Global exponential stability of impulsive high-order Hopfield type neural networks with delays

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Abstract

In this paper, we investigate the global exponential stability of impulsive high-order Hopfield type neural networks with delays. By establishing the impulsive delay differential inequalities and using Lyapunov method, two sufficient conditions that guarantee global exponential stability of these networks are given, and the exponential convergence rate is also obtained. An numerical example is given to demonstrate the validity of the results.

Key words: Impulsive high-order Hopfield type neural networks, exponential stability, Lyapunov function, delay

1 Introduction

Hopfield neural networks have been extensively studied and developed in recent years, and there has been considerable attention in the literature on Hopfield neural networks with time delays,(See,e.g.,[1-7]).Continous-time and discrete-time Hopfield-type neural networks have been applied to model identification, optimization, etc. However, there are many impulsive phenomena in biological systems, economics systems, control systems, telecommunication

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systems and engineering applications, etc., which can be well described by impulsive systems. Impulsive neural networks have been considered in [8-12], and the stability, existence of the equilibrium of such networks have been investigated. Because the fact that high-order neural networks have stronger approximation property, faster convergence rate, greater storage capacity, higher fault tolerance than lower order neural networks, we consider impulsive high-order Hopfield type neural networks with delays in the present paper. Lyapunov method and M-matrix theory are employed to investigate the sufficient conditions for the global exponential stability. This paper is organized as follows. In Section 2, impulsive high-order Hopfield type neural networks with delays model is described. Impulsive delay differential inequalities are established in Section 3. Based on the Lyapunov stability theory, in combination with the obtained results in Section 3, two global exponential stability criteria for neural networks are derived in Section 4. An example and conclusions are given in Section 5 and 6, respectively.

2 Systems description and preliminaries

We consider the impulsive high-order Hopfield type neural networks with delays described by the following impulsive differential equations:

$$\begin{cases} C_i\dot{u}_i(t) = -\dfrac{u_i(t)}{R_i} + \sum\limits_{j=1}^{n} T_{ij} g_j(u_j(t-\tau_j)) \\ \qquad\qquad + \sum\limits_{j=1}^{n}\sum\limits_{l=1}^{n} T_{ijl} g_j(u_j(t-\tau_j)) g_l(u_l(t-\tau_l)) + I_i, t \neq t_k \\ \Delta u_i(t) = d_i u_i(t^-) + \sum\limits_{j=1}^{n} W_{ij} h_j(u_j(t^- - \tau_j)) \\ \qquad\qquad + \sum\limits_{j=1}^{n}\sum\limits_{l=1}^{n} W_{ijl} h_j(u_j(t^- - \tau_j)) h_l(u_l(t^- - \tau_l)), t = t_k \end{cases} \tag{1}$$

where $i = 1, 2, \cdots, n$.

$\Delta u_i(t_k) = u_i(t_k) - u_i(t_k^-), u_i(t_k^-) = \lim\limits_{t\to t_k^-} u_i(t), k \in Z = \{1, 2, \cdots\}$, the time sequence $\{t_k\}$ satisfies $0 < t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$, and $\lim\limits_{k\to\infty} t_k = \infty; C_i > 0, R_i > 0$, and I_i are, respectively, the capacitance, the resistance, and the external input of the ith neuron; T_{ij}, W_{ij} and T_{ijl}, W_{ijl} are, respectively, the first and second order synaptic weights of the neural networks; τ_i $(i = 1, 2, \cdots, n)$, is the transmission delay of the ith neuron such that $0 \leq \tau_i \leq \tau$, where τ is a constant.

The initial condition for (1) is given by $u_i(s) = \psi_i(s),\ s \in [t_0 - \tau, t_0]$, where $\psi_i : [t_0 - \tau, t_0] \to \Re,\ (i = 1, 2, \cdots, n)$, is a continuous function.

Throughout this paper,we assume that the neuron activation function $g_i(u)$, $h_i(u), i = 1, 2, \cdots, n$, satisfies the following condition:

$$|g_i(u_i)| \leq M_i, 0 \leq \frac{g_i(u_i) - g_i(v_i)}{u_i - v_i} \leq K_i, \forall u_i \neq v_i, u_i, v_i \in \Re \tag{2a}$$

$$|h_i(u_i)| \leq N_i, 0 \leq \frac{h_i(u_i) - h_i(v_i)}{u_i - v_i} \leq L_i, \forall u_i \neq v_i, u_i, v_i \in \Re \tag{2b}$$

where $i = 1, 2, \cdots, n$.

Let $u^* = (u_1^*, u_2^*, \cdots, u_n^*,)^T$ be an equilibrium point of (1), and set $x_i(t) = u_i(t) - u_i^*, f_i(x_i(t-\tau_i)) = g_i(u_i(t-\tau_i)) - g_i(u_i^*), \varphi_i(x_i(t-\tau_i)) = h_i(u_i(t-\tau_i)) - h_i(u_i^*), d_i u_i^* + \sum\limits_{j=1}^{n} W_{ij} h_j(u_j^*) + \sum\limits_{j=1}^{n}\sum\limits_{l=1}^{n} W_{ijl} h_j(u_j^*) h_l(u_l^*) = 0, i = 1, 2, \cdots, n$. Then for each $i = 1, 2, \cdots, n$,

$$|f_i(z)| \leq K_i|z|, z f_i(z) \geq 0, |\varphi_i(z)| \leq L_i|z|, z\varphi_i(z) \geq 0, \forall z \in \Re. \tag{3}$$

System (1) may be written as follows:

$$\begin{cases} C_i \dot{x}_i(t) = -\dfrac{x_i(t)}{R_i} + \sum\limits_{j=1}^{n} \left(T_{ij} + \sum\limits_{l=1}^{n} (T_{ijl} + T_{ilj}) \zeta_l \right) f_j(x_j(t - \tau_j)), t \neq t_k \\ \Delta x_i(t) = d_i x_i(t^-) + \sum\limits_{j=1}^{n} \left(W_{ij} + \sum\limits_{l=1}^{n} (W_{ijl} + W_{ilj}) \xi_l \right) \varphi_j(x_j(t^- - \tau_j)), t = t_k \end{cases} \tag{4}$$

where $i = 1, 2, \cdots, n$; ζ_l is between $g_l(u_l(t - \tau_l))$ and $g_l(u_l^*)$ and ξ_l is between $h_l(u_l(t_k^- - \tau_l))$ and $h_l(u_l^*)$.

The initial condition for (4) is given by $\phi_i(t) = \psi_i(t) - u_i^*,\ t \in [t_0 - \tau, t_0],\ i = 1, 2, \cdots, n$.
Lemma 1. System (1) admits at least one equilibrium point.

The proof of Lemma 1 is similar to that given in [7, Theorem 1]. An additional difference is the consideration of the impulse effect.

We denote by $\Re^+$ the set of nonnegative real numbers, $\Re^n$ the $n-$dimensional Euclidean space. $C = \text{diag}(C_1, C_2, \cdots, C_n), R = \text{diag}(R_1, R_2, \cdots, R_n)$, and $K = \text{diag}(K_1, K_2, \cdots, K_n)$.

For $t \in \Re$ and $\tau,\ \tau_i \in \Re^+,\ i = 1, 2, \cdots, n$, define
$C([t - \tau, t], \Re) = \{\psi : [t - \tau, t] \to \Re \mid \psi \text{ is continuous on } [t - \tau, t]\}$,
$C^n = C([t - \tau_1, t], \Re) \times C([t - \tau_2, t], \Re) \times \cdots \times C([t - \tau_n, t], \Re)$,
$G(t, x, y) \in \{G : \Re^+ \times \Re^n \times C^n \to \Re^n \mid G \text{ is continuous on } \Re^+ \times \Re^n \times C^n\}$.
Definition 1[3]**.** We said that $G(t, x, y)$ is belong to the functions class H_n, if the following conditions are satisfied:

(i) $\forall\ t \in I,\ \forall\ x \in \Re^n,\ \forall\ y^{(1)},\ y^{(2)} \in C^n$. $G(t,x,y^{(1)}) \leq G(t,x,y^{(2)})$ for $y^{(1)} \leq y^{(2)}$ (i.e., $y_i^{(1)} \leq y_i^{(2)},\ i = 1,2,\cdots,n$);

(ii) $\forall\ t \in I,\ \forall\ y \in C^n,\ \forall\ x^{(1)} \leq x^{(2)} \in \Re^n$. If $x^{(1)} \leq x^{(2)}$, but for some $i,\ x_i^{(1)} = x_i^{(2)}$.

Then for these i, $g_i(t,x^{(1)},y) \leq g_i(t,x^{(2)},y)$.

3 Impulsive delay differential inequality

We can extend the Lemma 2 and 3 in [3] as follows:
Lemma 2. Assume that $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$, and $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T$ are $n-$dimensional continuous functions.

Let $\overline{x}(t) = (\overline{x}_1(t), \overline{x}_2(t), \cdots, \overline{x}_n(t))^T$, $\overline{y}(t) = (\overline{y}_1(t), \overline{y}_2(t), \cdots, \overline{y}_n(t))^T$, where $\overline{x}_i(t) = \sup\limits_{t-\tau_i \leq s \leq t} x_i(s)$, $\overline{y}_i(t) = \sup\limits_{t-\tau_i \leq s \leq t} y_i(s)$, $i = 1,2,\cdots,n$.

If the following conditions are satisfied:

(i) $x_i(\theta) < y_i(\theta)$ for all $\theta \in [-\tau_i, 0],\ i = 1,2,\cdots,n$;

(ii) $D^+y_i(t) > g_i(t, y(t), \overline{y}(t))$ for all $t \geq 0,\ i = 1,2,\cdots,n$,

$D^+x_i(t) \leq g_i(t, x(t), \overline{x}(t))$ for all $t \geq 0,\ i = 1,2,\cdots,n$.

Then $x(t) < y(t)$ for all $t > 0$, where
$G(t, x(t), \overline{x}(t)) = (g_1(t, x(t), \overline{x}(t)), g_2(t, x(t), \overline{x}(t)), \cdots, g_n(t, x(t), \overline{x}(t)))^T \in H_n$.
Lemma 3 Assume that the following conditions are satisfied:

(i) $D^+x_i(t) \leq \sum\limits_{j=1}^{n} a_{ij}x_j(t) + \sum\limits_{j=1}^{n} b_{ij}\overline{x}_j(t),\ i = 1,2,\cdots,n,$
where $\overline{x}_j(t) = \sup\limits_{t-\tau_j \leq s \leq t} x_j(s),\ a_{ij} \geq 0$ for $i \neq j,\ b_{ij} \geq 0,\ i,j = 1,2,\cdots,n$, $\sum\limits_{j=1}^{n} \overline{x}_j(t_0) > 0$;

(ii) $M = -(a_{ij} + b_{ij})_{n\times n}$ is an M-matrix.

Then there exist constants $\alpha > 0, \gamma_i > 0, i = 1,2,\cdots,n$, such that
$x_i(t) \leq \gamma_i[\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t-t_0)}$ for $t \geq t_0, i = 1,2,\cdots,n$.

Lemma 4. For differential inequality

$$\begin{cases} D^+x_i(t) \le \sum\limits_{j=1}^n a_{ij}x_j(t) + \sum\limits_{j=1}^n b_{ij}\overline{x}_j(t), t \ne t_k \\ x_i(t) \le \sum\limits_{j=1}^n c_{ij}^{(k)}x_j(t^-) + \sum\limits_{j=1}^n d_{ij}^{(k)}\overline{x}_j(t^-), t = t_k \end{cases} i = 1, 2, \cdots, n, \tag{5}$$

where $x_j(t_k^-) = \lim\limits_{t\to t_k^-} x_j(t), \overline{x}_j(t) = \sup\limits_{t-\tau_j\le s\le t} x_j(s), \overline{x}_j(t_k^-) = \sup\limits_{t_k-\tau_j\le s<t_k} x_j(s)$, $x_i(t_0+\theta)$ is continuous for all $\theta \in [t_0-\tau_i, t_0],\ 0 < \tau_i \le \tau,\ i = 1, 2, \cdots, n$. $a_{ij} \ge 0$ for $i \ne j,\ b_{ij} \ge 0,\ c_{ij}^{(k)} \ge 0,\ d_{ij}^{(k)} \ge 0,\ i, j = 1, 2, \cdots, n,\ k \in Z,\ \sum\limits_{j=1}^n \overline{x}_j(t_0) > 0$.

The sequence of impulse time $\{t_k\}$ is assumed to satisfy $0 < t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ and $\lim\limits_{k\to\infty} t_k = \infty$.

Assume that

(i) $\Phi = -(a_{ij} + b_{ij})_{n\times n}$ is an M-matrix;

(ii) $\inf\limits_{k\in Z}(t_k - t_{k-1}) > \tau\delta,\ \delta > 1$ and there exist constants $\alpha > 0, \gamma > 0$, $M_i > 0, \gamma_i > 0, i = 1, 2, \cdots, n$ such that $\rho_1\rho_2\cdots\rho_k\rho_i^{(k+1)}e^{k\alpha\tau} \le M_ie^{\gamma(t_k-t_0)}$, where $\rho_k = \sum\limits_{i=1}^n \rho_i^{(k)},\ \rho_i^{(k)} = \max\Big\{\gamma_i, \sum\limits_{j=1}^n (c_{ij}^{(k)} + d_{ij}^{(k)}e^{\alpha\tau_j})\gamma_j\Big\}$.

Then $x_i(t) \le M_i \sum\limits_{j=1}^n \overline{x}_j(t_0)e^{-(\alpha-\gamma)(t-t_0)}$ for all $t \ge t_0, i = 1, 2, \cdots, n$.

Proof. For $t \in [t_0, t_1)$, by assumption (i) and Lemma 3, there exist constants $\alpha > 0, \gamma_i > 0, i = 1, 2, \cdots, n$, such that $x_i(t) \le \gamma_i[\sum\limits_{j=1}^n \overline{x}_j(t_0)]e^{-\alpha(t-t_0)}, i = 1, 2, \cdots, n$.

Hence

$$\begin{aligned} x_i(t_1) &\le \sum_{j=1}^n c_{ij}^{(1)}x_j(t_1^-) + \sum_{j=1}^n d_{ij}^{(1)}\overline{x}_j(t_1^-) \\ &\le \sum_{j=1}^n c_{ij}^{(1)}\gamma_j[\sum_{j=1}^n \overline{x}_j(t_0)]e^{-\alpha(t_1-t_0)} + \sum_{j=1}^n d_{ij}^{(1)}\gamma_j[\sum_{j=1}^n \overline{x}_j(t_0)]e^{\alpha\tau_j}e^{-\alpha(t_1-t_0)} \\ &= \Big[\sum_{j=1}^n (c_{ij}^{(1)} + d_{ij}^{(1)}e^{\alpha\tau_j})\gamma_j\Big][\sum_{j=1}^n \overline{x}_j(t_0)]e^{-\alpha(t_1-t_0)} \\ &\le \rho_i^{(1)}[\sum_{j=1}^n \overline{x}_j(t_0)]e^{-\alpha(t_1-t_0)}, i = 1, 2, \cdots, n. \end{aligned}$$

For $t \in [t_1, t_2), D^+x_i(t) \leq \sum\limits_{j=1}^{n} a_{ij}x_j(t) + \sum\limits_{j=1}^{n} b_{ij}\overline{x}_j(t)$ and
$x_i(t_2) \leq \sum\limits_{j=1}^{n} c_{ij}^{(2)} x_j(t_2^-) + \sum\limits_{j=1}^{n} d_{ij}^{(2)} \overline{x}_j(t_2^-), i = 1, 2, \cdots, n.$

By $x_i(t) \leq \rho_i^{(1)}[\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t-t_0)}$ for $t \in [t_0, t_1)$, and
$x_i(t_1) \leq \rho_i^{(1)}[\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t_1-t_0)}, i = 1, 2, \cdots, n$, we get

$\overline{x}_i(t) \leq \rho_i^{(1)}[\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{\alpha\tau_i}e^{-\alpha(t_1-t_0)}$ for $t \in [t_1 - \tau_i, t_1], i = 1, 2, \cdots, n.$

Let $y_i^{(1)}(t) = \rho_i^{(1)}[\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t-t_0)}$ for $t \in [t_1 - \tau_i, t_1], i = 1, 2, \cdots, n$, then
$x_i(t) \leq y_i^{(1)}(t)$ for $t \in [t_1 - \tau_i, t_1]$ and
$\overline{y}_i^{(1)}(t_1) = \sup\limits_{t_1-\tau_i \leq s \leq t_1} y_i^{(1)}(s) = \rho_i^{(1)}[\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t_1-t_0)}e^{\alpha\tau_i}, i = 1, 2, \cdots, n.$

Consider the following system
$\dot{y}_i^{(1)}(t) = \sum\limits_{j=1}^{n} a_{ij}y_j^{(1)}(t) + \sum\limits_{j=1}^{n} b_{ij}\overline{y}_j^{(1)}(t)$ for $t \in [t_1, t_2), i = 1, 2, \cdots, n.$

Since $\sum\limits_{i=1}^{n} \overline{y}_j^{(1)}(t_1) = [\sum\limits_{i=1}^{n} \rho_i^{(1)}e^{\alpha\tau_i}][\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t_1-t_0)} > 0$, by Lemma 3, we obtain

$$
\begin{aligned}
y_i^{(1)}(t) &\leq \gamma_i[\sum_{j=1}^{n} \overline{y}_j^{(1)}(t_1)]e^{-\alpha(t-t_1)} \\
&= \gamma_i[\sum_{i=1}^{n} \rho_i^{(1)}e^{\alpha\tau_i}][\sum_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t-t_0)} \text{for} t \in [t_1, t_2), i = 1, 2, \cdots, n.
\end{aligned}
$$

Assume that $\theta > 1$ be any constant, then $x_i(t) < \theta y_i^{(1)}(t)$ for $[t_1 - \tau_i, t_1], i = 1, 2, \cdots, n.$

We claim that $x_i(t) < \theta y_i^{(1)}(t)$ for $t \in [t_1, t_2), i = 1, 2, \cdots, n.$

If it is not true, there exist some $l, 1 \leq l \leq n$ and $r \in (t_1, t_2)$ such that

$$
x_l(r) = \theta y_l^{(1)}(r) \tag{6}
$$

Hence, we have

$$
\begin{aligned}
D^+x_l(r) &\le \sum_{j=1}^{n} a_{lj}x_j(r) + \sum_{j=1}^{n} b_{lj}\overline{x}_j(r) \\
&< \sum_{j=1}^{n} a_{lj}\theta y_j^{(1)}(r) + \sum_{j=1}^{n} b_{lj}\theta \overline{y}_j^{(1)}(r) = \theta \dot{y}_l^{(1)}(r),
\end{aligned}
$$

i.e., $D^+x_l(r) < \theta \dot{y}_l^{(1)}(r)$, this is incompatible with (6).

Hence, $x_i(t) < \theta y_i^{(1)}(t)$ for $\theta > 1, t \in [t_1, t_2), i = 1, 2, \cdots, n$.

Let $\theta \to 1$, we obtain
$x_i(t) \le y_i^{(1)}(t) \le \gamma_i[\sum\limits_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t-t_0)}$ for $[t_1, t_2), i = 1, 2, \cdots, n$ and

$$
\begin{aligned}
x_i(t_2) &\le \sum_{j=1}^{n} c_{ij}^{(2)} x_j(t_2^-) + \sum_{j=1}^{n} d_{ij}^{(2)} \overline{x}_j(t_2^-) \\
&\le \sum_{j=1}^{n} c_{ij}^{(2)} \gamma_j [\sum_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t_2-t_0)} \\
&\quad + \sum_{j=1}^{n} d_{ij}^{(2)} \gamma_j [\sum_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum_{j=1}^{n} \overline{x}_j(t_0)]e^{\alpha\tau_j} e^{-\alpha(t_2-t_0)} \\
&= \Big[\sum_{j=1}^{n} (c_{ij}^{(2)} + d_{ij}^{(2)} e^{\alpha\tau_j})\gamma_j\Big][\sum_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t_2-t_0)} \\
&\le \rho_i^{(2)} [\sum_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t_2-t_0)}, i = 1, 2, \cdots, n.
\end{aligned}
$$

Suppose that, for each $i = 1, 2, \cdots, n,\ t \in [t_{k-1}, t_k)$,
$x_i(t) \le \gamma_i[\sum\limits_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum\limits_{i=1}^{n} \rho_i^{(2)} e^{\alpha\tau_i}] \cdots [\sum\limits_{i=1}^{n} \rho_i^{(k-1)} e^{\alpha\tau_i}][\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t-t_0)}$, and
$x_i(t_k) \le \rho_i^{(k)}[\sum\limits_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum\limits_{i=1}^{n} \rho_i^{(2)} e^{\alpha\tau_i}] \cdots [\sum\limits_{i=1}^{n} \rho_i^{(k-1)} e^{\alpha\tau_i}][\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t_k-t_0)}$.

Let $y_i^{(k)}(t) = \rho_i^{(k)}[\sum\limits_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum\limits_{i=1}^{n} \rho_i^{(2)} e^{\alpha\tau_i}] \cdots [\sum\limits_{i=1}^{n} \rho_i^{(k-1)} e^{\alpha\tau_i}][\sum\limits_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t-t_0)}$,
for $t \in [t_k - \tau_i, t_k], i = 1, 2, \cdots, n$, then $x_i(t) \le y_i^{(k)}(t)$ for $t \in [t_k - \tau_i, t_k]$ and

$$
\begin{aligned}
\overline{y}_i^{(k)}(t_k) &= \sup_{t_k-\tau_i \le s \le t_k} y_i^{(k)}(s) \\
&= \rho_i^{(k)}[\sum_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum_{i=1}^{n} \rho_i^{(2)} e^{\alpha\tau_i}] \cdots [\sum_{i=1}^{n} \rho_i^{(k-1)} e^{\alpha\tau_i}] \\
&\qquad \cdot [\sum_{j=1}^{n} \overline{x}_j(t_0)]e^{-\alpha(t_k-t_0)} e^{\alpha\tau_i}, i = 1, 2, \cdots, n.
\end{aligned}
$$

Consider the following system
$\dot{y}_i^{(k)}(t) = \sum\limits_{j=1}^{n} a_{ij} y_j^{(k)}(t) + \sum\limits_{j=1}^{n} b_{ij} \overline{y}_j^{(k)}(t)$ for $t \in [t_k, t_{k+1}), i = 1, 2, \cdots, n.$

In view of
$\sum\limits_{i=1}^{n} \overline{y}_j^{(k)}(t_k) = [\sum\limits_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum\limits_{i=1}^{n} \rho_i^{(2)} e^{\alpha\tau_i}] \cdots [\sum\limits_{i=1}^{n} \rho_i^{(k)} e^{\alpha\tau_i}][\sum\limits_{j=1}^{n} \overline{x}_j(t_0)] e^{-\alpha(t_k - t_0)} > 0,$
by Lemma 3, we have

$$\begin{aligned} y_i^{(k)}(t) &\leq \gamma_i [\sum_{j=1}^{n} \overline{y}_j^{(k)}(t_k)] e^{-\alpha(t-t_k)} \\ &= \gamma_i [\sum_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum_{i=1}^{n} \rho_i^{(2)} e^{\alpha\tau_i}] \cdots [\sum_{i=1}^{n} \rho_i^{(k)} e^{\alpha\tau_i}][\sum_{j=1}^{n} \overline{x}_j(t_0)] e^{-\alpha(t-t_0)}, \end{aligned}$$

for $t \in [t_k, t_{k+1}), i = 1, 2, \cdots, n.$

Similar to the above proof, we know that, for each $i = 1, 2, \cdots, n,$
$x_i(t) \leq \gamma_i [\sum\limits_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum\limits_{i=1}^{n} \rho_i^{(2)} e^{\alpha\tau_i}] \cdots [\sum\limits_{i=1}^{n} \rho_i^{(k)} e^{\alpha\tau_i}][\sum\limits_{j=1}^{n} \overline{x}_j(t_0)] e^{-\alpha(t-t_0)},$
for $t \in [t_k, t_{k+1})$, and
$x_i(t_{k+1}) \leq \rho_i^{(k+1)} [\sum\limits_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum\limits_{i=1}^{n} \rho_i^{(2)} e^{\alpha\tau_i}] \cdots [\sum\limits_{i=1}^{n} \rho_i^{(k)} e^{\alpha\tau_i}][\sum\limits_{j=1}^{n} \overline{x}_j(t_0)] e^{-\alpha(t_{k+1} - t_0)}.$

By mathematical induction, we can conclude that

$$\begin{aligned} x_i(t) &\leq \rho_i^{(k+1)} [\sum_{i=1}^{n} \rho_i^{(1)} e^{\alpha\tau_i}][\sum_{i=1}^{n} \rho_i^{(2)} e^{\alpha\tau_i}] \cdots [\sum_{i=1}^{n} \rho_i^{(k)} e^{\alpha\tau_i}][\sum_{j=1}^{n} \overline{x}_j(t_0)] e^{-\alpha(t-t_0)} \\ &\leq \rho_i^{(k+1)} \rho_1 \rho_2 \cdots \rho_k e^{k\alpha\tau} [\sum_{j=1}^{n} \overline{x}_j(t_0)] e^{-\alpha(t-t_0)}, \text{for } t \in [t_k, t_{k+1}], \\ &\quad i = 1, 2, \cdots, n. \end{aligned}$$

Noticing that $\rho_1 \rho_2 \cdots \rho_k \rho_i^{(k+1)} e^{k\alpha\tau} \leq M_i e^{\gamma(t_k - t_0)}$, we obtain

$$x_i(t) \leq M_i [\sum_{j=1}^{n} \overline{x}_j(t_0)] e^{-(\alpha-\gamma)(t-t_0)}, \text{ for all } t \geq t_0, i = 1, 2, \cdots, n.$$

This completes the proof.
Lemma 5. For differential inequality (5), if the following conditions are satisfied:

(i) $\Phi = -(a_{ij} + b_{ij})_{n\times n}$ is an M-matrix;

(ii) $\inf\limits_{k \in Z} (t_k - t_{k-1}) > \tau\delta,\ \delta > 1$ and there exist constants $\alpha > 0, \gamma_i > 0, i =$

$1, 2, \cdots, n$ such that $\sum_{i=1}^{n} \mu_i e^{\alpha\tau} > 1$, where $\mu_i = \sup_{k \in Z} \left\{ \gamma_i, \sum_{j=1}^{n} (c_{ij}^{(k)} + d_{ij}^{(k)} e^{\alpha\tau_j}) \gamma_j \right\}$.

Then

$$x_i(t) \leq (\sum_{i=1}^{n} \mu_i)(\sum_{j=1}^{n} \overline{x}_j(t_0)) e^{-\left[\alpha - \frac{\ln (e^{\alpha\tau} \sum_{i=1}^{n} \mu_i)}{\delta\tau}\right](t-t_0)}, \text{for all } t \geq t_0, i = 1, 2, \cdots, n.$$

Proof. It follows from the proof of Lemma 4 that $\rho_i^{(k)} \leq \mu_i$ for all $k \in Z$, and $\rho_l = \sum_{i=1}^{n} \rho_i^{(l)} \leq \sum_{i=1}^{n} \mu_i = \rho$.

For any $t \in \Re^+$, there exist $k \in Z$ such that $t \in [t_k, t_{k+1})$.

Hence, by $\rho e^{\alpha\tau} = \sum_{i=1}^{n} \mu_i e^{\alpha\tau} > 1$, we obtain

$$\begin{aligned} x_i(t) &\leq \rho_i^{(k+1)} (\rho e^{\alpha\tau})^k [\sum_{j=1}^{n} \overline{x}_j(t_0)] e^{-\alpha(t-t_0)} \\ &\leq \rho_i^{(k+1)} e^{\frac{\ln(\rho e^{\alpha\tau})}{\delta\tau}(t_k - t_0)} [\sum_{j=1}^{n} \overline{x}_j(t_0)] e^{-\alpha(t-t_0)} \\ &\leq \rho_{k+1} [\sum_{j=1}^{n} \overline{x}_j(t_0)] e^{-\left[\alpha - \frac{\ln (\rho e^{\alpha\tau})}{\delta\tau}\right](t-t_0)}, \ \forall\ t \in [t_k, t_{k+1}), i = 1, 2, \cdots, n. \end{aligned}$$

Hence, $x_i(t) \leq \rho[\sum_{j=1}^{n} \overline{x}_j(t_0)] e^{-\left[\alpha - \frac{\ln (\rho e^{\alpha\tau})}{\delta\tau}\right](t-t_0)}, \ \forall\ t \geq t_0, i = 1, 2, \cdots, n.$

This completes the proof.

4 Exponential stability

In this section, we shall obtain two sufficient conditions for global exponential stability of the impulsive high-order Hopfield type neural networks. If $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T$ is an equilibrium point of (1), then $x^* = (0, 0, \cdots, 0)^T$ is an equilibrium point of (4). To prove the global exponential stability of the equilibrium point u^* of (1), it is sufficient to prove the global exponential stability of the trivial solution of (4).
Theorem 1. Assume that

(i) $C^{-1}R^{-1} - C^{-1}\Theta K$ is an M-matrix, where $\Theta = (\theta_{ij})_{n\times n}$, $\theta_{ij} = |T_{ij}| + \sum\limits_{l=1}^{n} |T_{ijl} + T_{ilj}|M_l$;

(ii) There exists a constant δ satisfying $\delta > \dfrac{\ln(\rho e^{\alpha\tau})}{\alpha\tau}$ such that $\inf\limits_{k\in Z}(t_k - t_{k-1}) > \tau\delta$, where $\rho = \sum\limits_{i=1}^{n} \rho_i > 1$, $\rho_i = \max\Big\{\gamma_i, |1+d_i|\gamma_i + \sum\limits_{j=1}^{n}\Big(|W_{ij}| + \sum\limits_{l=1}^{n}|W_{ijl} + W_{ilj}|N_l\Big)L_j e^{\alpha\tau_j}\gamma_j\Big\}, \alpha > 0$, and $\gamma_i > 0, i = 1, 2, \cdots, n$.

Then the equilibrium point u^* of (1) is globally exponentially stable with convergence rate $\alpha - \dfrac{\ln(\rho e^{\alpha\tau})}{\delta\tau}$.

Proof. For $t \neq t_k$, we compute the Dini derivative of $|x_i(t)|$ along the trajectories of (4), by (2) and (3), we obtain

$$\begin{aligned}
D^+|x_i(t)| &= -\frac{1}{R_iC_i}|x_i(t)| + \sum_{j=1}^{n}\Big[T_{ij} + \sum_{l=1}^{n}(T_{ijl} + T_{ilj})\zeta_l\Big]\frac{f_j(x_j(t-\tau_j))}{C_i}\mathrm{sgn}(x_i(t)) \\
&\le -\frac{1}{R_iC_i}|x_i(t)| + \sum_{j=1}^{n}\theta_{ij}\frac{K_j}{C_i}|x_j(t-\tau_j)| \\
&\le -\frac{1}{R_iC_i}|x_i(t)| + \sum_{j=1}^{n}\theta_{ij}\frac{K_j}{C_i}|\overline{x}_j(t)|,
\end{aligned}$$

where $|\overline{x}_j(t)| = \sup\limits_{t-\tau_i\le s\le t}|x_j(s)|$.

By (2) and (3), we get in view of (4)

$$\begin{aligned}
|x_i(t_k)| &\le |1+d_i||x_i(t_k^-)| + \sum_{j=1}^{n}\omega_{ij}|\varphi_j(x_j(t_k^- - \tau_j))| \\
&\le |1+d_i||x_i(t_k^-)| + \sum_{j=1}^{n}\omega_{ij}L_j|x_j(t_k^- - \tau_j)| \\
&\le |1+d_i||x_i(t_k^-)| + \sum_{j=1}^{n}\omega_{ij}L_j|\overline{x}_j(t_k^-)|
\end{aligned}$$

where $\omega_{ij} = |W_{ij}| + \sum\limits_{l=1}^{n}|W_{ijl} + W_{ilj}|N_l$ and $|\overline{x}_j(t_k^-)| = \sup\limits_{t_k-\tau_i\le s<t_k}|x_j(s)|$.

By Lemma 5, we obtain

$$|x_i(t)| \le \rho\Big(\sum_{j=1}^{n}|\overline{x}_j(t_0)|\Big)e^{-\left[\alpha - \frac{\ln(\rho e^{\alpha\tau})}{\delta\tau}\right](t-t_0)},\ \forall\, t \ge t_0, i = 1, 2, \cdots, n.$$

This completes the proof.
Remark. For the Hopfield-type neural networks with impulses

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \sum\limits_{j=1}^{m} b_{ij} f_j(x_j(t-\tau_{ij})) + c_i, t \neq t_k, \\ \Delta x_i(t) = -\gamma_{ik}(x_i(t) - x_i^*), t = t_k, k = 1, 2, \cdots, m, 0 < \gamma_{ik} < 2, \end{cases} \tag{7}$$

where $i = 1, 2, \cdots, m$, $f_i(\cdot)$ satisfies $|f_i(x) - f_i(y)| \leq L_i|x-y|$ for all $x, y \in \Re$, and $|f_i(x)| \leq M_i, x_i \in \Re$, for some constant $M_i > 0$.

System (7) is a special case of (1) for $T_{ijk} = 0, W_{ijk} = 0, W_{ij} = 0, \tau_j(t) = \tau_{ij}, i, j, k = 1, 2, \cdots, m$, and appropriate selected $d_i, i = 1, 2, \cdots, m$.

The Theorem 1 and Theorem 2.4 in [9] give some sufficient conditions for the global exponential stability of system (7) are, respectively

$$a_i > 0, a_i - L_i \sum_{j=1}^{m} |b_{ji}| > 0, i = 1, 2, \cdots, m,$$

and $A - BL$ be an M-matrix, where $A = diag(a_1, a_2, \cdots, a_m), B = (|b_{ij}|)_{m\times m}$.

Obviously, $a_i > 0, a_i - L_i \sum\limits_{j=1}^{m} |b_{ji}| > 0, i = 1, 2, \cdots, m$, are sufficient conditions of $A - BL$ be an M-matrix. Hence, for system (7), Theorem 1 includes Theorem 2.4 in [9] as a special case.

From theorem 1 we don't know the value of α, in the following we shall obtain an estimation of α, in despite of it is conservative.

Theorem 2. Assume that

(i) There exist a constant $\lambda > 0$ such that $C^{-1}R^{-1} - \lambda I - e^{\lambda\tau}C^{-1}\Theta K$ is an M-matrix, where $\Theta = (\theta_{ij})_{n\times n}, \theta_{ij} = |T_{ij}| + \sum\limits_{l=1}^{n} |T_{ijl} + T_{ilj}|M_l$;

(ii) There exist a constant δ satisfying $\delta > \dfrac{\ln(\rho e^{\lambda\tau})}{\lambda\tau}$ such that $\inf\limits_{k\in Z}(t_k - t_{k-1}) > \tau\delta$, where $\rho = \sum\limits_{i=1}^{n} \rho_i > 1$, $\rho_i = \max\Big\{\gamma_i, |1+d_i|\gamma_i + \sum\limits_{j=1}^{n}\Big(|W_{ij}| + \sum\limits_{l=1}^{n} |W_{ijl} + W_{ilj}|N_l\Big)L_j e^{2\lambda\tau_j}\gamma_j\Big\}$ and $\gamma_i > 0$, $i = 1, 2, \cdots, n$.

Then the equilibrium point u^* of (1) is globally exponentially stable with convergence rate $2\lambda - \dfrac{\ln(\rho e^{\lambda\tau})}{\delta\tau}$.
Proof. Let $y_i(t) = e^{\lambda t}x_i(t), i = 1, 2, \cdots, n$. Then for $t \neq t_k$, we compute the

Dini derivative of $|y_i(t)|$ along the trajectories of (4), it follows from the proof of theorem 1 that

$$\begin{aligned} D^+|y_i(t)| &= e^{\lambda t}D^+|x_i(t)| + \lambda e^{\lambda t}|x_i(t)| = e^{\lambda t}(D^+|x_i(t)| + \lambda|x_i(t)|) \\ &\le e^{\lambda t}\Big[\Big(\lambda - \frac{1}{R_iC_i}\Big)|x_i(t)| + \sum_{j=1}^{n}\theta_{ij}\frac{K_j}{C_i}|x_j(t-\tau_j)|\Big] \\ &\le \Big(\lambda - \frac{1}{R_iC_i}\Big)|y_i(t)| + e^{\lambda\tau}\sum_{j=1}^{n}\theta_{ij}\frac{K_j}{C_i}|\overline{y}_j(t)|, \end{aligned}$$

where $|\overline{y}_j(t)| = \sup\limits_{t-\tau_j\le s\le t}|y_j(s)|$, and

$$\begin{aligned} |y_i(t_k)| &\le |1+d_i||y_i(t_k^-)| + \sum_{j=1}^{n}\omega_{ij}L_je^{\lambda t_k}|x_j(t_k^- - \tau_j)| \\ &\le |1+d_i||x_i(t_k^-)| + \sum_{j=1}^{n}\omega_{ij}L_je^{\lambda\tau_j}|\overline{y}_j(t_k^-)|, \end{aligned}$$

where $\omega_{ij} = |W_{ij}| + \sum\limits_{l=1}^{n}|W_{ijl} + W_{ilj}|N_l$ and $|\overline{y}_j(t_k^-)| = \sup\limits_{t_k-\tau_j\le s<t_k}|y_j(s)|$.

By Lemma 5 we obtain

$$|y_i(t)| \le \rho\Big(\sum_{j=1}^{n}|\overline{y}_j(t_0)|\Big)e^{-\left[\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau}\right](t-t_0)},\ \forall\ t\ge t_0, i=1,2,\cdots,n.$$

Hence, we get

$$|x_i(t)| \le \rho\Big(\sum_{j=1}^{n}|\overline{x}_j(t_0)|\Big)e^{-\left[2\lambda - \frac{\ln(\rho e^{\lambda\tau})}{\delta\tau}\right](t-t_0)},\ \forall\ t\ge t_0, i=1,2,\cdots,n.$$

This completes the proof.

5 Numercal example

Consider the following impulsive high-order Hopfield type neural network with delays:

$$\begin{cases} C_i\dot{u}_i(t) = -\dfrac{u_i(t)}{R_i} + \sum\limits_{j=1}^{3} T_{ij}g_j(u_j(t-\tau_j)) \\ \qquad\qquad + \sum\limits_{j=1}^{3}\sum\limits_{l=1}^{3} T_{ijl}g_j(u_j(t-\tau_j))g_l(u_l(t-\tau_l)), t \neq t_k \\ \Delta u_i(t) = d_iu_i(t^-) + \sum\limits_{j=1}^{3} W_{ij}h_j(u_j(t^- -\tau_j)) \\ \qquad\qquad + \sum\limits_{j=1}^{3}\sum\limits_{l=1}^{3} W_{ijl}h_j(u_j(t^- -\tau_j))h_l(u_l(t^- -\tau_l)), t = t_k \end{cases} \quad i = 1,2,3(8)$$

where $g_1(u_1) = \tanh(0.25u_1), g_2(u_2) = \tanh(0.31u_2), g_3(u_3) = \tanh(0.37u_3)$, $h_1(u_1) = \tanh(0.28u_1)$, $h_2(u_2) = \tanh(0.92u_2), h_3(u_3) = \tanh(0.87u_3)$, $\tau_1 = 0.12, \tau_2 = 0.15, \tau_3 = 0.11, \tau = 0.15, d_1 = -1.14, d_2 = -1.32, d_3 = -1.58$, $C = \mathrm{diag}(C_1, C_2, C_3) = \mathrm{diag}(3.46, 2.35, 1.11)$, $R = \mathrm{diag}(R_1, R_2, R_3) = \mathrm{diag}(2.55, 0.35, 0.45)$,

$$(T_{ij})_{3\times3} = \begin{bmatrix} 0.12 & 0.97 & -0.60 \\ -0.08 & 0.47 & 0.04 \\ -0.21 & -0.72 & 1.13 \end{bmatrix}, (T_{1ij})_{3\times3} = \begin{bmatrix} -0.07 & 0.06 & 0.03 \\ 0.08 & 0.01 & 0.06 \\ -0.01 & -0.04 & -0.01 \end{bmatrix},$$

$$(T_{2ij})_{3\times3} = \begin{bmatrix} -0.08 & -0.03 & -0.01 \\ 0.04 & 0.01 & 0.06 \\ -0.03 & 0.05 & -0.03 \end{bmatrix}, (T_{3ij})_{3\times3} = \begin{bmatrix} -0.05 & 0.08 & -0.02 \\ -0.03 & -0.04 & 0.06 \\ 0.01 & -0.03 & 0.01 \end{bmatrix},$$

$$(W_{ij})_{3\times3} = \begin{bmatrix} -0.08 & -0.37 & -0.41 \\ 0.25 & 0.21 & 0.10 \\ 0.29 & -0.11 & -0.13 \end{bmatrix}, (W_{1ij})_{3\times3} = \begin{bmatrix} 0.03 & -0.01 & -0.04 \\ 0.02 & 0.01 & -0.02 \\ 0.07 & 0.03 & -0.02 \end{bmatrix},$$

$$(W_{2ij})_{3\times3} = \begin{bmatrix} -0.05 & -0.03 & 0.06 \\ 0.01 & -0.06 & -0.06 \\ -0.01 & 0.01 & 0.02 \end{bmatrix}, (W_{3ij})_{3\times3} = \begin{bmatrix} -0.05 & 0.01 & -0.01 \\ 0.01 & -0.05 & 0.04 \\ -0.01 & -0.08 & 0.05 \end{bmatrix}.$$

In this case $M_i = N_i = 1, i = 1,2,3, L_1 = 0.28, L_2 = 0.92, L_3 = 0.87, K = \mathrm{diag}(0.25, 0.31, 0.37)$, $u^* = (0,0,0)^T$ is an equilibrium point of system (8).

By direct computation, it follows that the constants $\gamma_1 = 0.3, \gamma_2 = 0.26, \gamma_3 = 0.35, \alpha = 0.05$ such that $\rho = 1.0001 > 1$, and the matrix $C^{-1}R^{-1} - C^{-1}\Theta K$ in theorem 1 is an M-matrix. Let $\delta = 1.1 > \dfrac{\ln(\rho e^{\alpha\tau})}{\alpha\tau} = 1.0095$. Then by theorem

1, we see that the equilibrium point u^* of system (8) is globally exponentially stable with convergence rate 0.0041 for $\inf_{k \in Z}\{t_k - t_{k-1}\} > 0.165$.

There exist a constant $\lambda = 0.0749$ such that $C^{-1}R^{-1} - \lambda I - e^{\lambda\tau}C^{-1}\Theta K$ is an M -matrix, and constants $\gamma_1 = 0.32, \gamma_2 = 0.27, \gamma_3 = 0.32$ such that $\rho = 1.0011 > 1$ in theorem 2. Thus by theorem 2, we see that the equilibrium point u^* of system (8) is globally exponentially stable with convergence rate 0.0752 for $\inf_{k \in Z}\{t_k - t_{k-1}\} > 0.165$, if we let $\delta = 1.1 > \frac{\ln(\rho e^{\lambda\tau})}{\lambda\tau} = 1.0957$.

6 Conclusions

The problems of global exponential stability analysis for impulsive high-order Hopfield type neural networks with delays have been discussed in this paper. Impulsive delay differential inequalities are established, and some global exponential stability criteria have been derived by means of these inequalities and appropriate Lyapunov functions. These criteria are easy to verify and may be used to analysis the dynamics of biological neural systems.

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References

[1] Y. Zhang, S. M.Zhong, Z. L. Li, Periodic Solutions and Stability of Hopfield Neural Networks With Variable Delays, Int.J.Syst.Sci. 27 (1996) 895-901.

[2] P. Den Driessche, X. Zou, Global Attractivity in Delayed Hopfield Neural Networks model, SIAM Journal of Applied Mathematics. 58 (1998) 1878-1890.

[3] X. X. Liao, D. M. Xiao, Global Exponential Stability of Hopfield Neural Networks With Time-Varying Delays, Acta Electronica Sinica. 28 (2000) 87-90 (in Chinese)

[4] X. F. Liao, K. -W. Wong, Z. F. Wu, et al. Novel Robust Stability Criteria for Interval-Delayed Hopfield Neural Networks, IEEE Trans. on Circuits and Systems I. 48 (2001) 1355-1359.

[5] J. G. Peng, H. Qiao, Z. -B. Xu, A New Approach to Stability of Neural Networks With Time-Varying Delays, Neural Networks. 15 (2002) 95-103

[6] Z. G. Zeng, J. Wang, X. X. Liao, Global Exponential Stability of Neural Networks With Time-Varying Delays, IEEE Transactions on Circuits and Systems-I. 50 (2003) 1353-1358.

[7] B. J. Xu, X. Z. Liu, X. X. Liao, Global Asymptotic Stability of High Order Hopfield Type Neural Networks with Time Delays, Computers and Mathematics with Applications. 45 (2003) 1729-1737.

[8] K. Gopalsamy, Stability of Artificial Neural Networks With Impulses, Applied Mathematics and Computation. 154 (2004) 783-813.

[9] H. Acka, R. Alassar, V. Covachev, et al. Continuous-Time Additive Hopfield-Type Neural Networks With Impulses, Journal of Mathematical Analysis and Applications. 290 (2004) 436-451.

[10] X. Z. Liu, K. L. Teo, B. J. Xu, Exponential stability of impulsive high-order Hopfield-type neural networks with time-varying delays, IEEE Trans. on Neural Networks. 16 (2005) 1329–1339.

[11] Daniel W. C. Ho, J. L. Liang, J. Lam, Global exponential stability of impulsive high-order BAM neural networks with time-varying delays, Neural Networks. 19 (2006) 1581- -1590

[12] J. L. Qiu, Exponential stability of impulsive neural networks with time-varying delays and reaction-diffusion terms, Neurocomputing. 70 (2007) 1102–1108