

# Optimal control of piecewise affine systems with piecewise affine state feedback<sup>1</sup>

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## Abstract

In this paper, we show that the switching numbers of piecewise affine (PWA) system is finite if the PWA state feedback control is limited to be continuous at the switching boundaries. Then, this problem is transformed into a discrete valued optimal control problem. For this discrete valued optimal control problem, we introduce CPET (control parametrization enhancing technique) to solve it. A numerical example is presented to illustrate the proposed method.

**Keywords:** Piecewise affine system; CPET; State feedback; Optimal control

## 1 Introduction

PWA systems is, in fact, a special class of hybrid systems, *i.e.*, systems with a continuous-time state and discrete-event state. For PWA systems, the discrete-event state is associated with discrete modes and the continuous state is associated with the affine dynamics valid within each discrete modes. Many classical systems, such as saturated systems, relay systems, hysteresis systems, *etc.*, can be classified into this class systems. In addition, many other nonlinear system can also be approximated by PWA systems. Thus, in the past decade, this class systems have been received widely researched.

A number of significant results are available on analysis and control of such PWA systems. In [2], the piecewise quadratic Lyapunov function is used to stabilize a class of piecewise affine systems. In [4], the synthesis of a state feedback controller for a class of PWA system is discussed. The design of such a controller is formulated as a convex optimization problem with a parameterized set of linear matrix inequalities (LMI) that can be relaxed to a finite set of LMIs. In [3], the design of state and dynamic output feedback controllers for PWA is considered. This problem is formulated as an optimization problem subject to linear constraints and a bilinear matrix inequality. The sub-optimal solutions for this optimization problem is obtained by extension of iterative algorithms. This method greatly reduce the numerical complexity and yields good closed-loop performance. In [5], the optimal control of discrete PWA system is discussed. Based on the theory of the multi-parametric programming for the linear constrained quadratic optimization problem, the optimal solution is derived to be the form of a time-varying PWA feedback law. An off-line procedure to synthesize this control is developed which is based on the minimization of quadratic and linear performance indices subject to linear constraints on inputs and states. This procedure is based on a combination of dynamic programming and multi-parametric quadratic programming. However, because of the high complexity of the multi-parametric programs involved and the large number of multi-parametric programs which need to be solved when a controller is computed in a dynamic programming fashion, this procedure quickly becomes prohibitive for larger problems. To overcome this problem, a two algorithms to compute low complexity feedback controllers for constrained PWA systems are presented in [6].

Though the results for the controller design of PWA systems and for the optimal control of discrete PWA systems are abundant, there are rare results for the optimal control of continuous PWA systems. In this paper, the optimal control of a class of continuous PWA systems are considered. We suppose that the control takes the form of a PWA state feedback law. We show that if the control is restricted to be continuous on the switching boundary, the number of switchings is finite. This fact is very important in applications and numerical simulations because the fast switchings may destroy the systems and cannot be efficient computed by numerical method. Then, an efficient algorithm based on CPET is developed for the numerical computation of this PWA

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state feedback law. The numerical results show that our method is efficient to solve this optimal control problem.

## 2 Problem Formulation

Let

$$\mathcal{X}_i = \{x \in \mathbb{R}^n : H_i^T x - g_i \leq 0\}, \quad i \in \mathcal{I} = \{1, \dots, M\}, \quad (1)$$

and assume that

$$\bigcup_{i=1}^M \mathcal{X}_i = \mathbb{R}^n, \quad \text{int}(\mathcal{X}_i) \cap \text{int}(\mathcal{X}_j) = \emptyset, \quad (2)$$

where  $H_i$  and  $g_i$  are, respectively, matrices and vectors with appropriate dimensions, and  $\text{int}(\mathcal{R}_i)$  denotes the set which contains all the interior points of  $\mathcal{R}_i$ . From [1], a parametric description of the boundaries is given by

$$\mathcal{X}_i \cap \mathcal{X}_j \subseteq \{x = b_{i,j} + F_{i,j}s \mid s \in \mathbb{R}^{n-1}\}, \quad i, j \in \mathcal{I}, \quad (3)$$

where  $b_{i,j} \in \mathbb{R}^n$ ,  $i, j \in \mathcal{I}$ , are  $n$ -vectors, while  $F_{i,j} \in \mathbb{R}^{n \times (n-1)}$ ,  $i, j \in \mathcal{I}$ , are full rank matrices.

We now consider a class of PWA systems defined below.

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad \text{if } x \in \mathcal{X}_i \quad (4)$$

with initial condition

$$x(t_0) = x_0, \quad (5)$$

where  $A_i$  and  $B_i$  are matrices with appropriate dimensions,  $u : \mathbb{R} \rightarrow \mathbb{R}^m$  is the control vector,  $x : \mathbb{R} \rightarrow \mathbb{R}^n$  is the state vector. For this system, we assume that the continuity of the state vector field of the system (4) is satisfied, *i.e.*,

$$x \in \mathcal{X}_i \cap \mathcal{X}_j \Rightarrow A_i x = A_j x, \quad i, j \in \mathcal{I}. \quad (6)$$

We assume that the controls to be chosen are in the PWA state feedback form, *i.e.*,

$$u_i = K_i x, \quad \text{if } x \in \mathcal{X}_i, \quad (7)$$

where  $K_i$  are matrices with appropriate dimension. Furthermore, we assume that the following constraints are satisfied.

$$(B_i K_i - B_j K_j) F_{i,j} = 0, \quad (8)$$

$$(B_i K_i - B_j K_j) b_{i,j} = 0. \quad (9)$$

Such a control is said to be an admissible PWA state feedback control. Let  $\mathcal{U}$  denote the set of all such admissible PWA state feedback controls.

Note that for any  $u \in \mathcal{U}$ , it is continuous on the switching boundaries.

**Definition 1:**  $x(t)$  is said to be a trajectory of the system (4) and (5) on  $[t_0, t_f]$  corresponding to a given control  $u(t) \in \mathcal{U}$ , if it is absolutely continuous, satisfies the initial condition (5) at  $t = t_0$  and, for almost all  $t \in [t_0, t_f]$ , the equation (4) holds for  $x(t) \in \mathcal{X}_i$ .

**Definition 2:** Let  $x(t)$  be a trajectory of the system (4) and (5).  $\tau$  is called a switching point of  $x(t)$  if there exist  $\Delta > 0$  and  $i, j$  ( $i \neq j$ )  $\in \mathcal{I}$ , such that  $x(t) \in \mathcal{X}_i$  but  $x(t) \notin \mathcal{X}_j$  for  $t \in [\tau - \Delta, \tau]$  and  $x(t) \in \mathcal{X}_j$  but  $x(t) \notin \mathcal{X}_i$  for  $t \in [\tau, \tau + \Delta]$ .

We may now state our optimal control problem formally as follows.

Given the dynamic system (4) and (5), find a PWA state feedback gain matrix  $K$  such that the cost functional

$$J(K) = x^T(t_f) P x(t_f) + \int_0^{t_f} x^T(t) Q x(t) + u^T(t) R u(t) dt, \quad (10)$$

is minimized subject to the constraints (8), (9) and the boundedness constraint

$$\|K\|_\infty \leq M_0, \quad (11)$$

where  $K = [K_1, \dots, K_M] \in \mathbb{R}^{m \times M n}$ , and  $\|K\|_\infty$  denotes the largest absolute value among all the elements of the matrix  $K$ . Let this problem be referred to as Problem 1.

### 3 Problem Transformation

Note that the system (4) and (5) with the control  $u(t)$  taken from  $\mathcal{U}$  can be written in the following equivalent form, where the dynamic equation is given by

$$\dot{x}(t) = \sum_{i=1}^M \chi_{\{\mu(t)=i\}} (A_i + B_i K_i) x(t) \quad (12)$$

with the initial condition (5) and subject to the constraint

$$\sum_{i=1}^M \chi_{\{\mu(t)=i\}} [H_i^T x(t) - g_i] \leq 0. \quad (13)$$

Here,

$$\chi_{\{\mu(t)=i\}} = \begin{cases} 1, & \text{if } \mu(t) = i, \\ 0, & \text{if } \mu(t) \neq i. \end{cases}$$

Problem 1 can be re-stated equivalently as Problem 2 given below.

**Problem 2.** Given the system (12) and (2), find a feedback gain matrix  $\mathbf{K}$  such that the cost function (10) is minimized subject to the constraints (8), (9), (11) and (13).

The main objective of this paper is to develop an efficient numerical solution method for solving Problem 2. For this, it is important to know whether or not Problem 2 admits a solution with infinite number of switchings. The answer is given in the following theorem.

**Theorem 1:** Let  $x(t)$  be the trajectory of the system (4) and (2) corresponding to a given admissible PWA state feedback control  $u \in \mathcal{U}$ . Then,  $x(t)$  has only a finite number of switchings in  $[t_0, t_f]$ .

The proof of this theorem is rather lengthy. Thus, it is given in the appendix so as not to interrupt the flow of the reading of the paper.

**Remark 1:** In view of Theorem 1, we see that for each given  $K$ , the system (4) and (2) has only a finite number of switchings. In fact, all software packages currently available, such as [11] and [12], to model such a class of dynamical systems are developed for situations where no infinite number of switchings exists. The reason is quite clear. For if this is not the case, then there would exist a  $K$  such that the corresponding trajectory  $x(t)$  will have an infinite number of switchings in a finite interval  $I \subset [t_0, t_f]$ . This phenomena of fast switching will cause a major difficulty in numerical simulation. Clearly, it is also not allowed in many applications as the fast switching may cause serious damage to the systems [13]. Now, by virtue of Theorem 1, we can rule out this undesirable phenomenon of fast switching. From our numerical simulation studies, it appears that for a given problem, there exists a uniform bound on the number of switchings with respect to all  $u \in \mathcal{U}$ . However, this uniform bound has yet to be obtained.

To solve Problem 2, the number of switchings is required to be fixed. In the numerical simulation, we use a heuristic approach to determine this number. We start by fixing the number of switchings to be some fixed integer, say  $N$ . We solve the corresponding Problem 2. Then, we increase the number of switchings from  $N$  to  $N + d$ , where  $d$  is an integer. We solve the corresponding Problem 2 again. If there is no decrease in the optimal cost, we take the previous value of  $N$  to be the number of switchings.

Let  $N$  be the number of switchings. The corresponding Problem 2 is actually a mixed integer nonlinear optimization problem with continuous inequality constraints. Let  $J = (N + 1)M$ . To determine the switching sequence, we construct a sequence  $\{z_i\}_{i=1}^J$ , where  $z_i = (i - 1) \bmod M + 1$ . Let

$$\mu(t) = z_i, t \in [\tau_{i-1}, \tau_i), i = 1, 2, \dots, J, \quad (14)$$

where  $t_0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_{J-1} \leq \tau_J = t_f$ . Note that the determination of the switching sequence is equivalent to the determination of  $\tau_i$ ,  $i = 1, 2, \dots, J - 1$ . However, it is known [7] that direct determination of  $\tau_i$ ,  $i = 1, 2, \dots, J - 1$ , is problematic numerically. To overcome this difficulty, we shall use the time scaling transform, which is known as the Control Parametrization Enhancing Transform (CPET) in [7]. Define

$$\frac{dt}{ds} = \nu(s), t(0) = t_0, \quad (15)$$

$$\bar{\mu}(s) = z_i, \quad s \in [i-1, i), \quad i = 1, 2, \dots, J, \quad (16)$$

where  $\bar{\mu}(s) = \mu(t(s))$ , and

$$v(s) = \sum_{i=1}^J \chi_{[i-1, i)} \delta_i, \quad \delta_i \geq 0, \quad i = 1, 2, \dots, J. \quad (17)$$

Let  $\boldsymbol{\delta} = [\delta_1, \delta_2, \dots, \delta_J]^T$ . Using this transform, the system (12), (13) and the cost function (10) are, respectively, transformed into

$$\dot{y}(s) = \sum_{i=1}^M \chi_{\{\bar{\mu}(s)=i\}} (A_i + B_i K_i) y(s) v(s), \quad (18)$$

$$\sum_{i=1}^M \chi_{\{\bar{\mu}(s)=i\}} [H_i^T y(s) - g_i] \leq 0, \quad (19)$$

and

$$\bar{J}(K, \boldsymbol{\delta}) = y^T(L) P y(L) + \int_0^J \left[ y^T(s) Q y(s) + (K_i y(s))^T R K_i y(s) \right] v(s) ds, \quad (20)$$

where  $y(s) = x(t(s))$ .

We now obtain the following equivalent transformed optimal control problem.

**Problem 3.** Given the system (18), with the initial condition  $y(0) = x_0$ , find a pair  $(K, \boldsymbol{\delta})$  such that the cost function (20) is minimized subject to the constraints (8), (9), (11), (19) and (17).

We have the following theorem.

**Theorem 2:** If  $K^*$  is the optimal solution of Problem 1 with  $N$  being the number of switchings. Then, Problem 1 and Problem 3 are equivalent.

**Proof.** The proof is similar to that given for Theorem 3 in [14].

By Theorem 2, we see that solving Problem 1 is equivalent to solving Problem 3. The advantage of solving Problem 3 is that all the switching points are now pre-fixed at appropriate integer points. For the continuous state inequality constraints (19), they are handled by the  $\varepsilon - \tau$  constraint transcription method [8]. Then, the optimal control software package, MISER 3 [9], can be used.

## 4 Illustrative Example

Consider Problem 1 with  $n = 3$ ,  $m = 3$ ,  $M = 2$ ,  $M_0 = 100$ ,

$$A_1 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 1.1 & 0.2 \\ 0.1 & 0.5 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0.5 & -0.2 \\ -0.1 & 0.7 & -0.2 \\ 0 & 0.4 & 0.9 \end{bmatrix},$$

$$\mathcal{X}_1 = \left\{ x = [x_1, x_2, x_3]^T : x_1 + x_2 - x_3 \geq 0 \right\},$$

$$\mathcal{X}_2 = \left\{ x = [x_1, x_2, x_3]^T : x_1 + x_2 - x_3 \leq 0 \right\},$$

$B_1$  and  $B_2$  being the  $3 \times 3$  identity matrix, and  $x(0) = [0.1, -2, 0.1]^T$ . Clearly,

$$\mathcal{X}_1 \cap \mathcal{X}_2 \subseteq \left\{ x : x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} s, \quad s \in \mathbb{R}^2 \right\},$$

$$i.e., \quad v_{1,2} = [0, 0, 0]^T, \quad F_{1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Our objective is to find a  $K$  such that the cost function

$$J(K) = \sum_{i=1}^3 (x_i(1))^2,$$

is minimized subject to the constraints (8), (9) and (11), where  $K$  is defined in (7).

Choose  $N = 3$ . The optimal control software package, MISER 3.2, is used to solve the corresponding problem. After 33 iterations, there is only one switching detected. The optimal cost is  $2.70812 \times 10^{-9}$ , the switching point is 0.01084050, the corresponding switching sequence is 2, 1, and the optimal feedback matrices are

$$K_1 = \begin{bmatrix} -85.84418247 & -95.90217004 & -1.92872406 \\ 50.03363503 & -48.03363504 & -89.27927807 \\ -98.41709420 & -93.61236219 & -99.46292000 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -95.03985591 & -86.70649661 & -99.99599414 \\ -98.06727008 & -57.18315114 & -83.61231828 \\ -92.75013442 & -99.27932198 & -90.26724657 \end{bmatrix}.$$

The optimal trajectory is depicted in Figure 1.

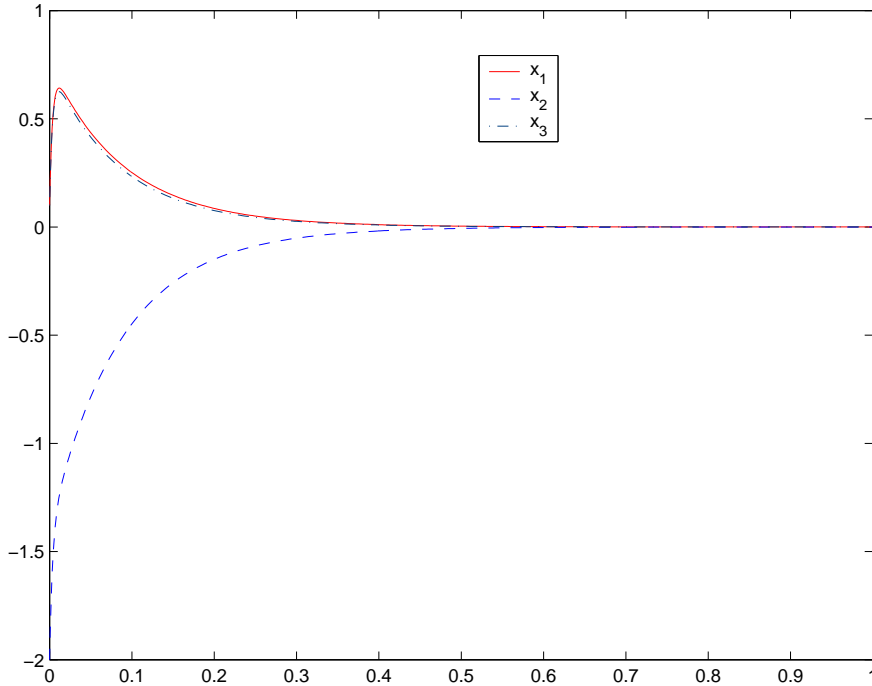


Figure 1: The profile of the optimal trajectory

(You need to say something about what has happened when you increase the number of switchings to, say,  $N + D$ .)

## 5 Conclusion

In this paper, we have discussed the optimal control of PWA systems. We have showed that if the control is limited to be continuous on the switching boundary, then this PWA system has only finite switchings. Then, a computational approach is developed based on the CPET. To illustrate the efficiency, a numerical example was presented.

## Appendix

For the brevity of notation, let

$$\tilde{A}_i = A_i + B_i K_i.$$

Then, the system (4) can be re-written as

$$\dot{x}(t) = \tilde{A}_i x(t). \quad (21)$$

From (6), (8) and (9), it follows that

$$x \in \mathcal{X}_i \cap \mathcal{X}_j \Rightarrow \tilde{A}_i x = \tilde{A}_j x, \quad i, j \in \mathcal{I}. \quad (22)$$

To proceed further, we need the following definition.

**Definition 3.** For any solution  $x(t)$  of the system (21) defined on the interval  $I \subset [t_0, t_f]$ , we call  $(\mathcal{X}_i, \mathcal{X}_j)$ ,  $1 \leq i \neq j \in \mathcal{I}$ , a switching pair if there exist a  $\tau \in I$  and an  $\varepsilon > 0$  such that  $x(t) \in \mathcal{X}_i$  for  $t \in [\tau - \varepsilon, \tau] \subset I$  and  $x(t) \in \mathcal{X}_j$  for  $t \in [\tau, \tau + \varepsilon] \subset I$ .

Let  $\Sigma_I(x)$  be the set of all such  $(\mathcal{X}_i, \mathcal{X}_j)$ . Clearly, the number of elements in  $\Sigma_I(x)$  is no more than  $M!(M-1)!$ . Let

$$\Gamma_I(x) = \left\{ \phi : \phi \in \left\{ \left[ \tilde{A}_i - \tilde{A}_j \right]_k \right\}_{k=1}^n, (\mathcal{X}_i, \mathcal{X}_j) \in \Sigma_I(x) \right\}, \quad (23)$$

where  $\left[ \tilde{A}_i - \tilde{A}_j \right]_k$  denotes the  $k$ -th row of the matrix  $\tilde{A}_i - \tilde{A}_j$ . Then, the number of elements in  $\Gamma_I(x)$  is no more than  $nM!(M-1)!$ . Let

$$\Omega_I(x) = \{i : \exists t \in I \text{ such that } x(t) \in \mathcal{X}_i, 1 \leq i \leq M\}. \quad (24)$$

We need the following lemma, which is Lemma A1 of [10].

**Lemma 1. (Lemma A1. [10]).** Let  $\bar{M} > 0$  and let  $n$  be a positive integer. Let  $T > 0$  be such that

$$T < \min \left( 1, \frac{e^{-n\bar{M}}}{n^{3/2}\bar{M}} \right). \quad (25)$$

Suppose that  $\varphi_1, \dots, \varphi_n$  are absolutely continuous functions on an interval  $I$  with length  $T$ , and that they satisfy a system of linear differential equations

$$\dot{\varphi}_i = \sum_{j=1}^n \alpha_{i,j} \varphi_j, \quad (26)$$

with coefficients  $\alpha_{i,j}$  being measurable real-valued functions on  $I$  such that

$$|\alpha_{i,j}(t)| \leq \bar{M} \quad \text{for } 1 \leq i, j \leq n, t \in I.$$

Then, either one of the following two statements is valid.

- (i) All the  $\varphi_i$  vanish identically, or
- (ii) At least one  $\varphi_i$  has no zeros on  $I$ .

**Lemma 2.** Let

$$\begin{aligned} S_1 &= \Gamma_I(x), \\ S_2 &= \left\{ \phi \tilde{A}_i : i \in \Omega_I(x), \phi \in S_1 \right\}, \\ &\dots, \\ S_{n+1} &= \left\{ \phi \tilde{A}_i : i \in \Omega_I(x), \phi \in S_n \right\}. \end{aligned}$$

Then, for any  $\phi_i \in S_i$ , where  $i = 1, \dots, n+1$ , there exist  $k$ ,  $2 \leq k \leq n+1$ , and coefficients  $\beta_1, \dots, \beta_{k-1}$  such that

$$\phi_k = \sum_{j=1}^{k-1} \beta_j \phi_j. \quad (27)$$

**Proof.** Since  $\phi_1, \dots, \phi_{n+1}$  are  $n+1$  vectors of  $\mathbb{R}^n$ , they must be linearly dependent. Thus, (27) is satisfied. ■

For each  $\phi_i \in S_i$ , where  $i = 1, \dots, n+1$ , we choose the first  $k$ , where  $2 \leq k \leq n+1$ , such that  $\phi_k$  is a linear combination of  $\phi_1, \dots, \phi_{k-1}$ . Let  $L_{\phi_1} = \{\phi_1, \dots, \phi_{k-1}\}$ . We call  $L_{\phi_1}$  a derived sequence of  $\phi_1$ . For each  $\phi \in \Gamma_I(x)$ , there may exist one or more  $M^n$  other than  $L_\phi$ . In the sequel, let  $L_\phi$  denote one of them. Define

$$H = \{\phi : \phi \in L_\psi, \psi \in \Gamma_I(x)\}. \quad (28)$$

Clearly, the number of elements in the set  $H$  is no more than  $n^2 M! (M-1)!$ . Thus, the number of all the coefficients that are used to generate linear combinations of  $\phi_k$  is finite. Let  $M_{coe}$  denote the coefficient with the largest absolute value amongst them. We have the following lemma.

**Lemma 3.** Let  $x : I \rightarrow \mathbb{R}^n$  be a solution of the system (21), and let  $L_{\phi_1} = \{\phi_1, \dots, \phi_{k-1}\}$  be a derived sequence of  $\phi_1$ . Then, for any  $\phi \in L_{\phi_1}$ ,

$$\frac{d}{dt}(\phi x(t)) = \sum_{\psi \in L \cup \Gamma_I(x)} \alpha_{\phi, \psi}(t) \psi x(t). \quad (29)$$

**Proof.** From the definition of  $L_{\phi_1}$ , there exists a  $k_0 \in \Omega_I(x)$  such that  $\phi_k = \phi_{k-1} A_{k_0}$ . For each  $\phi \in L_{\phi_1}$ , we have

$$\begin{aligned} \frac{d}{dt}(\phi x(t)) &= \sum_{i \in \Omega_I(x)} h_i(x(t)) \phi A_i x(t) \\ &= \phi A_{k_0} x(t) + \sum_{i \in \Omega_I(x), A_i \neq A_{k_0}} h_i(x(t)) \phi (A_i - A_{k_0}) x(t), \end{aligned}$$

where

$$h_i(x(t)) = \begin{cases} 1, & \text{if } x(t) \in \mathcal{X}_i, \\ 0, & \text{otherwise.} \end{cases} \quad (30)$$

Note that for any  $i, k_0 \in \Omega_I(x)$ , where  $i \neq k_0$ , there exists an integer  $r_{i, k_0}$  and a sequence  $\{A_k\}_{k=1}^{r_{i, k_0}}$  such that  $A_1 = A_i$ ,  $A_{r_{i, k_0}} = A_{k_0}$ ,  $A_k \in \Omega_I(x)$ , and  $(\mathcal{X}_k, \mathcal{X}_{k+1}) \in \Sigma_I(x)$ ,  $k = 1, \dots, r_{i, k_0} - 1$ . Since

$$\begin{aligned} &\sum_{i \in \Omega_I(x), i \neq k_0} h_i(x(t)) \phi (A_i - A_{k_0}) x(t) \\ &= \sum_{i \in \Omega_I(x), i \neq k_0} \sum_{j=1}^{r_{i, k_0} - 1} h_i(x(t)) \phi (A_j - A_{j+1}) x(t) \\ &= \sum_{j=1}^{r_{i, k_0} - 1} \left( \sum_{i \in \Omega_I(x), i \neq k_0} h_i(x(t)) \phi \right) (A_j - A_{j+1}) x(t), \end{aligned}$$

that is,

$$\frac{d}{dt}(\phi x(t)) = \phi A_{k_0} x(t) + \sum_{j=1}^{r_{i, k_0} - 1} \left( \sum_{i \in \Omega_I(x), i \neq k_0} h_i(x(t)) \phi \right) (A_j - A_{j+1}) x(t). \quad (31)$$

From (31) and the fact that all the rows of  $A_j - A_{j+1}$  are contained in  $\Gamma_I(x)$ ,  $\frac{d}{dt}(\phi x(t))$  can be written as a linear combination of the elements of  $L_{\phi_1} \cup \Gamma_I(x)$  for any  $\phi \in \{\phi_1, \dots, \phi_{k-2}\}$ . For  $\phi = \phi_{k-1}$ ,  $\phi A_{k_0} x(t) = \phi_k x(t)$ . Thus,  $\phi A_{k_0} x(t)$  can be written as a linear combination of  $\phi_1, \dots, \phi_{k-1}$ . That is, (29) is also satisfied for  $\phi = \phi_{k-1}$ . This completes the proof. ■

**Lemma 4.** Let  $x : I \rightarrow \mathbb{R}^n$  be a solution of the system (??). Then, there exists an  $\tilde{M} > 0$  such that for each  $\phi \in H$ ,

$$\frac{d}{dt}(\phi x(t)) = \sum_{\psi \in H} \alpha_{\phi, \psi}(t) \psi x(t), \quad (32)$$

where  $|\alpha_{\phi,\psi}(t)| \leq \tilde{M}$  for all  $\phi, \psi \in H$ .

**Proof.** From Lemma 3, it follows that (32) is satisfied. Thus, we only need to prove the existence of an  $M > 0$  such that  $|\alpha_{\phi,\psi}(t)| \leq M$  for all  $\phi, \psi \in H$ . By virtue of the argument given for the proof of Lemma 3, we can choose

$$\tilde{M} = M_{coe} + 1 + nr \max \{ \|\phi\| : \phi \in H \}, \quad (33)$$

where  $r = \max \{ r_{i,k_0} : i \in \Omega_I(x) \}$ , and  $r_{i,k_0}$  is as defined in the proof of Lemma 3. The proof is complete. ■

Now we choose an  $T > 0$  such that  $T < 1$  and  $T < e^{-rM} / (r^{-3/2} \tilde{M})$ , where  $\tilde{M}$  is defined by (33).

**Lemma 5.** Let  $x(t)$  be a solution of the system (21) on  $I$ . Suppose that for any interval  $I_{sub} \subset I$ , the length of  $I_{sub}$  is no more than  $T$  and that for any  $\phi \in \Gamma_I(x)$ ,  $\phi x(t)$  has at least  $(1+M)^n$  zeros in the interval  $I_{sub}$ . Then,  $\phi x(t)$  vanishes identically on  $I_{sub}$  for any  $\phi \in \Gamma_{I_{sub}}(x)$ .

**Proof.** By Lemma 1 and Lemma 4, we only need to show that for any  $\phi \in \Gamma_{I_{sub}}(x)$ , there exists a derived sequence  $L_\phi = \{\phi_1, \dots, \phi_{k-1}\}$  of  $\phi$  such that  $\phi_1 = \phi$  and that for each  $i = 1, \dots, k-1$ ,  $\phi_i x(t)$  has a zero in the interval  $I_{sub}$ .

Suppose that for any  $\phi \in \Gamma_{I_{sub}}(x)$ ,  $\phi x(t)$  has at least  $(1+M)^n$  zeros in the interval  $I_{sub}$ . Since  $\phi x(t)$  has at least  $(1+M)^n$  zeros in the interval  $I_{sub}$  and  $\phi x(t) \in C^1$ , it follows that  $d/dt(\phi x(t))$  has at least  $(1+M)^n - 1$  zeros in the interval  $I_{sub}$ . Furthermore,

$$\frac{d}{dt}(\phi x(t)) = \sum_{i \in \Omega_I(x)} h_i(x(t)) \phi A_i x(t),$$

and

$$M \left( (1+M)^{n-1} - 1 \right) \leq (1+M)^n - 1.$$

Thus, there exists an  $i_2 \in \Omega_{I_{sub}}(x)$  such that  $\phi A_{i_2} x$  has at least  $(1+M)^{n-1}$  zeros in the interval  $I_{sub}$ . Denote  $\phi_1 = \phi$  and  $\phi_2 = \phi A_{i_2}$ . Let  $\phi_2$  denote the new  $\phi$ . Then, by the same argument, we can find an  $i_3 \in \Omega_{I_{sub}}(x)$  such that  $\phi A_{i_3} x$  has at least  $(1+M)^{n-2}$  zeros in the interval  $I_{sub}$ . Denote  $\phi_3 = \phi A_{i_3} x$ . Repeat this process, we can find  $i_j$  such that  $\phi_j = \phi_j A_{i_j}$  and  $\phi_j x$  has at least  $(1+M)^{n+1-j}$  zeros in the interval  $I_{sub}$ ,  $j = 2 \dots, n$ . We complete the proof. ■

Now, we return to the proof of Theorem 1.

**Proof of Theorem 1.** We suppose the conclusion of Theorem 1 is not valid. Then, there exists a solution  $x(t)$  of the system (21) and a sequence of switching points  $\{\tau_i\}_{i=1}^\infty$  and a  $\tilde{\tau} \in I$  such that

$$\tau_1 < \tau_2 < \dots < \tau_N < \dots, \quad (34)$$

and

$$\lim_{i \rightarrow \infty} \tau_i = \tilde{\tau}. \quad (35)$$

It suffices to consider the behavior of  $x(t)$  on the interval  $[t_0, \tilde{\tau}]$ . We suppose that the length of the interval  $[t_0, \tilde{\tau}]$  is no more than  $T$ . If this is not the case, we can replace the interval  $[t_0, \tilde{\tau}]$  with  $[\tilde{\tau} - T/2, \tilde{\tau}]$ .

For any  $\phi \in \Gamma_{[t_0, \tilde{\tau}]}(x)$ , there are only two cases: i).  $\phi x$  has a finite number of zeros in  $[t_0, \tilde{\tau}]$ ; and ii)  $\phi x$  has an infinite number of zeros in  $[t_0, \tilde{\tau}]$ . We partition  $\Gamma_{[t_0, \tilde{\tau}]}(x)$  into two parts  $\Gamma_{[t_0, \tilde{\tau}]}^1(x)$  and  $\Gamma_{[t_0, \tilde{\tau}]}^2(x)$  as follows:  $\Gamma_{[t_0, \tilde{\tau}]}^1(x)$  is the set of all the rows of  $A_i - A_j$ , which  $\phi$  is one of the rows such that  $\phi x$  has a finite number of zeros in  $[t_0, \tilde{\tau}]$ .  $\Gamma_{[t_0, \tilde{\tau}]}^2(x)$  is the set of  $\phi$  contained in  $\Gamma_{[t_0, \tilde{\tau}]}(x)$  but not in  $\Gamma_{[t_0, \tilde{\tau}]}^1(x)$ . Clearly, any  $\phi \in \Gamma_{[t_0, \tilde{\tau}]}^2(x)$ ,  $\phi x$  has an infinite number of zeros in  $[t_0, \tilde{\tau}]$ . Let  $\Sigma_{[t_0, \tilde{\tau}]}^1(x)$  and  $\Sigma_{[t_0, \tilde{\tau}]}^2(x)$  be the respective parts of  $\Sigma_{[t_0, \tilde{\tau}]}(x)$ . From the continuity property (22) of the system (21), and noting that the number of  $\Sigma_{[t_0, \tilde{\tau}]}^1(x)$  is finite, it follows that the total number of switchings for the switching pair contained in  $\Sigma_{[t_0, \tilde{\tau}]}^1(x)$  is finite. Thus, there exists a  $\bar{\tau}$ ,  $t_0 < \bar{\tau} < \tilde{\tau}$ , such that if  $x(t)$  is restricted on  $[\bar{\tau}, \tilde{\tau}]$ , there is no switching occurrence for the switching pair contained in  $\Sigma_{[t_0, \tilde{\tau}]}^1(x)$ . That is, if  $(\mathcal{X}_i, \mathcal{X}_j) \in \Sigma_{[t_0, \tilde{\tau}]}^1(x)$ , then  $(\mathcal{X}_i, \mathcal{X}_j) \notin \Sigma_{[\bar{\tau}, \tilde{\tau}]}(x)$ . Thus, for any  $\phi \in \Gamma_{[\bar{\tau}, \tilde{\tau}]}(x)$ ,  $\phi x$  has an infinite number of zeros in  $[t_0, \tilde{\tau}]$ . Suppose that  $\phi x$  also has an infinite number of zeros in  $[\bar{\tau}, \tilde{\tau}]$  for all  $\phi \in \Gamma_{[\bar{\tau}, \tilde{\tau}]}(x)$ . (If this is not the case, we repeat the above process



until the result is obtained. This process can only be repeated a finite number of times, as  $\Gamma_{[\bar{\tau}, \bar{\tau}]}(x)$  has only a finite number of elements.) From Lemma 5, it follows that restricting  $x(t)$  to the interval  $[\bar{\tau}, \bar{\tau}]$ ,  $\phi x(t) \equiv 0$  for any  $t \in [\bar{\tau}, \bar{\tau}]$  and any  $\phi \in \Gamma_{[\bar{\tau}, \bar{\tau}]}(x)$ . For any switching pair  $(\mathcal{X}_i, \mathcal{X}_j) \in \Sigma_{[\bar{\tau}, \bar{\tau}]}(x)$ , we have  $A_i x(t) = A_j x(t)$  for all  $t \in [\bar{\tau}, \bar{\tau}]$  since  $\phi x(t) \equiv 0$ . Note that  $\dot{x}(t) = A_i x(t) = A_j x(t)$  for any  $t \in [\bar{\tau}, \bar{\tau}]$ . Thus, the number of switchings for any  $(\mathcal{X}_i, \mathcal{X}_j) \in \Sigma_{[\bar{\tau}, \bar{\tau}]}(x)$  is at most one according to Definition 2. Since  $\Sigma_{[\bar{\tau}, \bar{\tau}]}(x)$  has a finite number of elements, the total number of switchings in  $[\bar{\tau}, \bar{\tau}]$  is finite. However, the total number of switchings in  $[\bar{\tau}, \bar{\tau}]$  is infinite by (34) and (35). This is a contradiction. We complete the proof.

## References

- [1] A. Hassibi and S.P. Boyd, Quadratic stabilization and control of piecewise-linear systems, Proceedings of the American Control Conference, Philadelphia, 3659-3664, 1998.
- [2] M. Johansson and A. Rantzer, Computation of piecewise quadratic Lyapunov functions for hybrid systems, IEEE Transactions on Automatic Control, 43, 555-559, 1998.
- [3] L. Rodrigues and J.P. How, Observer-based control of piecewise-affine systems, International Journal of Control, 76, 459-477, 2003.
- [4] L. Rodrigues and S. Boyd, Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization, Systems and Control Letters, 54, 835-853, 2005.
- [5] F. Borrellia, M. Baotic, A. Bemporad and M. Moraria, Dynamic programming for constrained optimal control of discrete-time linear hybrid systems, Automatica 41, 1709 – 1721, 2005.
- [6] P. Grieder, M. Kvasnica, M. Baotic and M. Morari, Stabilizing lowcomplexity feedback control of constrained piecewise affine systems, Automatica, 41, 1683 – 1694, 2005.
- [7] H.W.J. Lee, K.L. Teo, V. Rehbock and L.S. Jennings, Control parametrization enhancing technique for optimal discrete-valued control problems, Automatica, 35, 1401-1407, 1999.
- [8] K.L. Teo, C.J. Goh, K.H. Wong, A unified computational approach to optimal control problems, Longman Scientific & Technical, Essex, England, (1991).
- [9] L.S. Jennings, K.L. Teo, M.E. Fisher and C.J. Goh, MISER version 3, optimal control software, theory and user manual. <http://www.maths.uwa.edu.au/~les/miser3.3.html>. Department of Mathematics, University of Western Australia, (2005).
- [10] H.J. Sussmann, Bounds on the number of switching for trajectories of piecewise analytic vector fields, Journal of Differential Equations, 43, 399-418, 1992.
- [11] H. Elmqvist, Dymola-Dynamic Modeling Language, User's Manual, Dynasim AB, Sweden, 1994.
- [12] A. Deshpande, A. Gollu, L. Semenzato, The SHIFt programming language for dynamic networks of hybrid automata, IEEE Transactions on Automatic Control, 43, 4283-4288 1998.
- [13] K.H. Johansson, M. Egerstedt, J. Lygeros and S. Sastry, On the regularization of Zeno hybrid automata, System and Control Letters, 38, 141-150, 1999.
- [14] H.W.J. Lee, K.L. Teo and A.E.B. Lim, Sensor scheduling in continuous time, Automatica, 2017-2023, 2001.