceptions which may be held, it should be noted that internalised conflict can be aroused simply by listening to the other students discussing their thoughts. Students are also more likely to express their own ideas if they don't know the correct answer. It is expected that not all students in each group will have misconceptions being addressed; in fact often there may be one student in a group or even none. This does not matter since even students who do not have a misconception can benefit from having to put forward a reasonable argument for their understanding of a concept. Indeed, this will undoubtedly consolidate their conceptual understanding.

Briefly summarised, the diagnostic teaching sessions described above typically begin with the challenge that exposes learners' exisiting ways of thinking. Cognitive conflicts occur when the learner recognises inconsistencies between existing beliefs and observed events. This happens, for example, when a learner completes a task using more than one method and arrives at conflicting answers. Activities are carefully designed so that such conflicts are likely to occur. Students are encouraged to openly discuss their errors, many of which they can discover on their own in a non-threatening environment. Because the teacher at no time says what is right and what is wrong, many of the students will not know whether they are in fact, right or wrong. In the group discussion situation they are forced to argue their case; why they think their opinion is right. This can lead to students not being afraid of discussing their opinions – attention is focused on the method rather than the answer. Research has shown that such conflicts, when resolved through reflective discussion, lead to more permanent learning than conventional, incremental teaching methods, which seek to avoid learners making mistakes (Swan, 2005).

3.3   Instrumentation and Data Collection Procedures

Two instruments were utilised in this research to facilitate the collection of data. They are: Algebra Diagnostic Test which assesses students' understanding and achievement; and the Test for Mathematics-Related Attitudes (TOMRA), which measures students' attitudes towards mathematics. These two survey instruments were administered to students in paper-and-pencil format.

This study utilises a paradigm of choices (Patton, 2002) incorporating a mutli-methods approach of both quantitative and qualitative techniques to generate answers to the research questions. Commenting on this dichotomy, Patton (2002) indicated that the collection of qualitative data in conjunction with quantitative data contributes to methodological rigour and by combining methods observers "can achieve the best of each while overcoming their unique deficiencies" (Denzin, 1978, p. 302). Besides, using an interview format allowed the researcher to obtain information beyond students' scores.

Various types of quantitative and qualitative data were collected as a means of responding to the research questions posed in this study. The instruments used for data collection in this study is not totally new but adapted from previous instruments.The next section will describe these instruments: the Algebra Diagnostic Test and Test of Mathematics-Related Attitudes (TOMRA) together with the types of items to be used in this study.

3.3.1   Algebra Diagnostic Test

Research Question 1 attempted to determine if students' achievement would be enhanced by implementing the intervention strategy of diagnostic teaching while Research Question 2 attempted to determine what difficulties and misconceptions do Form 2 Malaysian students have with algebra. In order to ascertain students' difficulties, misconceptions and achievement in algebra, the researcher found it necessary to either locate an exisiting instrument or to construct a new instrument

that would serve the purpose stated above. An algebra diagnostic test was considered by the researcher to be the most efficient means of identifying misconceptions and measure learning gains in terms of time available. Due consideration must be given to ensure that the test would be of valid use by additional researchers and that the algebra diagnostic test would achieve the threefold purpose intended:

1. To determine which misconceptions in algebra predominate;
2. To determine the proportion of students who possess particular misconceptions; and
3. To confirm for each student, the level of attained understanding (or students' achievement) of particular categories of algebraic concepts and as an indication of conceptual change in student ideas.

This search resulted in the location of two instruments, Blessing's (2004) Algebraic Thinking Content Knowledge Test for Students and Perso's (1991) Diagnostic Test that would adhere to the guidelines previously established by the researcher. As a result, Perso's (1991) Diagnostic Test and Blessing's (2004) Algebraic Thinking Content Knowledge Test for Students were adapted and given a name change, Algebra Diagnostic Test.

The purpose of adapting a ready-made instrument was the validity of the instrument. Validation for the use of Algebra Diagnostic Test for students as a pretest and posttest was derived from Perso (1991) and Blessing (2004) along with his recommendation for use of Algebraic Thinking Content Knowledge Test for Students. It was found that the item descriptors were relevant to the algebra content or concepts the researcher had in mind to assess the students. Once the item descriptors were determined, it then became necessary to construct items that were reflective of these descriptors that would achieve the threefold purpose intended. For this purpose, the bulk of the test items used for the algebra diagnostic test were amended from Perso's (1991) Diagnostic Test (16 items) and Blessing's (2004) Algebraic Thinking Content Knowledge Test for Students (8 items). A breakdown of the source of the Algebra Diagnostic Test items may be found in Appendix B.

Following the test amendment and construction, several mathematics teachers in the school were asked to review items for correctness. Each reviewer completed an assessment of the items based on two criteria, one being readability, and the other being the alignment of the test item with the item descriptor. Comments from the reviewers were collected, and items that showed consistency on the part of the reviewers were amended.

After all items of concern had been amended, the draft version of the test was administered to approximately 300 students thoughout the district. Each item scored one point with a maximum of 24 points. Tests were scored and items that demonstrated a high degree of incorrect responses, or items that showed a large degree of correct responses, were amended and analysis of the results that the test was valid for use with Form 2 students. Careful attention was also paid to the time taken by students to complete the test, as the researcher was aware of the time constraints within the various classes to be tested. This trial was conducted as a means of establishing face validity of the test.

Following this process, the final form of this instrument resulted in an assessment with seven descriptors containing three items in four of the descriptors and four items in three descriptors. The final version of the Algebraic Diagnostic Test can be found in Appendix C and a list of the item descriptors can be found in Appendix D.

3.3.1.1 *The test administration*

The test was administered twice, one as a pretest before the teaching intervention and another as a posttest after the 6-week diagnostic teaching experiement. This was done in order to minimise testing material that the students merely remembered from their previous instruction; it was intended that the instrument would test the students' understanding of algebra. Students were given 40 minutes in which to complete the test, but more time was given if the teacher felt it was necessary. The extended time was allowed since the test had not been designed to see how quickly the students could complete the tasks set but how much they understood of the concepts being tested. The students were instructed to take their time and to consider their choice of

answer carefully, rather then 'rushing' to try and finish the test. They were advised to avoid random guessing of an answer.

The researcher marked all papers. Individual responses by each student for every question on the test paper were recorded on the spreadsheet.

3.3.2 Test of Mathematics-Related Attitudes (TOMRA)

Research Question 4 asked if students' attitudes about algebra were enhanced after a teaching intervention. As a means of answering this question, this study chose to use a modified version of the Test of Science-Related Attitudes (TOSRA), renamed the Test of Mathematis-Related Attiudes (TOMRA) which is shown in Appendix A.

As a means of assessing whether the diagnostic teaching intervention had a positive effect on students' attitudes about mathematics, an amended form of the Test of Science-Related Attitudes (TOSRA) was used. In its original form, the TOSRA was used to measure the science-related attitudes among secondary school students in seven districts (Fraser, 1978). This form of assessment, with its seven scales totaling 70 items, was determined to be too difficult in wording and the total number of items was too many to be valid for use with Form 2 students. Furthermore, the purpose of this study deals with measuring student attitudes towards mathematics as opposed to science. As a result, the amended form of the TOSRA resulted in a test renamed the Test of Mathematics-Related Attitudes (TOMRA).

In order to assess students' attitudes towards mathematics in my study, two scales from the TOSRA (Fraser, 1981) were used. The scales chosen were Enjoyment of Mathematics Lessons and Attitude to Mathematics Inquiry. Each of these scales contained rewritten mathematics form of the original 10 items of the two scales mentioned above. For the new instrument question such as "I dislike science lessons" were changed to "I dislike mathematics lessons." Odd-numbered items and even-numbered items were used to represent Enjoyment of Mathematics Lessons and Attitude to Mathematics Inquiry respectively. Each scale contains 10 questions; half of the items are designated as positive, and another half of the items are designated

negative. Participants responded using a five-point Likert scale ranging from 'strongly agree' to 'strongly disagree.' Postive items are scored by allotting 5 for 'strongly agree' and 1 for 'strongly disagree' responses. Negative items are scored by allotting 1 for 'strongly agree' and 5 for 'strongly disagree' responses.

For the purpose of the actual study, the TOMRA was administered to students prior to the inception of diagnostic teaching as a pretest. The teacher researcher was present during the administration of the instrument; however, the researcher refrained from interacting with students during the administration. Following the diagnostic teaching intervention, the TOMRA was again administered to students under the same conditions as the intial administration as a posttest. Validation for the use of the TOMRA as a pretest and posttest was derived from Fraser (1981) and his recommendation for the use of the TOSRA.

3.4   Data Analysis Procedures

In analyzing the data the researcher strove for methodological rigour (Clement, 2000). But before discussing data analysis procedures it was necessary to briefly describe the relationship between reliability and validity of the research instruments used in this study. While conducting research we make lots of different inferences or conclusions on matters related to the process of doing research. Like the bricks that go into a building wall, these intermediate process and methodological propositions provide the foundation for the substantive conclusions that we wish to address which involve measurement or observation (Trochim, 2000). And according to the author (Trochim, 2000), whenever we measure or observe we are concerned with whether or not we are measuring what we intended to measure and/or do we have a dependable measure or observation in a research context. We reach conclusions about the quality and consistency of our measures – conclusions that will play an important role in addressing the broader substantive issues of our study. For this purpose the ideas of validity and reliability will be discussed in the subsequent section prior to the discussion on qualitative and quantitative data.

3.4.1 Validity and Reliability

Reliability and validity are two main psychometric characteristics used to assess the quality of the measuring instruments used in research. The central concept in measurement is reliability which essentially implies consistency or repeatability. The two main components to consistency are stability which refers to consistency over time and internal consistency (Punch, 1998). From the constructivist position, reliability in interpretive research is replaced by the idea of dependability (Guba & Lincoln, 1989).

Validity is the second central concept in measurement. The meaning of concept validity can be found in answer to the question: how do we know that the instrument is measuring what we think or designed it to measure? A second view focuses on whether the interpretations we make from the measurements are defensible, concentrating on the interpretation of data rather than the measurement instrument (Punch, 1998). Validity can be classified as internal or external. Internal validity is defined from a positivist paradigm as the degree to which variations in an outcome or dependable variable can be attributed to controlled variation in an independent variable. External validity defined from a positivist perspective is the approximate validity with which we infer that the presumed cause-effect relationship can be generalized to different individuals, situations, or time (Guba & Lincoln, 1989).

In qualitative research, validity relates to whether findings of the study are "true and certain" (Guion, 2002, p. 1). "True" in the sense of the results accurately reflecting the real conditions and "certain" in the sense that the findings are supported by evidence. "Certain" also implies that the "weight of the evidence" supports the conclusions and there is no reason to doubt the results (Guion, 2002, p. 1).

Among the threats to internal validity the researcher identified in this study were of maturation and testing. The maturation threat can operate when biological or psychological changes (over the six weeks) occur within the students and these changes may account for in past or in total for effects discerned in the study. The testing threat may occur when changes in the test scores occur not because of the

teaching intervention but rather because of repeated testing. This is of particular concern when identical pretest and posttest were administered in this study.

3.4.2 Qualitative Data

Qualitative data for this study were collected from student interviews. A total of 9 students were selected purposefully to be interviewed. Student interviews were recorded and transcribed verbatim in order to allow the researcher the ability to thoroughly assess student conceptual understanding (difficulties, misconceptions and conceptual change) of mathematics. For the purpose of the interview the researcher chose to use what Fontana and Frey (1994) characterised as a semi-structured interview. The semi-structured form of interview allows the researcher to gain the most amount of insight into the difficulties and misconceptions students encountered, examine whether or not conceptual change took place or determine the status of a student's conceptions; while at the same time allowing the interviewer to use structured questions as a means of keeping the interview focus within certain succinct areas of importance. The maximum length of each interview was 15 – 20 minutes.

It was decided that an efficient way of checking the validity of the Algebra Diagnostic Test was to interview students from the study group. The misconceptions previously identified would no doubt be revealed in the errors made by the students interviewed. By examining the thought processes of students in this way, it was also hoped that any misconceptions which had previously not been revealed, would be identified. It was also necessary for the interviewer to use probes to facilitate student problem solving (e.g., if students do 'this,', then prompt for "that"). Students were asked to explain their thinking and selection of solution strategy (e.g., "Tell me more about your thinking on this problem"). Questions proposed by Hewson and Thorley (1989, p. 550) like, "How would you explain that to a friend?" were included to encourage students to reflect on their own conceptions and thus to elicit status information (Treagust et al., 1996). It was hoped that through students'reasoning and explanation, the researcher can draw accurate conclusion on whether students have

understood the concept in question and as an indication of successful conceptual change.

The researcher was particularly interested in student participation during class, student perceptions of the diagnostic teaching unit taught, and student behavior throughout the teaching of the unit and the testing procedure. All of the qualitative data were then analysed and combined with quantitative data as a means of answering the research questions.

3.4.3 Quantitative Data

The Algebra Diagnostic Test and TOMRA data were analysed to verify the internal consistency of each scale and discriminant validity. The internal consistency of each scale on the Algebra Diagnostic Test and TOMRA were determined using the Cronbach (1951) alpha coefficient using the student as the unit of analysis.

The internal consistency, mean, standard deviation, and t-value were statistical analyses used to evaluate and interpret data obtained from the diagnostic test and the TOMRA scales. The average item means (scale mean divided by the number of items) were used enabling important camparisons of the scales. Since these achievement measures involved a pretest-posttest design, it was necessary to use statistical t-tests for paired samples with the resulting p-value to form a decision. If the p-value is .05 or greater, then the relationship between variables is considered to be inconclusive. Then, there is no need to compute Cohen's d (effect size index) to provide information about how strongly the variables are related. However, if the p-value is less than the traditional value of .05, the results are statistically significant, and support is inferred for the relationship under study (Gigerenzer, Swijtink, Porter, Daston, Beatty, & Kruger, 1995). In such a situation, effect size index (the difference in means expressed in standard deviation, σ, units) were calculated to provide information about the magnitudes of the effect. The estimates of magnitude of effect or effect size tell us how strongly the variables are related, or how large the difference between variables is. Its calculation enables immediate comparison to increasingly larger numbers of published studies. Cohen (1992) proposes as a convention, the arbitrary effect size (*d*) values of .20 as small, .50 as medium, and

.80 as large enables us to compare an experiment's effect-size results to known benchmark showing how far and in what direction a given treatment pushes performance along the normal distribution curve (Cohen, 1988). Abelson (1995) predicts that "as social scientists move gradually from the reliance on single studies and obsession with null hypothesis testing, effect size measures will become more and more popular" (p. 47).

3.5   Summary

This study involved four research questions. The nature of the research questions mandated that various instruments be used in order to gain answers relevant to the questions being asked. As this study used Form 2 students (Grade 9 equivalent) as the sample population, and as instruments needed to collect data from this sample population were not readily available, the researcher found it necessary to either design new instruments for data collection, or to amend presently existing surveys. For this reason, the researcher amended various instruments. These instruments included an amended form of the Algebraic Thinking Test for students renamed the Algebra Diagnostic Test and amended form of the Test of Science-Related Atiitudes (TOSRA) renamed the Test of Mathematics-Related Attitudes (TOMRA). As a result, the preceding section has given a description of the various instruments used in this study.

This chapter has given a comprehensive description of diagnostic teaching methodology and how it was to be implemented. A detailed description of data collection and analysis procedures including quantitative and qualitative data was also provided.

Chapter 4

Findings and Discussion

4.0   Introduction

The previous sections have presented an analysis of the quantitative and qualitative data collected during this study. The following section will include a discussion of the findings based on these data. The purpose of this chapter is to report the results of this study based on the analysis of the findings. The findings are grouped into four sections based on the four research questions. While the results of the study confirm a number of findings of other researchers, the findings go beyond those discussed in the literature.

4.1   Response to Research Question 1

Are there learning gains in understanding algebra concepts evident after the six weeks intervention?

The effectiveness of diagnostic teaching on student achievement or learning gains is addressed in this section. In order to investigate this association, data from a diagnostic test administered prior to the beginning of the teaching experiment and posttest at the conclusion of the teaching experiment were collected and analysed for the 78 Form two students. Data from all 78 students were used as no one exited the school throughout the period. The algebra diagnostic test posttest was administered (October 2009) at the end of the teaching experiment to determine students' growth in knowledge. Students' scores on the tests were recorded and the average or mean scores were calculated to ascertain each student's achievement in algebra. The diagnostic test consists of 24 one-point items for a maximum total score of 24 points. The total score on the test was used as the criterion measure to evaluate instructional effects of diagnostic teaching on students' understanding of algebra concepts. This

section presents the results of the pretest and posttest given to students who participated in the study.

One measure of the appropriateness of the test involved calculation of the difficulty and discrimination indices for the test items. Difficulty index or item facility is defined as the proportion of students who answered a particular item correctly. A discrimination index (or item discrimination) is the ability of the item to differentiate those students with the knowledge from those with less. To calculate the discrimination index, subtract the number of students in the lower group (typically divided into thirds with the middle group excluded) that got an item correct from those of the upper group, and divide by the number of students that made up the the upper or lower group. The range of difficulties on the pretest ranged from the easiest item with a difficulty of .91, to the most difficult item with a difficult index of .21. On the posttest, the range of difficulty ranged from the easiest item with a difficulty index of .91, to the most difficult item with a difficulty index of .21. The easiest and the most difficult items remained consistent between the pretest and posttest. Analysis of the item difficulty index that the pretest contained 4 items rated easy, 9 items rated moderate, and 11 items rated difficult. The posttest resulted in 6 items easy, 11 items moderate, and 7 items rated difficult. Overall, these results indicate that the tests were skewed toward a preponderance of moderate to difficult items with the mean difficulty indices being .54 and .63 for the pretest and posttest respectively. A summary of the data for the difficulty and discrimination indices is shown in Table 2.

Additionally, when analysing the item discrimination index, the results indicated overall that the discrimination on the pretest resulted in 5 items being rated good, 13 items rated as fair, and 6 items rated poor. The posttest rendered somewhat better discrimination with 9 items rated as good, 9 items rated as fair, and 6 items rated as poor. The mean discrimination indices from the pretest and posttest were .31 and .36 respectively. Most psychometricians would say that items yielding positive discrimination index values of 0.30 and above are quite good discriminators and worthy of retention for future exams. Most experts consider a discrimination index greater than .40 to be acceptable without further revision, discrimination index between .20 and .39 to be worthy of retention but to be revised and items with a

discrimination index less than .19 to be deleted (Aiken, 1997; Cohen, Swerdlik & Smith, 1992).

Table 2　　　　Difficulty and discrimination indices for the items in the pretest and posttest of the Algebra Diagnostic Test (N=78)

| | Indices for | | | |
| | Pretest | | Posttest | |
Item	Difficulty	Discrimination	Difficulty	Discrimination
1	.64	.50	.67	.54
2	.81	.50	.86	.42
3	.33	.33	.53	.69
4	.81	.08	.87	-.08
5	.79	.33	.81	.38
6	.74	.31	.78	.27
7	.32	.38	.36	.42
8	.91	.19	.91	.19
9	.69	.42	.65	.54
10	.50	.31	.56	.69
11	.33	.00	.37	.35
12	.27	.00	.44	.19
13	.78	.58	.79	.35
14	.45	.35	.49	.27
15	.21	.35	.28	.15
16	.81	.31	.83	.38
17	.55	.58	.49	.61
18	.72	.19	.81	.19
19	.40	.04	.56	.15
20	.21	.35	.21	.27
21	.32	.35	.62	.31
22	.45	..35	.68	.58
23	.51	.38	.71	.38
24	.51	.31	.64	.46
Total	13.06	7.49	14.92	8.70
Average/Mean	.54	.31	.62	.36

The internal consistency of the algebra diagnostic test pre and posttest were established by calculating Cronbach's alpha coefficient (Cronbach, 1951). Cronbach's coefficient alpha estimates the reliability of a scale by determining the internal consistency of the average/mean correlation of the items in the test. It is a measure of the squared correlation between observed scores and actual scores, embedded in a theory that the observed score is equal to the trus score plus the measurement error. The internal consistency for the 24 items on the algebra diagnostic Tests are shown in Table 3 using the individual as the unit of analysis, the alpha coefficient for the pretest was .63 and .72 for the posttest. As expected, the

small sample size (78) resulted in a low alpha reliability values. Based on Nunnally (1967, 1978), .7 has been indicated as representative of acceptable reliability. According to this accepted value, the Cronbach Alpha reliability for the pretest and the posttest determined that each test had an acceptable reliabilty as indicated in Table 3.

Table 3   Mean, standard deviation and Cronbach Alpha Reliability for pretest and posttest of the Algebra Diagnostic Test (N=78).

Test	No. of items	Mean	Standard Deviation	Cronbach Alpha Reliability
Pretest	24	13.06	3.54	.63
Posttest	24	14.91	3.96	.72

The significant contribution for each one of the variables and interactions is now discussed in some detail. In this connection, one must remember that the number of students in this study was rather small (n=78) and that it is relatively difficult to obtain statistically significant differences using a p-value with a small n is because the p-value (significance test) depends on essentially the size of effect and the size of the sample (Cohen, 1994; Harlow, Mulaik, & Steigler, 1997; Thompson, 1999) However, what is important here is not whether these differences are statistically significant but rather that they are educationally significant (Begle, 1972). It is assumed in the ensuing discussion that statistical significance is necessary but not a sufficient condition for educational significance.

The average item mean, average item standard deviation and t-test for paired samples for differences between pretest and posttest scores on the algebra diagnostic test using the individual as the unit of analysis are reported in Table 4.

Table 4   Descriptive statistics and comparison of mean scores, pretest and posttest responses and gain score on the Algebra Diagnostic Test

Descriptive Statistics	N	Pretest	Posttest	Gain score	t-value	Effect size
Mean	78	13.06	14.91	1.85	9.27*	.49
Standard Deviation	78	3.54	3.96			

*p< .05 (there is statistical difference between pre-and-posttest mean scores)

The pretest average mean score was 13.06 while the posttest average mean was 14.91. The standard deviations ranged from 3.58 to 3.96 for the pre and post algebra diagnostic test. These data for both the pretest and posttest indicate that the scores were clustered around the mean and stability of the results. The results show that there were statistically significant differences between the pretest and posttest (t=9.27, p<0.05) indicating that students' gain in understanding improved at a statistically significant level. In order to estimate the degree of differences in addition to statistical significance, the effect size was determined as recommended by Thompson (1998) and Cohen (1988). The effect size is the mean difference between groups in standard deviation form or the ratio of the difference between means to the standard deviation. An effect size is equivalent to the "z-score" of the standard normal distribution. The effect size for algebra diagnostic test pretest and posttest was .49 with the individual as the unit of analysis, suggesting a moderately educationally significant difference between the pretest and posttest scores for the test. Therefore, an effect size of .49 as shown for the pre and post test means that the score of the average person in the posttest group is 0.49 standard deviations above the average student in the posttest group indicating that the mean of the posttest (treated) group is at the 69th percentile of the pretest (untreated) group (refer to Cohen (1988) for an interpretation of Cohen's *d* as the average percentile standing of the average treated participant relative to the average untreated participant).

Comparison of pre-and-post test scores for each question

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
ALGpre	50	63	26	63	62	58	25	71	54	39	26	21	61	35	16	63	43	56	31	16	25	35	40	40
ALGpost	52	67	41	68	63	61	28	71	51	44	29	34	62	38	22	65	38	63	44	16	48	53	55	50
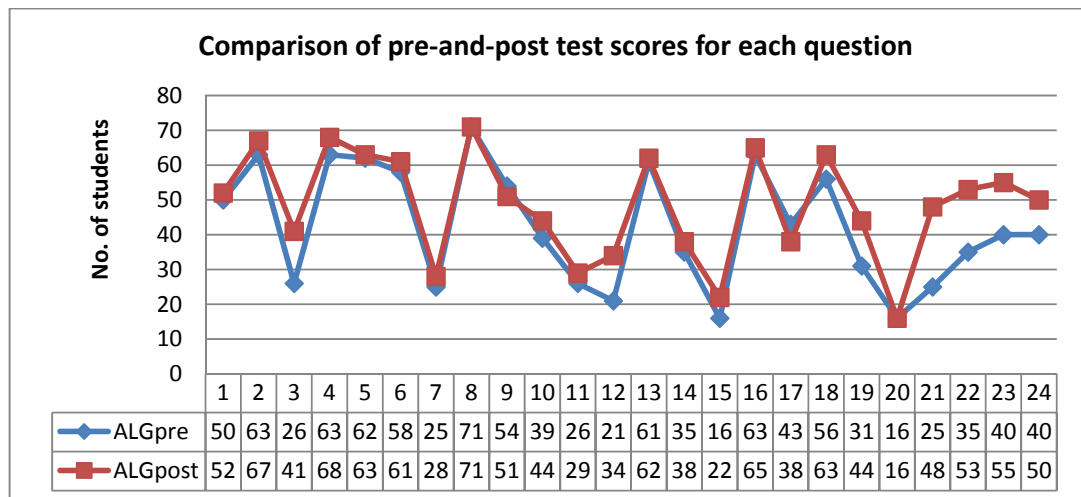
Figure 1   Comparison of pre-and-post test scores for each question

Totals obtained by 78 students for the pre-and-post test (raw scores) for each of the 24 questions were plotted to produce the composite graph in Figure 1. Except for questions 8 and 20 (where the number of students answered correctly remained the same, 71 and 16 respectively) and questions 9 and 17 (a drop of 3 and 5 students respectively), this chart reveals a trend of increasing number of students with more questions correct for the post test with the most significant improvement for question numbers (Items) 4, 12, 19, 21, 22, 23 and 24.

For the observed scores, students performed better in the post test. As expected, the comparison charts of Figures 2 and 3 reveal an increasing distribution of students getting more questions correct for the post test as compared to pre test. The range for the the number of students pre test score was from 4 to 20 whereas the post test score was from 6 to 22 with significant increase in number of students scoring 18, 19, 20, 21 and 22 questions correct out of a total of 24 questions. For example, the number of students scoring 18 questions correct increased from 2 to 12, an increase of 10 students. The number of students getting 20, 21 and 22 questions correct increased by 4, 3 and 2 respectively after the intervention programme (refer to Figure 2).

Comparison of overall pre-and-post test achievement

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
ALGpre				2		2	2	5	6	5	5	6	10	5	9	9	9	2	2	1				
ALGpost						1	1	6	2	3	2	4	7	9	7	5	6	12	3	5	3	2		
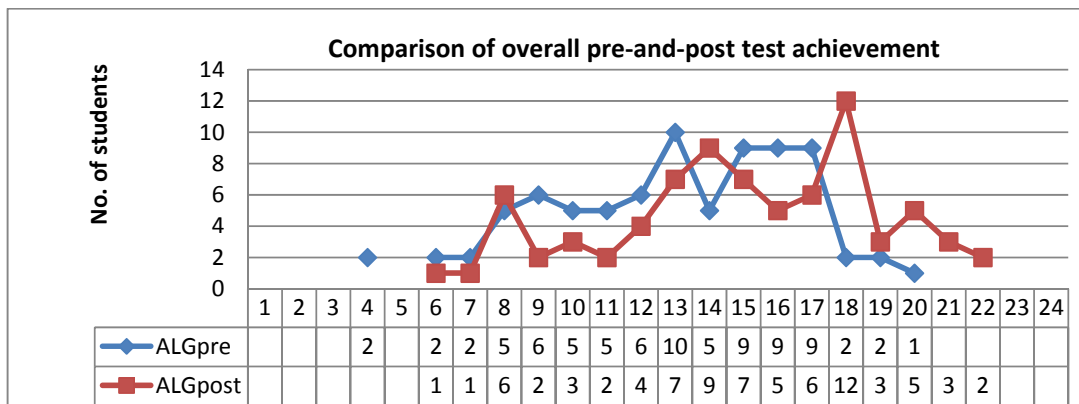
Figure 2   Comparison of overall pre-and-post test achievement

A graphical representation of the comparison of overall pre-and-post test achievement is shown in Figure 3.
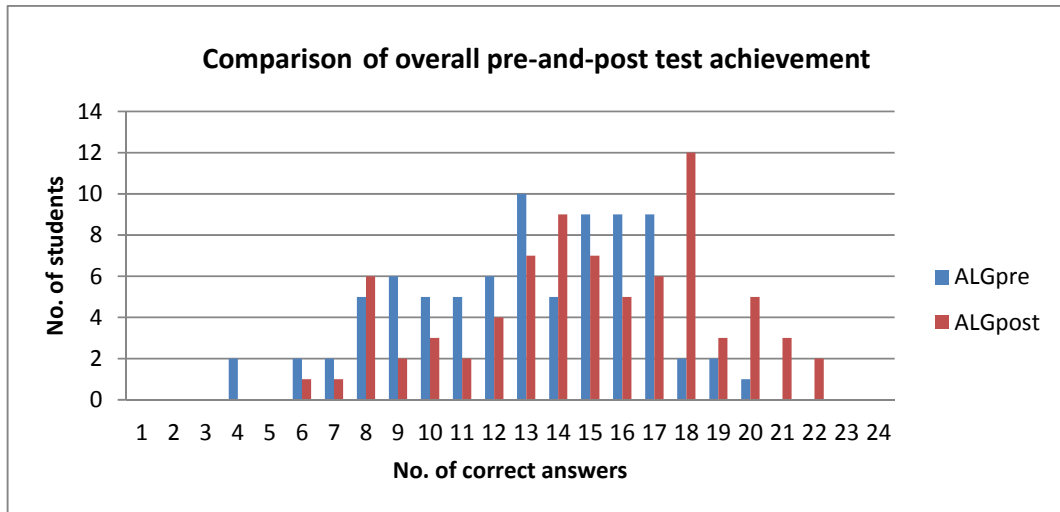
Figure 3   Comparison of overall pre-and-post test achievement

In sum, Figures 2 and 3 display the comparison of overall distribution of pre-and-post test student achievement as a function of number of correct answers. One sees a general increase in the number of students getting 18 to 22 out of 24 questions correct indicating significant gain in achievement of post test scores.

A comparison of pre-and-post test scores for each individual student is displayed in Figure 4. Figure 4 shows the comparison of pre-and post test scores for each individual student. In evaluating the results, there was a drop in performance of seven students (student nos. 34, 35, 41, 46, 48, 57and 60) and another seven students (student nos. 3, 23, 32, 54, 59, 68, 69 and 70) with no improvement in their scores. A total of 64 (about 82%) students improved their post test scores after the teaching intervention indicating significant learning gains in algebra concepts after the diagnostic (conflict) teaching intervention.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78
ALGpre	16	15	17	18	13	14	16	19	16	17	14	16	17	15	20	17	15	14	17	15	17	13	17	14	15	13	16	18	14	16	16	15	16	15	19	17	15	13	16	16	15	13	12	13	12	13	10	12	11	10	11	13	12	13	11	12	10	9	9	9	11	10	13	4	7	12	4	8	8	9	8	9	7	10	9	11	8	8
ALGpost	20	18	17	19	17	18	20	22	18	18	15	21	16	17	22	19	17	15	20	16	18	18	13	19	18	17	20	14	16	21	18	16	18	15	13	18	18	21	20	18	12	15	16	15	14	8	14	10	14	15	14	14	17	13	15	14	8	10	9	8	14	12	13	6	8	14	7	8	8	9	12	11	10	11	12	13	13	13

Figure 4   Comparison of pre-and-post test scores for each individual student

4.2   Response to Research Question 2

What conceptual difficulties and misconceptions do Form 2 Malaysian students have with algebra?

The previous chapter contained a discussion of the development of a test designed to diagnose the algebra misconceptions held by secondary school students. The test was administered to a sample of 78 students from two classes, the scripts marked by the researcher and the results entered onto a spreadsheet in such a way that every student's response to each of the 24 questions in the test was recorded. The results were tabulated and the number or percentage of students selecting each response is shown in Table 5.

Table 5   Results of questions: Number and corresponding percentage of students
                   selecting response or option, A, B, C or D (N=78)

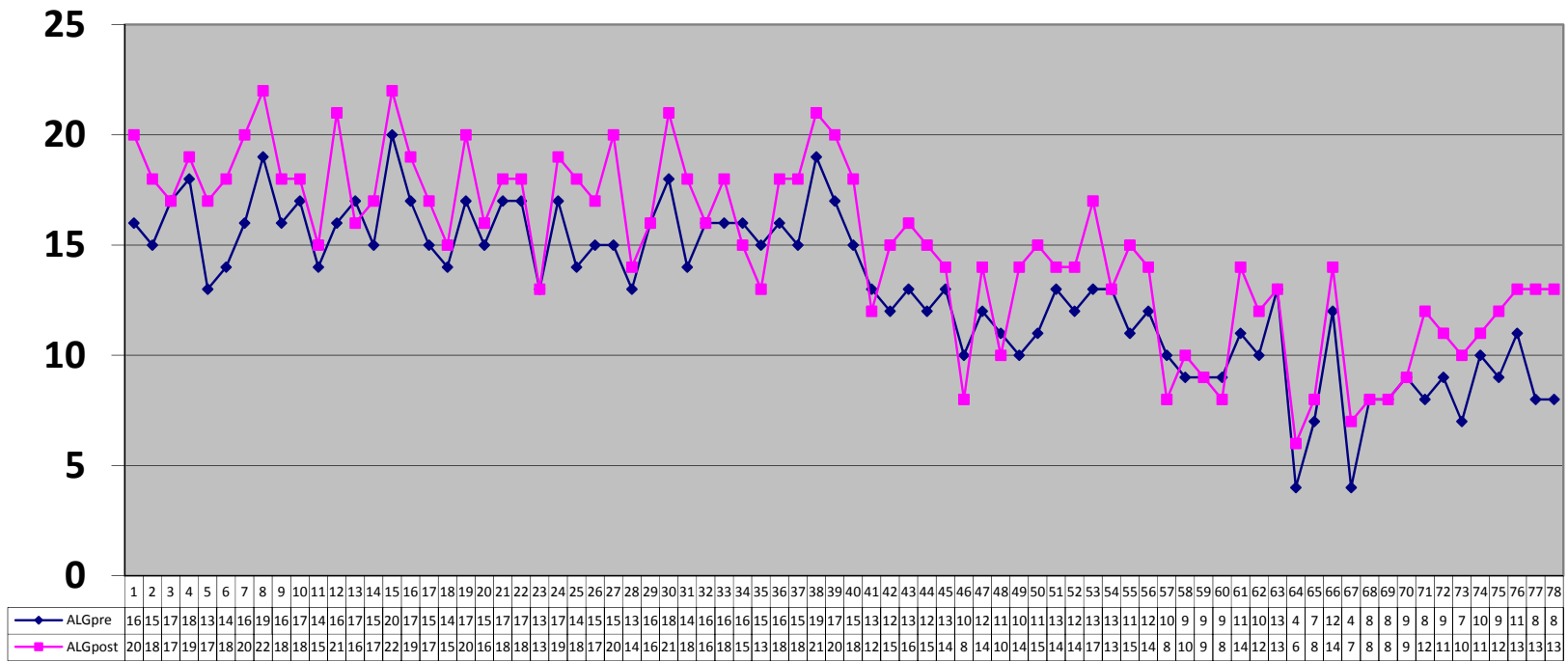| Item No.	Ans. *	A	%	B	%	C	%	D	%	N
1	D	2	2.56	4	5.13	20	25.64	52	66.67	78
2	D	4	5.13	7	8.97	0	0.0	67	85.90	78
3	D	32	41.03	2	2.56	3	3.85	41	52.56	78
4	C	6	7.69	1	1.28	68	87.18	3	3.85	78
5	B	12	15.38	63	80.77	2	2.56	1	1.28	78
6	A	61	78.21	4	5.13	2	2.56	10	12.82	78
7	C	1	1.28	46	58.97	28	35.90	9	11.54	78
8	A	71	01.03	2	2.56	2	2.56	3	3.85	78
9	D	11	14.10	5	6.41	11	14.10	51	65.38	78
10	C	3	3.85	24	30.77	44	56.41	7	8.97	78
11	B	26	33.33	29	37.18	6	7.69	17	21.79	78
12	D	6	7.69	8	10.26	30	38.46	34	43.59	78
13	A	62	79.49	14	17.95	1	1.28	1	1.28	78
14	B	25	32.05	38	48.72	7	8.97	8	10.26	78
15	D	44	56.41	7	8.97	5	6.41	22	28.21	78
16	B	7	8.97	65	83.33	3	3.85	3	3.85	78
17	A	38	48.72	26	33.33	9	11.54	5	6.41	78
18	B	3	3.85	63	80.77	7	8.97	5	6.41	78
19	A	44	56.41	9	11.54	11	14.10	13	16.67	78
20	C	41	52.56	19	24.36	16	20.51	2	2.56	78
21	D	4	5.13	20	25.64	6	7.69	48	61.54	78
22	B	13	16.67	53	67.95	9	11.54	3	3.85	78
23	A	55	70.51	9	11.54	10	12.82	4	5.13	78
24	C	13	16.67	14	17.94	50	64.10	1	1.28	78

Ans.* Indicating the correct response to the question (item)

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This table displays the number and corresponding percentage of students' responses on each item of the Algebra Diagnostic Test. For example, for Item 1 (question 1), the correct response is 'D' with 52 students out of 78 students or about 67% of the students selecting this response; 2 students or about 3% selecting response 'A', and so on. This section provides a description and explanation of the difficulties and misconceptions identified in this study. However, the conceptual difficulties and misconceptions identified are limited by the items used in the test.

In order to ascertain the algebraic difficulties and misconceptions made by the students, results from the algebra diagnostic test, interview transcripts (of 5 students) and classroom notes taken during the teaching experiment were analysed. The interview transcripts are attached in Appendix E. The difficulties and misconceptions in algebra described in the previous chapter (Section 2.2) provided the basis for this categorisation. On close examination of the difficulties and misconceptions, they were found to fall roughly into five broad areas:

1.   Basic understanding of letters and their place in algebra;
2.   The manipulation of these letters, symbols or variables;
3.   Use of rules of manipulation to solve equations;
4.   Use of the knowledge of algebraic structure and syntax to form equations; and
5.   Use and understanding of algebraic notations in the context of the generalisaton of patterns.

Consequently, the misconceptions were arranged into five categories as shown in Table 6. The difficulties and misconceptions of each category are discussed with reference to the results in Table 5. The first category to be discussed is the misinterpretation of literal symbols as variables in algebra.

Table 6   Classification of difficulties and misconceptions

Category		Difficulty and Misconceptions
1	Interpretation of the use of literal symbols as variables in algebra	Different letters always represent different numbers Treating letters as labels for objects and/or words Letters have no meaning Letters standing alone equals 1 Letters are sequential and represent numerical position
2	Manipulation of operator symbols in algebra	A '+' or '-' sign must produce a closed answer An '=' sign means action, not equivalence Variables, numbers and signs are detached
3	Use of rules of algebraic manipulation	Order of operation is unimportant Neglecting the use of brackets when needed Over-generalisation of mathematical rules
4	Formulation of algebraic expression or equation from information presented in words	Replacing key words by mathematical symbols sequentially from left to right
5	Use of standard algebraic symbolism to express generality	Production of general rule to express a relationship

4.2.1   Interpretation of Literal Symbols as Variables in Algebra

The major interpretations students have in relation to the use of letters identified in this study confirm the findings of previous studies are: (1) each letter has a unique value or variables cannot take on multiple values; (2) letters stand for objects rather than numbers; (3) letters have no meaning; (4) a letter standing alone equals 1; and (5) letters are sequential and represent numerical position in the alphabet. The misinterpretations of the use of literal symbols as variables in algebra are briefly discussed with reference to the results inTable 5.

4.2.1.1   *Different letters have different values*

Analyses of the algebraic diagnostic test items and interviews revealed that students misinterpreted that every different letter must stand for a different number. They believed that different letters within an equation cannot take on the same numerical value consistent with findings of Stephens (2005), thinking that letters always have

one specific value. For example, some students believe that when the literal symbol changes, then the value that it represents also changes. They are unwilling to accept that different symbols could stand for the same value. This view, however, is applicable to natural numbers, where each number has a unique symbolic representation and where different symbols stand for different numbers. This belief was clearly exhibited when students were asked when is $'a + b + c = a + z + c?'$ (Item 10). Many responded incorrectly. In fact, 24 out of 78 students answered never by selecting response 'B' (refer to Table 5). The results indicated that about 31% of students believed that each letter has a unique value. When Sam (S3) was interviewed and asked to explain his rationale, he made the claim that b and z cannot be the same answer in the following excerpt: (Interviewer: Teacher (T))

T: When is $a + b + c = a + z + c$? What you you think?
S3:   I think they are never the same.
T: Why do you think that?
S3:   No, … It's not that, (pause) 'cause b can't be equal to z.
T: Why?
S3:   Because they're different letters.

It was clear that Sam (S3) (one of the 24 students) experienced difficulties in understanding and using variables as general number misinterpreting that every different letter must stand for a different number.

4.2.1.2   Letters represent labels for objects rather than numbers

The label or object misconception has been researched heavily since Kuchemann in the 1970s. For example, often students will interpret 3a as 3 apples instead of 3 times the number of apples. They perceived letters as representing objects rather then numbers. A very small proportion of the students, only 2 out of 78 students (from Item 5 in Table 5), 6 out of 78 (from Item 11) and about 8% of the students (Item 12) seemed to have this misconception. Further evidence from student's (Pete: S4) interview confirms this misconception. Pete was asked what answer he selected for this question (Item 5) "Plums cost 8 cents each and bananas cost 5 cents each. If 'p'

stands for the number of plums and 'b' stands for 'the number of bananas' bought, what does 8p + 5b stand for?"

T: What answer did you select?

S4:   I chose C. [The answer for C is 8 plums and 5 bananas].

T: Why did you choose that?

S4:   Well, it's just that it would have cents on it if it was the cost … it would equal so many cents. Yeah, I still think it's 8 plums and 5 bananas, ... 'p' is for plums and 'b' is for bananas.

By selecting response 'C' for Item 5, Pete apparently perceived letters as labels for objects. Very often during the course, a very small proportion of the students make this error. They see a letter as standing for an object, or acting as a label, in which the letter, rather than clearly being a placeholder for a number, is regarded as being an object (Kuchemann, 1981). In this case, the letter is manipulated without being evaluated. When students view variables as objects, they are often used as labels for something. Variables are often used in geometry for points and lines. Moreover, variables are used in arithmetic to refer to units, such as 'm' for meters and 'c' for cents. When students enter algebra, they are suddenly referring to 'm' as the number of meters and 'c' as the number of cents.

4.2.1.3   Letters have no meaning

Students simply believe letters have no meaning in the realm of numeracy and that they belong to the literal realm (Perso, 1991). Sometimes, they would simply disregard the letters and solve the problem without taking them into account. Surprisingly, there were 17 out of 78 (students) or approximately 22% of the students (all from the low-achieving class) believed that letters have no meaning in mathematics by selecting 'nothing' (response 'D') to test item (item 11) 'In the expression $a + 5$, $'a'$ stands for' (refer to Table 5).

4.2.1.4   *A letter standing alone equals 1*

Mathematically speaking, the variable does not have a specific sign of its own. The values of a variable are determined when specific numbers are substituted for the literal symbol. Variables can stand for either positive or negative numbers (Vlassis, 2004). For example, the variable which is represented by the variable '*x*' can stand for positive and also for negative numbers: '$-x$' can stand for positive or negative numbers as well (this happens because $-(-5) = 5$). When students are introduced to negative numbers they learn that the presence of the negative sign means 'negative value'. Therefore, students would tend to interpret '*x*' to stand for positive number and '$-x$' to stand for negative numbers.

Besides, students also saw a one-to-one correspondence between letters and numbers resulting in the misconception that *x* standing alone must be equal to 1. Results from the test (item 11) indicated that 26 out of 78 students who appeared to have this misconception. Approximately 33% of the students responded by selecting '1' (i.e., response 'A') to the question what does $'a'$ stands for in the expression $a + 5$? (refer to Table 5).

4.2.1.5   *Letters are sequential and represent numerical position in the alphabet*

One of the problems that some students have in what Warren (2003) discovered was the assigning of numerical values to letters according to their rank in the alphabet. When students do this, they will often assume that the variable '*a*' is equal to 1, with the variable '*b*' being equal to 2, and so on. Findings of prior studies on students' difficulties with the use of the literal symbols in algebra are consistent with this view. There is evidence that some students associate literal symbols with natural numbers, in the sense that they respond as if there is a correspondence between the linear ordering of the alphabet and that of the natural number system. By examining the transcript of interview made with respect to this question, it appeared that the student used the letters as labels. For example Ramiah (S8) assigned a numerical value 5 to

the literal symbol *'c'* to the following question (Item 1) "If $a + b = 7$, then $a + b + c = $?" She failed to realise that the meaning or value of a variable is not dependent on the letter used as evident in the following excerpt:

T: If $a + b = 7$, then $a + b + c = $ what?

S8:   I think it's 12.

T: Why?

S8:   Well, if $a + b = 7$, a could equal 3, b could equal 4 and c could equal 5, in a sequence.

T: Why do you think it would be a sequence, because they are alphabetical?

S8:   Yeah, that's it, $a + b + c$ *is* $3 + 4 + 5 = 12$.

This discourse suggests that Ramiah (S8) assigned a numerical value 5 to the letter c to get the answer 12. She is one of the four (about 5% in total) of the students that associated the letter *'c'* with the natural number 5 by selecting response 'B' for Item 1 (Table 5).

In sum, results showed that students had difficulty with letter usage in algebra and persistently misinterpreted letters as (1) representing different values, that is, different letters have different values; (2) abbreviated words or labels for objects; (3) having no meaning or ignoring letters; (4) standing alone equals 1; and (5) sequential and representing numerical position in the alphabet. Next, category 2 misconceptions related to the manipulation of operator symbols in algebra are discussed.

4.2.2   Manipulation of Operator Symbols in Algebra

In the process of simplifying algebraic expression or solving an algebraic equation students are required to apply a succession of transformation rules in their manipulation of symbols involved [such as the equal sign (=), operation signs (+, −, × or ÷)], variables (x, y), and a variety of types of numbers (5, ¾, .23) which may appear as constants, coefficients or other roles in the equation. These rules involve rewriting expressions (combining like terms, factoring, expanding) and applying the

same operation on both sides of the equation. Within the domain of equation solving, a number of misconceptions dealing with the manipulation of operator symbols in algebra have been identified. For example, students believe that (1) an answer should not contain an operator symbol such as $+$, $-$, $\times$ or $\div$; (2) the equal sign ($=$) is an indicator of operations to be performed; and (3) variables, numbers and signs are detached.

4.2.2.1   An answer should not contain an operator symbol

Results from the diagnostic test and lesson observations revealed that students interpret the '$+$' and '$-$' signs in terms of the action to be performed and tend to conjoin or 'finish' open algebraic expressions. The results from Item 23 showed that an alarming large proportion of students selecting response 'C' ($7a$) to "Add 5 unto $a + 2$". Perhaps the wording of the question confused them or they could not make sense of the question. About 29% of (i.e, 23 out of 78) students (refer to Table 5) were found to have difficulty accepting an unclosed expression such as $'a + 7'$ as a legitimate answer. To these students $a + 7$ does not look like an answer. The presence of the operator symbol, $+$, makes the answer appear unfinished. In short, students see the plus symbol ($+$) as invitations to do something, and if something is still to be done, then they ought to do it. The need for 'closure' is a major obstacle. This kind of misconception greatly distorts perception of the function of the operator symbols.

4.2.2.2   An equal sign is a sign connecting the answer to the problem

Consistent with findings of previous research, lower-achieving students tend to see the equal sign as a procedural marking telling them "to do something" (makes), or as a symbol that separates a problem its answer, rather than a symbol of equivalence (Warren & Cooper, 2005; Molina & Ambrose, 2008). Such an 'operational' view is consistent with students' belief that the number immediately to the right of an equal sign needed to be the answer to the calculation on the left hand side. This kind of misconception greatly distorts perception of the equation and the immediate goals

necessary for its solution. Thus, an incomplete or incorrect understanding of the role of the equal sign will be detrimental to students' performance and learning of equation-solving procedures. The results from Item 3 indicated that 41% of the students simply did not understand the equivalence role of the equal sign. There were 32 out of 78 students selecting the response 'A' ($x = y + 7$) to the question $'x = y + 2,$ what happens to x if 5 is added to $y?'$ apparently forgetting to apply the rule 'do the same to both sides' of the equation.

4.2.2.3   Numbers, signs and variables are detached

Students frequently have difficulty that some signs are attached to numbers (or variables) but not others. Students believed that negative signs represent only the subtraction operation and do not modify terms (Vlassis, 2004). For example, -3 mean that the '$-$' is part of the 3 whereas in 'x 3' or '÷ 3' the signs are detached. This confusion may lead to students totally ignoring signs, particularly negative signs, in an algebraic expression or equation. It is difficult to understand how a variable can have a negative. What does '$-y$' mean? This misconception is also related to the fact that many students are unwilling to work with fractions and negative numbers and will consequently ignore any ties which appear to exist between numbers, signs and variables.

Three students (4%) seemed to ignore the negative sign and just multiply to get $6p$ selecting response 'A' for Item 18 (What is $-3p \times 2$?). This answer is consistent with the misconception that negatives can enter and exit phrases without consequence and that their locations (and connections to numbers or variables in the problem) are detached or not significant. Failing to tie the negative sign to the term it modifies or to understand how changing or moving a negative sign impacts the equation resulting in incorrect strategies for solving algebraic equations.

Similarly, in Item 19, about 17% of students who selected response 'D' appeared to be under the impression that numbers, signs and variables are deteached when they subtracted '$j + m$' instead of subtracting 'j' and subtracting 'm'. Not only does the failure to understand the negative sign but also the failure to apply the correct rules

of algebraic manipulation will result in incorrect strategies for solving algebraic equations. Following, category 3 misconceptions concerning the use of rules of algebraic manipulation to solve equations is discussed next.

4.2.3 Use of Rules of Algebraic Manipulation to Solve Equations

As mentioned in Subsection 4.2.2, students are required to apply a succession of transformation rules in their manipulation of letters, symbols or variables involved in the process of simplifying algebraic expression or solving an algebraic equation. In the process, students frequently fail to realise that formulas in mathematical symbol systems have an intrinsic structure where expressions are structured explicitly by the use of parentheses, and implicitly by assuming conventions for order in which we perform arithmetic operations. Students commit errors as a result of their inattention to the structure of expressions and equations. Among the misconceptions identified were: (1) order of operation is unimportant; (2) parentheses don't mean anything; and (3) over-generalisation of some mathematical rules while ignoring others.

4.2.3.1   Order of operation is unimportant

The order of operation is very important when simplifying expressions and equations. The failure to use the order of operation can result in a wrong answer to a problem. For example, in Item 20, to find the value of $3 + y \times 2$, without the order of operation one might decide to simplify the problem working left-to-right. The resulting answer would be $6y$. In contrast, the correct answer is $3 + 2y$.

Alarmingly, there were 41 students (out of 78) choosing to ignore the need for a rule in this item (Item 20, Table 5).These students simply did not see a need for rules presented within the order of operation. Instead, they attempted to perform multiple-operations in any order which appears possible – they will attempt any part of an expression which they think they can do, and in any order. Hence, solving the expression based on how the items are listed, in a left-to-right fashion, consistent with their cultural tradition of reading and writing.

4.2.3.2   Parentheses don't mean anything

Students are found neglecting to use brackets when needed. Brackets are an essential element of mathematical notation in both arithmetic and algebra. Algebra, however, requires students to have a much more flexible understanding of brackets. In arithmetic, brackets are generally used as a static signal telling students which operation to perform first (or to specify the order of operation). Although symbolising the grouping of two terms (in an additive situation) is one important usage, students need to understand that brackets can also be used as a multiplicative operator (Linchevski, 1995). Although this dual usage is vital in understanding algebraic equivalence, Gallardo (1995) found that students were unable to correctly apply both usages of brackets to disapprove the number sentence: $20 - (7 - 8) = (20 - 7) - 8$.

Many students, however, accept the need for brackets in arithmetic but the rule: 'do the the brackets first' often is impossible in algebra. Therefore, students simply ignore it. To some students, parentheses don't mean anything in algebra. This is related to the misconception that working is always from left to right; the way we read. For example, simplify the expression $2\,x\,(a\,+\,b)$. Responses such as $2a\,+\,b$ and $2ab$ are not uncommon instead of the correct answer $2a + 3b$. The results of Item 22 clearly indicated that there were approximately 17% of the students who selected the response such as $2a + b$ and roughly 11% of them selecting the response such as $2ab$, indicating that they were either compelled to either 'close' their answer or to ignore the parentheses and work from left to right.

Another question concerning the non-use of parentheses was selecting the incorrect expression representing the total number of newspaper Lee delivers in 5 days (Item 16). Approximately 9% of the students (selected option A, $5 \times \square + 3$ instead of $(5 \times \square) + 3$) were either ignorant of the use of brackets or chose to ignore the use of brackets mainly because they considered them unnecessary and took it for granted that operations are performed from left-to-right.

4.2.3.3   *Over-generalisation of mathematical rules*

Chua and Wood (2005) argued that among the common misconceptions students have in algebra, the most significant were errors due to over-generalisation of concepts and rules. These authors attribute these errors to misconceptions that students have actively constructed when they use their existing schema to interpret data which are grounded in faulty understanding. Students' misconceptions of over-generalisation and/or inability to generalise mathematical rules discussed in this subsection are: (a) to remove a term from the equation, students subtract it from both sides of the equation; and, (b) dividing the larger number by the smaller rather than respecting the order of inversing.

4.2.3.3 (a)   A common strategy students have for solving equations is that if they want to remove a term from the equation, they subtract it from both sides of the equation irrespective of the adjoining operator symbol ($+\ or\ -$). This works just fine for removing 3 from the equation $y + 3 = 9$. However, when they encounter equations like $y - 3 = 9$, many students still try to subtract 3 from both sides to solve the problem. Booth and Koedinger (2008) attribute this misconception to students' inability to process the fact that the negative sign modifies the 3 and is a necessary part of the 'term' they are trying to remove. Students were found to often ignoring it to the detriment of their goal of solving the problem.

Results from Item 15 "What is "$'k'\ if\ k - 12\ = 4?$" revealed the presence of this misconception. A significant proportion (56%) of the students tried to remove -12 from the equation ended up with $k - 12 - 12\ = 4 - 12$ resulting in $k\ = -8$. By selecting option 'A', these 44 students appeared to be unable to process the fact that the negative sign modifies the 12.

4.2.3.3 (b)   The misconception of dividing the larger number by the smaller rather than following the order of inversing is related to students' inablility to generalise because of a lack of understanding of arithmetic operations. It also assumes that frequently students do not know which operation will undo another operation. For

example, results from Item 17 concerning this misconception as "If $5 = 9y,$ then $y =$." Twenty six students answered this question wrongly by selecting option B, $9 \div 5$, instead of $5 \div 9$.

Here, it is obvious that students perceived the need to isolate the variable, but were unsure which operation was necessary to inverse the one given. Thirty three percent of the students selecting option B appeared to be aware of the need to perform the operation of division but exhibit a lack of understanding of the arithmetic operator. They did not consider that the inverse operation for multiplying by 9 was dividing by 9. It was simply seen that it is easier to divide a smaller number into a larger one, and so this was done. Generalisation from early experiences that you can't divide smaller numbers by larger ones is thought to be the cause of this misconception.

Misconceptions due to students' inablility to use knowledge of algebraic structure and syntax to generate equations are discussed next.

4.2.4 Formulation of Algebraic Expression or Equation from Information Presented in Words

Word or 'story' algebra problems are difficult. Even students who are good at mathematics often loathe them, and neither teachers nor textbooks know how to teach them; typically they give such well-meaning advice as to "read and reread the problem until what is stated is clear" (Fuller, 1977, p. 115), and otherwise are satisfied to provide students with an opportunity to practice. Conceptual problem representation must be constructed from the text according to arithmetic or algebraic schemata. To this schematic problem representation, calculational operators are then applied, in that both the derivation of an equation as well as its solution may require multiple steps. To be able to produce this schematic problem representation, students are required to use the knowledge of algebraic structure and syntax to form the equation. A significant proportion of students were unable to unable to formulate the right equation to represent trading cards in Item 7. Only 28 out of 78 or 34% of the student population answered correctly by selecting option 'C' (refer to Table 5). A

misconception identified in this category is words and letters are matched from left to right when transposing literal sentences to form algebraic expressions or equations.

4.2.4.1   Words and letters are matched from left to right

When transposing literal sentences to form algebraic expressions, this left-to-right tendency can create problems, even for experienced mathematicians. Word problems – problems that involve a narrative description of quantitative relationships among variables – seem to give students the great difficulty. Item 6 was designed to test the misconception that words and letters are matched from left to right. Ten students selected response 'D' which implied that these students either possessed the misconception or misinterpreted the arithmetic term 'subtracted from'.

Interestingly, about 15% of the students (12 out of 78) selected response 'A' for Item 5 in Table 5 apparently replacing key words from left-to-right and used 'p' as the 'cost of plums' rather than the 'number of plums'.
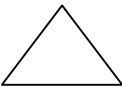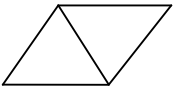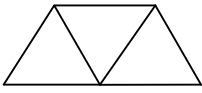
Finally, students' use of standard algebraic symbolism to produce the general term (a rule) for repetitive patterns or sequences of shapes is discussed.

4.2.5   Use of Standard Algebraic Symbolism to Express Generality

While the four categories discussed in the preceding subsections above are mainly concerned with misconceptions, this category discussed the difficulty that students encountered as they engage in generalising a pattern algebraically. In the process of the production of a general rule to express a relationship, students are found to be using an iterative (additive) rule instead of multiple rule and have great difficulty in generating a symbolic expression for the general term of a geometric-numeric pattern as illustrated in the following task:

(1) Make a table for the number of toothpicks for several triangular figures and find out how many toothpicks are in figure 10 and figure 25? (2) Find a direct formula

that expresses the total number of toothpicks for figure *n* (in terms of the triangular figure). Explain how you arrive at the formula.

Table 7   Toothpick pattern

| Figure 1 | Figure 2 | Figure 3 |

(Adapted from Radford, 2006)

In a classroom activity students were asked to perform an arithmetical investigation to find how many toothpicks are in figure 10 and figure 25 for the triangle toothpick pattern given in Table 7 before proceeding to find an expression for figure n. Two students [Megat (S1) and Suhaila (S2)] were interviewed to gather qualitative data on their ability to generate a direct rule that expresses the total number of toothpicks for figure *n*.

As evidenced by their work, Megat (S1) and Suhaila (S2) did not have much trouble calculating the number of toothpicks in the concrete figure 10. They did so by counting the number of toothpicks, figure after figure up to figure 10 and then proceeded laboriously to figure 25. Megat was (found) using the recursive formula: "It's always n + 2. Yes, sure. It's like figure 4 + 2 equals figure 5." This is indication of using the iterative (additive) rule to express the general term of this triangular pattern activity (i.e., the relationship between the figure number and the number of toothpicks).

However, Suhaila found the formula '$n \times 2 + 1$' (where '*n*' stands for the figure number) through a process of generalisation of numerical actions in the form of an operation scheme that, according to Radford (2006), remains bound to the numerical level. When asked what does *n* times 2 means? She replied: "*n* two." An excerpt from the interview when asked by the researcher how she arrived at the formula and explain why she added '1' to $n \times 2$ follows:

T   So how do you establish a formula? How did you come up with $2 \times n + 1$?

S2   Cos you do 2 times the box plus 1. [S2 interpreted a triangle as a box.]

T I don't get that. Why you add '1' to $n \times 2$?

S2 Ahm, you know figure number, like figure 10. You times 10 by 2 equals 20 and then you add 1 equals 21 and you get the number of toothpicks. Cos when you count it, you only count 1 extra [at the beginning] and then you kinda … keep adding 2 extra sides. Yea, it works! (At this point, Megat (S1) interjected).

S1 But no, that would really be the number of the figure times 2 plus 1. Because, look. 2 times 2, 4, plus 1, 5. One times … 1 times 2, 2, plus 1, 3! Yes. That would work!"

S2 So it's times 2 plus 1, right? And, to calculate the number of toothpicks in figure number, 25, it's … figure 25, so, yes, it's 25 times 2 plus 1 , 51. Yeah, 51 toothpicks!

The interview indicated that these two students were not using the standard algebraic symbolism. Results from test Item 14 showed that a high proportion of the students, about 51% had difficulty in providing the correct rule to determine the following number pattern: $2, 5, 11, 23, 47, ...$ This proved to be extremely difficult for the students as only 49% of the students were able to select the correct response to this item (Item 14, Table 5). In addition, results from Item 24 provided further evidence of this difficulty. About 36% of the students selected the wrong response.

In sum, the difficulties and misconceptions identified in this study are by no means exhaustive. Many more could be given but only those identified in this study were reported and discussed. From the researcher's point of view, when students take new knowledge and try to make sense of it using previously learned schema, they are sometimes successful and sometimes they are not. But when they develop misconceptions they are not without some sort of logic and the researcher believed that the roots of such flawed understandings are impossible to establish as they are likely a combination of students' own ideas, imprecise statements made by the teacher, friends' explanations that are incomplete, media, and so on. A realisation that such misconceptions are common and teachers must specifically take the likelihood of their occurrence into account because errors can be used as "springboards for inquiry" (Borasi, 1994) to address misconceptions during teaching. In addition, errors can be used as springboards for cognitive conflicts to provoke

students' thinking and to guide them to correct the errors or misconceptions themselves. The results of an intervention strategy using cognitive conflict to facilitate conceptual change by provoking and guiding them to correct their misconceptions are discussed next.

4.3   Response to Research Question 3

Is there any evidence of students' conceptual change in algebra concepts following the teaching intervention (diagnostic teaching)?

Both quantitative and qualitative data from the Algebra Diagnostic test, interviews and notes taken from classroom observations of students' discussion with each other and the teacher were used to judge for conceptual change in students' learning or changes in students' ideas, that is, from unintelligible to intelligible, intelligible to plausible and from plausible to fruitful (as indicators of changes in students' conception).

In order to effectively ascertain the status of the student's conceptions, data were obtained from the interview to elicit the various status of the concepts (intelligible, plausible or fruitful), a conceptual change model developed by Posner et al. (1982), and researched comprehensively by Hewson and Thorley (1989) was used (see Figure 1) in this study to investigate the role that diagnostic teaching play in students' conceptual understanding of algebra. This model describes learning as a process that involves the interaction between new and existing conceptions with the outcome being dependent on the nature of interaction. Learners use their existing knowledge to determine whether a new conception is intelligible, plausible, and fruitful. If the new conception satisfies all three criteria and is integrated with existing conceptions, then conceptual change learning has taken place by a process of assimilation (Treagust et al., 1996). It is possible for the new conception to conflict with existing conceptions; the learner becomes dissatisfied with the old conception which is exchanged for the new one (Harrison & Treagust, 1996). Conceptual change of this nature is termed conceptual exchange (Hewson, 1981, Hewson & Thorley, 1989), a process described by Posner et al. (1982) as accommodation. The status of a

person's conceptions is determined by the extent to which the conception meets the three conditions of intelligibility, plausibility and fruitfulness. The more conditions it meets, the higher is its status.

Intelligible	Is the conception intelligible to the learner? That is, does the learner know what it means? Is the learner able to find a way of representing the conceptions?
Plausible	Is the conception plausible to the learner? That is, if a conception is intelligible to the learner, does s/he also believe that it is true? Is it consistent with and able to be reconciled with other conceptions accepted by the learner?
Fruitful	Is the conception fruitful for the learner? That is, if a conception is intelligible and plausible to the learner, does it achieve something of value for him/her? Does it solve otherwise insoluble problems? Does it suggest new possibilities, directions or ideas?

(Adapted from Hewson and Hewson, 1992.)

Figure 5   The conditions that have to be met for conceptual change to occur

4.3.1   Classifying the Status of Students' Conceptions

The interview questions with 4 students [Henry (S5), Gita (S6), Sue (S7) and Terry (S9)] focused on algebraic concept of change between two variables and how that change can be represented with symbolic equations. Students were required to construct a table of values showing the distance travelled over several hours at an average rate of 55 kilometres per hour in the following task:

Information

A car is travelling at a constant speed of 55 kilometres an hour for the whole journey

Task:

(a)   You are required to complete the following table by filling in the average speed, time, and the distance travelled.

(b)   Write a rule to explain to your friend how to determine the total distance travelled as a function of speed and time.

Table 8   Speed, time and distance travelled

Speed (kilometres an hour)	Time (hours)	Distance travelled (kilometres)
	1	
	3	
	5	
	7	
	10	
	n	

The learning goal is to write an equation to represent distance as a function of time multiplied by speed. These students are expected to write a rule expressing the relationship of distance to elapsed time and speed but this process of iteratively filling out the table may not support the development of this rule (distance travelled is a function of speed and time, i.e., $D = S \times T$, where D refers to distance travelled, S refers to speed or how fast the vehicle is travelling per hour and T refers to the total number of hours of travel).

To assess students' conceptual change subsequent to the teaching intervention, factors related to the status were identified from the interview transcripts to help in the process of classifying each student's conception of change in moving from additive to multiplicative reasoning or recursive to explicit reasoning with the anticipation of culminating to write a symbolic equation expressing the relationship of distance to elapsed time and speed. For this purpose, descriptors described by Hewson and Hennessey (1992) were used as a guide during this process.

It is necessary and important for the reader to note that the status of students' conceptions for intelligibility, plausibility and fruitfulness was arbitrarily determined by the researcher for the sole purpose of this study. For a concept to be intelligible, students must know what the concept means and should be able to describe it in their own words. In this exercise, students must be able to explain to the interviewer what it means to keep a car at 55 kilometres an hour for an entire hour. For a concept to be plausible the concept must first be intelligible and students must believe that this is how the world actually is and that it must fit in with other ideas or concepts that students know about and believe. In this instance, students must be able to explain what it means to keep a vehicle at 55 kilometres an hour for an entire hour. Besides, students must be able to construct a table of values showing the distance travelled over several hours at an average rate of 55 kilometres an hour. Furthermore, students must also be able to explain how they compute the distance travelled for 3, 7 and 10 hours and able to write a symbolic equation expressing the relationship of distance to elapsed time and speed. It is here where the researcher will be able to identify whether the students were using the iterative additive process or the multiplicative process to complete the task. In other words, whether students determine total distance by summing distance from row to row (vertically by adding 55 for each entry) or horizontally, which would more easily translate into symbolic equation form of distance equal to rate times time, in its explicit form. Both situations were considered as plausible.

Finally, for a concept to be fruitful, it must first be intelligible and plausible and should be seen as something useful to solve problems or as a better way of explaining things. Beside being able to use the standard algebraic symbolism to express generality (in this case, being able to generate a symbolic equation of $D = S \times T$), students must be able to manipulate formula to determine how long it will take to travel one kilometre at the rate of 50 kilometres per hour (as indication of conceptual understanding of the task). In addition, students were asked "how long will it take a vehicle to travel 125 kilometres at the rate of 50 kilometre per hour?' Students were asked to sketch, draw or verbalise their predictions.

An excerpt from the transcript of a student (Henry: S5) whose conception of the distance travelled is a function of speed and time was given the status of intelligible proceeded as follows:

T: I think you usually think about speed in terms of how fast or slow it is, and I need you to start to think about speed in terms of how far it's going to get you. If you keep that car at 55 kilometres an hour for an entire hour, do you know how far that'll get you?

S5:   Fifty-five kilometres. That's like, that you would be, you could travel, if you're travelling at 55 kilometres an hour, then you might get … 55 kilometres.

T: So, can you explain to a friend what does it mean when a car is travelling at seventy kilometres per hour?

S5:   That's like, that you would be, you could travel, if you're travelling at 70 kilometres an hour, then you might get … 70 kilometres. I mean, it means you'll be seventy kilometres away from where you started every hour that you can drive like that.

T: Now, there are two cars, one is travelling at a speed of 70 kilometre an hour and the other is travelling at 55 kilometres an hour. Which car do you think is travelling faster?

S5:   Er … I think the one, the 70 kilometres … one is faster.

The student [Henry (S5)] demonstrated that the idea of distance, speed and time was intelligible to him. He knew what it meant and was able to find a way to represent the notion of travelling at speed of 55 kilometres an hour. He was also able to distinguish the vehicle travelling at 70 kilometres an hour was faster than the other travelling at 55 kilometres an hour.

Another student [Gita (S6)], whose conception was classified as being intelligible and plausible, talked about how she determined total distance travelled from row to row adding fifty-five kilometres each time to the current running total to get the successive distance travelled without reference to the total number of hours travelled as follows:

T: How did you come up with this column (referring to the table 5 column)? What did you do to get to this column?

S6:   I did, umm, fifty-five each time and kept going down.

T: Okay, you kept adding 55 each time?

S6:   Yes. The next one would be 55 plus 55, 110. The 110 plus 55, er, 165.

T: So, for like 3 hours you come up with 165 kilometres? How can I get for five then?

S6:   Err, okay, three hours I get one hundred sixty-five, so four is plus 55 (calculating on a piece of paper), writes 220. Four, 220, plus 55, two hundred and seventy-five. It's 275.

The above excerpt indicates that the student was determining the total distance by summing distance travelled from row to row. This is as an indication that she may not be explicitly connecting number of hours travelled to total distance travelled. The next step was to help the student to discover and acknowledge the multiplicative nature of rate times time as useful for computing distance. Here, at length, is an excerpt of that process:

T: Okay, go beyond what you see and what you're doing. How can I get for three if I didn't have two and one?

S6:   Okay, let me see (referring to the constructed table of values) … inaudible. Oh, yes, one, fifty-five, 55, 2 fifty-five, 110, yes, yes, (excited by the new discovery). I know, I know, it's 55 times three.

T: So for like 3 hours how did you come up with one 165?

S6:   55 times three.

T: Very good, three times 55. So, for three hours the car would have traveld a distance of 165 kilometres. Well, how far do I get for five hours then?

S6:   Five times 55, 275. It's 275 kilometres.

T: Excellent! How do I get eight and ten?

S6:   It's eight times 55, (manipulating the calculator this time), four hundred forty, yes, 440. Ten times 55, 550.

T: Excellent. So, what do these numbers represent?

S6:   Distance.

T: Represent distance but represent distance in what?

S6:   Kilometres

T: Kilometres? Is that on here? (Student adds units). Kilometres, okay. And you
        have time with hours and speed in kilometres an hour. Now, can you write a
        formula using letters of the alphabet to represent distance travelled with time
        and speed?

S6:   What? I don't understand.

T: Let's say a friend of yours want to know the distance a car can travel in two
        hours if you were driving at a speed of 50 kilometres an hour. Can you write
        down a formula to help him find the distance travelled connecting the number
        of hours and the speed of the car? Let's assume D to represent the distance
        travelled, S refers to speed and T refers to the number of hours. Can you write
        the formula?

S6:   (Hesitates) I'm not sure, but I'll try, … can it be distance travelled, er, D,
        (pauses, silent for a while).

T: Yes, D for distance travelled. Then, how did you get this distance? Can you
        recall what you did just now, you use something times something? What was
        that?

S6:   Er, three times 55, five times 55, … (refers to the table of values) three hours,
        five hours. Yes, yes, hours, hours for T. That's it, time for T, and … fifty-
        five, fifty-five standing for … (pauses).

T: And, 55 for what? Fifty-five stands how fast or slow your car is travelling?

S6:   Oh, I see, fast, (pauses a while) aha! Yes, it's speed, speed isn't it?

T: Absolutely correct. Fifty-five kilometres an hour is the speed. Now can you
        write the formula using these three letters, D, T and S?

S6:   I'll try. Let me think, … (speaking to himself, inaudible). I think, time times
        speed is equal to distance, (writes) $T \times S = D$. Am I right?

To this student, the notion of distance is a function of time and speed was plausible
because she was able to determine the distance travelled (D) by multiplying the other
two related variables, time (T) and speed (S). Statements like 'Three times 55 is 165'
and 'Ten times 55 is 550' indicated that she believe this is how to derive the distance
travelled (how the phenomenon is) and fitted with her picture of the phenomenon.
The student was able to develop a new conception of the multiplicative nature of the

task and ultimately able to write a rule expressing the relationships of distance to elapsed time and speed.

Another student [Sue (7)], whose conception was classified as intelligible, plausible but not fruitful, talked about her solution strategy in solving the problem 'How long would it take to travel one kilometre at a rate of 50 kilometres per hour':

S7:   Fifty kilometres? That's like, that you would be, you could travel a kilometre in like, (pause) … in like. (inaudible) Uhhh, it's like if you're doing 50 kilometres an hour you can cover kilometres an hour. (Not sure of the answer).

T: Say that one more time.

S7:   Say if, if you're travelling 50 kilometres an hour, then you might get … 50 kilometres in one hour.

T: Right.

S7:   Like say if you add one it's like a kilometre a minute.

T: It's very close to doing that. Yeah. Would you say it's more than a kilometre a minute or less than a kilometre a minute? If you're doing 50 kilometres in an hour?

S7:   It's probably less.

T: Well tell me this; if you could do a kilometre per minute now, how far would you get in an hour?

S7:   Sixty kilometres.

T: Right. But you're doing 50 so that means?

S7:   [Inaudible.]

T: Let's go back to how far they're going every minute. So, if they can put in sixty kilometres in an hour, that means they're doing a kilometre a minute. They're just doing 50. So, are they doing more than a kilometre a minute or less?

S7:   Less.

T: A little bit less.

To this student, statements like 'Uhhh, it's like if you're doing 50 kilometres an hour you can cover kilometres an hour' and 'Like say if you add one it's like a kilometre

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a minute' (hesitation) indicated the her uncertainty and hence, her inability to use the rules of algebraic manipulation successfully to manipulate the symbolic equation, $D = S \times T$. Furthermore, this student was unable to find the solution for the problem 'How long will it take to travel 125 kilometres at a rate of 50 kilometres per hour?'

Yet, another student [Terry, (S9)] whose conception was classified as intelligible, plausible and fruitful explained the strategy used to solve the problem 'How long will it take to travel 125 kilometres at a rate of 50 kilometres per hour?':

S9:   (Substituting D for 125 and S for 50, student calculates). Distance divide by speed equals hour, no, no, equals time, right? So, 125 divided by 50 (manipulates the calculator given), …, equals 2 point 5 (2.5).

T: 2.5 what?

S9:   2.5 hours. It's two-and-half hours, correct?

T: How many minutes are there in an hour?

S9:   Sixty. So, half of sixty, sixty into half, yes, yes, sixty into two, it's thirty. Oh, it's 2 hours and thirty minutes.

Excerpts such as the ones described above were used to classify the status of students' conceptions in a task to write a rule expressing the relationship of distance to elapsed time and speed. It was found that the process of iteratively filling out the table by students was not able to support the development of this rule. During the interview, Gita (S6) abandoned an alternative conception she had already expressed or changed from using the iterative additive process to employing multiplicative nature of rate times time as useful for computing distance. This transformation from additive to multiplicative reasoning requires reconceptualisation of the notion of unit (Hiebert & Behr, 1988). Hence, she (S6) was able to write the formula $D = S \times T$. Such shift was identified as evidence of her showing dissatisfaction with the process of vertically adding 55 to employing an alternative view of multiplicative nature of rate times time for computing distance as evidence of conceptual change or change in students' understanding (from intelligible to plausible).

Responses to the task that this researcher believe were indicative of the status of the student's conceptions were those statement in which the student provided answers

and explanations that were, as far as the researcher could determine, intelligible (two students: S5, S6), plausible (one student: S7) and/or fruitful (one student: S9) from his viewpoint. Thus the responses the author used to arrive at each student's (S5, S6, S7 or S9) status were as Hewson and Thorley recommend, descriptive 'comments about the conception' (Treagust et al, 1996). Hence, during the interviews, for these four students the highest status of their new conception, one student (S9) was assessed as being fruitful, another student (S7) the status was assessed plausible while for the other two (S5 & S6), the status was assessed as intelligible.

4.4 Response to Research Question 4

Are students' attitudes towards algebra enhanced after the six weeks intervention?

One would agree with Ruffell et al. (1998) that attitudes can "flip' from negative to positive and in particular that 'good teaching' can have such an effect, but one disagrees that this means that monitoring attitude will not prove fruitful for mathematics education research. Since attitudes can be affected by recent experience, a series of experiences promoting positive or negative attitude can indeed contribute to the development of more persistent attitudes and even beliefs which are deeply held and strongly influence future behavior (Pierce, Stacey & Barkatsas, 2007).

As a means of determining students' attitudes about mathematics, the researcher chose to use the Test of Mathematics-Related attitudes (TOMRA) which was an amended form of the Test of Science-Related Attitudes (TOSRA). The scales chosen were Enjoyment of Mathematics Lessons and Attitude to Mathematics Inquiry. Each scale contains 10 questions; half of the items are designated as positive, and another half of the items are designated negative. Participants responded using a five-point Likert scale ranging from 'strongly agree' to 'strongly disagree.' Positive items are scored by allotting 5 for 'strongly agree' and 1 for 'strongly disagree' responses. Negative items are scored by allotting 1 for 'strongly agree' and 5 for 'strongly disagree' responses. The mean values for both the scales ranges from 1 (low) to 5 (high).

Analyses of the two scales used for this study – Inquiry of Mathematics Lessons and Enjoyment of Mathematics Lessons – each had an acceptable reliability measure as shown in Table 9.

When the pretest and posttest were analysed independently, data indicated that the pretest results, as shown on Table 10 showed a non-significant decrease on the Enjoyment of Mathematics Lessons scale, while showing a significant increase on the Inquiry of Mathematics scale which was educationally significant.

Table 9   Cronbach Alpha Reliability of scales for the pretest and posttest of Test of
                 Mathematics-Related Attitudes (TOMRA)

Scale	Cronbach Alpha	
	Pretest (n=78)	Posttest (n=78)
Attitude to Math Inquiry	.64	.76
Enjoyment of Math Lessons	.93	.86

Students' attitude towards inquiry of mathematics lessons showed significant positive improvement [(pretest: M = 3.39, SD = .49), (Posttest: M = 3.65, SD = .58); t = 3.30, p<.05, effect size = 1.09]. Enjoyment remained high (mean at 4.02 out of 5) even though enjoyment of mathematics lessons showed no change [(Pretest: M = 4.03, SD = .73), (Posttest: M = 4.02, SD = .59); t = .11, p>.05, effect size = .02]. The results suggest that pre-and-post enjoyment of mathematics lessons did not differ.

Table 10   Descriptive statistics and a comparison of students' pretest and posttest
                 responses on TOMRA scales after teaching intervention (n=78)

	Pretest		Postest		Differences	
Scale	Mean	SD	Mean	SD	t-value	Effect Size
Inquiry of Math Lesson	3.39	.49	3.65	.58	3.30**	1.09
Enjoyment of Math Lesson	4.03	.73	4.02	.59	0.11	0.02

**sig. level (2-tailed) at .001, i.e., p<0.05 (statistically significant)

A graphical representation of data appears below.

Comparison of mean score for pre-and-post attitude to mathematics inquiry

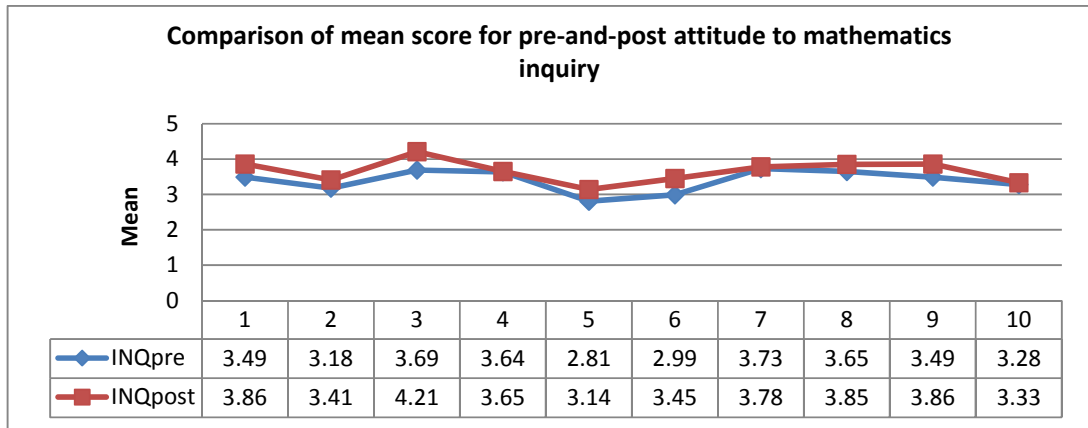|	1	2	3	4	5	6	7	8	9	10
INQpre	3.49	3.18	3.69	3.64	2.81	2.99	3.73	3.65	3.49	3.28
INQpost	3.86	3.41	4.21	3.65	3.14	3.45	3.78	3.85	3.86	3.33

Figure 6   Comparison of mean score for pre-and-post attitude to mathematics inquiry

Comparison of mean score for pre-and-post enjoyment to mathematics lessons

	1	2	3	4	5	6	7	8	9	10
ENJpre	4.18	4.28	3.33	4.04	4.12	4.59	3.91	4	3.85	4.01
ENJpost	4.24	4.29	2.81	4.15	4.1	4.72	4.04	3.94	3.74	4.15
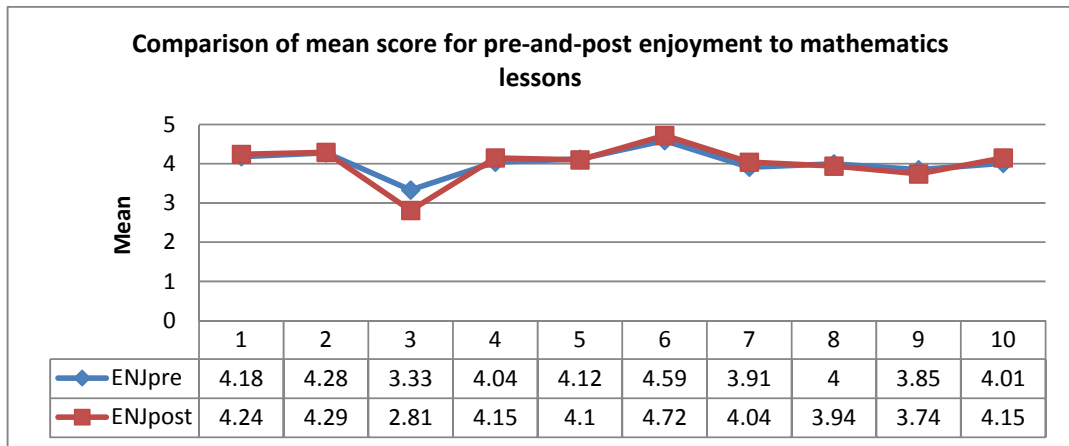
Figure.7   Comparison of mean score for pre-and-post enjoyment to mathematics lessons

The diagnostic teaching experiment emphasises the need for 'conceptual conflict' as a focal point of the programme. It would then become somewhat apparent that students who had been properly exposed to the programme, and who had been presented with an opportunity to engage in the programme accordingly, would confront their existing schema and realise that those are no longer tenable; they need to do something new in order to resolve the conflict. Hopefully, they would demonstrate a greater degree of mathematics inquiry and enjoyment.

4.5   Summary

The previous sections have presented an analysis of the quantitative and qualitative data collected during this study. It was found that diagnostic or conflict teaching was, in general, a useful and valuable means of identifying, overcoming, or 'treating' misconceptions in algebra held by students in the classes. Overall, the conceptual change theory used to frame the study suggests possible effectiveness of conflict teaching in achieving conceptual gain and changes in students' attitudes. The following chapter, as well as providing a summary of the entire study, will present implications of the study, outline the limitations of the research and make recommendations for further research.

Chapter 5

Conclusions

5.1   Introduction

This section concludes the thesis by addressing all the important areas incorporated into the study. Section 5.2 provides a brief overview of the background, objectives, and methodology of the study. Section 5.3 reviews the major findings of the study by relating these findings to each research questions. Section 5.4 addresses the implications that this study has. Section 5.5 focuses on the limitations that evolved during the study. Chapter five concludes in Section 5.6 with suggestions for further research.

5.2   Overview of the Scope of the Thesis

Chapter 1 of the thesis began with the rationale for this study. As stated in that chapter, many students have well developed informal and intuitive mathematical competence before they start formal education. As students learn mathematics in a formal setting, the sense they make of what they are presented with can differ from what the teachers might expect. The concepts can be counterintuitive and often many students do not understand the fundamental ideas or basic concepts covered in the mathematics class. To the students, some critical concepts become troublesome knowledge where they get stuck and mathematics stopped making sense along the way. As a result, these students gave up on mathematics as hopelessly baffling and difficult and developed a negative perception on mathematics. Such luminal or stuck places prevent the student from undergoing a transformation that could extend their understanding of formal mathematical concepts and allow them to make an irreversible and integrative change (Meyer & Land, 2005). To this, Driver and Bell (1986) suggest that, for students to move out of the luminal or stuck places, they need a 'conceptual conflict' where they are forced to confront their existing scheme

and realise that those are no longer tenable; they need to do something new in order to resolve the conflict. As a researcher, I took up the challenge to implement an intervention strategy using conflict teaching to help students cross the cognitive threshold, enabling them to step through an open door within their mind to an 'Ah – Ha' moment (Liljegahl, 2005) and theorised that a conceptual change instructional programme involving cognitive conflict would facilitate students' understanding and improve students' attitudes towards learning mathematics and achievement in mathematics.

Chapter 2 developed the conceptual framework for this study by reviewing relevant literature that addressed the questions being asked. This review of literature evolved from the four questions that formed the bases for this study, namely: (1) Are there learning gains in understanding algebra concepts evident after the six weeks intervention?; (2) What conceptual difficulties and misconceptions do Form 2 Malaysian students have with algebra?; (3) Is there any evidence of students' conceptual change in algebra concepts following the teaching intervention?; and (4) Are students' attitudes towards algebra enhanced after the six weeks intervention?

As a means of reviewing information that would be relevant to answering the above questions, the framework for the literature review focused on five areas, these being: students' conceptual difficulties, misconceptions and what research says about current trends in mathematics instructional strategy; research dealing with student attitudes and their relationship to mathematics achievement; teaching for conceptual change; research on diagnostic teaching using cognitive conflict; and the use of qualitative and quantitative data.

An explanation of the methodology used in this study was the focus of Chapter 3. It contained a brief description of the population in this study, a detailed discussion of the diagnostic teaching methodology, a description of the various instruments used for data collection, a description of how they were amended from their original versions. In addition to the amended instruments used to gather data on student attitudes and misconceptions, the Algebra Diagnostic Test and the Test of Mathematics-Related Attitudes (TOMRA) were described in this section. The final section of Chapter 3 gave a description of data analysis procedures.

For the purpose of this study, the researcher included the findings and discussion in Chapter 4. Each research question was addressed using charts and/or figures based on the quantitative data, as well as qualitative data that had been gathered from a representative sample of students who participated in the study. The researcher chose to use a combination of quantitative and qualitative data as a means of reaching consensus from the two modes of analysis.

5.3   Major Findings of the Study

The major findings of this study are organized around the four research questions presented throughout the study. These findings are presented in sub-headings related to each of the questions.

5.3.1   Student Learning Gain or Achievement

The results showed that there was significant improvement in students' achievement in mathematics [(Pretest: M = 13.06, SD = 3.54), (Posttest: M = 14.91, SD = 3.94); t = 9.27, p<.05, effect size = .49]. There was also a general increase in the number of students scoring 18 to 22 correct answers indicating a significant gain in achievement in posttest scores after the teaching intervention. In addition, the average mean percentage gain on the posttest was 30%. These data support the notion that diagnostic teaching seem to enhance students understanding and achievement in mathematics. In brief, analysis of Algebra Diagnostic Test data indicated that the effect of cognitive conflict teaching on students' achievement was statistically significant (Refer to Table 4, and Figures 1, 2, 3 and 4).

Results from this study indicated that students did in fact, show increased scores on the posttest over the pretest (see Table 4). Results did indicate, however, that the majority of the items of the test were of moderate to high degree of difficulty, with the post test showing a slight increase in difficulty. The discrimination indices showed the pretest to be in need of a revision of eight items showing low

discrimination, 15 items showing medium discrimination and one item showing high discrimination. The discrimination on the posttest came close to being acceptable with no revisions showing only two items with low discrimination, eight items with medium discrimination, and 14 items with high discrimination. Overall, the mean discrimination for the pretest and posttest was .23 and .38 respectively. Based on the categorical make-up of the pretest and posttest, only one category, solve problems by identifying a predictable visual or numerical pattern, result in a discrimination value too low to be acceptable (see Section 4.1, Table 2).

5.3.2   Student Algebraic Conceptual Difficulties and Misconceptions

The results of the study indicated that students' difficulties and misconceptions from both classes fell into five broad areas: (1) basic understanding of letters and their place in mathematics; (2) manipulation of these letters or variables; (3) use of rules of manipulation to solve equations and inequalities; (4) use of knowledge of algebraic structure and syntax to form equations; and (5) use and understanding of algebraic notations in the context of generalisation of patterns.

This study categorised student algebraic conceptual difficulties and misconceptions into five categories (see Table 6). As one might expect, these types of misconceptions are detrimental to students' performance on equation solving tasks: students who hold misconceptions about critical features in algebraic equations tend to solve fewer problems correctly. Such misconceptions also hinder students' learning of new material. According to Booth and Koedinger (2008), students who begin an equation-solving lesson with misconceptions learn less from a typical algebra lesson than students with more sound conceptual knowledge. One might ask "Why might this be the case?" Brown (1992), Chiu and Liu (2004) argue that one of the reasons is highly related to abundant research in science education which demonstrates the importance of engaging and correcting students' preconceptions about scientific topics before presenting new information. If these preconceptions are not engaged, teachers are merely attempting to pile more information on top of the flawed foundation built on persistent misconceptions. In this case, students will not achieve full comprehension of the new material (Kendeou & van den Broek, 2005);

rather, they may reject the new information that does not fit with their prior conception or try in vain to integrate the new information into their flawed or immature conceptions, resulting in a confused understanding of the content (Linn & Eylon, 2006). How can students be expected to learn what the teacher intends if they are not correctly viewing, let alone interpreting, the instructional materials? Hence, eliminating student misconceptions should be a major goal for successful mathematics teaching. There is no better way than to begin by identifying and analysing the difficulties and misconceptions students faced before embarking on an instructional strategy to help students overcome these difficulties and misconceptions. And this precisely was the first task set in this study and consequently examined and interpretted student algebraic difficulties and misconceptions into five categories mentioned above. The presence of student misconceptions suggests teachers need to identify and target misconceptions and build up accurate conceptual knowledge while still providing students with enough instruction and practice on the wealth of procedural skills that are required to succeed in mathematics.

5.3.3 Student Conceptual Change

Meaningful learning in mathematics entails conceptual understanding rather than rote memorization. Students who develop conceptual understanding of certain concepts typically construct well-connected and hierarchically arranged conceptual frameworks, as opposed to having isolated pieces of information (Mintzes, Wandersee & Novak, 1998). Some researchers view learning of a new concept as either integration into an existing knowledge framework (conceptual growth/assimilation) or fundamental reorganization of existing knowledge to fit the new concept into the frasmework (conceptual change/accomodation) (Posner et al., 1982; Treagust & Duit, 2008). The central commitment of the conceptual change learning is that learning is a coherent action that can be defined as coming to understand and accept ideas because they are seen as rational (Posner et al., 1982; Suping, 2003). According to Posner et al. (1982), there are four conditions that facilitate conceptual change. They are dissatisfaction with the existing conceptions, intelligibility, plausibility and fruitfulness. This situation was observed in this study.

From the analysis and interpretation of the data of the Algebra Diagnostic Test and of the student interviews, the results do appear to lend credibility to the idea of conceptual change being engendered by the implementation of an instructional strategy that involves cognitive conflict. As predetermined earlier in Section 4.3.1, changes in students' understanding from unintelligible to intelligible, intelligible to plausible, plausible to fruitful illustrated the extent of changes in their conceptions. Excerpts discussed in Section 4.3.1 provide evidence of such changes. This research has shown that increased status of a conception is possible by means of cognitive conflict teaching. The idea of status of a conception serves to show the degree to which students understand, believe and can apply their algebraic knowledge to otherwise unsolved problems.

5.3.4 Student Attitude

To determine whether diagnostic teaching had a positive impact on student attitudes towards algebra two scales of the Test of Mathematics Related Attitudes (TOMRA), namely Attitude to Mathematics Inquiry and Enjoyment of Mathematics Lessons were measured.

Results of these data indicated that attitude to mathematics inquiry showed a statistically significant increase in the attitudes of the students (see Section 4.4, Table 10). Enjoyment remained high (mean of 4.02 out of 5) even though the enjoyment of mathematics lessons showed no change (see Section 4.4, Table10, Figures 6 and 7).

These results would seem to indicate that there is a possibility that diagnostic teaching intervention did have a positive impact among the students, particularly on mathematical inquiry.

5.4   Implications of the Study

This study attempted to evaluate the efficacy of a conceptual change instructional programme involving cognitive conflict in (1) facilitating Form two (grade 8) students' understanding of algebra concepts, and (2) assessing changes in students' attitudes towards learning mathematics. The hypothetical assertion of this research is that the implementation of diagnostic teaching strategy in Form two mathematics' class will challenge misconceptions and alternative conceptions, enhance students' attitudes and perceptions of the mathematics learning, and facilitate conceptual change and achievement in algebra.

Findings of the study suggest potentially important implications for the teaching and learning of mathematics. First, the teaching approach used to foster conceptual change in student's mathematics learning involving cognitive conflict can be considered for other similar classrooms to promote conceptual change or learning progression among students. More importantly, cognitive conflict teaching places students in an environment that encourages them to confront their preconceptions and then work toward resolution and conceptual change. It helps students to learn by actively identifying and challenging their existing conceptions and the views of their classmates. In the process, students are encouraged to come up with and explore multiple ways to approach a problem rather than just following strict instructions that may present one way of doing something.   As noted in this study, changes in students' ideas (i.e., from unintelligible to intelligible, intelligible to plausible, plausible to fruitful) are indicative of changes in conception (Hewson & Thorley, 1989).

Second, mathematics teachers should be aware of the possible intuitive ideas (presuppositions) students may hold. For example, the prior knowledge that negative signs represent only the subtraction operation and do not modify terms (Vlassis, 2004; Booth & Koedinger, 2008) might be an obstacle to learning to solve simple algebraic equations. These alternative conceptions or gaps in conceptual knowledge may inhibit students' performance and learning. To become more effective in nurturing conceptual change, teachers should seek to understand students' naïve

conceptions so they can be addressed directly by instruction. Additionally, since students organise their lives arounds views that they hold about phenomena, so some conceptual change that teachers consider desirable may be highly resistant to change, and potentially threatening to students.

Third, teachers should realise that conceptual change is multi-faceted, interactive and theoretically complex (Sinatra & Mason, 2008, Taber, 2011), a slow process (Vosniadou, 2008a), "where any observed apparent sudden changes are hard-won and simply offer the surface evidence of extended, preconcious processes influenced by many months of classroom experience" (Taber, 2011, p. 13). And concepts are not static. They change in many ways from the most simple – as in cases where a new instance is added on to an existing concept – to the most radical – as in cases that involve belief revision, ontological category shifts and changes in causality. Therefore, teachers should realise that instruction designed to promote conceptual change usually requires substantial time and effort on the part of teachers and learners. Hence, the durability of students' algebraic conceptions constructed in the context of a particular instruction appears to be critical for supporting the development of new algebraic conceptions. Therefore, teachers should provide opportunity for students to think about their intuitive ideas and use alternative teaching methods to attract students' interest and also eliminate students' misconceptions.

Finally, though there remains a range of views on the process of conceptual change, progress has been made in identifying instructional methods that promote conceptual change. One of the ways to promote conceptual change is to create cognitive conflict among the students by presenting them with meaningful contradictions (leading students through a process of construction of meaning or the mechanisms of conceptual change). Nevertheless, according to research, there are situations where contradiction exists but students do not experience the unbalanced state of cognitive conflict. On the contrary, in some situations, some students ignore the contradiction (Niaz, 1995), and in others, students recognise the contradiction but act in an inconsistent manner (Vinner, 1990; Wilson, 1990). Therefore, teachers need to be aware of the two contrasting sides of conflict strategy. It can either enhance or inhibit learning. For this reason, Clement (2008) warns of problems with teaching through

setting up cognitive conflict/dissonance, because although "the brighter, more successful students reacted enthusiastically to 'cognitive conflicts', the unsuccessful students developed negative attitudes and tried to avoid conflicts" (p. 423). However, the results of this research seem to assert that when effective conflict teaching strategy is integrated in the mathematics classroom, there are significant correlations to an increase in student achievement in mathematics and attitudes towards mathematics.

5.5   Limitations

The results and conclusions generated in this study refer specially to the sample groups involved in the study and this short period of six weeks in which this study was conducted. Generalisations of the findings to all secondary students in Malaysia must be considered with caution due to the nature, limited size of the sample and the short time frame. Due to these constraints, the effect of a conceptual change instructional programme involving cognitive conflict on students' understanding, achievement, attitudes and perceptions, and conceptual change cannot be established with absolute assurance. Potential effects of the students' learning styles, the attitude of the students towards the learning of mathematics, the classroom climate/environment, as well as the effect of different teachers who taught the students, their teaching and management styles were not explored in this research. Though a small number of Malaysian students were involved in the development and administration of diagnostic teaching intervention, the author is confident that the findings presented here, if not strictly speaking from the representative sample, are nevertheless relevant to all mathematics teachers of secondary schools in Malaysia.

5.6   Suggestions for Future Research

Inferring from the findings of the study and relying on the past literature (Buck, Johnson, Fischler, Peukert, & Seifert, 2001; Vosniadou, 2003), it seems evident that without explicit metacognitive reflection, promoting stable radical changes in

students' conceptual framework is difficult. In fact, the diagnostic teaching instructional programme to foster conceptual change did not explicitly teach or emphasise the development of metacognitive strategies. Modification of the current diagnostic instruction by integrating explicit teaching of metacognitive strategies and offering more opportunities for student metacognitive reflection might further promote the construction of students' understandings of the relevant algebra concepts.

Moreover, conceptual change not only involves cognitive process but also is stimulated by a range of affective dimensions such as student interest, motivation, and beliefs. Treagust and Duit (2008) claimed that affective dimension "play(s) an important role in supporting conceptual change on the level of science content knowledge" (p. 300). This study did not inquire about how instruction impacts students' interest or motivation to learn mathematics, more specifically algebra. Further research should simultaneously explore the contribution of the instruction to the improvement of affective dimensions in mathematics learning and the construction of mathematics understandings. Such research may provide a more complete portrayal of students' conceptual learning with multidimensional evidence from cognitive and affective domains of student learning.

Different pedagogies can affect how conceptual change and challenge of misconceptions occurs. Therefore, knowledge of the origin of different types of misconceptions can be useful in selecting more effective pedagogical techniques for challenging particular misconceptions. Also, for teachers to create an effective learning experience they must be aware of and acknowledge students' prior knowledge acquired from academic settings and from everyday previous personal experience. Since all learning involves transfer from prior knowledge and previous experiences, an awareness and understanding of a student's intial conceptual framework and/or topic can be used to formulate more effective teaching strategies. If this idea is taken a step further, it could be said that, since misconceptions comprise part of a conceptual framework, then understanding origins of misconceptions would further facilitate development of effective teaching strategies. Understanding the source of errors carries important consequences for how teachers

address misconceptions. In order to facilitate more effective student construction of new knowledge, students' misconceptions need to be characterised and addressed. Additionally, further research is needed to help teachers understand how students experience conflict, how students feel when they experience cognitive conflict, and how those experiences are related to their final responses because cognitive conflict has both constructive and destructive potential. Thus, by being able to recognise, interpret and manage cognitive conflict, a teacher can then successfully interpret his/her students' cognitive conflict and be able to make conceptual change more likely or guide and lead students to have meaningful learning experiences in secondary school algebra.

5.7   Summary of the Thesis

This section has completed this study by presenting the conclusion obtained through careful analysis of the data. It is the researcher's intention that this research can be used as an addition to the existing body of knowledge regarding students' difficulties and misconceptions in algebra, the teaching and learning of algebra using a conceptual change instructional programme involving cognitive conflict, its impact on students' understanding of algebra concepts, and its impact on student attitudes of mathematics and achievement in mathematics.

In sum, the reliance on the skill of the teacher cannot be overemphasised. Evidence indicates that diagnostic lessons clearly promote learning, but they cannot be used every lesson – the students need time to consolidate; the teacher needs time to recuperate. Thus in teaching toward understanding of major concepts in algebra and achieving conceptual change for students, it is first necessary to understand students' prior knowledge, examine it, identify confusions, and then provide opportunities for old and new ideas to collide. In teaching toward conceptual change, it is counterproductive to simply cover more material and present an extensive list of new ideas without engaging students in their own metacognitive analysis. As advocated by many mathematics education reform documents (Bell et al., 1986; Fujii, 2003; Swan, 2005; Vosniadou, 2007a), diagnostic teaching may be seen as one strategy for

teaching toward conceptual change, in that conflict engages students in the exact same questioning of one's preconceptions and challenging of one's own knowledge that is characteristic of both conceptual change and scientific habits of mind. In this sense, working toward conceptual change is fundamentally what scientists do in the laboratory every day, yet it is not generally the norm of what students are doing in the classroom. If teachers are to be successful in changing the way students think about how certain mathematical concepts work, in the same way mathematicians continue to revolutionise ideas about the same subject, then students and teachers together must access prior knowledge and uncover misunderstandings and incomplete understandings. Perhaps paradoxically, students' "wrong answers" may be our best tool for crafting learning experiences that will move them toward the "right" answers, at least "right" in the sense that they are better aligned with current mathematical evidence.

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TOMRA

TEST OF MATHEMATICS-RELATED ATTITUDES

NAME: _____

CLASS: _____

Directions: This test contains a number of statements about mathematics lessons. You will be asked what you think about these statements. There are no "right" or "wrong" answers. Your opinion is what is wanted.

All answers should be given on the test sheet.

For each statement, draw a circle around the answer that best describes your opinion.

     SA   if you STRONLY AGREE with the statement,

     A   if you AGREE with the statement,

     N   if you are NOT SURE about the statement,

     D   if you DISAGREE with the statement,

     SD   if you STRONGLY DISAGREE WITH THE STATEMENT.

Practice Item:

It would be interesting to learn about cars.

Suppose that you AGREE with this statement, then you would circle A on your survey sheet like this:

     SA   (A)   N   D   SD

If you change your mind about an answer, cross it out and circle another one.

Although some sentences in this test are fairly similar to other statements, you are asked to indicate your opinion about all statements

1. Math lessons are fun.   SA   A   N   D   SD

2. I would prefer to find out why something happens   SA   A   N   D   SD
   by solving a mathematics problem than by being
   told.

3. I dislike mathematics lessons.   SA   A   N   D   SD

4. Doing mathematics problems is not as good as   SA   A   N   D   SD
   finding out from teachers.

5. Schools should have more math lessons each   SA   A   N   D   SD
   week.

6. I would prefer to do mathematics problems than   SA   A   N   D   SD
   read about them.

7. Mathematics lessons bore me.   SA   A   N   D   SD

8. I would rather agree with other people than to do   SA   A   N   D   SD
   a mathematics problem to find out for myself.

9. Mathematics one of the most interesting school   SA   A   N   D   SD
   subjects.

10. I would prefer to do my own mathematics   SA   A   N   D   SD
    problems than to find out information from a
    teacher.

11. Mathematics lessons are a waste of time.   SA   A   N   D   SD

12. I would rather find out about things by asking an   SA   A   N   D   SD
    expert than by doing a mathematics problem.

13. I really enjoy going to mathematics lessons. SA A N D SD

14. I would rather solve a problem by doing SA A N D SD
    mathematics than be told the answer.

15. The material covered in math lessons is SA A N D SD
    uninteresting.

16. It is better to ask the teacher the answer than to SA A N D SD
    find it out by doing a mathematics problem.

17. I look forward to mathematics lessons. SA A N D SD

18. I would prefer to do a mathematics problem on a SA A N D SD
    topic than to read about it in a mathematics
    magazine.

19. I would enjoy school more if there were no SA A N D SD
    mathematics lessons.

20. It is better to be told mathematics facts than to SA A N D SD
    find them out from doing a mathematics problem.

Appendix B

Source of Algebra Diagnostic Test items

Item	Source from
1	Perso, 1991
2	Blessing, 2004
3	Perso, 1991
4	Blessing, 2004
5	Perso, 1991
6	Perso, 1991
7	Blessing, 2004
8	Blessing, 2004
9	Blessing, 2004
10	Perso, 1991
11	Perso, 1991
12	Perso, 1991
13	Perso, 1991
14	Blessing, 2004
15	Perso, 1991
16	Perso, 1991
17	Perso, 1991
18	Perso, 1991
19	Perso, 1991
20	Perso, 1991
21	Blessing, 2004
22	Perso, 1991
23	Perso, 1991
24	Blessing, 2004

Appendix C

ALGEBRA DIAGNOSTIC TEST

Instructions:

This paper consists of 24 items that evaluate your understanding of algebra concepts that you have studied in your mathematics course.

Circle one of the **choices (A, B, C or D)** to indicate what you consider to be the most appropriate answer.

1   If $a + b = 7,$ then $a + b + c =$

    A   8

    B   12

    C $7 + b$

    D $7 + c$

2   What must be done to each number in Column A to get the number in Column B?

Column A	Column B
6	4
12	8
15	10
24	16

    A   Subtract 3 from the number in Column A and add 1 to the answer

    B   Subtract 2 from the number in Column A

    C   Add 2 to the number in Column A

    D   Divide the number in Column A by 3 and multiply the answer by 2

3   $x = y + 2$, what happens to $'x'$ if 5 is added to $'y'$?

     A $x = y + 7$

     B $x = 5y + 2$

     C   x is decreased by 3

     D   x is increased by 3
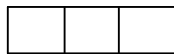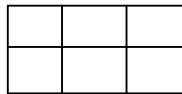
4   Here is the beginning of a pattern of tiles.
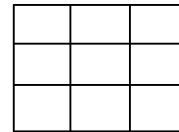
Figure 1   Figure 2   Figure 3

If this pattern continues, how many tiles will be in Figure 6?

     A   12

     B   15

     C   18

     D   21

5   Plums cost 8 cents each and bananas cost 5 cents each. If 'p' stands for the number of plums bought and 'b' stands for the number of bananas bought, what does 8p + 5b stand for?

     A   8 times the cost of plums plus 5 times the cost of bananas

     B   the cost of plums plus the cost of bananas

     C   8 plums and 5 bananas

     D   89 cents

6   Half a number, subtracted from four is fifteen. The equation best representing
     this statement is

A $4 - \frac{1}{2}x = 15$

B $x - \frac{1}{2}x = 15$

C $\frac{1}{2} - 4 = 15$

D $\frac{1}{2}x - 4 = 15$

7   Sue has some trading cards. Pam has three times as many trading cards as Sue.
     They have 36 trading cards in all. Which of these equations represents their
     trading card collection?

A $3x = 36$

B $x + 3 = 36$

C $x + 3x = 36$

D $3x + 36 = x$

8   N stands for the number of stamps that Clarita had. She gave 12 stamps to her
     sister and then purchased 8 more at the post office. Which expression tells how
     many stamps Clarita has now?

A $(N - 12) + 8$

B $(N - 4) + 12$

C $(N + 8) - 12$

D $(N - 8) + 12$

9   For every $15 that William spends at the grocery store, he receives three free store Discount coupons. If William spent $90 on groceries last week, how many free coupons did he receive?

   A   6
   B   75
   C   45
   D   18

10   When is $a + b + c = a + z + c$?

   A   always
   B   never
   C   when $b = z$
   D   none of these

11   In the expression $a + 5$, 'a' stands for

   A   1
   B   any number
   C   apple
   D   nothing

12   $5b + 4b =$

   A $9\ bananas$
   B $5b + 4b$
   C $9b^2$
   D $9b$

13 What can you say about 'a' if a + 5 = 4a?

A $a = \frac{5}{3}$

B $4a + 5$

C $a = \frac{3}{5}$

D $a = 0$

14 What rule can be used to determine the number pattern below?

2, 5, 11, 23, 47, ...

A $N + 3$

B $2N + 1$

C $N - 1$

D $3N - 1$

15 What is $'k'$ if $k - 12 = 4$?

A -8

B 3

C 8

D 16

16 □ represents the number of newspapers that Lee delivers each day. Which of the following expressions represents the total number of newspapers that Lee delivers in 5 days if on the fifth day he delivers an additional three newspapers?

A $5 \times □ + 3$

B $(5 \times □) + 3$

C $(□ \div 5) + 3$

D $3 + (5 - □)$

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17   If $5 = 9y$, then $y =$

    A $5 \div 9$

    B $9 \div 5$

    C $5 - 9$

    D 5×9

18   What is $-3p \times 2$?

    A $6p$

    B $-6p$

    C $5p$

    D $-5p$

19   What can be said about 'k' if $k = j + m$ and $k + j + m = 12$

    A $k = 6$

    B $k = 3$

    C $k = 12 + j + m$

    D $k = 12 - j + m$

20   Find the value of $3 + y \times 2$

    A $6y$

    B $5 \times y$

    C $2y + 3$

    D   none of these

21 Daryl earned $75 mowing lawns. He wants to buy a bicycle costing $163. How much more money will Daryl have to earn in order to purchase the bicycle? Which number sentence below can be used to solve this problem?

  A $163 + $75 = □

  B   □ − $75 = $163

  C $163 + □ = $75

  D $163 − □ = $75

22 Another way of writing $2 \times (a + b)$ would be

  A $2a + b$

  B $2a + 2b$

  C $2ab$

  D   none of these

23 Add 5 onto $a + 2$

  A $a + 7$

  B $5a + 2$

  C $7a$

  D   none of these

24 What rule can be used to fill in the blanks in the number pattern below?

INPUT	1	2	3	5	9
OUTPUT	1	4	9	__	__

  A $INPUT + OUTPUT$

  B $INPUT \times 2$

  C $INPUT \times INPUT$

  D $INPUT \div 2$

Algebra Diagnostic Test – Descriptors

Category	Item Descriptor	Item #
1	Basic understanding of letters and their place in algebra	10, 11, 13, 23
2	Manipulation of letters, symbols or variables	3, 15, 18
3	The use of appropriate rules of manipulation to solve equations	1, 2, 20, 22
4	Solves problems by identifying a predictable visual or numerical pattern	4, 7, 17
5	Translates words into algebraic expressions	5, 6, 16, 21
6	Analyses and generalises number patterns and states the general rule for the relationship	8, 14, 24
7	Solves problems involving simple equations using symbolic expressions or written phrases	9, 12, 19

Adapted from Blessing (2004)

Appendix E

Interview Transcripts

T = Teacher Interviewer

S = Student

Sam (S3)

T: When is $a + b + c = a + z + c$? What you you think?

S3:   I think they are never the same.

T: Why do you think that?

S3:   No, … It's not that, (pause) 'cause b can't be equal to z.

T: Why?

S3:   Because they're different letters.

(Pete: S4) was asked what answer he selected for this question (Item 5) "Plums cost 8 cents each and bananas cost 5 cents each. If 'p' stands for the number of plums and 'b' stands for 'the number of bananas' bought, what does 8p + 5b stand for?"

T: What answer did you select?

S4:   I chose C. [The answer for C is 8 plums and 5 bananas].

T: Why did you choose that?

S4:   Well, it's just that it would have cents on it if it was the cost … it would equal so many cents. Yeah, I still think it's 8 plums and 5 bananas, .. 'p' is for plums and 'b' is for bananas.

Ramiah (S8)

T: If $a + b = 7$, then $a + b + c =$ what?

S8:   I think it's 12.

T: Why?

S8:   Well, if $a + b = 7$, a could equal 3, b could equal 4 and c could equal 5, in a sequence.

T: Why do you think it would be a sequence, because they are alphabetical?

S8:   Yeah, that's it, $a + b + c$ is $3 + 4 + 5 = 12$

Megat (S1) and Suhaila (S2)

T   So how do you establish a formula? How did you come up with 2 x n + 1?

S2   Cos you do 2 times the box plus 1. [S2 interpreted a triangle as a box.]

T   I don't get that. Why you add '1' to $n \: x \: 2$?

S2   Ahm, you know figure number, like figure 10. You times 10 by 2 equals 20 and then you add 1 equals 21 and you get the number of toothpicks. Cos when you count it, you only count 1 extra [at the beginning] and then you kinda … keep adding 2 extra sides. Yea, it works! (At this point, Megat (S1) interjected).

S1   But no, that would really be the number of the figure times 2 plus 1. Because, look. 2 times 2, 4, plus 1, 5. One times … 1 times 2, 2, plus 1, 3! Yes. That would work!"

S2   So it's times 2 plus 1, right? And, to calculate the number of toothpicks in figure number, 25, it's … figure 25, so, yes, it's 25 times 2 plus 1 , 51. Yeah, 51 toothpicks!

(Henry: S5)

T: I think you usually think about speed in terms of how fast or slow it is, and I need you to start to think about speed in terms of how far it's going to get you. If you keep that car at 55 kilometres an hour for an entire hour, do you know how far that'll get you?

S5:   Fifty-five kilometres. That's like, that you would be, you could travel, if you're travelling at 55 kilometres an hour, then you might get … 55 kilometres.

T: So, can you explain to a friend what does it mean when a car is travelling at seventy kilometres per hour?

S5:   That's like, that you would be, you could travel, if you're travelling at 70 kilometres an hour, then you might get … 70 kilometres. I mean, it means you'll be seventy kilometres away from where you started every hour that you can drive like that.

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T: Now, there are two cars, one is travelling at a speed of 70 kilometres an hour and the other is travelling at 55 kilometres an hour. Which car do you think is travelling faster?

S5:   Er … I think the one, the 70 kilometres … one is faster.

Gita (S6)

T: How did you come up with this column (referring to the table 5 column)? What did you do to get to this column?

S6:   I did, umm, fifty-five each time and kept going down.

T: Okay, you kept adding 55 each time?

S6:   Yes. The next one would be 55 plus 55, 110. The 110 plus 55, er, 165.

T: So, for like 3 hours you come up with 165 kilometres? How can I get for five then?

S6:   Err, okay, three hours I get one hundred sixty-five, so four is plus 55 (calculating on a piece of paper), writes 220. Four, 220, plus 55, two hundred and seventy-five. It's 275.

T: Okay, go beyond what you see and what you're doing. How can I get for three if I didn't have two and one?

S6:   Okay, let me see (referring to the constructed table of values) … inaudible. Oh, yes, one, fifty-five, 55, 2 fifty-five, 110, yes, yes, (excited by the new discovery). I know, I know, it's 55 times three.

T: So for like 3 hours how did you come up with one 165?

S6:   55 times three.

T: Very good, three times 55. So, for three hours the car would have traveld a distance of 165 kilometres. Well, how far do I get for five hours then?

S6:   Five times 55, 275. It's 275 kilometres.

T: Excellent! How do I get eight and ten?

S6:   It's eight times 55, (manipulating the calculator this time), four hundred forty, yes, 440. Ten times 55, 550.

T: Excellent. So, what do these numbers represent?

S6:   Distance.

T: Represent distance but represent distance in what?

S6:   Kilometres

T: Kilometres? Is that on here? (Student adds units). Kilometres, okay. And you have time with hours and speed in kilometres an hour. Now, can you write a formula using letters of the alphabet to represent distance travelled with time and speed?

S6:   What? I don't understand.

T: Let's say a friend of yours want to know the distance a car can travel in two hours if you were driving at a speed of 50 kilometres an hour. Can you write down a formula to help him find the distance travelled connecting the number of hours and the speed of the car? Let's assume D to represent the distance travelled, S refers to speed and T refers to the number of hours. Can you write the formula?

S6:   (Hesitates) I'm not sure, but I'll try, … can it be distance travelled, er, D, (pauses, silent for a while).

T: Yes, D for distance travelled. Then, how did you get this distance? Can you recall what you did just now, you use something times something? What was that?

S6:   Er, three times 55, five times 55, … (refers to the table of values) three hours, five hours. Yes, yes, hours, hours for T. That's it, time for T, and … fifty-five, fifty-five standing for … (pauses).

T: And, 55 for what? Fifty-five stands how fast or slow your car is travelling?

S6:   Oh, I see, fast, (pauses a while) aha! Yes, it's speed, speed isn't it?

T: Absolutely correct. Fifty-five kilometres an hour is the speed. Now can you write the formula using these three letters, D, T and S?

S6:   I'll try. Let me think, … (speaking to himself, inaudible). I think, time times speed is equal to distance, (writes) T x S = D. Am I right?

Sue (7)

S7:   Fifty kilometres? That's like, that you would be, you could travel a kilometre in like, (pause) … in like. (inaudible) Uhhh, it's like if you're doing 50 kilometres an hour you can cover kilometres an hour. (Not sure of the answer).

T: Say that one more time.

S7:   Say if, if you're travelling 50 kilometres an hour, then you might get … 50 kilometres in one hour.

T: Right.

S7:   Like say if you add one it's like a kilometre a minute.

T: It's very close to doing that. Yeah. Would you say it's more than a kilometre a minute or less than a kilometer a minute? If you're doing 50 kilometres in an hour?

S7:   It's probably less.

T: Well tell me this; if you could do a kilometre per minute now, how far would you get in an hour?

S7:   Sixty kilometres.

T: Right. But you're doing 50 so that means?

S7:   [Inaudible.]

T: Let's go back to how far they're going every minute. So, if they can put in sixty kilometres in an hour, that means they're doing a kilometre a minute. They're just doing 50. So, are they doing more than a kilometre a minute or less?

S7:   Less.

T: A little bit less.

Terry (S9)

S9:   (Substituting D for 125 and S for 50, student calculates). Distance divide by speed equals hour, no, no, equals time, right? So, 125 divided by 50 (manipulates the calculator given), …, equals 2 point 5 (2.5).

T: 2.5 what?

S9:   2.5 hours. It's two-and-half hours, correct?

T: How many minutes are there in an hour?

S9:   Sixty. So, half of sixty, sixty into half, yes, yes, sixty into two, it's thirty. Oh, it's 2 hours and thirty minutes.