A Comparison of TCAR, CIR and LAMBDA GNSS Ambiguity Resolution

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BIOGRAPHY

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ABSTRACT

With the envisioned introduction of three-carrier GNSS's (modernized GPS, Galileo), new methods of ambiguity resolution have been developed. In this contribution we will compare two important candidate methods for triple-frequency ambiguity resolution with the already existing LAMBDA (Least-squares Ambiguity Decorrelation Adjustment) method: the TCAR (Three-Carrier Ambiguity Resolution) method, and the CIR (Cascading Integer Resolution) method.

It will be shown that for their estimation principle, both TCAR and CIR rely on integer bootstrapping, whereas LAMBDA is based on integer least-squares, of which optimality has been proven, that is, highest probability of success. In TCAR and CIR pre-defined ambiguity transformation are used, whereas LAMBDA exploits the information content of the full ambiguity variance-covariance matrix, with statistical decorrelation the objective in constructing the ambiguity transformation. For the aspect of resolving the ambiguities, TCAR and CIR are designed for use with the geometry-free model. LAMBDA can intrinsically handle any GNSS model with integer ambiguities and thereby utilize satellite geometry to its benefit in geometry-based models.

INTRODUCTION

The TCAR, CIR and LAMBDA methods for GNSS ambiguity estimation will be compared at three different levels. We first compare the three methods at the conceptual level. This reveals the basic assumptions involved in either method and shows conceptual similarities and differences. The consequences and significance of the differences are also described. Next the three methods are compared numerically. This involves the complexity of the computational steps of either method. Finally the performance of the methods is compared. The probability of correct integer estimation will be the performance indicator used. In order to compare the methods on those aspects, the three methods will be applied to various GNSS data processing models and the differences in performance will be explained.

This comparison is of importance in order to understand the strengths and weaknesses of these three methods. It will help developers, users and practitioners alike in making their choice between the different methods for their particular application at hand.

1 THE GENERAL GNSS MODEL

Every GNSS model in which carrier phase observables are included can be cast in the following system of linear(ized) observation equations:

$$y = Aa + Bb + e \tag{1}$$

where y is the given m-dimensional GNSS data vector, a and b are the unknown parameter vectors respectively of order n and p, and where e is the noise vector. The data vector y will usually consist of the 'observed minus computed' single-, dual- or triple-frequency double-difference (DD) phase and/or pseudorange (code) observations, accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in Z^n$. The entries of vector b will consist of the remaining unknown parameters, such as DD ranges in case of the geometry-free model or baseline components (coordinates) in case of the geometry-based model, possibly together with atmospheric delay parameters (troposphere, ionosphere) and/or other parameters of interest. The entries of b are known to be real-valued, $b \in R^p$.

The procedure which is usually followed for solving the GNSS model can be divided into three steps. In the first step a model according equation (1) is setup, with the aim of ambiguity resolution in mind. At first, one simply disregards the integer constraints $a \in Z^n$ on the ambiguities and performs a standard least-squares adjustment. As a result one obtains the (real-valued) estimates of a and b, together with their variance-covariance (vc-) matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \;;\quad \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2}$$

This solution is referred to as the 'float' solution. Since the fact that the ambiguities are integers has not yet been exploited, this float solution is not as precise as possible. Therefore in a second step the 'float' ambiguity estimate $\hat{a}$ is used to compute the corresponding integer ambiguity estimate $\check{a}$. This implies that a mapping $M : R^n \mapsto Z^n$, from the n-dimensional space of reals to the n-dimensional space of integers, is introduced such that:

$$\check{a} = M(\hat{a}) \;, \quad \check{a} \in Z^n \;, \quad \hat{a} \in R^n \tag{3}$$

In the final and third step, the fixed ambiguities $\check{a}$ are now assumed to be determined, and they are used to estimate the final parameters of interest with high precision. In case the model in the first step was chosen in such a way that it is (also) suitable for determining these parameters of interest, one can simply improve the float solution in order to find the fixed solution

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \tag{4}$$

Note that once the ambiguities are solved, it is also possible to setup a different model, according the general model given in equation (1). Assuming the ambiguities to be deterministic, this model can be solved directly using a least squares approach. Here lies the first distinction between TCAR and CIR on the one hand, and LAMBDA on the other. Ambiguity resolution using TCAR and CIR is based on the use of the *geometry-free* model, whereas in case of LAMBDA any model can be used.

Since in most GPS applications one of the aims is positioning, the parameters of interest usually include baseline coordinates. This implies that when using TCAR or CIR, one has to deal with the geometry-free model in the first step, and with the geometry-based model in the third, while when using LAMBDA, the geometry-based model can be used from the outset. For those cases the full process for any of the three methods under consideration can be schematized as shown in figure 1.

In the present contribution we will focus our attention on ambiguity resolution and compare the LAMBDA method (Least-squares AMBiguity Decorrelation Adjustment), as introduced in Teunissen (1993), with two more recently proposed methods of ambiguity resolution, namely TCAR as proposed for Galileo, Three Carrier Ambiguity Resolution, Forsell *et al.* (1997), Vollath *et al.* (1998), and CIR as proposed for modernized GPS, Cascade Integer Resolution, Jung *et al.* (2000) and Hatch *et al.* (2000).

It should be noted however that ambiguity resolution is not a goal in itself. The sole purpose of taking the integerness of the ambiguities in the parameter estimation into account is to reach a significant improvement in the estimated parameters of interest. Whether or not this is the case depends on the structure of the covariance matrix $Q_{\hat{a}\hat{b}}$ in figure 1, and therefore on the particular application at hand. In the ideal case one would like ambiguity resolution to be optimized in relation to the parameters of interest. Given the parameters of interest, this would then automatically provide an answer to the question which ambiguities or admissible functions thereof need to be chosen so as to obtain the largest possible improvement in these parameters through successful ambiguity resolution. Since this topic is outside the scope of the present contribution, we will restrict our comparison to the level of evaluating the respective ambiguity success rates of the different methods.

In order to make such a comparison possible we first need to present some results from the theory of integer inference. We therefore briefly summarize some properties of two important integer estimation principles, namely of integer bootstrapping and of integer least-squares.

2 INTEGER ESTIMATION

Various methods for mapping $\hat{a}$ into $\check{a}$ have been proposed in the literature. For a discussion of some of these methods we refer to standard GNSS textbooks such as Hofmann-Wellenhof *et al.* (2001), Leick (1995), Enge and Misra (2001), Parkinson and Spilker (1996), Strang and Borre (1997) and Teunissen and Kleusberg (1998). This section briefly summarizes the underlying theory as far as relevant to understanding the TCAR, CIR and LAMBDA methods.

2.1 Integer rounding

The simplest way to obtain an integer vector from the real-valued 'float' solution is to round each of the entries of $\hat{a}$ to its nearest integer. The corresponding integer estimator would then read as

$$\check{a}_R = ([\hat{a}_1], \ldots, [\hat{a}_n])^T \tag{5}$$

where '[.]' denotes rounding to the nearest integer. Note that this estimator does not take correlation between the elements of the integer ambiguity vector into account. Since elements of the ambiguity vector are known to be highly correlated, rounding can not be considered a serious estimator for the purpose of integer ambiguity resolution.

2.2 Conditional integer rounding or "Bootstrapping"

A more advanced but still relatively simple integer ambiguity estimator is the bootstrapped estimator. This estimator still makes use of integer rounding, but it also takes some of the correlation between the ambiguities into account. The bootstrapped estimator follows from a sequential conditional least-squares adjustment and is computed

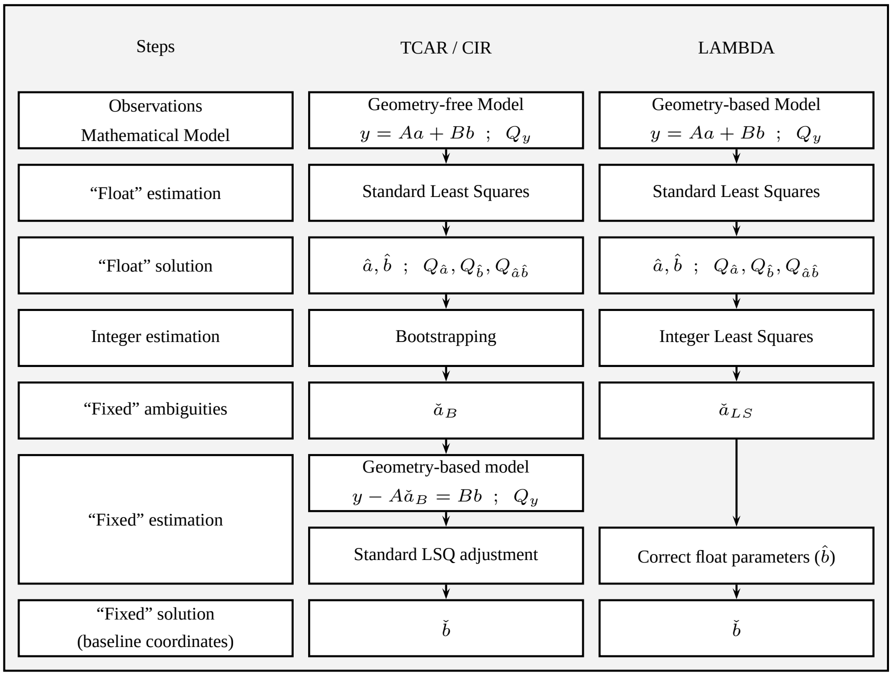

Fig. 1. Examples of procedures to obtain high-precision parameters of interest including baseline-coordinates, using integer ambiguity estimation techniques.

as follows. If n ambiguities are available, one starts with the first ambiguity $\hat{a}_1$, and rounds its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining $n-2$ ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is continued until all ambiguities are considered. In the sequel the bootstrapped estimator will be denoted by $\check{a}_B$. Note that the bootstrapped estimator is not unique. Changing the order in which the ambiguities appear in vector $\hat{a}$ will already produce a *different* bootstrapped estimator. Note also that it is advisable to order the ambiguities according to their conditional standard deviations. For a more extensive review of the theory of integer bootstrapping we refer to Teunissen (2001a).

2.3 Integer Least Squares

Let $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n)^T \in R^n$ be the ambiguity 'float' solution and let $\check{a}_{LS} \in Z^n$ denote the corresponding integer least-squares solution. Then

$$\check{a}_{LS} = \arg \min_{z \in Z^n} (\hat{a} - z)^T Q_{\hat{a}}^{-1} (\hat{a} - z) \tag{6}$$

This type of least-squares problem was first introduced in Teunissen (1993) and has been coined with the term '*integer least-squares*'. It is a nonstandard least-squares problem due to the integer constraints $z \in Z^n$. In contrast to integer rounding and integer bootstrapping, an integer search is needed to compute the solution $\check{a}_{LS}$, making the use of this estimator more elaborate from a computational point of view. For an extensive review of the theory of integer least-squares we refer to Teunissen (2001b).

2.4 Performance

In order to compare the different estimators, one needs a criterion to judge them by. The most obvious choice is the probability of estimating the *correct* integers, also referred to as the success-rate. The higher this probability, the *better* the performance of the estimator. This success-rate is given as:

$$P(\check{a} = a) = \int_{S_a} p_{\hat{a}}(x) dx \tag{7}$$

with $p_{\hat{a}}(x)$ the probability density function of the float ambiguities and S_a the pull-in region, or area around the correct integer for which any float solution gets “pulled” towards the correct fixed solution. It is stressed that the success rate should be used as measure for predicting the success of ambiguity resolution instead of e.g. the standard deviations of the ambiguities. Note that the success rate depends on three factors, being the functional model (observation equations), the stochastic model (distribution and precision of the observables), which govern the distribution $p_{\hat{a}}$ and the chosen method of integer estimation, which governs the shape of the pull-in region S_a. A more extensive description of the successrate is given in for example Joosten and Tiberius (2000).

One might expect that in case the shape of the pull-in region S_a has more resemblence to the probability density function, the successrate is higher. In figure 2 two-dimensional examples of pull-in regions for the three different estimators are shown, while the probability density function is represented by an ellipse.

In general, the integral in equation (7) is difficult to evaluate, however in case of the bootstrapping estimator, the probability of correct integer estimation is given explicitly as:

$$P(\check{a}_B = a) = \prod_{i=1}^{n}[2\Phi(\frac{1}{2\sigma_{i|I}}) - 1] \quad (8)$$

with

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}v^2\}dv$$

The conditional standard deviations $\sigma_{i|I}$ as needed in (8) can be obtained directly as the square-roots of the entries of the diagonal matrix D in the triangular decomposition of the variance-covariance (vc-) matrix $Q_{\hat{a}} = LDL^T$.

In Teunissen (1999) it was proven that the integer least-squares estimator maximizes the probability of correct integer estimation. Consequently, the successrate for the integer least squares estimator is equal to or higher than the successrate when using the bootstrapping estimator. This proof gives a probabilistic justification for using the integer least-squares estimator. For GNSS ambiguity resolution one is therefore better off using the integer least-squares estimator than any other admissible integer estimator, including the bootstrapping estimator

3 AMBIGUITY TRANSFORMATIONS

Up to know we have presented the results in terms of the DD ambiguity vector a. Ambiguities, however, can be reordered and transformed. Think for instance of the non-uniqueness in the DD definition in which different satellites can be taken as the reference satellite. Hence, there is no single unique definition of an ambiguity vector. It is possible however to give an unequivocal description of the type of transformation which links all the admissible definitions of an ambiguity vector. This class of transformations is referred to as the class of admissible ambiguity transformations and it is defined as follows (Teunissen (1995a)).

Let the integer ambiguity vector a be transformed as $z = Z^T a$. The $n \times n$ transformation matrix Z is then said to be admissible when all the entries of both Z and its inverse Z^{-1} are integers.

The above two conditions on Z together ensure that the integerness of the ambiguities does not get lost in the transformation. Thus z is an integer vector when all the entries of a are integer, and vice versa, a is an integer vector when all the entries of z are integer. One can show that the above two conditions are equivalent to stating that the integer matrix Z must be volume preserving ($\det Z = \pm 1$).

It is now of interest to know how integer bootstrapping and integer least-squares behave under admissible ambiguity transformations. We therefore consider the integer solutions as well as the corresponding success rates. Let the transformed 'float' solution and its vc-matrix be given as

$$\hat{z} = Z^T \hat{a}\ ,\ Q_{\hat{z}} = Z^T Q_{\hat{a}} Z \quad (9)$$

in which Z is an arbitrary, but admissible ambiguity transformation. One can then show that for the bootstrapped solution and success rate

$$\check{z}_B \neq Z^T \check{a}_B\ ,\ P(\check{z}_B = z) \neq P(\check{a}_B = a) \quad (10)$$

while for the integer least-squares solution and success rate

$$\check{z}_{LS} = Z^T \check{a}_{LS}\ ,\ P(\check{z}_{LS} = z) = P(\check{a}_{LS} = a) \quad (11)$$

Hence, the integer least-squares principle is invariant against Z-transformations, while bootstrapping is not. This implies that with bootstrapping one has some degrees of freedom left for improving the success rate by choosing an appropriate Z-transformation. This freedom is absent in case of integer least-squares, since integer least-squares already produces the highest possible success rate of all admissible integer estimators.

4 COMPARING TCAR, CIR AND LAMBDA

Next to the LAMBDA method for ambiguity resolution, special methods have been proposed in the context of upcoming multi-frequency systems like modernized GPS and the future Galileo. The basic idea is to maximize the wavelength of combinations of (each time) two carrier phase observations, while at the same time achieving a not too poor precision Forsell *et al.* (1997). Combinations alike are referred to as wide-lane, extra wide-lane and super wide-lane.

4.1 TCAR

Both TCAR and CIR use the geometry-free model for ambiguity resolution. For an arbitrary frequency f_α and a single epoch i, the DD phase and code observation equations of the geometry-free model read

$$\begin{array}{rcl} \phi_\alpha(i) & = & \rho(i) - \nu_\alpha I(i) + \lambda_\alpha a_\alpha \\ p_\alpha(i) & = & \rho(i) + \nu_\alpha I(i) \end{array} \quad (12)$$

where $\phi_\alpha(i)$ and $p_\alpha(i)$, in units of range, are respectively the DD phase and code observable on f_α, $\rho(i)$ is the DD form of the unknown receiver-satellite range, ν_α is the known frequency dependent coefficient, $I(i)$ is the DD form of the unknown ionospheric delay, λ_α is the known wavelength of f_α, and a_α is the unknown but time-invariant

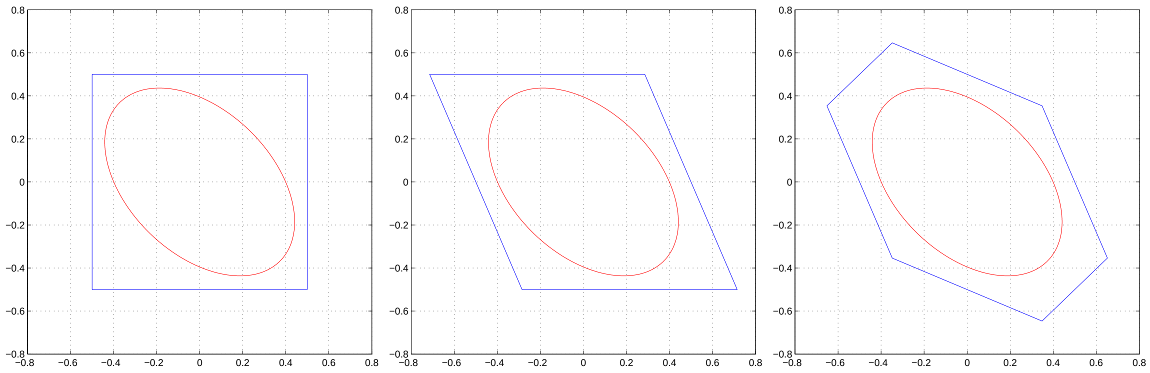

Fig. 2. Example of 2-dimensional pull-in regions for integer rounding (left), bootstrapping (middle) and integer least squares (right). The ellipse (locations all with the same probability density) represents the shape of the ambiguities' probability density function.

integer DD ambiguity of frequency f_α. For dual-frequency systems (current GPS), we have $\alpha = 1, 2$, while for triple-frequency systems, $\alpha = 1, 2, 3$ (Galileo, modernized GPS). Differential ionospheric delays are usually assumed absent in case of short baselines. For long baselines they need to be included as unknown parameters. Note that the geometry-free model is linear from the outset and that it fits the general GNSS model (1).

Three Carrier Ambiguity Resolution as described in Vollath *et al.* (1998) is based on a three frequency envisioned Galileo system as shown in table 1.

TCAR can be described as follows. One starts with the long widelane carrier phase combination of $E1$ and $E2$, with ambiguity $z_1 = a_1 - a_2$. This combination has a wavelength of 10.47 m and the ambiguity z_1 is resolved by rounding its 'float' solution to the nearest integer. With this information the widelane combination of $E1$ and $E4$, with ambiguity $z_2 = a_1 - a_3$ and a wavelength of 0.90 m, is resolved in the second step through rounding. Using the information of the first two steps, the third and last step amounts to resolving the ambiguity on any one frequency (usually $z_3 = a_1$) through rounding.

With the theory of the previous section in mind, it will be clear that TCAR is an example of geometry-free integer bootstrapping. It operates on transformed ambiguities using the pre-set Z-transformation

$$Z^T_{TCAR} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \qquad (13)$$

It is easily verified that this transformation is indeed admissible, the determinant equals $+1$.

4.2 CIR

CIR as proposed in Jung *et al.* (2000) is based on the modernized three frequency GPS system as given in table 1.

The approach taken by CIR is essentially the same as that of TCAR. One starts with the extra widelane carrier phase combination of $L2$ and $L5$, with ambiguity $z_1 = a_2 - a_3$. This combination has a wavelength of 5.86 m and the ambiguity z_1 is resolved by rounding its 'float' solution to the nearest integer. With this information the widelane combination of $L1$ and $L2$, with ambiguity $z_2 = a_1 - a_2$ and wavelength of 0.86 m, is resolved in the second step through rounding. Using the information of the first two steps, the third and last step amounts to resolving the ambiguity on any one frequency (usually $z_3 = a_3$) through rounding. Hence, also CIR is an example of geometry-free integer bootstrapping. It operates on the pre-set admissible Z-transformation

$$Z^T_{CIR} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (14)$$

Also in this case it is easily verified that this transformation is admissible, the determinant equals -1.

4.3 LAMBDA

The LAMBDA method essentially consists of an efficient implementation of integer least squares estimation, where part of the efficiency is caused by performing a decorrelating ambiguity transformation. One starts by defining the ambiguity search space

$$\Omega_a = \{a \in Z^n \mid (\hat{a} - a)^T Q_{\hat{a}}^{-1} (\hat{a} - a) \leq \chi^2\} \qquad (15)$$

with χ^2 a to be chosen positive constant. The boundary of this search space is ellipsoidal. It is centred at $\hat{a}$, its shape is governed by the vc-matrix $Q_{\hat{a}}$ and its size is determined by χ^2. In case of GNSS, the search space is usually extremely elongated, due to the high correlations between the ambiguities. Since this extreme elongation hinders the computational efficiency of the search, the search space is first transformed to a more spherical shape,

$$\Omega_z = \{z \in Z^n \mid (\hat{z} - z)^T Q_{\hat{z}}^{-1} (\hat{z} - z) \leq \chi^2\} \qquad (16)$$

	Galileo			Modernized GPS		
amb	ID	Frequency [MHz]	Wavelength [m]	ID	Frequency [MHz]	Wavelength [m]
a_1	E1	1589.742	0.1886	L1	1575.42	0.1903
a_2	E2	1561.098	0.1920	L2	1227.60	0.2442
a_3	E4	1256.244	0.2386	L5	1176.45	0.2548

Table 1. Frequencies for Galileo as given in Vollath *et al.* (1998) and for Modernized GPS as given in Jung *et al.* (2000)

using the admissible ambiguity transformation $\hat{z} = Z_\Lambda^T \hat{a}$, $Q_{\hat{z}} = Z_\Lambda^T Q_{\hat{a}} Z_\Lambda$. In order for the transformed search space to become more spherical, the Z_Λ matrix is constructed so as to decorrelate the ambiguities as much as possible.

In order for the search to be efficient, one not only would like the vc- matrix $Q_{\hat{z}}$ to be as close as possible to a diagonal matrix, but also that the search space does not contain too many integer grid points. This requires the choice of a small value for χ^2, but one that still guarantees that the search space contains at least one integer grid point. Since the bootstrapped estimator is so easy to compute and at the same time gives a good approximation to the integer least-squares estimator once properly decorrelated, the bootstrapped solution is an excellent candidate for setting the size of the ambiguity search space. Following the decorrelation step $\hat{z} = Z_\Lambda^T \hat{a}$, the LAMBDA-method therefore uses, as one of its options, the bootstrapped solution $\check{z}_B$ for setting the size of the ambiguity search space as

$$\chi^2 = (\hat{z} - \check{z}_B)^T Q_{\hat{z}}^{-1} (\hat{z} - \check{z}_B) \qquad (17)$$

In this way one can work with a very small search space and still guarantee that the actual integer least-squares solution we are after is contained in it.

Using the triangular decomposition of $Q_{\hat{z}}$, the left-hand side of the quadratic inequality in (16) is then written as a sum-of-squares:

$$\sum_{i=1}^{n} \frac{(\hat{z}_{i|I} - z_i)^2}{\sigma_{i|I}^2} \leq \chi^2 \qquad (18)$$

On the left-hand side one recognizes the conditional least-squares estimator $\hat{z}_{i|I}$, which follows when the conditioning takes place on the integers $z_1, z_2, \ldots, z_{i-1}$ and the variance of $\hat{z}_{i|I}$ in the denominator. Using the sum-of-squares structure, one can finally set up the n intervals which are used for the search. These sequential intervals are given as

$$\begin{aligned}
(\hat{z}_1 - z_1)^2 &\leq \sigma_1^2 \chi^2 \\
(\hat{z}_{2|1} - z_2)^2 &\leq \sigma_{2|1}^2 \left(\chi^2 - \frac{(\hat{z}_1 - z_1)^2}{\sigma_1^2} \right) \\
&\vdots \\
(\hat{z}_{n|N} - z_n)^2 &\leq \sigma_{n|N}^2 \left(\chi^2 - \sum_{j=1}^{n-1} \frac{(\hat{z}_{j|J} - z_j)^2}{\sigma_{j|J}^2} \right)
\end{aligned} \qquad (19)$$

Once the search has completed, one can either output the transformed integer least-squares solution $\check{z}_{LS}$ or, by using the back-transform $\check{a}_{LS} = Z_\Lambda^{-T} \check{z}_{LS}$, output the integer least-squares solution of the original ambiguities.

For more information on the LAMBDA method, we refer to e.g. Teunissen (1993), Teunissen (1995b) and de Jonge and Tiberius (1996b) or to the textbooks Hofmann-Wellenhof *et al.* (2001), Enge and Misra (2001), Strang and Borre (1997) and Teunissen and Kleusberg (1998). Practical results obtained with it can be found, for example, in Boon and Ambrosius (1997), Boon *et al.* (1997), Cox and Brading (1999), de Jonge and Tiberius (1996a), Han (1995), Jonkman (1998), Peng *et al.* (1999), Tiberius and de Jonge (1995) and Tiberius *et al.* (1997).

4.4 Conceptual comparison

The three methods differ essentially in the following aspects. First, the ambiguity resolution in case of TCAR and CIR is always based on the geometry-free model, whereas in case of the LAMBDA method ambiguity resolution is based on whatever model the user believes is suitable for his/her problem. This implies that in case of the LAMBDA method all available information in the model is actually used. This is important, since in most cases the user is finally interested in geometric information (baseline coordinates), and using the relative satellite-receiver geometry can greatly enhance the performance of ambiguity resolution.

All three methods perform a decorrelation step, although the reason to do this is different. In TCAR and CIR, the decorrelation is necessary to increase the probability of obtaining the correct result. LAMBDA, since it is based on the integer least squares estimator, will always find the most likely candidate. Here the decorrelation step is necessary to optimize on computational efficiency. In general the LAMBDA decorrelation will achieve (much) better decorrelation than the pre-defined decorrelations as used in TCAR and CIR.

The main conceptual differences between the three methods are summarized in table 2.

5 RESULTS

5.1 Geometry-free model

To judge the performance of TCAR, we will compare the TCAR success rate with the bootstrapped success rate of

Method	Mathematical Model	Ambiguity transformation	Estimator
TCAR	Geometry-free Galileo	Pre-defined, Based on frequencies	Bootstrapping, Per satellite-pair
CIR	Geometry-free Modernized GPS	Pre-defined, Based on frequencies	Bootstrapping, Per satellite-pair
LAMBDA	Any GNSS model	Determined for actual observation scenario, Based on entire model	Integer Least Squares Whole ambiguity vector

Table 2. Conceptual comparison of the three methods

the original DD ambiguities and with the bootstrapped success rate of the LAMBDA transformed ambiguities. This implies that for TCAR and CIR, the actual successrate is computed, while for LAMBDA a lower bound for this successrate is obtained. Thus in all three cases bootstrapping is used, be it that different Z-transformations are used. The results will be shown for the geometry-free short baseline case (ionosphere assumed absent) and for the geometry-free long baseline case (ionosphere assumed present, and unknown, which corresponds to baseline lengths of hundreds of kilometers, or longer). The standard deviations of the original undifferenced phase and code observations were set at $\sigma_\phi = 3mm$ and $\sigma_p = 30cm$ respectively. A similar set-up is used for the comparison with CIR. The results are shown in tables 3 and 4 respectively. Shown are the Z-transformations, the conditional standard deviations of the transformed ambiguities, the correlation coefficients and the bootstrapped success rates. In case of the original ambiguities bootstrapping starts with z_3, then z_2 and eventually z_1; the ordering — the Z^T matrix involves just a permutation — is after decreasing wavelength, see table 1.

For the short baseline case we see that in case of Galileo both TCAR and LAMBDA return a much higher success rate than the one based on the original DD ambiguities. The same holds true for CIR and LAMBDA in case of modernized GPS. But we also see that in both cases the LAMBDA ambiguities perform better than the TCAR or CIR ambiguities. The two pairs of triple-ambiguities are related as

$$z_\Lambda(Gal) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -5 & 1 \\ -4 & 1 & 0 \end{bmatrix} z_{TCAR}$$

$$z_\Lambda(GPS) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -3 & 1 \\ -5 & 1 & 0 \end{bmatrix} z_{CIR}$$

The reason for the difference between LAMBDA on the one hand and TCAR and CIR on the other hand, lies in the fact that the TCAR and CIR ambiguities are selected only on the basis of the values of their carrier frequencies, while the selection of the LAMBDA ambiguities is based on the information content of the complete ambiguity vc-matrix.

For the long baseline case two conclusions can be drawn from tables 3 and 4. First we note that the bootstrapped success rate of the LAMBDA ambiguities is identical to that of TCAR for Galileo and also identical to that of CIR for modernized GPS. At the same time there is no significant improvement of any set of transformed ambiguities when compared to the original DD ambiguities. This last result is due to the fact that no Z-transformation exists which is capable to produce a small enough value for the third conditional standard deviation when the long baseline, geometry-free model is used; model and data are simply too weak to allow for full ambiguity resolution, based on just a single epoch of data. The first result can be explained when one compares the different sets of transformed ambiguities. For long baselines the two pairs of triple-ambiguities are related as

$$z_\Lambda(Gal) = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -162 & 15 & 1 \end{bmatrix} z_{TCAR}$$

$$z_\Lambda(GPS) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 123 & -24 & 1 \end{bmatrix} z_{CIR}$$

Note that both matrices have a unit lower triangular structure. This implies that they will not alter the conditional variances and therefore also not the success rate of integer bootstrapping.

5.2 Geometry-based model

So far, results had been based on the use of the geometry-free model. It was shown that the LAMBDA method performs only marginally better than the TCAR and CIR methods in case of short baselines, and equally poor in case of long baselines. However, since most users are interested in geometry, figure 3 shows results for the geometry based model. All further assumptions are identical to the ones made before. The figure shows the performance of single-epoch ambiguity resolution over an arbitrary chosen period of 24 hours, based on the current GPS satellite constellation. For Galileo a satellite constellation as outlined in Salgado *et al.* (2001) was used, while maintaining the frequency scheme of table 1. In both cases a cutoff elevation of 15 degrees was chosen.

Figure 3 shows a serious difference between the perfor-

	Short baseline - No ionosphere			Long baseline - Ionosphere present		
	Original	TCAR	LAMBDA	Original	TCAR	LAMBDA
Z^T	$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}1&-1&0\\1&0&-1\\1&0&0\end{bmatrix}$	$\begin{bmatrix}1&-1&0\\-2&-2&5\\-3&4&-1\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}1&-1&0\\1&0&-1\\1&0&0\end{bmatrix}$	$\begin{bmatrix}1&-1&0\\-2&3&-1\\-146&162&-15\end{bmatrix}$
σ_1	1.452	0.056	0.056	14.915	0.060	0.060
$\sigma_{2\|1}$	0.044	0.298	0.155	0.362	0.340	0.340
$\sigma_{3\|2,1}$	0.039	0.151	0.290	0.046	12.246	12.246
$\rho_{1,2}$	1.000	0.640	-0.099	1.000	0.507	0.056
$\rho_{1,3}$	1.000	0.606	0.086	1.000	0.454	-0.000
$\rho_{2,3}$	1.000	0.996	-0.233	1.000	-0.065	0.001
$P(z_b = z)$	0.270	0.906	0.914	0.022	0.028	0.028

Table 3. Comparison of single epoch bootstrapping using DD, TCAR and LAMBDA ambiguities.

	Short baseline - No ionosphere			Long baseline - Ionosphere present		
	Original	CIR	LAMBDA	Original	CIR	LAMBDA
Z^T	$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&-1\\1&-1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&1&-1\\-3&1&3\\1&-6&5\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&-1\\1&-1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&1&-1\\1&-3&2\\-24&147&-122\end{bmatrix}$
σ_1	1.360	0.068	0.068	12.981	0.080	0.080
$\sigma_{2\|1}$	0.035	0.219	0.125	0.072	0.431	0.431
$\sigma_{3\|2,1}$	0.039	0.122	0.214	0.197	5.342	5.342
$\rho_{1,2}$	1.000	0.841	0.121	1.000	0.373	0.033
$\rho_{1,3}$	1.000	0.861	0.007	0.999	-0.434	-0.002
$\rho_{2,3}$	1.000	0.995	0.199	1.000	0.582	0.011
$P(z_b = z)$	0.287	0.978	0.980	0.030	0.056	0.056

Table 4. Comparison of single epoch bootstrapping using DD, CIR and LAMBDA ambiguities.

mance of the LAMBDA method on the one hand and TCAR/CIR on the other. In fact it can be said that under the current assumptions single epoch ambiguity resolution with LAMBDA is feasible, while single epoch ambiguity resolution using TCAR or CIR is not. These results can be explained as follows. In practice one will have some 5-10 satellites available simultanously. When the amount of satellite-pairs increases over just a single pair, there will be more ambiguities to be determined. This has the disadvantageous effect of decreasing the successrate, as can easily be seen from equation (8), since the success-rate is a product of probabilities all smaller then or at best equal to 1.

However when using the geometry-based model, one also has the advantageous effect of the geometry. Since this last effect is by far dominating, a method that utilizes this geometry will perform significantly better than one that does not. Since both TCAR and CIR resolve the ambiguities by means of the geometry-free model, they don't take advantage of the geometry. Instead they only suffer from the increased dimension of the ambiguity vector. LAMBDA on the other hand suffers from this increased dimension as well, but also utilizes the geometry.

5.3 Partial fixing

From the results shown in tables 4 and 3 it is clear that in case of long baselines instantaneous ambiguity resolution is not feasible. Use of the geometry gives some improvement, but certainly not enough to allow for a reliable integer solution. As an alternative of resolving the complete vector of ambiguities, one might consider resolving only a subset of the ambiguities. From the tables mentioned above, one can see that estimating only the first two linear combinations of ambiguites might be feasible, as the (conditional) standard deviations of those combinations are quite small. When applying equation (8) to those conditional standard deviations, the success-rate for TCAR/Galileo is computed as 0.8586, while the success-rate for CIR/GPS is computed as 0.7540. This indicates that reliable resolution of those combinations of ambiguities would be feasible within a couple of epochs. Note that these number are related to a single satellite pair. In order to estimate baseline coordinates, one would need to use at least 3 satellite pairs, which reduces the success-rates to 0.6330 and 0.4287 respectively (the success-rates for a single satellite-pair to the power of 3).

Figure 4 shows the success-rates for estimating the above-mentioned combinations for all the available satellite pairs using the TCAR/CIR methods (blue lines) and the LAMBDA method (red lines). The black lines in the figures

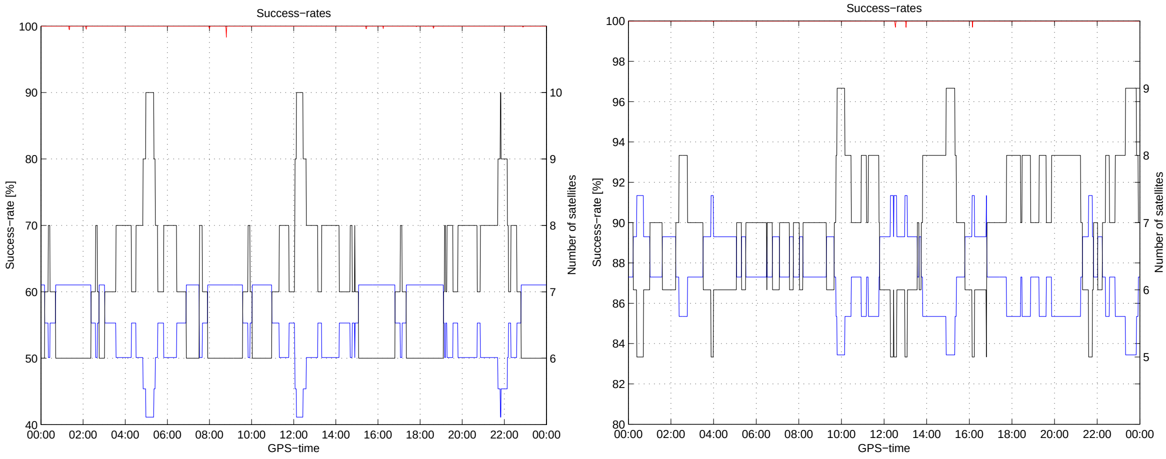

Fig. 3. Results for the geometry-based model, short baseline, cut-off elevation 15 degrees, for a period of 24 hours, at location Delft, the Netherlands. Results for LAMBDA are in red, results for TCAR (left) and CIR (right) are in blue. The number of satellites is indicated in black. Note the different scale of the Y-axis.

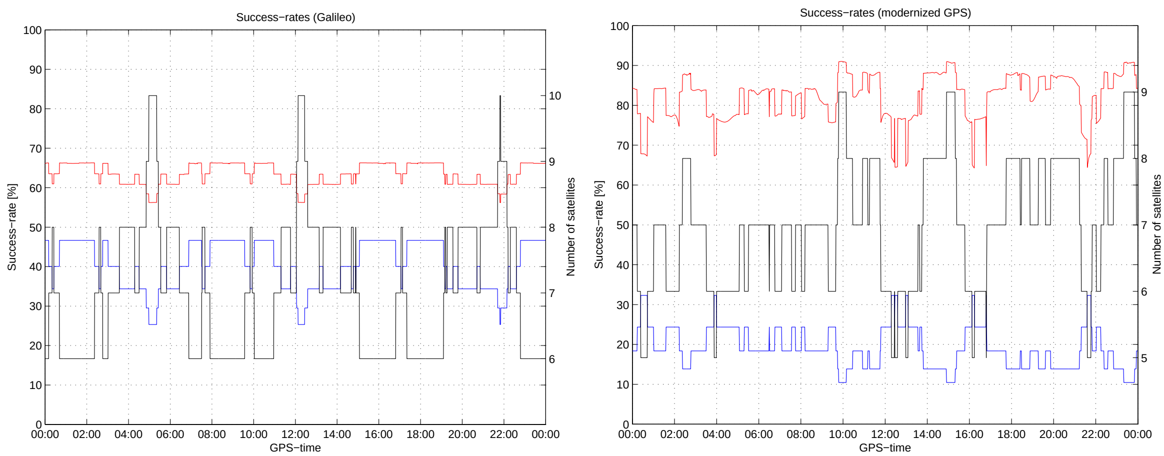

Fig. 4. Results for the geometry-based model, long baseline, partial fixing, cut-off elevation 15 degrees, for a period of 24 hours, at location Delft, the Netherlands. Results for LAMBDA are in red, results for TCAR (left) and CIR (right) are in blue. The number of satellites is indicated in black.

indicate the number of available satellites. Note that in the case of Galileo, with the frequencies taken from table 1 the use of the geometry hardly influences the success-rate. The improvement the LAMBDA method brings over TCAR, is caused by taking the correlations between the satellite pairs into account as well as by parameterization in baseline coordinates instead of in receiver-satellite distances. In case of GPS, the graph at right, the use of the geometry improves the success-rates even more clearly.

6 CONCLUSIONS

Based on the theory and analysis of the previous sections we are now in a position to formulate our conclusions. The conclusions pertain to the integer estimation principles used, the general applicability of the methods and their performance.

1. The integer estimation principles of TCAR and CIR are both examples of integer bootstrapping, whereas LAMBDA is based on the integer least squares estimator.
2. Therefore LAMBDA will always perform better or at least as good as TCAR and CIR, in a sense that it offers the highest probability of success.
3. Both TCAR and CIR make use of a pre-set admissible ambiguity transformation to improve the ambiguity success rate. Their ambiguity transformations,

Z_{TCAR} and Z_{CIR} respectively, are tuned to the specific values of the carrier frequencies of the two systems.

4. Geometry-free, integer bootstrapping based on the LAMBDA ambiguities performs better or at least as good as TCAR and CIR. In contrast to TCAR and CIR, the selection of the LAMBDA ambiguities is based on the information content of the full ambiguity vc-matrix.

5. In case of geometry-based GNSS models, an increasing number of satellites implies a better geometry (positive effect) as well as more ambiguities to resolve (negative effect). The first (positive) effect is far more pronounced. LAMBDA utilizes the improved geometry, whereas TCAR and CIR do not.

6. LAMBDA is computationally more intensive for two reasons. First, the decorrelating Z-transformation has to be derived, while for TCAR and CIR they are known on beforehand. Secondly, the integer search procedure needs to applied, *after* performing bootstrapping in order to set the size of the search ellipsoid.

7. TCAR and CIR reach their respective solutions in a pre-determined number of steps, whereas LAMBDA does not. In theory this can be a problem in fast real-time applications. In practice this has not been a problem.

8. The LAMBDA method applies to the whole suit of phase-based GNSS models one can think of, while ambiguity resolution with TCAR and CIR is restricted to the geometry-free model. The LAMBDA method applies, for instance, to single-, dual-, triple-, or even m-frequency systems, to geometry-free as well as to geometry-based models, to phase and code based systems or to phase-only systems, to GPS, modernized GPS, Galileo, or even to a future integration of GPS and Galileo. The only input the LAMBDA method requires is the 'float' solution $\hat{a}$ and its vc- matrix.

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