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Practical Stability and Controllability for Nonlinear Discrete Time-delay Systems

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Abstract— **In this paper we study the practical asymptotic stability for a class of discrete-time time-delay systems via Razumikhin-type Theorems. Further estimations of the solution boundary and arrival time of the solution are also investigated based on practical stability. In addition, the proposed theorems are used to analyze the practical controllability of a general class of nonlinear discrete systems with input time delay. Some easy testing criteria for the uniform practical asymptotical stability are derived via Lyapunov function and Razumikhin technique. Finally a numerical example is given to illustrate the effectiveness of the proposed results.**

I. INTRODUCTION

Since Lasalle first introduced the concept of practical stability in [1], it attracts much attention in control community. Many works on practical stability have been published with broad applications in different areas. Being much different from stability in terms of Lyapunov functions, practical stability, which stabilizes a system into a region of phase space, is a significant performance specification from engineering point of view, and are satisfactory in many applications for quality analysis. In practice, a system is actually unstable, just because the stable domain or the domain of the desired attractor is not large enough; or sometimes, the desired state of a system may be mathematically unstable, yet the system may oscillate sufficiently near to a state, in which the performance is still acceptable, i.e., it is stable in practice. For example, in practical communication or digital control systems, the signals of controller states, measurement outputs, and control inputs are quantized and then coded for transmission. A feedback law, which global asymptotically stabilizes a given system without quantization, will in general fail to guarantee global asymptotic stability of the closed-loop system, which arises in the presence of a quantizer with a finite number of values. Instead of global asymptotic stability, the practice stability can be obtained, where there is a region of attraction in the state and the steady state converges to a small limit cycle [2], [3], [4], [5], [6].

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On the other hand, it is well known that for more than one hundred years Lyapunov's direct method has been the primary technique for dealing with stability problems in difference equations. However, the construction of Lyapunov's function is much more difficult for time-delay systems than for non-delay systems. Such difficulties can be overcome via using Lyapunov functions and Razumikhin techniques. It should be pointed out that the Razumikhin-type method could deal with the time-delay problem effectively and are easier to apply in general, therefore such a method has been a main technique for analyzing the stability for time-delay systems [7], [8], [9], [10]. To the best of our knowledge, the studies of Razumikhin-type method on practical stability for discrete time-delay systems has not been investigated. Motivated by results in [9], we will study the Razumikhin-type theorem on practical asymptotic stability for a class of discrete time-delay system in this paper. Also estimations of the solution boundary and arrival time of the solution are discussed. Consequently, the proposed theorems are used to study the practical controllability of a general class of nonlinear discrete systems with input time delay. Some easy-testing criteria for the uniform practical asymptotical stability are obtained via Lyapunov function and Razumikhin technique.

This paper is organized as follows. In Section II, some definitions and preliminaries are introduced. In Section III, some criteria for uniform practical asymptotical stability of discrete-time systems with finite delay are derived via Lyapunov functions and Razumikhin-technique. In Section IV, estimation of the solution boundary and arrival time of the solution are investigated in terms of practical stability. In Section V, the proposed theorems are used to analyze the practical controllability for a general class of nonlinear discrete systems with input time delay. In Section VI, a numerical example is given to illustrate the effectiveness of main results obtained from the Section V. The last section gives some conclusions.

Notation: $\mathbb{R}^N$ denotes the N-dimensional Euclidean space, $\mathbb{Z}^+$ is the set of nonnegative integer. Let $I_d = \{-d, -d+1, \ldots, -1, 0\}$ with some integer $d \geq 0$, $I_d^1 = I_d \cup \{1\}$, $\Upsilon(I_d, \mathbb{R}^N) = \{\varphi_d = (\varphi^T(0), \varphi^T(-1), \ldots, \varphi^T(-d))^T \,|\, \varphi : I_d \to \mathbb{R}^N\}$. For all $\varphi_d \in \Upsilon(I_d, \mathbb{R}^N)$, define the norm of φ_d as $\|\varphi_d\| = \max_{s \in I_d} |\varphi(s)|$, where $|\cdot|$ stands for any norm in $\mathbb{R}^N$. Let $\Upsilon_B(I_d, \mathbb{R}^N) = \Upsilon(I_d, \mathbb{R}^N) \cap \{\varphi_d : \varphi(s) \in B, s \in I_d\}$, where B is an open ball.

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II. PRELIMINARIES

Consider a general class of nonlinear discrete-time systems with finite delay as follows:

$$X(k+1) = F(k, X_d(k)), \quad k \in \mathbb{Z}^+, \tag{1}$$

where $d \geq 0$ is an integer, $X(k) \in \mathbb{R}^N$, $X_d(k) = (X^T(k), X^T(k-1), \ldots, X^T(k-d))^T$, $F : \mathbb{Z}^+ \times \Upsilon_B(I_d, \mathbb{R}^N) \rightarrow \mathbb{R}^N$. We assume that F satisfies certain conditions to guarantee the global existence and uniqueness of solutions, and $F(k,0) = 0$ for $k \in \mathbb{Z}^+$. Thus system (1) has zero solution $X(\cdot) \equiv 0$. For any $k_0 \in \mathbb{Z}^+$ and any given initial function $X_0 \in \Upsilon_B(I_d, \mathbb{R}^N)$, the solution of the systems (1) denoted by $X(k; k_0, X_0)$ satisfies (1) for all integers $k \geq k_0$, and $X(k_0+s; k_0, X_0) = X_0(s)$ for all $s \in I_d$. We further assume that there exists a constant $L > 0$ such that for all $\varphi_d \in \Upsilon_B(I_d, \mathbb{R}^N)$,

$$|F(k, \varphi_d)| \leq L\|\varphi_d\|, \quad \forall k \in \mathbb{Z}^+. \tag{2}$$

We introduce the following definitions.

Definition 2.1: [9] A wedge function is a continuous strictly increasing function $W : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $W(0) = 0$.

Definition 2.2: System (1) is called to be

(PS_1) practically stable (P.S.) if for given (α, β) with $0 < \alpha < \beta$ and some $k_0 \in \mathbb{Z}^+$, we have $\|X_0\| < \alpha$ implies $|X(k; k_0, X_0)| < \beta$, $k \geq k_0$;

(PS_2) uniformly practically stable (U.P.S.) if P.S. holds for all $k_0 \in \mathbb{Z}^+$;

(PS_3) practically asymptotic stable (P.A.S.) if P.S. holds, and for each $\varepsilon \in (0, \beta)$, there exists a positive number $K = K(k_0, \alpha, \varepsilon)$ such that $\|X_0\| < \alpha$ implies $|X(k; k_0, X_0)| < \varepsilon$, $k \geq k_0 + K$;

(PS_4) uniformly practically asymptotic stable (U.P.A.S.) if P.A.S. holds for all $k_0 \in \mathbb{Z}^+$.

Definition 2.3: For a function $V : \mathbb{Z}^+ \times \mathbb{R}^N \rightarrow \mathbb{R}^+$, define:

$$\begin{aligned}\Delta V(k, X(k)) &\triangleq V(k+1, X(k+1)) - V(k, X(k)) \\ &= V(k+1, F(k, X_d(k))) - V(k, X(k)).\end{aligned}$$

III. RAZUMIKHIN-TYPE THEOREMS

In this section we will prove the Razumikhin-type theorems with an aim of analyzing the uniformly practical asymptotic stability (U.P.A.S.) for a general class of nonlinear discrete-time systems with finite delay. We present the following result first for system (1). For the sake of brevity, we denote $X = X(k)$ and $X_d = X_d(k)$.

Theorem 3.1: Given positive scalars α and β. Assume that scalars π_1, π_2, π_3 with $0 < \pi_1 \leq \pi_2$, $\pi_3 > 0$ are all arbitrary. If there exist a scalar $\gamma > 0$, a Lyapunov function $V : \mathbb{Z}^+ \times \mathbb{R}^N \rightarrow \mathbb{R}^+$, and wedge functions $W_i(\cdot)(i = 1,2,3)$, such that

(i) $W_1(|X|) \leq V(k, X) \leq W_2(|X|)$;

(ii) $\Delta V(k, X) \leq -W_3(|X(k+1)|) + \pi_3$ for $\varepsilon_0 \leq \|X_d\| \leq \rho_0$,

provided $\varepsilon_0 \leq \rho_0$, $V(k+s, X(k+s)) \leq \min\{\pi_2, V(k+1, X(k+1)) + \gamma\}$ for $s \in I_d^1$, and $\pi_1 \leq V(k+1, X(k+1))$. Where $\varepsilon_0 = L^{-1}\alpha$, $\rho_0 = \max\{\beta, W_1^{-1}(W_2(\alpha))\}$, L is defined by (2). Then, for the balls:

$$B_1 = \{X : V(k, X) < W_1(\beta)\}; \quad B_2 = \{X : V(k, X) < W_2(\alpha)\}.$$

We have (1) B_2 is an invariable set; (2) If $W_2(\alpha) < W_1(\beta)$, then B_1 is an invariable set and there exists a positive number $K = K(\alpha, \beta)$ such that for any $k_0 \in \mathbb{Z}$, $X_0 \in \Upsilon_{B_1}(I_d, \mathbb{R}^N)$ implies $\forall\ k \geq k_0 + K$, $X(k; k_0, X_0) \in B_2$.

Proof. (1) For each $X_0 \in \Upsilon_{B_2}(I_d, \mathbb{R}^N)$, we have $X(k; k_0, X_0) \in B_2$ for $k_0 - d \leq k \leq k_0$. We claim that for all $k \geq k_0$, $X = X(k; k_0, X_0) \in B_2$.

Suppose this is not true. Then there exist some $k^1 \geq k_0$ such that $X \in B_2$ for all $k_0 - d \leq k \leq k^1$, and

$$V(k^1+1, X(k^1+1)) \geq W_2(\alpha), \tag{3}$$

and consequently,

$$\Delta V(k^1, X(k^1)) = V(k^1+1, X(k^1+1)) - V(k^1, X(k^1)) > 0.$$

On the other hand, by condition (i), we have $W_1(|X|) < W_2(\alpha)$ for $k_0 - d \leq k \leq k^1$, which implies $\|X_d\| \leq \rho_0$ for $k_0 \leq k \leq k^1$. It follows from (2), (3) and condition (i) that $\alpha \leq |X(k^1+1)| \leq L\|X_d(k^1)\| \leq L\rho_0$, which implies $\varepsilon_0 \leq \|X_d(k^1)\| \leq \rho_0$, $\varepsilon_0 \leq \rho_0$. Let $0 < \pi_1 \leq W_2(\alpha) \leq W_2(L\rho_0) \leq \pi_2$, and $0 < \pi_3 < W_3(\alpha)$. Then, it follows from (3) that $\pi_1 \leq V(k^1+1, X(k^1+1))$, and for $\gamma > 0$, $\forall\ s \in I_d^1$, there holds

$$\begin{aligned}&\left\{\begin{array}{l} V(k^1+s, X(k^1+s)) < \pi_2 \\ V(k^1+s, X(k^1+s)) < V(k^1+1, X(k^1+1)) + \gamma \end{array}\right. \\ \Longrightarrow\quad & V(k^1+s, X(k^1+s)) \\ & \leq \min\{\pi_2, V(k^1+1, X(k^1+1)) + \gamma\}.\end{aligned}$$

By condition (ii), we have

$$\Delta V(k^1, X(k^1)) \leq -W_3(|X(k^1+1)|) + \pi_3 < 0.$$

This is a contradiction. Thus for all $k \geq k_0$, $X \in B_2$, i.e., B_2 is an invariable set.

(2) If $W_2(\alpha) < W_1(\beta)$, we first prove that B_1 is an invariable set. In fact, $\rho_0 = \beta$, and $\varepsilon_0 = L^{-1}\alpha < L^{-1}W_2^{-1}(W_1(\beta))$. Similar to the proof of (1), one can derive that, $X_0 \in \Upsilon_{B_1}(I_d, \mathbb{R}^N)$ implies $X \in B_1$ for all $k \geq k_0$.

Next, we will find an integer $K = K(\alpha, \beta) > 0$ such that for all $k_0 \in \mathbb{Z}^+$, $X_0 \in \Upsilon_{B_1}(I_d, \mathbb{R}^N)$ implies $X(k; k_0, X_0) \in B_2$ for all $k \geq k_0 + K$.

Assume that $0 < \pi_1 \leq W_2(\alpha) < W_1(\beta) \leq \pi_2$, $0 < \pi_3 < (1/2)W_3(\alpha)$. Let $\hat{N}$ be the first positive integer such that

$$W_1(\beta) < W_2(\alpha) + \hat{N}\gamma. \tag{4}$$

For each $i \in \{0, 1, \ldots, \hat{N}\}$, let

$$k_i = k_0 + i(d + \left[\frac{W_1(\beta)}{\pi_3}\right]),$$

where $[\cdot]$ denotes the greatest integer function, γ is depended on π_1 and π_3. We show that for all $i \in \{0, 1, \ldots, \hat{N}\}$,

$$V(k, X) < W_2(\alpha) + (\hat{N} - i)\gamma, \quad \forall\ k \geq k_i. \tag{5}$$

Obviously, it follows (4) that (5) holds for $i=0$ since $X \in B_1$ for all $k \geq k_0$. Suppose (5) holds for some $i \in \{0,1,\ldots,\hat{N}-1\}$, we aim to show that (5) also holds for $i+1$, i.e.,

$$V(k,X) < W_2(\alpha)+(\hat{N}-i-1)\gamma, \quad \forall\, k \geq k_{i+1}.$$

We decompose our proof into 2 steps.

Step 1. We show that there does exist some $k' \in [k_i+d, k_{i+1}]$ such that

$$V(k', X(k')) < W_2(\alpha)+(\hat{N}-i-1)\gamma. \tag{6}$$

Suppose this is not true, for all $k \in [k_i+d, k_{i+1}]$, we would have

$$V(k,X) \geq W_2(\alpha)+(\hat{N}-i-1)\gamma. \tag{7}$$

Noting the assumption that (5) holds for some $i \in \{0,1,\ldots,\hat{N}-1\}$, then, for all $k \in [k_i+d, k_{i+1}-1]$, $s \in I_d^1$, from (7) we have

$$\begin{aligned} V(k+s, X(k+s)) &< W_2(\alpha)+(\hat{N}-i)\gamma \\ &\leq V(k+1, X(k+1))+\gamma. \end{aligned}$$

On the other hand, for all $k \in [k_i+d, k_{i+1}-1]$, it follows from condition (i), (2) and (7) that $W_2(\alpha) \leq V(k+1, X(k+1)) \leq W_2(|X(k+1)|)$, which implies that $\alpha \leq |X(k+1)| \leq L\rho_0$, $\varepsilon_0 \leq \|X_d(k)\| \leq \rho_0$, $\varepsilon_0 \leq \rho_0$. Then, for all $k \in [k_i+d, k_{i+1}-1]$, $V(k+s, X(k+s)) \leq \pi_2$, $s \in I_d^1$, and it follows from (7) that $V(k+1, X(k+1)) \geq \pi_1$. By condition (ii), for all $k \in [k_i+d, k_{i+1}-1]$,

$$\Delta V(k,X) \leq -W_3(|X(k+1)|)+\pi_3 < -\pi_3.$$

Hence, we have

$$\begin{aligned} V(k_{i+1},X) &\leq V(k_i+d, X(k_i+d))-\pi_3(k_{i+1}-k_i-d) \\ &< W_1(\beta)-\pi_3\left[\frac{W_1(\beta)}{\pi_3}\right] < 0. \end{aligned}$$

This is a contradiction to the definition of Lyapunov function V. Thus, there does exist some $k' \in [k_i+d, k_{i+1}]$ such that (6) holds.

Step 2. We show that

$$V(k,X) < W_2(\alpha)+(\hat{N}-i-1)\gamma, \quad \forall\, k \geq k'. \tag{8}$$

In fact, suppose this is not true, there must be some $k_1' \geq k'$ such that

$$\begin{aligned} &V(k_1', X(k_1')) < W_2(\alpha)+(\hat{N}-i-1)\gamma, \quad \text{and} \\ &V(k_1'+1, X(k_1'+1)) \geq W_2(\alpha)+(\hat{N}-i-1)\gamma, \end{aligned} \tag{9}$$

and hence we have $\Delta V(k_1', X(k_1')) > 0$. On the other hand, $\pi_1 \leq W_2(\alpha) \leq V(k_1'+1, X(k_1'+1))$, $V(k_1'+s, X(k_1'+s)) \leq \pi_2$. Noting the assumption that (5) holds for some $i \in \{0,1,\ldots,\hat{N}-1\}$, then, for $s \in I_d^1$, we have

$$\begin{aligned} V(k_1'+s, X(k_1'+s)) &< W_2(\alpha)+(\hat{N}-i)\gamma \\ &\leq V(k_1'+1, X(k_1'+1))+\gamma. \end{aligned}$$

From condition (i), (2) and (9), we have $W_2(\alpha) \leq V(k_1'+1, X(k_1'+1)) \leq W_2(|X(k_1'+1)|)$, and hence, $\alpha \leq |X(k_1'+1)| \leq L\rho_0$, $\varepsilon_0 \leq \|X_d(k_1')\| \leq \rho_0$, $\varepsilon_0 \leq \rho_0$. With condition (ii), one can derive that

$$\Delta V(k_1', X(k_1')) \leq -W_3(|X(k_1'+1)|)+\pi_3 \leq -\pi_3 < 0.$$

This is a contradiction again to the definition of Lyapunov function V. Thus (8) holds, and consequently, (5) holds for all $i \in \{0,1,\ldots,\hat{N}\}$. Therefore, we obtain that $X \in B_2$ for all $k \geq k_{\hat{N}} = k_0+K$, where $K = \hat{N}(d+\left[\frac{W_1(\beta)}{\pi_3}\right])$ is independent of k_0 and X_0. □

From Theorem 3.1, we have following corollary.

Corollary 3.2: Given positive scalars α and β. Assume that $\hat{P}(s) \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\hat{P}(s) > s$ for $s > 0$. If there exist a Lyapunov function $V : \mathbb{Z}^+ \times \mathbb{R}^N \to \mathbb{R}^+$, and wedge functions $W_i(\cdot)(i=1,2,3)$, satisfying the conditions (i) in Theorem 3.1 and the following condition (ii)$'$:

$$\text{(ii)}' \quad \Delta V(k,X) \leq -W_3(|X(k+1)|) \text{ for } \varepsilon_0 \leq \|X_d\| \leq \rho_0,$$

provided $\varepsilon_0 \leq \rho_0$, $V(k+s, X(k+s)) < \hat{P}(V(k+1, X(k+1)))$ for $s \in I_d^1$. Where $\varepsilon_0 = L^{-1}\alpha$, $\rho_0 = \max\{\beta, W_1^{-1}(W_2(\alpha))\}$, L is defined by (2). Then, the conclusion of Theorem 3.1 still holds.

Proof. For any $0 < \pi_1 \leq \pi_2$, and any $\pi_3 > 0$, choose $\gamma \in (0, \inf\{\hat{P}(s)-s : \pi_1 \leq s \leq \pi_2\})$. Then, if $V(k+s, X(k+s)) \leq \min\{\pi_2, V(k+1, X(k+1)+\gamma)\}$ for $s \in I_d^1$, and $\pi_1 \leq V(k+1, X(k+1))$, we have

$$\begin{aligned} V(k+s, X(k+s)) &\leq V(k+1, X(k+1))+\gamma \\ &< \hat{P}(V(k+1, X(k+1))), \end{aligned}$$

for $s \in I_d^1$. Hence, by condition (ii)$'$, we have

$$\Delta V(k,X) \leq -W_3(|X(k+1)|) \leq -W_3(|X(k+1)|)+\pi_3.$$

Then, the condition (i) and (ii) in Theorem 3.1 are both satisfied. Therefore, the result follows. □

By employing Theorem 3.1 and Corollary 3.2, we obtain the following Razumikhin-type theorems for the U.P.A.S. of the zero solution of systems (1).

Theorem 3.3: For given scalar pair (α, β) with $0 < \alpha < \beta$, $\varepsilon \in (0, \beta)$ is arbitrary. Assume that scalars π_1, π_2, π_3 with $0 < \pi_1 \leq \pi_2$, $\pi_3 > 0$ are all arbitrary, $\hat{P}(s) \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\hat{P}(s) > s$ for $s > 0$. If there exist a Lyapunov function $V : \mathbb{Z}^+ \times \mathbb{R}^N \to \mathbb{R}^+$, wedge functions $W_i(\cdot)(i=1,2,3)$, satisfying

(i) $W_2(\alpha) \leq W_1(\beta)$;

(ii) $W_1(|X|) \leq V(k,X) \leq W_2(|X|)$;

and either the following conditions (iii)$_a$ or(iii)$_b$ for $\varepsilon_0 \leq \|X_d\| \leq \rho_0$, $\varepsilon_0 \leq \rho_0$:

(iii)$_a$ $\Delta V(k,X) \leq -W_3(|X(k+1)|)+\pi_3$, provided $V(k+s, X(k+s)) \leq \min\{\pi_2, V(k+1, X(k+1))+\gamma\}$ for $s \in I_d^1$, and $\pi_1 \leq V(k+1, X(k+1))$;

(iii)$_b$ $\Delta V(k,X) \leq -W_3(|X(k+1)|)$, provided for $s \in I_d^1$, $V(k+s, X(k+s)) < \hat{P}(V(k+1, X(k+1)))$.

Where $\varepsilon_0 = L^{-1}W_2^{-1}(W_1(\varepsilon))$, $\rho_0 = \beta$, L is defined by (2). Then the zero solution of systems (1) is U.P.A.S..

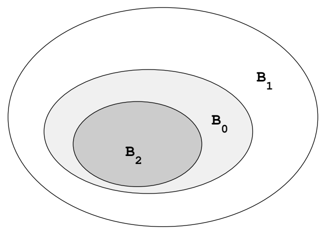

Fig. 1. The relationship of the balls B_0, B_1 and B_2

Proof. Define the following balls:

$$\begin{aligned} B_0 &= \{X: V(k,X) < W_2(\alpha)\}; \\ B_1 &= \{X: V(k,X) < W_1(\beta)\}; \\ B_2 &= \{X: V(k,X) < W_1(\varepsilon)\}. \end{aligned}$$

By condition (i), $B_0 \subseteq B_1$, as shown in Fig 1. Since $\varepsilon_0 = L^{-1}W_2^{-1}(W_1(\varepsilon)) < L^{-1}W_2^{-1}(W_1(\beta))$, then, by Theorem 3.1 and Corollary 3.2, we can assert that, both B_1 and B_2 are invariant sets, and there exists a positive number $K = K(\alpha, \varepsilon)$ such that for any $k_0 \in \mathbb{Z}$, $X_0 \in \Upsilon_{B_0}(I_d, \mathbb{R}^N)$ implies $\forall\ k \geq k_0 + K$, $X(k; k_0, X_0) \in B_2$. By condition (ii), $|X| < \alpha$ implies $X \in B_0$; $X \in B_1$ implies $|X| < \beta$; $X \in B_2$ implies $|X| < \varepsilon$. Then, for any $k_0 \in \mathbb{Z}$, $\|X_0\| < \alpha$ implies $\forall\ k \geq k_0 + K$, $|X(k; k_0, X_0)| < \varepsilon$, i.e., the zero solution of the systems (1) is U.P.A.S.. $\square$

IV. ESTIMATION OF SOLUTION BOUNDARY AND ARRIVAL TIME

In the previous section, without condition (i), for the balls $B_0 = \{X : V(k,X) < W_2(\alpha)\}$ and $B_1 = \{X : V(k,X) < W_1(\beta)\}$, let $\varepsilon_0 = L^{-1}W_2^{-1}(W_1(\beta))$, $\rho_0 = \max\{\beta, W_1^{-1}(W_2(\alpha))\}$. From conditions (ii), (iii)$_a$∼(iii)$_b$, we can obtain the conclusion that B_1 is an invariant set and the trajectory of the solution of system (1), which starts from B_0, will fall into B_1 in finite time. In addition, with the assumption of condition (i), all trajectories of the considered solutions, which exit from the ball B_0, will take the ball B_1 to be their boundary. Thus, in the proposed theorems, as long as $\varepsilon_0 \leq L^{-1}W_2^{-1}(W_1(\beta))$, and $\Delta V(k,X) \leq 0$ in conditions (iii)$_a$∼(iii)$_b$, the system is U.P.S.. Along the light of the above analysis, it is more conveniently to apply Theorem 3.1 and Corollary 3.2 in this paper to estimate relations between balls B_1 and B_2 by utility of information on $\varepsilon_0 \leq \|X_d(k)\| \leq \rho_0$, which are not mentioned in Theorem 1, Corollary 1 and Corollary 2 in [9]. We give the following theorem to estimate both the boundary of the solution of system (1) and arrival time K, after which the solution exists from the given ball $\{X : \|X\| < \alpha\}$ falls into the region $\{X : \|X\| < \varepsilon\}$, where $0 < \varepsilon < \alpha$.

Theorem 4.1: Given scalars α, ε with $0 < \varepsilon < \alpha$, $\sigma_1 > 1$. If there exist a Lyapunov function $V : \mathbb{Z}^+ \times \mathbb{R}^N \to \mathbb{R}^+$, wedge functions $W_i(\cdot)(i = 1, 2, 3)$, satisfying

(i) $W_1(|X|) \leq V(k,X) \leq W_2(|X|)$;

(ii) $\Delta V(k,X) \leq -W_3(|X(k+1)|)$ for $\|X_d\| \leq \rho_0$, provided $V(k+s, X(k+s)) < \sigma_1(V(k+1, X(k+1)))$ for $s \in I_d^1$.

Then
(1) $\hat{\beta} = W_1^{-1}(W_2(\alpha))$;
(2) $K = k_0 + \bar{N}_1\,(d + \bar{N}_2)$.
Where

$$\bar{N}_1 = \begin{cases} \frac{W_2(\alpha)+(\sigma_1-2)W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)}, & \frac{W_2(\alpha)-W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)} \text{ is integer;} \\ \left[\frac{W_2(\alpha)-W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)}\right], & \text{otherwise,} \end{cases}$$

$$\bar{N}_2 = \begin{cases} \frac{2W_2(\alpha)}{W_3(W_2^{-1}(W_1(\varepsilon)))} + 1, & \frac{2W_2(\alpha)}{W_3(W_2^{-1}(W_1(\varepsilon)))} \text{ is integer;} \\ \left[\frac{2W_2(\alpha)}{W_3(W_2^{-1}(W_1(\varepsilon)))}\right], & \text{otherwise,} \end{cases}$$

$[\cdot]$ denotes the greatest integer function, $\rho_0 = W_1^{-1}(W_2(\alpha))$, $\hat{\beta}$ is the estimation of the solution boundary of system (1), and K is the time that the solution exists from the given ball $\{X : \|X\| < \alpha\}$ and falls into the region $\{X : \|X\| < \varepsilon\}$.

Proof. (1) In Theorem 3.1, we let $W_1(\hat{\beta}) = W_2(\alpha)$. Then, $\varepsilon_0 = L^{-1}\alpha$, $\rho_0 = \hat{\beta}$, and $B_1 = B_2 = \{X : V < W_1(\hat{\beta})\}$. It follows from Theorem 3.1 that the solution starts from B_2 can not exits from B_1, which implies that the solution starting from set $\{X : \|X\| < \alpha\}$ will have a boundary $\hat{\beta} = W_1^{-1}(W_2(\alpha))$.
(2) In Theorem 3.1, we let $B_1 = \{X : V(X) < W_2(\alpha)\}$, and $B_2 = \{X : V(X) < W_1(\varepsilon)\}$. Noting that $\varepsilon_0 = L^{-1}W_2^{-1}(W_1(\varepsilon))$ and $\rho_0 = W_1^{-1}(W_2(\alpha))$. Let $\hat{P}(s) = \sigma_1 s$, then $\hat{P}(s)$ has the required property in Corollary 3.2. There exist scalars $\delta_1 > 0$ and $\delta_2 \in (0, 1/2)$, such that $\frac{W_2(\alpha)-W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)} < \frac{W_2(\alpha)-W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)-\delta_1} < \bar{N}_1$, and $\frac{2W_2(\alpha)}{W_3(W_2^{-1}(W_1(\varepsilon)))} < \frac{W_2(\alpha)}{\delta_2 W_3(W_2^{-1}(W_1(\varepsilon)))} < \bar{N}_2$. Under the similar process of Theorem 3.1 and Corollary 3.2, for $\varepsilon \leq \|X\| < \alpha$, let $\gamma = (\sigma_1 - 1)W_1(\varepsilon) - \delta_1 \in (0, \inf(\hat{P}(V) - V))$ and $\pi_3 = \delta_2 W_3(W_2^{-1}(W_1(\varepsilon)))$, one can derive the conclusion of (2). $\square$

V. PRACTICAL CONTROLLABILITY

In this section we will use the results in previous sections to study the practical controllability for a general class of nonlinear discrete systems with input time delay. Consider the following system:

$$x(k+1) = f(k, x(k)) + \sum_{i=0}^{d} B(k-i)u(k-i) \qquad (10)$$

where $f : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $i = 1, \ldots, d$, $u(k) \in \mathbb{R}^m$ is input, and is supposed to guarantee the existence and uniqueness of the solution. This type of model is generally studied in networked control systems (NCSs). We first introduce the following definitions:

Definition 5.1: System (10) is called to be

(PC$_1$) uniformly practically controllable (U.P.C.) with respect to (α, β), $0 < \alpha < \beta$, if there exist finite time K and a control $u(\cdot)$ defined on $[k_0, K]$ such that all the solutions $x(k) = x(k; k_0, x_0, u)$ that exit from $\{x \in \mathbb{R}^n : \|x_0\| < \alpha\}$

will enter into a bounded region $\{x \in \mathbb{R}^n : \|x_d\| < \beta\}$ at time K instant for all $k_0 \in \mathbb{Z}^+$;

(PC_2) uniformly practically asymptotic controllable (U.P.A.C.) with respect to (α, β), $0 < \alpha < \beta$, if U.P.C. holds, and for each $\varepsilon \in (0, \beta)$, there exists a positive number $K = K(k_0, \alpha, \varepsilon)$ such that $\|x_0\| < \alpha$ implies $|x(k; k_0, x_0, u)| < \varepsilon$ for all $k_0 \in \mathbb{Z}^+$.

For system (10), we have the following theorem on U.P.A.C. with respect to (α, β).

Theorem 5.2: Assume that there exists a control law $u(k)$ such that system (10) can be expressed by the form of (1), and the conditions of Theorem 3.3 are satisfied. Then, system (10) is U.P.A.C.with respect to (α, β).

In system (10), let $\hat{f}(k, x(k)) = f(k, x(k)) + B(k)u(k)$. Suppose that $\|\hat{f}(k, x(k))\| \le \|\Psi_0(k)\|\|x(k)\|$. Adopt the feedback control law $u(k) = F(k, x(k))x(k)$, and let $\Psi_i(k) = B(k-i)F(k-i, x(k-i))$, where $F(k, x(k))$ is the control gain matrix, $\Psi_0(k)$ and $\Psi_i(k)$ are of compatible dimensions. Consequently, the closed-loop system of (10) has the following form:

$$x(k+1) = \hat{f}(k, x(k)) + \sum_{i=1}^{d} \Psi_i(k)x(k-i), \qquad (11)$$

Let $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ be the maximum eigenvalue and the minimum eigenvalue of a real symmetric matrix, respectively. $\|\cdot\|_2$ stands for the Euclidean vector norm or the 2-norm of a matrix. Then, we have the following corollary.

Corollary 5.3: If there exists $F(k)$ such that

$$\sup_{k\in\mathbb{Z}} \sum_{i=0}^{d} \|\Psi_i(k)\|_2^2 < 1 - (\frac{\alpha}{\beta})^2 \qquad (12)$$

Then, the closed-loop system (11) is U.P.A.S., and system (10) is U.P.A.C. with respect to (α, β) with $0 < \alpha < \beta$.

Proof. In fact, by (12), noting that $0 < \alpha < \beta$, then, $\forall \varepsilon \in (0, \alpha^2/\beta^2)$, there exist scalars $\nu_1 \in [\alpha^2/\beta^2, 1]$ and $\nu_2 > 1$ such that

$$\sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2 + \nu_2 \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2 < \nu_1 - (\frac{\alpha}{\beta})^2 + \varepsilon < \nu_1.$$

Thus, there exists a positive definite matrix P such that $\lambda_{\min}(P) = \nu_1 \lambda_{\max}(P)$. Choose $V(k, x(k)) = x^T(k)Px(k)$, $W_1(|x(k)|) = \lambda_{\min}(P)x^T(k)x(k)$, and $W_2(|x(k)|) = \lambda_{\max}(P)x^T(k)x(k)$. It is obvious that

$$W_1(|x(k)|) \le V(k, x(k)) \le W_2(|x(k)|).$$

Let $\hat{P}(s) = \nu_2 s$ for $s \ge 0$. Then $\hat{P}(s) > s$ for $s \ge 0$. For all $i \in \{1, \ldots, d\}$, if $V(k-i, x(k-i)) < \hat{P}(V(k+1, x(k+1)))$, then, $\|x(k-i)\|_2^2 < \|x(k+1)\|_2^2 \nu_2/\nu_1$, and it follows (11) that

$$\begin{aligned}
\|x(k+1)\|_2^2 &\le \sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2 \|x(k)\|_2^2 \\
&\quad + \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2 \|x(k-i)\|_2^2 \\
&\le \sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2 \|x(k)\|_2^2 \\
&\quad + \frac{\nu_2 \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2}{\nu_1} \|x(k+1)\|_2^2.
\end{aligned}$$

Consequently,

$$-\|x(k)\|_2^2 \le \frac{\nu_2 \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2 - \nu_1}{\nu_1 \sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2} \|x(k+1)\|_2^2.$$

Let $\delta = \frac{\nu_1 - \sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2 - \nu_2 \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2}{\sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2}$. Since scalar $\varepsilon \in (0, \alpha^2/\beta^2)$ is arbitrary, thus, $\delta > \frac{\alpha^2}{\beta^2 - \alpha^2} > 0$, and

$$\begin{aligned}
\Delta V(k, X) &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\
&\le -\lambda_{\max}(P) \frac{\alpha^2}{\beta^2 - \alpha^2} \|x(k+1)\|_2^2.
\end{aligned}$$

Then, conditions (i), (ii) and $(\text{iii})_b$ of Theorem 3.3 are all satisfied, and hence, the conclusion follows. □

Remark 5.4: In Theorem 3.1, Corollary 3.2, Theorem 3.3 and Theorem 4.1, there is a relation between $V(k+s, x(k+s))(s \in I_d^1)$ and $V(k+1, x(k+1))$, namely, "provided $\mathscr{R}(V(k+s, x(k+s)), V(k+1, x(k+1)))$", where $\mathscr{R}(\cdot, \cdot)$ defines a relation. We call this relation as the $\mathscr{R}$-relation. The condition (ii), (ii)$'$, $(\text{iii})_a$ and $(\text{iii})_b$ describe the constraint on $\Delta V(k, X)$ under the $\mathscr{R}$-relation, but no constrain on $\Delta V(k, X)$ without $\mathscr{R}$-relation. Thus, the condition that the constraint on $\Delta V(k, X)$ holds not only with but also without the $\mathscr{R}$-relation, is more restrictive than the condition that the constraint on $\Delta V(k, X)$ holds only with the $\mathscr{R}$-relation. Therefore, we can obtain a class of particular cases of Theorem 3.3 with conditions (i), (ii), either $(\text{iii})_a$ or $(\text{iii})_b$, which in fact are corresponding to the well-known Lyapunov-like theorems.

VI. ILLUSTRATIVE NUMERICAL EXAMPLE

To illustrate the effectiveness of the obtained results in previous sections, we consider the following nonlinear discrete system with input time delay:

$$\begin{aligned}
x(k+1) &= 1.44x(k) - x^3(k) + 0.069u(k) \\
&\quad + 0.031u(k-1),\ x(k) \in [-1.2, 1.2]. \quad (13)
\end{aligned}$$

Assume that $\alpha = 0.45$ and $\beta = 0.60$. To obtain the zero solution $x(k) = 0$ in U.P.A.S with (α, β), adopt the following fuzzy control law:

R_1: IF x is about ± 1.2, THEN $u = F_1 x(k)$,

R_2: IF x is about 0, THEN $u = F_2 x(k)$.

The references on fuzzy control can be found in [11], [12]. Then, the overall control law is

$$u(k) = \sum_{i=1}^{2} \mu_i F_i x(k). \qquad (14)$$

where $\mu_1 = \frac{x^2}{1.44}$ and $\mu_2 = 1 - \mu_1$ are both membership functions, as shown in Fig. 2. The control gain matrices are designed to be $F_1 = -0.0694$ and $F_2 = -18.9114$. Then, the closed-loop system can be expressed as follows:

$$x(k+1) = \big(1.44 - x^2(k) + 0.069\sum_{i=1}^{2}\mu_i F_i\big)x(k) + 0.031\sum_{i=1}^{2}\mu_i F_i x(k-1).$$

Denote discriminant function by

$$g(x(k)) = \big(1.44 - x^2(k) + 0.069\sum_{i=1}^{2}\mu_i F_i\big)^2 + (0.031\sum_{i=1}^{2}\mu_i F_i)^2.$$

The profile of $g(x)$ is illustrated in Fig. 3. We can calculate that $g(x) \le 0.3619 < 1 - \alpha^2/\beta^2 = 0.4375$ for $x \in [-1.2, 1.2]$. by (12) and Corollary 5.3, system (13) is U.P.A.C. with respect to (α, β). The state curve with initial values $x(-1) = 0.3$, $x(0) = 0.4$ of system (13) with and without fuzzy controller (14) are shown in Fig. 4. Without fuzzy controller, i.e., $u(k) = 0$, the zero solution is unstable, and the nonlinear discrete system converges to $x(k) \approx 0.6633 > \beta$; whereas, with fuzzy controller (14), the closed-loop system is U.P.A.S. with (α, β).

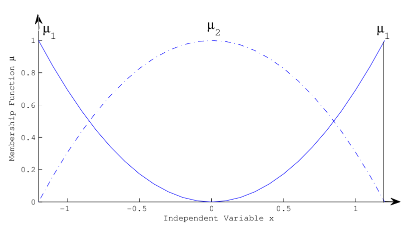

Fig. 2. The membership functions of μ_1 and μ_2

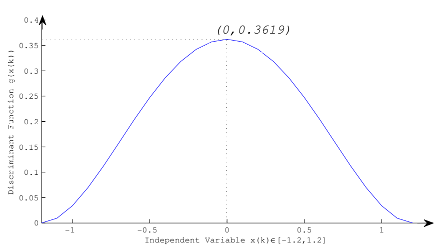

Fig. 3. The profile of $g(x(k))$

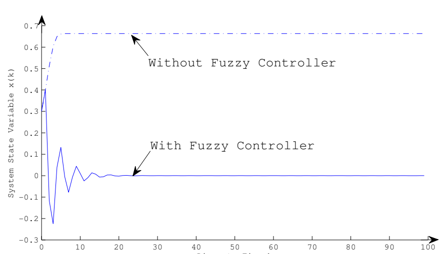

Fig. 4. The state curve of system (13) with and without fuzzy controller (14)

VII. CONCLUSIONS

Motivated by the idea in [9], we studied the Razumikhin-type theorems on practical asymptotic stability for a class of discrete time-delay system. Some easy testing criteria for the uniform practical asymptotical stability are derived via Lyapunov function and Razumikhin technique. Estimations of the solution boundary and arrival time of the solution are also investigated. In addition, the proposed theorems are used to study the practical controllability for a general class of nonlinear discrete systems with input time delay. Finally, a numerical example is present to illustrate the effectiveness of the proposed results. We believe the results in this paper are useful for the study networked control systems.

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