# Continuous-Time Singular Linear-Quadratic Control: Necessary and Sufficient Conditions for the Existence of Regular Solutions*

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**Abstract**

The purpose of this paper is to provide a full understanding of the role that the constrained generalized continuous algebraic Riccati equation plays in singular linear-quadratic (LQ) optimal control. Indeed, in spite of the vast literature on LQ problems, only recently a sufficient condition for the existence of a non-impulsive optimal control has for the first time connected this equation with the singular LQ optimal control problem. In this paper, we establish four equivalent conditions providing a complete picture that connects the singular LQ problem with the constrained generalized continuous algebraic Riccati equation and with the geometric properties of the underlying system.

**Keywords:** Riccati equations, singular LQ problem, regular solutions.

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# 1 Introduction

This paper addresses the continuous-time linear quadratic (LQ) optimal control problem when the matrix weighting the input in the cost function, traditionally denoted by $R$, is possibly singular. This problem has a long history. It has been investigated in several papers and with the use of different techniques, see [6, 13, 10, 9, 7] and the references cited therein. In particular, in the classical contributions [6] and [13] it was proved that an optimal solution of the singular LQ problem exists for all initial conditions if the class of allowable controls is extended to include distributions. In the discrete time, the solution of regular and singular finite and infinite-horizon LQ problems can be found resorting to the so-called *constrained generalized discrete algebraic Riccati equation*, see [3, 2] and also [11]. A similar generalization has been carried out for the continuous-time algebraic Riccati equation in [8], where the *constrained generalized Riccati equation* was defined in such a way that the inverse of $R$ appearing in the standard Riccati equation is replaced by its pseudo-inverse. On the other hand, until very recently this counterpart of the generalized discrete algebraic Riccati equation was only studied without any understanding of its links with the linear quadratic optimal control problem.

The recent paper [4] was the first attempt to provide a description of the role played by the constrained generalized continuous algebraic Riccati equation in singular LQ optimal control problems. Such role does not trivially follow from the analogy with the discrete case, as one can immediately realize by considering the fact that in the continuous time, whenever the optimal control involves distributions, none of the solutions of the constrained generalized Riccati equation is optimizing. In particular, in [4] it was shown that when the continuous-time constrained generalized Riccati equation possesses a symmetric solution, the corresponding LQ problem admits a *regular* (i.e. impulse-free) solution, and an optimal control can always be expressed as a state-feedback. This is just a single trait of a rich picture where necessary and sufficient conditions for the existence of regular solutions are given in terms of the algebraic and geometric structures of the underlying system. The purpose of this paper is to provide a full illustration of this picture which nicely complements the list of possible situations discussed in the pioneering work [13] (see p. 332).

**Notation.** The image and the kernel of matrix $M$ are denoted by $\operatorname{im} M$ and $\ker M$, respectively; the transpose and the Moore-Penrose pseudo-inverse of $M$ are denoted by $M^{\mathrm{T}}$ and $M^{\dagger}$, respectively. Given a system in state-space form, we denote by $\mathscr{V}^\star$ the corresponding largest output-nulling subspace, by $\mathscr{S}^\star$ the smallest input containing subspace, and by $\mathscr{R}^\star$ the largest reachability output-nulling subspace, see [12] for details.

## 1.1 Preliminaries

Let $Q, A \in \mathbb{R}^{n \times n}$, $B, S \in \mathbb{R}^{n \times m}$, $R \in \mathbb{R}^{m \times m}$. We make the following standing assumption:

$$\Pi \stackrel{\text{def}}{=} \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} = \Pi^{\mathrm{T}} \geq 0. \tag{1}$$

Thus, the *Popov matrix* $\Pi$ can be factorized in terms of two matrices $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$ as

$$\Pi = \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}. \tag{2}$$

We define $\Sigma$ to be the triple $(A, B, \Pi)$. The classic LQ optimal control problem associated to $\Sigma$ can be stated as follows.

**Problem 1** *Find a piecewise continuous control input $u(t)$, $t \geq 0$, that minimizes the performance index*

$$J_\infty(x_0, u) = \int_0^\infty \begin{bmatrix} x^{\mathrm{T}}(t) & u^{\mathrm{T}}(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{3}$$

*subject to the constraint*

$$\dot{x}(t) = A\,x(t) + B\,u(t), \qquad x(0) = x_0 \in \mathbb{R}^n. \tag{4}$$

We consider $u$ to be a solution of Problem 1 only if the corresponding value of the performance index is finite.[1]

It is well-known that when $R$ is positive definite, an optimal control exists (and is indeed unique) if and only if there exists a control input for which the performance index $J_\infty$ is finite. This is a very mild condition that admits an elegant characterization in terms of the system matrices (see Remark 1 below). If $R$ is only positive semidefinite, in general Problem 1 does not admits solutions. In fact, to guarantee existence, we need to consider a relaxed problem where the control input can contain distributions (Dirac delta distributions and its derivatives). To see this fact, consider the simple case where $n = m = 1$, $A = S = R = 0$, $Q = B = 1$. In this case, the feedback control $u_k(t) = -k\,x(t)$, $k \geq 0$, generates the performance index $J_\infty(x_0, u_k) = \frac{x_0^2}{2k}$. Clearly, for any given $x_0$, $J_\infty(x_0, u_k)$ can be made arbitrarily close to 0 by suitably choosing the constant $k$ to be sufficiently large. In this case 0 is not the minimum but only the infimum of the values of the performance index as the control input $u(t)$ varies among piecewise continuous functions. On the other hand if we are allowed to resort to distributional control input, it is easy

[1] We make this remark since, if the cost is unbounded for every control, one might alternatively say that all controls are optimal since they all lead to the same value of the performance index.

to see that the infimum is indeed a minimum as it can be attained by taking $u(t) = -x_0\delta(t)$, with $\delta(t)$ being the Dirac delta distribution.

We shall investigate the conditions under which Problem 1 admits solutions (which are, by definition, non-impulsive) in the general case where $R$ is allowed to be singular. To this end a key role will be played by the following matrix equation

$$X A + A^{\mathrm{T}} X - (S + X B) R^{\dagger} (S^{\mathrm{T}} + B^{\mathrm{T}} X) + Q = 0. \tag{5}$$

Eq. (5) is often referred to as the *generalized continuous algebraic Riccati equation* GCARE($\Sigma$), and represents a generalization of the classic continuous algebraic Riccati equation CARE($\Sigma$)

$$X A + A^{\mathrm{T}} X - (S + X B) R^{-1} (S^{\mathrm{T}} + B^{\mathrm{T}} X) + Q = 0, \tag{6}$$

arising in infinite-horizon LQ problems since in the present setting $R$ is allowed to be singular. Eq. (5) along with the additional condition

$$\ker R \subseteq \ker(S + X B), \tag{7}$$

is usually referred to as *constrained generalized continuous algebraic Riccati equation*, and is denoted by CGCARE($\Sigma$). Observe that from (1) we have $\ker R \subseteq \ker S$, which implies that (7) is equivalent to $\ker R \subseteq \ker(X B)$.

The following notation is used throughout the paper. We denote by $G \stackrel{\text{def}}{=} I_m - R^{\dagger} R$ the orthogonal projector that projects onto $\ker R$. Moreover, we consider a non-singular matrix $T = [\ T_1 \mid T_2\ ]$ where $\operatorname{im} T_1 = \operatorname{im} R$ and $\operatorname{im} T_2 = \operatorname{im} G$, and we define $B_1 \stackrel{\text{def}}{=} B T_1$ and $B_2 \stackrel{\text{def}}{=} B T_2$. Finally, to any $X = X^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ we associate the matrices

$$Q_X \stackrel{\text{def}}{=} Q + A^{\mathrm{T}} X + X A, \qquad S_X \stackrel{\text{def}}{=} S + X B, \tag{8}$$

$$K_X \stackrel{\text{def}}{=} R^{\dagger} (S^{\mathrm{T}} + B^{\mathrm{T}} X) = R^{\dagger} S_X^{\mathrm{T}}, \qquad A_X \stackrel{\text{def}}{=} A - B K_X, \tag{9}$$

$$\Pi_X \stackrel{\text{def}}{=} \begin{bmatrix} Q_X & S_X \\ S_X^{\mathrm{T}} & R \end{bmatrix}. \tag{10}$$

The CGCARE($\Sigma$) is strictly connected to the LMI

$$\Pi_X \geq 0. \tag{11}$$

Indeed, by taking the generalized Schur complement of $R$ in $\Pi_X$, is is easy to see that (11) is equivalent to the constrained generalized continuous algebraic Riccati inequality CGCARI($\Sigma$)

$$X A + A^{\mathrm{T}} X - (S + X B) R^{\dagger} (S^{\mathrm{T}} + B^{\mathrm{T}} X) + Q \geq 0, \quad \ker R \subseteq \ker(S + X B) \tag{12}$$

and the symmetric solutions of CGCARE($\Sigma$) are indeed the solutions of LMI (11) for which the rank of $\Pi_X$ is minimum.

# 2 Main result

The main result of this paper is the following theorem, whose proof will be developed in several steps in the sequel.

**Theorem 1** *The following statements are equivalent:*

***(A)*** *For every $x_0 \in \mathbb{R}^n$, Problem 1 has a solution;*

***(B)*** *There exists a symmetric and positive semidefinite solution of CGCARE(Σ);*

***(C)*** *There exists a symmetric solution of CGCARE(Σ), and for each $x_0 \in \mathbb{R}^n$, there exists $u_0(t)$ such that $J_\infty(x_0, u_0)$ is finite;*

***(D)*** *For any factorization (2), the subspaces $\mathscr{S}^\star$ and $\mathscr{R}^\star$ of the quadruple $(A, B, C, D)$ coincide, and for each initial state $x_0 \in \mathbb{R}^n$, there exists $u_0(t)$ such that $J_\infty(x_0, u_0)$ is finite.*

*If any of these conditions holds an optimal solution can be obtained by static state feedback and is therefore in $\mathscr{C}_\infty[0, \infty)$.*

**Remark 1** Existence, for each $x_0$, of a control function $u_0(t)$ such that $J_\infty(x_0, u_0)$ is finite is a very natural condition. Its testability, however, is not obvious. It has been shown in [5] that such condition is equivalent to the following neat and easily testable geometric condition:

$$\mathscr{V}^\star + \mathscr{R}(A, B) + \mathscr{X}_{\text{stab}} = \mathbb{R}^n,$$

where $\mathscr{V}^\star$ is the largest output-nulling subspace of the quadruple $(A, B, C, D)$, $\mathscr{R}(A, B)$ is the reachable subspace (i.e., the smallest $A$-invariant subspace containing the range of $B$), and $\mathscr{X}_{\text{stab}}$ is the $A$-invariant subspace corresponding to the asymptotically stable uncontrollable eigenvalues of $A$ (so that, in other words, the sum $\mathscr{R}(A, B) + \mathscr{X}_{\text{stab}}$ is the stabilizable subspace of the pair $(A, B)$).

# 3 Ancillary results and proof of main result

The main result of [4], which we now recall, establishes that when CGCARE(Σ) admits at least one symmetric solution, and the performance index can be rendered finite with a certain control function for every initial state, the corresponding LQ optimal control problem admits impulse-free controls.

**Proposition 1** *Suppose CGCARE(Σ) admits symmetric solutions, and that for every $x_0$ there exists an input $u(t) \in \mathbb{R}^m$, with $t \geq 0$, such that $J_\infty(x_0, u)$ in (3) is finite. Then:*

- *A solution $\overline{X} = \overline{X}^{\mathrm{T}} \geq 0$ of CGCARE(Σ) is obtained as the limit of the time varying matrix generated by integrating (forward in time) the matrix differential equation*

$$\dot{X}(t) = X(t)A + A^{\mathrm{T}}X(t) - (S + X(t)B)R^{\dagger}(S^{\mathrm{T}} + B^{\mathrm{T}}X(t)) + Q \tag{13}$$

  *with the zero initial condition $X(0) = 0$.*

- *The value of the optimal cost is $x_0^{\mathrm{T}}\overline{X}x_0$.*

- *$\overline{X}$ is the smallest positive semidefinite solution of CGCARE(Σ).*

- *The set of* all *optimal controls minimizing the cost in (3) can be parameterized as*

$$u(t) = -R^{\dagger}S_{\overline{X}}^{\mathrm{T}}x(t) + Gv(t), \tag{14}$$

  *where $v(t)$ is an arbitrary piecewise continuous function.*

It is easy to see that Proposition 1 proves that the implications ***(C)*** $\Rightarrow$ ***(B)*** and ***(C)*** $\Rightarrow$ ***(A)*** in Theorem 1 hold true. The following Proposition shows that ***(B)*** $\Rightarrow$ ***(C)*** as well. The idea of the proof is the same of the case $R > 0$, but it requires some additional care in dealing with the matrix products.

**Proposition 2** *If there exists a symmetric positive semidefinite solution $\overline{X} = \overline{X}^{\mathrm{T}} \geq 0$ of CGCARE(Σ), then for all initial states $x_0 \in \mathbb{R}^n$, there exists $u_0(t)$ such that $J_{\infty}(x_0, u_0)$ is finite.*

***Proof:*** Let $u_0(t) = -R^{\dagger}S_{\overline{X}}^{\mathrm{T}}x(t)$, where we recall that $S_{\overline{X}} = S + \overline{X}B$. We can write the state equation as $\dot{x}(t) = A_{\overline{X}}x(t)$, where $A_{\overline{X}} = A - BR^{\dagger}S_{\overline{X}}^{\mathrm{T}}$. This obviously implies that $x(t) = e^{A_{\overline{X}}t}x_0$. Consider the finite-horizon performance index: we have

$$\begin{aligned}
J_T(x_0, u_0) &\stackrel{\text{def}}{=} \int_0^T [\, x^{\mathrm{T}}(t) \;\; u^{\mathrm{T}}(t) \,] \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \\
&= \int_0^T x^{\mathrm{T}}(t) [\; I_n \;\; -S_{\overline{X}}R^{\dagger} \;] \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} I_n \\ -R^{\dagger}S_{\overline{X}}^{\mathrm{T}} \end{bmatrix} x(t)dt \\
&= \int_0^T x^{\mathrm{T}}(t)\big(Q - S_{\overline{X}}R^{\dagger}S_{\overline{X}}^{\mathrm{T}} + S_{\overline{X}}R^{\dagger}B^{\mathrm{T}}\overline{X} + \overline{X}BR^{\dagger}S_{\overline{X}}^{\mathrm{T}}\big)x(t)dt \\
&= \int_0^T x^{\mathrm{T}}(t)\big(-\overline{X}A - A^{\mathrm{T}}\overline{X} + S_{\overline{X}}R^{\dagger}B^{\mathrm{T}}\overline{X} + \overline{X}BR^{\dagger}S_{\overline{X}}^{\mathrm{T}}\big)x(t)dt \\
&= \int_0^T x_0^{\mathrm{T}} e^{A_{\overline{X}}^{\mathrm{T}}t}\big(-\overline{X}A_{\overline{X}} - A_{\overline{X}}^{\mathrm{T}}\overline{X}\big)e^{A_{\overline{X}}t}x_0\,dt \\
&= \int_0^T x_0^{\mathrm{T}} \frac{d}{dt}\big(-e^{A_{\overline{X}}^{\mathrm{T}}t}\overline{X}e^{A_{\overline{X}}t}\big)x_0\,dt \\
&= x_0^{\mathrm{T}}\big(\overline{X} - e^{A_{\overline{X}}^{\mathrm{T}}T}\overline{X}e^{A_{\overline{X}}T}\big)x_0 \leq x_0^{\mathrm{T}}\overline{X}x_0.
\end{aligned}$$

Hence, $J_T(x_0,u_0)$ is bounded; moreover, it is clearly a nondecreasing function of $T$. Thus the limit for $T$ going to infinity of $J_T(x_0,u_0)$ exists and is also bounded from above by the same bound. ■

The classical papers on singular LQ optimal control [6, 13] make the strong assumption of stabilizability of the pair $(A,B)$, even when the problem is formulated without a stability constraint on the state trajectory, just to the end of ensuring the convergence of the integral in the cost function. We want to remove this conservative assumption, and only ask for the very weak requirement that there exists a control function that renders the value of the cost function finite. The following classical result will be of key importance to accomplish this task. For the proof of such a result we refer the reader to [12, Lemma 10.11] (see also [12, Theorem 10.9]) where, however, it is assumed that $S=0$ and $R=I$. It is immediate to generalize this proof to our case, by considering a preliminary state feedback $u(t)=-R^{-1}S^{\mathrm{T}}x(t)+R^{-1}w(t)$, where $w(t)$ is an auxiliary input. This preliminary transformation normalizes $R$ to the identity and reduces $S$ to zero.

**Lemma 1** *Consider a regular LQ problem, i.e., with $R=R^{\mathrm{T}}>0$. If for every $x_0\in\mathbb{R}^n$ there exists a control function $u(t)\in\mathbb{R}^m$, with $t\geq 0$, such that $J_\infty(x_0,u)$ is finite, then there exist solutions $X=X^{\mathrm{T}}\geq 0$ of CARE(Σ). Among such solutions there is a smallest one $\overline{X}$ and the optimal control is given by $u^*(t)=-R^{-1}(S^{\mathrm{T}}+B^{\mathrm{T}}\overline{X})x(t)$.*

As already observed, Proposition 1 shows that the existence of symmetric positive semidefinite solutions of CGCARE(Σ) guarantees that the associated LQ optimal control problem admits an impulse-free solution.

For the converse implication we need a preliminary lemma.

**Lemma 2** *Assume that for every $x_0\in\mathbb{R}^n$, Problem 1 admits a solution. Then, there exists an optimal control $u^*$ that can be written as a static state feedback*

$$u^*(t)=-Kx(t). \tag{15}$$

***Proof:*** To show this result we invoke [13, Theorem 2]. There are, however, two delicate issues. First, [13, Theorem 2] was proved under the assumption of stabilizability of the pair $(A,B)$. We observe, however, that this assumption was only introduced to the end of exploiting [13, Proposition 10], dealing with the regular case, as taken from [6, Theorem 6.1]. Lemma 1 above generalizes [13, Proposition 10] by just requiring the weaker assumption that the performance index $J_\infty(x_0,u)$ can be rendered finite from any initial condition $x_0$ with a suitable control function $u(t)$, in place of the stabilizability of the pair $(A,B)$. Therefore, the proof of [13, Theorem 2] can be carried out verbatim with just the assumption of the existence of a control that renders $J_\infty(x_0,u)$ finite for any $x_0\in\mathbb{R}^n$. The second delicate point (that we initially missed and that was pointed out to us by an anonymous reviewer) is that [13, Theorem 2] only provides an optimal solution that can be written as the sum of a part generated by static state feedback and an impulsive part and this does not rule out the possibility that the only optimal regular solutions are not of the form of static state feedback. To overcome this difficulty, we need to enter in the proof of [13, Theorem 2]. The key idea of that proof is to consider a part of the state — $x_5$ in the notation of [13] — to be unconstrained (so that it can be considered as an auxiliary control input). With this relaxation the problem becomes regular so that a unique optimal solution exists. Therefore, equations (9) and (10) in [13], need to be satisfied by *any* optimal solution. If an optimal control exists that does not contain impulses, equation (10) of [13] implies that $x_5$ cannot be present. Therefore, it becomes apparent that equation (9) of [13] together with $u_2'^{*}=0$ is an optimal solution in the form of state feedback. ■

**Proposition 3** *Assume that for every $x_0 \in \mathbb{R}^n$, Problem 1 admits a solution. Then, CGCARE($\Sigma$) admits a symmetric positive semidefinite solution.*

***Proof:*** In view of Lemma 2, we can assume that an optimal control of the form (15) exists. We re-write (3) as

$$J_\infty(x_0,u) = \int_0^\infty y^{\mathrm{T}}(t)\, y(t)\, dt, \tag{16}$$

where $y(t) = C\,x(t) + D\,u(t)$ can be considered as a fictitious output function obtained by factorizing the Popov $\Pi$ matrix as in (2). The closed-loop system that corresponds to the application of the optimal control (15) is

$$\begin{cases} \dot{x}(t) \;=\; (A-BK)\,x(t) \\ y(t) \;=\; (C-DK)\,x(t) \end{cases} \tag{17}$$

Let $A_K \stackrel{\text{def}}{=} A-BK$ and $C_K \stackrel{\text{def}}{=} C-DK$. The optimal state is $x(t)=e^{A_K t}x_0$, and the corresponding output is $y(t)=C_K\,e^{A_K t}x_0$. Thus, the optimal cost is given by

$$J_\infty(x_0,u^*) = x_0^{\mathrm{T}} \left[\int_0^\infty e^{A_K^{\mathrm{T}} t}\, C_K^{\mathrm{T}} C_K\, e^{A_K t}\, dt\right] x_0.$$

Let $r$ be the rank of $R$. Consider a basis of the input space such that $D = [\; D_1 \;\; 0 \;]$ and $B = [\; B_1 \;\; B_2 \;]$, where $D_1$ is of full column-rank $r$. In this basis, we have $R = \left[\begin{smallmatrix} R_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $S = [\; S_1 \;\; 0 \;]$, where $R_1 \in \mathbb{R}^{r\times r}$ is invertible and $S_1$ has $r$ columns. Let us now consider $x_0 \in \operatorname{im} B_2$. Using a control $u^\circ = \left[\begin{smallmatrix} 0_r \\ u_2^\circ \end{smallmatrix}\right]$ such that $u_2^\circ(t)$ is allowed to contain impulses (i.e., Dirac deltas and its derivatives in the distributional sense), the state can be instantaneously driven to the origin, i.e., $x(0^+)=0$, and $J_\infty(x_0,u^*)=0$ because in this basis the second block of components of the control law are not weighted in the performance index. Thus, $\operatorname{im} B_2 \subseteq \ker(C_K\,e^{A_K t})$, so that

$$C_K\,e^{A_K t} B_2 = 0 \quad \forall t \ge 0, \tag{18}$$

which means that the transfer function $C_K(sI_n - A_K)^{-1}B_2$ is zero. Let $x_0 \in \mathbb{R}^n$, and $u^*$ be a corresponding optimal control. Let $u^*$ be partitioned as $u^*(t) = \begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix}$, conformably with the decomposition of the input space. Then, given any $\delta u_2(t)$, we can define the new input $\tilde{u}^*(t) \stackrel{\text{def}}{=} \begin{bmatrix} u_1^*(t) \\ u_2^*(t)+\delta u_2(t) \end{bmatrix}$. Thus, (18) guarantees that $y_{x_0,u^*}(t) = y_{x_0,\tilde{u}^*}(t)$, where $y_{x_0,u^*}(t)$ is the output that corresponds to $x_0$ and $u^*$ while $y_{x_0,\tilde{u}^*}(t)$ is the one that corresponds to $x_0$ and $\tilde{u}^*$, this in turn implies that $J^\star \stackrel{\text{def}}{=} J(x_0,u^*) = J(x_0,\tilde{u}^*)$. Hence, the (regular) LQ problem for the quadruple $(A,B_1,C,D_1)$, i.e., the one consisting of the minimization of the performance index

$$\hat{J}(x_0,u_1) \stackrel{\text{def}}{=} \int_0^\infty \begin{bmatrix} x^{\mathrm{T}}(t) & u_1^{\mathrm{T}}(t) \end{bmatrix} \begin{bmatrix} Q & S_1 \\ S_1^{\mathrm{T}} & R_1 \end{bmatrix} \begin{bmatrix} x(t) \\ u_1(t) \end{bmatrix} dt$$

subject to the constraint $\dot{x}(t) = A\,x(t) + B_1\,u_1(t)$ and $x(0) = x_0$, admits solutions for all $x_0$, and the corresponding optimal cost coincides with the optimal cost of the original LQ problem, which is $\hat{J}(x_0,u_1^*) = J^\star$. On the other hand, as already observed, since $R_1 = D_1^{\mathrm{T}}D_1$ is positive definite, this LQ problem for the quadruple $(A,B_1,C,D_1)$ is regular. The fact that it admits solutions for all $x_0$ implies that the corresponding algebraic Riccati equation

$$X\,A + A^{\mathrm{T}}X - (C^{\mathrm{T}}D_1 + X\,B_1)(D_1^{\mathrm{T}}D_1)^{-1}(D_1^{\mathrm{T}}C + B_1^{\mathrm{T}}X) + C^{\mathrm{T}}C = 0 \tag{19}$$

admits a solution $\overline{X} = \overline{X}^{\mathrm{T}} \geq 0$, and $J^\star = x_0^{\mathrm{T}}\,\overline{X}\,x_0$. Thus,

$$\overline{X} = \int_0^\infty e^{A_K^{\mathrm{T}}t}\,C_K^{\mathrm{T}}C_K\,e^{A_K t}\,dt. \tag{20}$$

We can re-write (19) in the form

$$X\,A + A^{\mathrm{T}}X - \begin{bmatrix} C^{\mathrm{T}}D_1 + X\,B_1 & X\,B_2 \end{bmatrix} \begin{bmatrix} D_1^{\mathrm{T}}D_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_1^{\mathrm{T}}C + B_1^{\mathrm{T}}X \\ B_2^{\mathrm{T}}X \end{bmatrix} + C^{\mathrm{T}}C = 0,$$

which is exactly the original GCARE(Σ)

$$X\,A + A^{\mathrm{T}}X - (C^{\mathrm{T}}D + X\,B)\,(D^{\mathrm{T}}D)^{\dagger}\,(D^{\mathrm{T}}C + B^{\mathrm{T}}X) + C^{\mathrm{T}}C = 0.$$

Thus, $\overline{X} = \overline{X}^{\mathrm{T}} \geq 0$ is a solution of GCARE(Σ). Moreover, from (18) we have $\mathrm{im}\,B_2 \subseteq \ker(C_K\,e^{A_K t})$ for all $t \geq 0$, which, together with (20), yields $\mathrm{im}\,B_2 \subseteq \ker\overline{X}$. It is easy to see that this means that $\ker R \subseteq \ker(S + \overline{X}\,B)$. Indeed, in the chosen basis this subspace inclusion reads as

$$\begin{aligned} \mathrm{im}\begin{bmatrix} 0 \\ I \end{bmatrix} &= \ker\begin{bmatrix} D_1^{\mathrm{T}}D_1 & 0 \\ 0 & 0 \end{bmatrix} \subseteq \ker\begin{bmatrix} C\,D_1 + \overline{X}\,B_1 & X\,B_2 \end{bmatrix} \\ &= \ker\begin{bmatrix} C\,D_1 + \overline{X}\,B_1 & 0 \end{bmatrix}, \end{aligned}$$

which is certainly satisfied. Thus, $\overline{X}$ is also a symmetric and positive semidefinite solution of CGCARE(Σ). ■

Notice that the previous Proposition proves the implication ***(A)*** $\Rightarrow$ ***(B)*** of Theorem 1. As a byproduct, in the so-called *cheap* case, i.e. when $R=0$, we have the following

**Corollary 1** *Let $R=0$. If Problem 1 admits a solution for any initial condition $x_0$ then the optimal cost is zero: $J^\star(x_0)=0$ for each $x_0 \in \mathbb{R}^n$.*

## 3.1 Geometric conditions

So far, we have proved that the statements ***(A)***, ***(B)*** and ***(C)*** in Theorem 1 are equivalent. In this section, we focus our attention on condition ***(D)*** of the same theorem, and we show that it is also equivalent to the other three conditions.

Consider the quadruple $(A,B,C,D)$, where $C$ and $D$ are matrices of suitable sizes such that (2) holds.

**Proposition 4** *Let CGCARE($\Sigma$) admit a solution $X=X^{\mathrm{T}}$. Then, $\mathscr{S}^\star=\mathscr{R}^\star$.*

***Proof:*** Let $X=X^{\mathrm{T}}$ be a solution of CGCARE($\Sigma$). Observe also that CGCARE($\Sigma$) can be re-written as

$$\left\{\begin{array}{l} X A_0 + A_0^{\mathrm{T}} X - X B R^\dagger B^{\mathrm{T}} X + Q_0 = 0 \\ \ker R \subseteq \ker(X B) \end{array}\right. \tag{21}$$

where $A_0 \stackrel{\text{def}}{=} A - BR^\dagger S^{\mathrm{T}}$ and $Q_0 \stackrel{\text{def}}{=} Q - SR^\dagger S^{\mathrm{T}}$. Recall that $G = I_m - R^\dagger R$, so that $B_2 = BG$, and (21) becomes

$$\left\{\begin{array}{l} X A_0 + A_0^{\mathrm{T}} X - X B R^\dagger B^{\mathrm{T}} X + Q_0 = 0 \\ X B G = 0 \end{array}\right. \tag{22}$$

It is easy to see that $\ker X \subseteq \ker Q_0$. Indeed, by multiplying the first of (22) on the left by $\xi^{\mathrm{T}}$ and on the right by $\xi$, where $\xi \in \ker X$, we get $\xi^{\mathrm{T}} Q_0 \xi = 0$. However, $Q_0$ is positive semidefinite, being the generalized Schur complement of $Q$ in $\Pi$. Hence, $Q_0 \xi = 0$, which implies $\ker X \subseteq \ker Q_0$. Since $XBG=0$, we get also $Q_0 BG = 0$. By post-multiplying the first of (22) by a vector $\xi \in \ker X$ we find $X A_0 \xi = 0$, which says that $\ker X$ is $A_0$-invariant. This means that $\ker X$ is an $A_0$-invariant subspace containing the image of $BG$. Then, the reachable subspace of the pair $(A_0, BG)$, denoted by $\mathscr{R}(A_0,BG)$, which is the smallest $A_0$-invariant subspace containing the image of $BG$, is contained in $\ker X$, i.e., $\mathscr{R}(A_0,BG) \subseteq \ker X$. Therefore also $\mathscr{R}(A_0,BG) \subseteq \ker Q_0$. Notice that $Q_0$ can be written as $C_0^{\mathrm{T}} C_0$, where $C_0 \stackrel{\text{def}}{=} C - DR^\dagger S^{\mathrm{T}}$. Indeed,

$$\begin{aligned} C_0^{\mathrm{T}} C_0 &= C^{\mathrm{T}} C - C^{\mathrm{T}} D R^\dagger S^{\mathrm{T}} - S R^\dagger D^{\mathrm{T}} C + S R^\dagger D^{\mathrm{T}} D R^\dagger S^{\mathrm{T}} \\ &= Q - SR^\dagger S - SR^\dagger S^{\mathrm{T}} + SR^\dagger S^{\mathrm{T}} = Q_0. \end{aligned}$$

Consider the two quadruples $(A,B,C,D)$ and $(A_0,B,C_0,D)$. We observe that the second is obtained directly from the first by applying the feedback input $u(t) = -R^\dagger S x(t) + v(t)$. We denote by $\mathscr{V}^\star$, $\mathscr{R}^\star$ the largest output-nulling and reachability subspace of $(A,B,C,D)$, and by $\mathscr{S}^\star$ the smallest input-containing subspace of $(A,B,C,D)$. Likewise, we denote by $\mathscr{V}_0^\star$, $\mathscr{R}_0^\star$, $\mathscr{S}_0^\star$ the same subspaces relative to the quadruple $(A_0,B,C_0,D)$. Thus, $\mathscr{V}^\star=\mathscr{V}_0^\star$, $\mathscr{R}^\star=\mathscr{R}_0^\star$, and $\mathscr{S}^\star=\mathscr{S}_0^\star$. The first two identities are obvious, since output-nulling subspaces can be made invariant under state-feedback transformations and reachability is invariant under the same transformation. The third follows from [12, Theorem 8.17]. There holds $\mathscr{R}^\star=\mathscr{R}(A_0,BG)$. Indeed, consider a state $x_1\in\mathscr{R}(A_0,BG)$. There exists a control function $u$ driving the state from the origin to $x_1$, and we show that this control keeps the output at zero. Since $\mathrm{im}(BG)=B\ker D$, such control can be chosen to satisfy $Du(t)=0$ for all $t\geq 0$. Moreover, as we have already seen, from $Q_0=C_0^{\mathrm{T}}C_0$ and $\mathscr{R}(A,BG)=\mathscr{R}(A_0,BG)$ we have $C_0\mathscr{R}(A_0,BG)=0$ since $\mathscr{R}(A,BG)$ lies in $\ker Q_0$. Therefore, the output is identically zero. This implies that $\mathscr{R}(A_0,BG)\subseteq\mathscr{R}^\star$. However, the reachability subspace of $(A_0,B,C_0,D)$ cannot be greater than $\mathscr{R}(A_0,BG)$, since $D^{\mathrm{T}}C_0=D^{\mathrm{T}}(I_m-D(D^{\mathrm{T}}D)^\dagger D^{\mathrm{T}})C=0$. Therefore, such control must necessarily render the output non-zero. The same argument can be used to prove that $\mathscr{S}^\star=\mathscr{R}(A_0,BG)$, where distributions can also be used in the allowed control, since $\mathscr{R}(A,BG)$ represents also the set of states that are reachable from the origin using distributions in the control law [12, p. 183]. Hence, $\mathscr{S}^\star=\mathscr{R}^\star$. ∎

**Remark 2** Proposition 4 proves a stronger result than the implication of ***(C)*** $\Rightarrow$ ***(D)*** in Theorem 1. On the other hand, it is easy to see that the converse of Proposition 4 does not hold in general. Indeed, consider an LQ problem where $A=\begin{bmatrix}0&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}0\\1\end{bmatrix}$, $Q=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, and $S=R=0$, so that $C=[\,1\quad 0\,]$ and $D=0$. In this case, it is found that $\mathscr{V}^\star=\mathscr{S}^\star=\mathscr{R}^\star=\mathrm{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$. Moreover, the CGCARE(Σ) reduces to the Lyapunov equation $XA+A^{\mathrm{T}}X+Q=0$. Partitioning $X$ as $X=\begin{bmatrix}x_1&x_2\\x_2&x_3\end{bmatrix}$, the Lyapunov equation becomes $\begin{bmatrix}1&x_2\\x_2&2x_3\end{bmatrix}=0$, which clearly does not admit solutions. However, in this example the state dynamics are $\dot{x}_1(t)=0$ and $\dot{x}_2(t)=x_2(t)+u(t)$ and the performance index is $J_\infty(x_0,u)=\int_0^\infty x_1^2(t)\,dt$, which is not finite if $x_1(0)\neq 0$.

The following result shows that ***(D)*** $\Rightarrow$ ***(A)***, completing the proof of Theorem 1.

**Proposition 5** *Let $\mathscr{S}^\star=\mathscr{R}^\star$, and assume that for every initial condition $x_0$ there exists a control $u$ such that $J_\infty(x_0,u)$ is finite. Then, there exists a non-impulsive optimal control.*

***Proof:*** Let $\mathscr{S}^\star=\mathscr{R}^\star$. Consider the decomposition in [13, p. 328]. If $\mathscr{S}^\star=\mathscr{R}^\star$, the fourth and the fifth block components of the state disappear, and the system dynamics reduce to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ 0 & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{12} \\ B_{13} \end{bmatrix} u'_1(t) + \begin{bmatrix} 0 \\ 0 \\ B_{23} \end{bmatrix} u'_2(t)$$

$$y_1(t) = u'_1(t)$$

$$y_2(t) = [\, C_{21} \;\; 0 \;\; 0 \,] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

In view of [13, Theorem 2], the only part of the state where there may be distributions in the optimal control is the third. On the other hand, the third block of coordinates of this basis span $\mathscr{R}^\star$. This implies that $x_3$ is arbitrary, in the sense that it is not penalized in the performance index. Thus, an optimal control such that there are distributions in $x_3$ continues to be optimal even when such distributions are removed. Therefore, the optimal control can be rendered regular. ■

**Remark 3** For a more intuitive understanding of Condition ***(D)***, we observe that under this condition the linear system described by the quadruple $(A,B,C,D)$ does not have zeros at infinity. To see this fact, we refer to [1, Theorem 4] where the orders of the zeros at infinity are shown to be related to the spaces $\mathscr{S}^i \subseteq \mathscr{S}^\star$, $i = 1,2,\ldots$ (that form an increasing sequence converging to $\mathscr{S}^\star$ in a finite number of steps). Under condition ***(D)***, $\mathscr{S}^i \subseteq \mathscr{S}^\star = \mathscr{R}^\star \subseteq \mathscr{V}^\star$, so that an immediate consequence of [1, Theorem 4] is that the linear system described by the quadruple $(A,B,C,D)$ does not have zeros at infinity.

# 4 Concluding remarks

In this paper, a full picture has been drawn illustrating the relationship that exists between the solvability of the so-called constrained generalized Riccati equation and the existence of non-impulsive optimal controls of the associated infinite-horizon LQ problem. This link has been examined both from an algebraic and a geometric angle. Now that this relationship has been clarified and explained, an important direction of future research aims at obtaining a full characterization of the set of solutions of the CGCARE that parallels the discrete time counterpart in [2, 3].

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