Output Stabilization of Boundary-controlled Parabolic PDEs via Gradient-based Dynamic Optimization*

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***Abstract*—This paper proposes a new control synthesis approach for the stabilization of boundary-controlled parabolic partial differential equations (PDEs). In the proposed approach, the optimal boundary control is expressed in integral state feedback form with quadratic kernel function, where the quadratic's coefficients are decision variables to be optimized. We introduce a system cost functional to penalize both state and kernel magnitude, and then derive the cost functional's gradient in terms of the solution of an auxiliary "costate" PDE. On this basis, the output stabilization problem can be solved using gradient-based optimization techniques such as sequential quadratic programming. The resulting optimal boundary control is guaranteed to yield closed-loop stability under mild conditions. The primary advantage of our new approach is that the costate PDE is in standard form and can be solved easily using the finite difference method. In contrast, the traditional control synthesis approaches for boundary-controlled parabolic PDEs (i.e., the LQ control and backstepping approaches) require solving non-standard Riccati-type and Klein-Gorden-type PDEs.**

I. INTRODUCTION

Parabolic partial differential equation (PDE) systems are an important type of distributed parameter system (DPS) describing a wide range of natural phenomena, including diffusion, heat transfer, and fusion plasma transport. Over the past few decades, control theory for the parabolic DPS has developed into a mature research topic at the interface of engineering and applied mathematics [1], [2], [10].

The linear quadratic (LQ) control framework is well-defined in infinite dimensional function spaces to deal with the parabolic DPS (e.g., [1], [2]). However, the LQ control framework requires solving Riccati-type differential equations, which are nonlinear parabolic PDEs of dimension one greater than the original parabolic PDE system. For example, to generate an optimal feedback controller for a scalar heat equation, a Riccati PDE defined over a rectangular domain must be solved [11]. Hence, the LQ approach does not actually solve the controller synthesis problem directly, but instead converts it into another problem (i.e., solve a Riccati-type PDE) that is still extremely difficult to solve from a computational point of view.

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One of the major advances in PDE control in recent years has been the so-called infinite dimensional Voltera integral feedback, or the backstepping method (e.g., [5], [9]). Instead of Riccati-type PDEs, the backstepping method requires solving the so-called kernel equations—linear Klein-Gorden-type PDEs for which the successive approach can be used to obtain explicit solutions. This method was originally developed for the stabilization of one dimensional parabolic DPS and then extended to fluid flows [16], [19], magnetohydrodynamic flows [17], and elastic vibration [4]. In addition, the backstepping method can also be applied to achieve full state feedback stabilization and state estimation of PDE-ODE cascade systems [13].

In this paper, we propose a new framework for control synthesis for boundary-controlled parabolic DPS. This new framework does not require solving Riccati-type or Klein-Gorden-type PDEs. Instead, it requires solving a so-called "costate" PDE, which is much easier to solve from a computational viewpoint. In fact, many numerical software packages, such as Comsol Multiphysics and MATLAB PDE ToolBox, can be used to generate numerical solutions for the costate PDE. The Riccati PDEs, on the other hand, are usually not in standard form and thus cannot be solved using off-the-shelf software packages. The approach proposed in this paper can be viewed as an extension of optimization-based PID tuning ideas (see [3], [6], [14], [18]) to infinite dimensional systems.

II. PROBLEM FORMULATION

A. Feedback Kernel Optimization

We consider the following parabolic PDE system:

$$\begin{cases} y_t(x,t) = y_{xx}(x,t) + cy(x,t), & \text{(1a)} \\ y(0,t) = 0, & \text{(1b)} \\ y(1,t) = u(t), & \text{(1c)} \\ y(x,0) = y_0(x), & \text{(1d)} \end{cases}$$

where $c > 0$ is a given constant and $u(t)$ is a boundary control. It is well known that the uncontrolled version of system (1) is unstable when the constant c is sufficiently large [5]. According to the LQ control [11] and backstepping synthesis approaches [5], the optimal stabilizing control law takes the following feedback form:

$$u(t) = \int_0^1 \mathscr{K}(1,\xi) y(\xi,t) d\xi, \tag{2}$$

where the feedback kernel $\mathscr{K}(1,\xi)$ is obtained by solving either a Riccati-type or a Klein-Gorden-type PDE. By introducing the new notation $k(\xi) = \mathscr{K}(1,\xi)$, we can write the

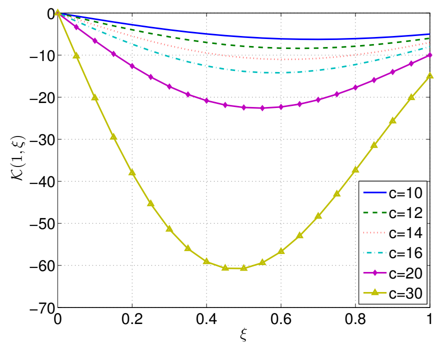

Fig. 1. The feedback kernel (4) for various values of c.

feedback control policy (2) in the following form:

$$u(t) = \int_0^1 k(\xi)y(\xi,t)d\xi.$$

The corresponding closed-loop system is

$$\begin{cases} y_t(x,t) = y_{xx}(x,t) + cy(x,t), & \text{(3a)} \\ y(x,0) = y_0(x), & \text{(3b)} \\ y(0,t) = 0, & \text{(3c)} \\ y(1,t) = \displaystyle\int_0^1 k(\xi)y(\xi,t)d\xi. & \text{(3d)} \end{cases}$$

In reference [5], the backstepping method is used to express the optimal feedback kernel as follows:

$$\mathcal{K}(1,\xi) = -c\xi \frac{I_1(\sqrt{c(1-\xi^2)})}{\sqrt{c(1-\xi^2)}}, \tag{4}$$

where I_1 is the first-order modified Bessel function given by

$$I_1(\omega) = \sum_{n=0}^{\infty} \frac{\omega^{2n+1}}{2^{2n+1}n!(n+1)!}.$$

The feedback kernel (4) is plotted in Figure 1 for different values of c. Note that its shape is similar to a quadratic function. Note also that $\mathcal{K}(1,\xi) = 0$ when $\xi = 0$. Accordingly, motivated by the quadratic behavior exhibited in Fig. 1, we express $k(\xi)$ in the following parameterized form:

$$k(\xi;\Theta) = \theta_1\xi + \theta_2\xi^2, \tag{5}$$

where $\Theta = (\theta_1, \theta_2)^\top$ is a parameter vector to be optimized.

Moreover, we assume that the parameters must satisfy the following bound constraints:

$$a_1 \leq \theta_1 \leq b_1, \quad a_2 \leq \theta_2 \leq b_2, \tag{6}$$

where a_1, a_2, b_1 and b_2 are given bounds.

Let $y(x,t;\Theta)$ denote the solution of the closed-loop system (3) with the parameterized kernel (5). The results in [15] ensure that such a solution exists and is unique. Our goal is to stabilize the closed-loop system with minimal energy input. Accordingly, we consider the following cost functional:

$$g_0(\Theta) = \frac{1}{2}\int_0^T \int_0^1 y^2(x,t;\Theta)dxdt + \frac{1}{2}\int_0^1 k^2(x;\Theta)dx. \tag{7}$$

We now state our kernel optimization problem formally as follows.

Problem 2.1: Given the PDE system (3) with the parameterized kernel (5), find an optimal parameter vector $\Theta = (\theta_1, \theta_2)^\top$ such that the cost functional (7) is minimized subject to the bound constraints (6).

B. Closed-Loop Stability

Since (7) is a finite-time cost functional, there is no guarantee that the optimized kernel (5) generated by the solution of Problem 2.1 stabilizes the closed-loop system (3) as $t \to \infty$. Nevertheless, we now show that, by analyzing the solution structure of (3), additional constraints can be added to Problem 2.1 to ensure closed-loop stability.

Using the separation of variables approach, we decompose $y(x,t)$ as follows:

$$y(x,t) = \mathscr{X}(x)\mathscr{T}(t). \tag{8}$$

Substituting (8) into (3a), we obtain

$$\mathscr{X}(x)\dot{\mathscr{T}}(t) = \mathscr{X}''(x)\mathscr{T}(t) + c\mathscr{X}(x)\mathscr{T}(t), \tag{9}$$

where

$$\dot{\mathscr{T}}(t) = \frac{d\mathscr{T}(t)}{dt}, \quad \mathscr{X}''(x) = \frac{d^2\mathscr{X}(x)}{dx^2}.$$

Furthermore, from the boundary conditions (3c) and (3d),

$$\mathscr{X}(0)\mathscr{T}(t) = 0,$$
$$\mathscr{X}(1)\mathscr{T}(t) = \int_0^1 k(\xi;\Theta)\mathscr{X}(\xi)\mathscr{T}(t)d\xi.$$

Thus, we immediately obtain

$$\mathscr{X}(0) = 0, \tag{10}$$

$$\mathscr{X}(1) = \int_0^1 k(\xi;\Theta)\mathscr{X}(\xi)d\xi. \tag{11}$$

Rearranging (9) gives

$$\frac{\mathscr{X}''(x) + c\mathscr{X}(x)}{\mathscr{X}(x)} = \frac{\dot{\mathscr{T}}(t)}{\mathscr{T}(t)} = \sigma, \tag{12}$$

where σ is a constant called the eigenvalue. Clearly,

$$\mathscr{T}(t) = T_0 \mathrm{e}^{\sigma t}, \tag{13}$$

where $T_0 = \mathscr{T}(0)$ is a constant to be determined.

To solve for $\mathscr{X}(x)$, we must consider three cases: (i) $c < \sigma$; (ii) $c = \sigma$; (iii) $c > \sigma$. In cases (i) and (ii), the general solutions of (12) are, respectively,

$$\mathscr{X}(x) = X_0 \mathrm{e}^{\sqrt{\sigma - c}x} + X_1 \mathrm{e}^{-\sqrt{\sigma - c}x},$$

and

$$\mathscr{X}(x) = X_0 + X_1 x,$$

where X_0 and X_1 are constants to be determined from the boundary conditions (10) and (11). Then the corresponding solutions of (3) are

$$y(x,t) = X_0 T_0 \mathrm{e}^{\sqrt{\sigma - c}x + \sigma t} + X_1 T_0 \mathrm{e}^{-\sqrt{\sigma - c}x + \sigma t},$$

and

$$y(x,t) = X_0 T_0 \mathrm{e}^{\sigma t} + X_1 T_0 x \mathrm{e}^{\sigma t}.$$

These solutions are clearly unstable because $0 < c \le \sigma$. Thus, the parameters θ_1 and θ_2 should be chosen so that the unique solution of (3) satisfies case (iii) instead of cases (i) and (ii).

In case (iii), the general solution of (12) is

$$\mathscr{X}(x) = X_0 \cos(\sqrt{c-\sigma}x) + X_1 \sin(\sqrt{c-\sigma}x), \tag{14}$$

where X_0 and X_1 are constants to be determined from the boundary conditions (10) and (11). Substituting (14) into (10), we obtain

$$\mathscr{X}(0) = X_0 = 0.$$

Hence,

$$\mathscr{X}(x) = X_1 \sin(\sqrt{c-\sigma}x). \tag{15}$$

To simplify the notation, we introduce a new variable $\alpha = \sqrt{c-\sigma}$. Substituting (15) into condition (11), we have

$$X_1 \sin\alpha = X_1 \int_0^1 \theta_1 \xi \sin(\alpha\xi) d\xi + X_1 \int_0^1 \theta_2 \xi^2 \sin(\alpha\xi) d\xi,$$

and thus

$$\sin\alpha = \int_0^1 \theta_1 \xi \sin(\alpha\xi) d\xi + \int_0^1 \theta_2 \xi^2 \sin(\alpha\xi) d\xi. \tag{16}$$

By evaluating the integrals on the right-hand side, equation (16) can be simplified to obtain

$$\begin{aligned}&(\theta_1\alpha^2 + \theta_2\alpha^2 - 2\theta_2)\cos\alpha \\ &\quad + (\alpha^3 - \theta_1\alpha - 2\theta_2\alpha)\sin\alpha + 2\theta_2 = 0.\end{aligned} \tag{17}$$

For any α satisfying (17), there exists a corresponding solution of (12) in the form (15). It can be shown that (17) has an infinite number of positive solutions when $\Theta = (\theta_1, \theta_2)^\top$ satisfies the following inequality:

$$\theta_1^2 + \theta_2^2 + 2\theta_1\theta_2 - 2\theta_1 - 4\theta_2 \ge 0.$$

This is demonstrated numerically in Section IV. A formal proof will be given in a forthcoming journal article [12]. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of positive solutions of (17). Then the general solution of (12) is

$$\mathscr{X}(x) = \sum_{n=1}^\infty A_n \sin(\alpha_n x),$$

where A_n are constants to be determined. The corresponding eigenvalues are

$$\sigma_n = c - \alpha_n^2, \quad n = 1, 2, 3, \ldots$$

Hence, using (13),

$$y(x,t) = \mathscr{X}(x)\mathscr{T}(t) = \sum_{n=1}^\infty T_0 A_n \mathrm{e}^{(c-\alpha_n^2)t} \sin(\alpha_n x). \tag{18}$$

By virtue of (10) and (11), this solution satisfies the boundary conditions (3c) and (3d). The constants T_0 and A_n must be selected appropriately so that the initial condition (3b) is also satisfied. To ensure stability as $t \to \infty$, each eigenvalue $\sigma_n = c - \alpha_n^2$ in (18) must be negative. Thus, we impose the following constraints on $\Theta = (\theta_1, \theta_2)^\top$:

$$\theta_1^2 + \theta_2^2 + 2\theta_1\theta_2 - 2\theta_1 - 4\theta_2 \ge 0, \tag{19a}$$

$$c - \alpha^2 \le -\varepsilon, \tag{19b}$$

$$\begin{aligned}&(\theta_1\alpha^2 + \theta_2\alpha^2 - 2\theta_2)\cos\alpha \\ &\quad + (\alpha^3 - \theta_1\alpha - 2\theta_2\alpha)\sin\alpha + 2\theta_2 = 0,\end{aligned} \tag{19c}$$

where ε is a given positive parameter and α is the smallest positive solution of (17). Note that α here is treated as an additional optimization variable. Constraint (19a) ensures that there are an infinite number of eigenvalues and thus the solution form (18) is valid. Constraints (19b) and (19c) ensure that the largest eigenvalue is negative, thus guaranteeing solution stability. Adding constraints (19) to Problem 2.1 yields the following modified problem.

Problem 2.2: Given the PDE system (3) with the parameterized kernel (5), choose $\Theta = (\theta_1, \theta_2)^\top$ and α such that the cost functional (7) is minimized subject to the bound constraints (6) and the nonlinear constraints (19).

The next result is concerned with the stability of the closed-loop system corresponding to the optimized kernel from Problem 2.2.

Theorem 2.1: Let (Θ^*, α^*) be an optimal solution of Problem 2.2, where α^* is the smallest positive solution of equation (19c) corresponding to Θ^*. Suppose that there exists a sequence $\{\alpha_n^*\}_{n=1}^\infty$ of positive solutions to equation (19c) corresponding to Θ^* such that $y_0(x) \in \mathrm{span}\{\sin(\alpha_n^* x)\}$. Then the closed-loop system (3) corresponding to Θ^* is stable.

Proof: Because of constraint (19a), the solution form (18) with $\alpha_n = \alpha_n^*$ is guaranteed to satisfy (3a), (3c) and (3d). If $y_0(x) \in \mathrm{span}\{\sin(\alpha_n^* x)\}$, then there exists constants $Y_n, n \ge 1$, such that

$$y_0(x) = \sum_{n=1}^\infty Y_n \sin(\alpha_n^* x).$$

Taking $Y_n = T_0 A_n$ ensures that (18) with $\alpha_n = \alpha_n^*$ also satisfies the initial conditions (3b), and is therefore the unique solution of (3). Since α^* is the first positive solution of equation (17), it follows from constraints (19b) and (19c) that for each $n \ge 1$,

$$c - (\alpha_n^*)^2 \le c - (\alpha^*)^2 \le -\varepsilon < 0.$$

This shows that all eigenvalues are negative. ■

Theorem 2.1 requires that the initial function $y_0(x)$ be contained within the linear span of sinusoidal functions $\sin(\alpha_n^* x)$, where each α_n^* is a solution of equation (17) corresponding to Θ^*. The good thing about this condition is that it can be verified numerically by solving the following optimization problem: choose span coefficients $Y_n, 1 \le n \le N$, to minimize

$$J = \int_0^1 \left| y_0(x) - \sum_{n=1}^N Y_n \sin(\alpha_n^* x) \right|^2 dx, \tag{20}$$

where N is a sufficiently large integer and each α_n^* is a solution of equation (17) corresponding to the optimal solution of Problem 2.2. If the optimal cost value for this optimization problem is sufficiently small, then the span condition in Theorem 2.1 is likely to be satisfied, and therefore closed-loop stability is expected. Based on our computational experience, the span condition in Theorem 2.1 is usually satisfied. In fact, as we show in [12], the solutions α_n^* of (17) converge to the integer multiples of π. Thus, it is reasonable to expect that the linear span of $\{\sin(\alpha_n^* x)\}$ is "approximately" the same as the linear span of $\{\sin(n\pi x)\}$, which is known to be a basis for the space of continuous functions defined on $[0,1]$.

III. NUMERICAL COMPUTATION

Problem 2.2 is an optimal parameter selection problem with decision parameters θ_1, θ_2 and α. In principle, such problems can be solved as nonlinear optimization problems using sequential quadratic programming or other nonlinear optimization methods. However, to do this, we need the gradients of the cost functional (7) and the constraint functions (19) with respect to the decision parameters.

Since the constraint functions in (19) are explicit functions of the decision variables, their gradients are easily derived using elementary differentiation. The cost functional (7), on the other hand, is an implicit function of Θ because it depends on the state trajectory $y(x,t)$. Thus, computing the gradient of (7) is a non-trivial task. We now develop a computational method, analogous to the costate method in the optimal control of ordinary differential equations [7], [8], [14], for computing this gradient.

We define the following costate PDE system:

$$\begin{cases} v_t(x,t)+v_{xx}(x,t)+cv(x,t) \\ \qquad + y(x,t;\Theta)-k(x;\Theta)v_x(1,t)=0, & \text{(21a)} \\ v(0,t)=v(1,t)=0, & \text{(21b)} \\ v(x,T)=0. & \text{(21c)} \end{cases}$$

Let $v(x,t;\Theta)$ denote the solution of the costate PDE system (21) corresponding to the parameter vector Θ. Then we have the following theorem.

Theorem 3.1: The gradient of the cost functional (7) is given by

$$\nabla_{\theta_1} g_0(\Theta) = -\int_0^T\int_0^1 x v_x(1,t)y(x,t)dxdt+\frac{1}{3}\theta_1+\frac{1}{4}\theta_2,$$
$$\nabla_{\theta_2} g_0(\Theta) = -\int_0^T\int_0^1 x^2 v_x(1,t)y(x,t)dxdt+\frac{1}{4}\theta_1+\frac{1}{5}\theta_2,$$

where $y(x,t)=y(x,t;\Theta)$ and $v_x(x,t)=v_x(x,t;\Theta)$.

Proof: For simplicity, we write $y(x,t;\Theta)$ as $y(x,t)$ and $k(x;\Theta)$ as $k(x)$. Let $\psi(x,t)$ be an arbitrary function satisfying

$$\psi(x,T)=0, \quad \psi(0,t)=\psi(1,t)=0. \tag{22}$$

Then we can rewrite the cost functional (7) in augmented form as follows:

$$g_0(\Theta)=\frac{1}{2}\int_0^T\int_0^1 y^2(x,t)dxdt+\frac{1}{2}\int_0^1 k^2(x)dx$$
$$+\int_0^T\int_0^1 \psi(x,t)\big\{-y_t(x,t)+y_{xx}(x,t)+cy(x,t)\big\}dxdt. \tag{23}$$

Using integration by parts and applying conditions (3b), (3c) and (22), we can simplify the augmented cost functional (23) to obtain

$$\begin{aligned} g_0(\Theta) &= \frac{1}{2}\int_0^T\int_0^1 y^2(x,t)dxdt+\frac{1}{2}\int_0^1 k^2(x)dx \\ &+\int_0^T\int_0^1 \big\{\psi_t(x,t)+\psi_{xx}(x,t)+c\psi(x,t)\big\}y(x,t)dxdt \\ &+\int_0^1 \psi(x,0)y_0(x)dx-\int_0^T \psi_x(1,t)y(1,t)dt. \end{aligned}$$

Now, consider a perturbation $\delta\rho$ in the parameter vector Θ, where δ is a constant of sufficiently small magnitude and ρ is an arbitrary vector. The corresponding perturbation in the state is,

$$y_\delta(x,t)=y(x,t)+\delta\langle\nabla_\Theta y(x,t),\rho\rangle+\mathscr{O}(\delta^2), \tag{24}$$

and the perturbation in the feedback kernel is,

$$k_\delta(x)=k(x)+\delta\langle\nabla_\Theta k(x),\rho\rangle+\mathscr{O}(\delta^2), \tag{25}$$

where $\mathscr{O}(\delta^2)$ denotes omitted second-order terms such that $\delta^{-1}\mathscr{O}(\delta^2)\to 0$ as $\delta\to 0$. For notational simplicity, we define $\eta(x,t)=\langle\nabla_\Theta y(x,t),\rho\rangle$. Obviously, $\eta(x,0)=0$, because the initial profile $y_0(x)$ is independent of the parameter vector Θ. Based on (24) and (25), the perturbed augmented cost functional takes the following form:

$$\begin{aligned} g_0(\Theta+\delta\rho) &= \frac{1}{2}\int_0^T\int_0^1 \big\{y(x,t)+\delta\eta(x,t)\big\}^2 dxdt \\ &+\int_0^T\int_0^1 \big\{\psi_t(x,t)+\psi_{xx}(x,t)\big\}\big\{y(x,t)+\delta\eta(x,t)\big\}dxdt \\ &+\int_0^T\int_0^1 c\psi(x,t)\big\{y(x,t)+\delta\eta(x,t)\big\}dxdt \\ &-\int_0^T \psi_x(1,t)\left[\int_0^1 k(x)\big\{y(x,t)+\delta\eta(x,t)\big\}dx\right]dt \\ &-\int_0^T \psi_x(1,t)\left[\int_0^1 \delta\langle\nabla_\Theta k(x),\rho\rangle y(x,t)dx\right]dt \\ &+\frac{1}{2}\int_0^1 \big\{k(x)+\delta\langle\nabla_\Theta k(x),\rho\rangle\big\}^2 dx \\ &+\int_0^1 \psi(x,0)y_0(x)dx+\mathscr{O}(\delta^2). \end{aligned} \tag{26}$$

Taking the derivative of (26) with respect to δ and setting $\delta=0$ gives

$$\begin{aligned} \langle\nabla_\Theta g_0(\Theta),\rho\rangle &= \left.\frac{dg_0(\Theta+\delta\rho)}{d\delta}\right|_{\delta=0} \\ &= \int_0^T\int_0^1 \big\{y(x,t)+\psi_t(x,t)+\psi_{xx}(x,t)+c\psi(x,t)\big\}\eta(x,t)dxdt \\ &-\int_0^T\int_0^1 \psi_x(1,t)k(x)\eta(x,t)dxdt \\ &-\int_0^T\int_0^1 \psi_x(1,t)\langle\nabla_\Theta k(x),\rho\rangle y(x,t)dxdt \\ &+\int_0^1 k(x)\langle\nabla_\Theta k(x),\rho\rangle dx. \end{aligned}$$

Since the perturbation ρ was selected arbitrarily, the theorem follows immediately by setting $\psi(x,t)=v(x,t;\Theta)$ and taking

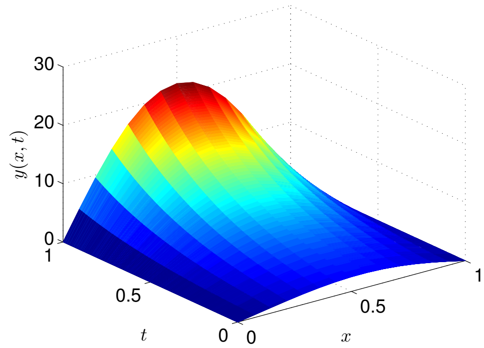

Fig. 2. Uncontrolled open-loop response.

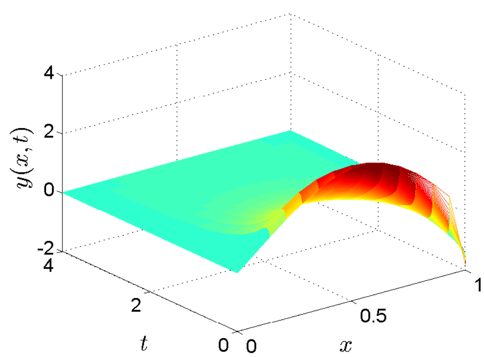

Fig. 3. Optimal closed-loop response.

ρ to be the standard unit basis vectors in $\mathbb{R}^2$. This completes the proof. ■

By combining the gradient formulas in Theorem 3.1 with a standard gradient-based optimization method (such as sequential quadratic programming), Problem 2.2 can be solved efficiently.

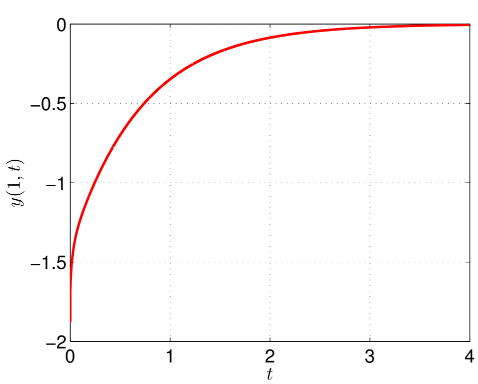

Fig. 4. Optimal boundary control.

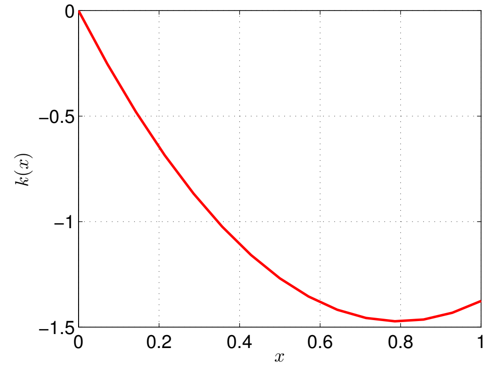

Fig. 5. Optimal kernel function.

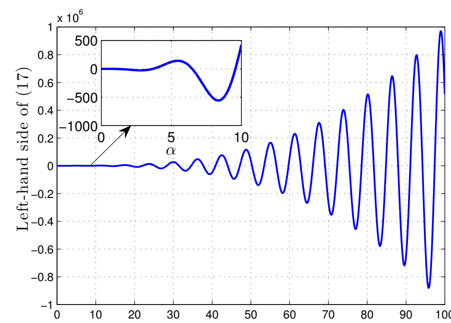

Fig. 6. Equation (17) has an infinite number of positive solutions.

IV. NUMERICAL EXAMPLE

We consider Problem 2.2 over a time horizon of $[0,T] = [0,4]$. To solve the problem, we wrote a MATLAB program that combines the FMINCON optimization function with the gradient formulas in Theorem 3.1. The state system (3) and the costate system (21) are solved numerically using the finite difference method (with 14 spatial intervals and 5000 temporal intervals). All numerical simulations were performed on a desktop computer with the following configuration: Intel Core i7-2600 3.40GHz CPU, 4.00GB RAM, 64-bit Windows 7 Operating System.

Consider the uncontrolled version of (3) in which $u(t) = 0$. In this case, the exact solution is [5]

$$y(x,t) = 2\sum_{n=1}^{\infty} C_n \mathrm{e}^{(c-n^2\pi^2)t} \sin(n\pi x)dx, \tag{27}$$

where C_n are the Fourier coefficients defined by

$$C_n = \int_0^1 y_0(x)\sin(n\pi x)dx.$$

The eigenvalues of (27) are $c - n^2\pi^2$, $n = 1,2,\ldots$ The largest eigenvalue is therefore $c - \pi^2$, which indicates that system (3) with $u(t) = 0$ is unstable for $c > \pi^2 \approx 9.8696$.

We choose $c = 12$ and $y_0(x) = (2+x)\sin(\pi x)$. The corresponding uncontrolled open-loop response (see equation (27)) is shown in Fig. 2. As we can see from Fig. 2, the state

TABLE I
SOLUTIONS OF EQUATION (17) AND CORRESPONDING SPAN COEFFICIENTS.

n	α_n^*	α_n^*/π	Y_n
1	3.6934	1.0801	−0.4211
2	6.4961	2.0678	1.5226
3	9.5794	3.0492	1.4629
4	12.6751	4.0346	−0.7636
5	15.7975	5.0285	0.4483
6	18.9223	6.0231	−0.3458
7	22.0544	7.0201	0.2681
8	25.1874	8.0173	0.9085
9	28.3233	9.0155	1.0726
10	31.4597	10.0139	0.9415
11	34.5975	11.0127	1.0418
12	37.7356	12.0116	0.9709
13	40.8745	13.0107	1.0156
14	44.0136	14.0099	0.9968
15	47.1532	15.0093	0.9908

of the uncontrolled system grows as time increases. For the feedback kernel optimization, we suppose that the lower and upper bounds for the optimization parameters are $a_i = -10$ and $b_i = 10$, respectively. We also choose $\varepsilon = 1$ in (19b). Starting from the initial guess $(\theta_1, \theta_2, \alpha) = (-1, 2, 0)$, our program terminates after 29 iterations and 21.0904 seconds. The optimal cost value is $g_0 = 1.2092$ and the optimal solution of Problem 2.2 is $(\theta_1^*, \theta_2^*, \alpha^*) = (-3.6977, 2.3220, 3.6934)$.

The spatial-temporal response of the controlled plant corresponding to (θ_1^*, θ_2^*) is shown in Fig. 3. The figure clearly shows that the controlled system (3) with optimized parameters (θ_1^*, θ_2^*) is stable. The corresponding optimal boundary control and kernel function are shown in Fig. 4 and Fig. 5, respectively.

Recall from Theorem 2.1 that closed-loop stability is guaranteed if $\alpha^* = 3.6934$ is the first positive solution of equation (17) and the initial function $y_0(x)$ is contained within the linear span of $\{\sin(\alpha_n^* x)\}$, where each α_n^* is a solution of equation (17) corresponding to (θ_1^*, θ_2^*). By viewing a plot of the left-hand side of equation (17), it can be easily verified that α^* is indeed the first positive solution; see Fig. 6. To verify the linear span condition, we use FMINCON in MATLAB to minimize (20) for $N = 20$. The first 15 positive solutions of (17) corresponding to the optimal parameters $\theta_1^* = -3.6977$ and $\theta_2^* = 2.3220$ are given in Table I. The optimal span coefficients that minimize (20) are also given. The optimal value of J in (20) is 7.7387×10^{-14}, which indicates that the span condition holds. Note also from Table I that α_n^*/π converges to an integer as $n \to \infty$.

V. CONCLUSIONS

In this paper, we have introduced a new gradient-based optimization approach for boundary stabilization of parabolic PDE systems. Our new approach involves expressing the boundary controller as an integral state feedback in which a kernel function needs to be designed judiciously. We do not determine the feedback kernel by solving Riccati-type or Klein-Gorden-type PDEs; instead, we approximate the feedback kernel by a quadratic function and then optimize the quadratic's coefficients using dynamic optimization techniques. This preliminary work has also raised several issues that require further investigation: (i) Can the proposed kernel optimization approach be applied to other classes of PDE plant models (i.e., 2D or 3D domains)? (ii) Is it possible to develop methods for minimizing cost functional (7) over an infinite time horizon? These issues will be explored in future work.

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Output Stabilization of Boundary-controlled Parabolic PDEs via Gradient-based Dynamic Optimization*

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***Abstract*—This paper proposes a new control synthesis approach for the stabilization of boundary-controlled parabolic partial differential equations (PDEs). In the proposed approach, the optimal boundary control is expressed in integral state feedback form with quadratic kernel function, where the quadratic's coefficients are decision variables to be optimized. We introduce a system cost functional to penalize both state and kernel magnitude, and then derive the cost functional's gradient in terms of the solution of an auxiliary "costate" PDE. On this basis, the output stabilization problem can be solved using gradient-based optimization techniques such as sequential quadratic programming. The resulting optimal boundary control is guaranteed to yield closed-loop stability under mild conditions. The primary advantage of our new approach is that the costate PDE is in standard form and can be solved easily using the finite difference method. In contrast, the traditional control synthesis approaches for boundary-controlled parabolic PDEs (i.e., the LQ control and backstepping approaches) require solving nonstandard Riccati-type and Klein-Gorden-type PDEs.**

I. INTRODUCTION

Parabolic partial differential equation (PDE) systems are an important type of distributed parameter system (DPS) describing a wide range of natural phenomena, including diffusion, heat transfer, and fusion plasma transport. Over the past few decades, control theory for the parabolic DPS has developed into a mature research topic at the interface of engineering and applied mathematics [1], [2], [10].

The linear quadratic (LQ) control framework is well-defined in infinite dimensional function spaces to deal with the parabolic DPS (e.g., [1], [2]). However, the LQ control framework requires solving Riccati-type differential equations, which are nonlinear parabolic PDEs of dimension one greater than the original parabolic PDE system. For example, to generate an optimal feedback controller for a scalar heat equation, a Riccati PDE defined over a rectangular domain must be solved [11]. Hence, the LQ approach does not actually solve the controller synthesis problem directly, but instead converts it into another problem (i.e., solve a Riccati-type PDE) that is still extremely difficult to solve from a computational point of view.

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One of the major advances in PDE control in recent years has been the so-called infinite dimensional Voltera integral feedback, or the backstepping method (e.g., [5], [9]). Instead of Riccati-type PDEs, the backstepping method requires solving the so-called kernel equations—linear Klein-Gorden-type PDEs for which the successive approach can be used to obtain explicit solutions. This method was originally developed for the stabilization of one dimensional parabolic DPS and then extended to fluid flows [16], [19], magnetohydrodynamic flows [17], and elastic vibration [4]. In addition, the backstepping method can also be applied to achieve full state feedback stabilization and state estimation of PDE-ODE cascade systems [13].

In this paper, we propose a new framework for control synthesis for boundary-controlled parabolic DPS. This new framework does not require solving Riccati-type or Klein-Gorden-type PDEs. Instead, it requires solving a so-called "costate" PDE, which is much easier to solve from a computational viewpoint. In fact, many numerical software packages, such as Comsol Multiphysics and MATLAB PDE ToolBox, can be used to generate numerical solutions for the costate PDE. The Riccati PDEs, on the other hand, are usually not in standard form and thus cannot be solved using off-the-shelf software packages. The approach proposed in this paper can be viewed as an extension of optimization-based PID tuning ideas (see [3], [6], [14], [18]) to infinite dimensional systems.

II. PROBLEM FORMULATION

A. Feedback Kernel Optimization

We consider the following parabolic PDE system:

$$\begin{cases} y_t(x,t) = y_{xx}(x,t) + cy(x,t), & \text{(1a)} \\ y(0,t) = 0, & \text{(1b)} \\ y(1,t) = u(t), & \text{(1c)} \\ y(x,0) = y_0(x), & \text{(1d)} \end{cases}$$

where $c > 0$ is a given constant and $u(t)$ is a boundary control. It is well known that the uncontrolled version of system (1) is unstable when the constant c is sufficiently large [5]. According to the LQ control [11] and backstepping synthesis approaches [5], the optimal stabilizing control law takes the following feedback form:

$$u(t) = \int_0^1 \mathscr{K}(1,\xi) y(\xi,t) d\xi, \tag{2}$$

where the feedback kernel $\mathscr{K}(1,\xi)$ is obtained by solving either a Riccati-type or a Klein-Gorden-type PDE. By introducing the new notation $k(\xi) = \mathscr{K}(1,\xi)$, we can write the

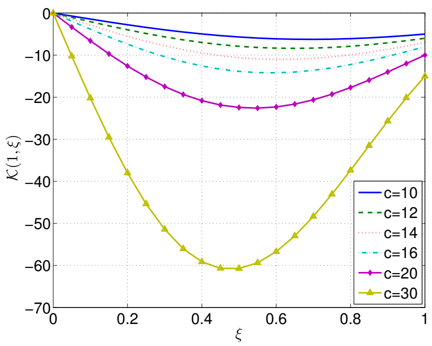

Fig. 1. The feedback kernel (4) for various values of c.

feedback control policy (2) in the following form:

$$u(t) = \int_0^1 k(\xi)y(\xi,t)d\xi.$$

The corresponding closed-loop system is

$$\begin{cases} y_t(x,t) = y_{xx}(x,t) + cy(x,t), & \text{(3a)} \\ y(x,0) = y_0(x), & \text{(3b)} \\ y(0,t) = 0, & \text{(3c)} \\ y(1,t) = \int_0^1 k(\xi)y(\xi,t)d\xi. & \text{(3d)} \end{cases}$$

In reference [5], the backstepping method is used to express the optimal feedback kernel as follows:

$$\mathscr{K}(1,\xi) = -c\xi\frac{I_1(\sqrt{c(1-\xi^2)})}{\sqrt{c(1-\xi^2)}}, \tag{4}$$

where I_1 is the first-order modified Bessel function given by

$$I_1(\omega) = \sum_{n=0}^{\infty}\frac{\omega^{2n+1}}{2^{2n+1}n!(n+1)!}.$$

The feedback kernel (4) is plotted in Figure 1 for different values of c. Note that its shape is similar to a quadratic function. Note also that $\mathscr{K}(1,\xi)=0$ when $\xi=0$. Accordingly, motivated by the quadratic behavior exhibited in Fig. 1, we express $k(\xi)$ in the following parameterized form:

$$k(\xi;\Theta) = \theta_1\xi + \theta_2\xi^2, \tag{5}$$

where $\Theta = (\theta_1,\theta_2)^\top$ is a parameter vector to be optimized.

Moreover, we assume that the parameters must satisfy the following bound constraints:

$$a_1 \le \theta_1 \le b_1, \quad a_2 \le \theta_2 \le b_2, \tag{6}$$

where a_1, a_2, b_1 and b_2 are given bounds.

Let $y(x,t;\Theta)$ denote the solution of the closed-loop system (3) with the parameterized kernel (5). The results in [15] ensure that such a solution exists and is unique. Our goal is to stabilize the closed-loop system with minimal energy input. Accordingly, we consider the following cost functional:

$$g_0(\Theta) = \frac{1}{2}\int_0^T\int_0^1 y^2(x,t;\Theta)dxdt + \frac{1}{2}\int_0^1 k^2(x;\Theta)dx. \tag{7}$$

We now state our kernel optimization problem formally as follows.

Problem 2.1: Given the PDE system (3) with the parameterized kernel (5), find an optimal parameter vector $\Theta = (\theta_1,\theta_2)^\top$ such that the cost functional (7) is minimized subject to the bound constraints (6).

B. Closed-Loop Stability

Since (7) is a finite-time cost functional, there is no guarantee that the optimized kernel (5) generated by the solution of Problem 2.1 stabilizes the closed-loop system (3) as $t\to\infty$. Nevertheless, we now show that, by analyzing the solution structure of (3), additional constraints can be added to Problem 2.1 to ensure closed-loop stability.

Using the separation of variables approach, we decompose $y(x,t)$ as follows:

$$y(x,t) = \mathscr{X}(x)\mathscr{T}(t). \tag{8}$$

Substituting (8) into (3a), we obtain

$$\mathscr{X}(x)\dot{\mathscr{T}}(t) = \mathscr{X}''(x)\mathscr{T}(t) + c\mathscr{X}(x)\mathscr{T}(t), \tag{9}$$

where

$$\dot{\mathscr{T}}(t) = \frac{d\mathscr{T}(t)}{dt}, \quad \mathscr{X}''(x) = \frac{d^2\mathscr{X}(x)}{dx^2}.$$

Furthermore, from the boundary conditions (3c) and (3d),

$$\mathscr{X}(0)\mathscr{T}(t) = 0,$$
$$\mathscr{X}(1)\mathscr{T}(t) = \int_0^1 k(\xi;\Theta)\mathscr{X}(\xi)\mathscr{T}(t)d\xi.$$

Thus, we immediately obtain

$$\mathscr{X}(0) = 0, \tag{10}$$

$$\mathscr{X}(1) = \int_0^1 k(\xi;\Theta)\mathscr{X}(\xi)d\xi. \tag{11}$$

Rearranging (9) gives

$$\frac{\mathscr{X}''(x)+c\mathscr{X}(x)}{\mathscr{X}(x)} = \frac{\dot{\mathscr{T}}(t)}{\mathscr{T}(t)} = \sigma, \tag{12}$$

where σ is a constant called the eigenvalue. Clearly,

$$\mathscr{T}(t) = T_0 e^{\sigma t}, \tag{13}$$

where $T_0 = \mathscr{T}(0)$ is a constant to be determined.

To solve for $\mathscr{X}(x)$, we must consider three cases: (i) $c<\sigma$; (ii) $c=\sigma$; (iii) $c>\sigma$. In cases (i) and (ii), the general solutions of (12) are, respectively,

$$\mathscr{X}(x) = X_0 e^{\sqrt{\sigma-c}x} + X_1 e^{-\sqrt{\sigma-c}x},$$

and

$$\mathscr{X}(x) = X_0 + X_1 x,$$

where X_0 and X_1 are constants to be determined from the boundary conditions (10) and (11). Then the corresponding solutions of (3) are

$$y(x,t) = X_0T_0 e^{\sqrt{\sigma-c}x+\sigma t} + X_1T_0 e^{-\sqrt{\sigma-c}x+\sigma t},$$

and

$$y(x,t) = X_0 T_0 \mathrm{e}^{\sigma t} + X_1 T_0 x \mathrm{e}^{\sigma t}.$$

These solutions are clearly unstable because $0 < c \leq \sigma$. Thus, the parameters θ_1 and θ_2 should be chosen so that the unique solution of (3) satisfies case (iii) instead of cases (i) and (ii).

In case (iii), the general solution of (12) is

$$\mathscr{X}(x) = X_0 \cos(\sqrt{c-\sigma}x) + X_1 \sin(\sqrt{c-\sigma}x), \tag{14}$$

where X_0 and X_1 are constants to be determined from the boundary conditions (10) and (11). Substituting (14) into (10), we obtain

$$\mathscr{X}(0) = X_0 = 0.$$

Hence,

$$\mathscr{X}(x) = X_1 \sin(\sqrt{c-\sigma}x). \tag{15}$$

To simplify the notation, we introduce a new variable $\alpha = \sqrt{c-\sigma}$. Substituting (15) into condition (11), we have

$$X_1 \sin\alpha = X_1 \int_0^1 \theta_1 \xi \sin(\alpha\xi) d\xi + X_1 \int_0^1 \theta_2 \xi^2 \sin(\alpha\xi) d\xi,$$

and thus

$$\sin\alpha = \int_0^1 \theta_1 \xi \sin(\alpha\xi) d\xi + \int_0^1 \theta_2 \xi^2 \sin(\alpha\xi) d\xi. \tag{16}$$

By evaluating the integrals on the right-hand side, equation (16) can be simplified to obtain

$$\begin{aligned}(\theta_1\alpha^2 + \theta_2\alpha^2 - 2\theta_2)\cos\alpha \\ + (\alpha^3 - \theta_1\alpha - 2\theta_2\alpha)\sin\alpha + 2\theta_2 = 0.\end{aligned} \tag{17}$$

For any α satisfying (17), there exists a corresponding solution of (12) in the form (15). It can be shown that (17) has an infinite number of positive solutions when $\Theta = (\theta_1, \theta_2)^\top$ satisfies the following inequality:

$$\theta_1^2 + \theta_2^2 + 2\theta_1\theta_2 - 2\theta_1 - 4\theta_2 \geq 0.$$

This is demonstrated numerically in Section IV. A formal proof will be given in a forthcoming journal article [12]. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of positive solutions of (17). Then the general solution of (12) is

$$\mathscr{X}(x) = \sum_{n=1}^\infty A_n \sin(\alpha_n x),$$

where A_n are constants to be determined. The corresponding eigenvalues are

$$\sigma_n = c - \alpha_n^2, \quad n = 1, 2, 3, \ldots$$

Hence, using (13),

$$y(x,t) = \mathscr{X}(x)\mathscr{T}(t) = \sum_{n=1}^\infty T_0 A_n \mathrm{e}^{(c-\alpha_n^2)t} \sin(\alpha_n x). \tag{18}$$

By virtue of (10) and (11), this solution satisfies the boundary conditions (3c) and (3d). The constants T_0 and A_n must be selected appropriately so that the initial condition (3b) is also satisfied. To ensure stability as $t \to \infty$, each eigenvalue $\sigma_n = c - \alpha_n^2$ in (18) must be negative. Thus, we impose the following constraints on $\Theta = (\theta_1, \theta_2)^\top$:

$$\theta_1^2 + \theta_2^2 + 2\theta_1\theta_2 - 2\theta_1 - 4\theta_2 \geq 0, \tag{19a}$$

$$c - \alpha^2 \leq -\varepsilon, \tag{19b}$$

$$\begin{aligned}&(\theta_1\alpha^2 + \theta_2\alpha^2 - 2\theta_2)\cos\alpha \\ &\quad + (\alpha^3 - \theta_1\alpha - 2\theta_2\alpha)\sin\alpha + 2\theta_2 = 0,\end{aligned} \tag{19c}$$

where ε is a given positive parameter and α is the smallest positive solution of (17). Note that α here is treated as an additional optimization variable. Constraint (19a) ensures that there are an infinite number of eigenvalues and thus the solution form (18) is valid. Constraints (19b) and (19c) ensure that the largest eigenvalue is negative, thus guaranteeing solution stability. Adding constraints (19) to Problem 2.1 yields the following modified problem.

Problem 2.2: Given the PDE system (3) with the parameterized kernel (5), choose $\Theta = (\theta_1, \theta_2)^\top$ and α such that the cost functional (7) is minimized subject to the bound constraints (6) and the nonlinear constraints (19).

The next result is concerned with the stability of the closed-loop system corresponding to the optimized kernel from Problem 2.2.

Theorem 2.1: Let (Θ^*, α^*) be an optimal solution of Problem 2.2, where α^* is the smallest positive solution of equation (19c) corresponding to Θ^*. Suppose that there exists a sequence $\{\alpha_n^*\}_{n=1}^\infty$ of positive solutions to equation (19c) corresponding to Θ^* such that $y_0(x) \in \mathrm{span}\{\sin(\alpha_n^* x)\}$. Then the closed-loop system (3) corresponding to Θ^* is stable.

Proof: Because of constraint (19a), the solution form (18) with $\alpha_n = \alpha_n^*$ is guaranteed to satisfy (3a), (3c) and (3d). If $y_0(x) \in \mathrm{span}\{\sin(\alpha_n^* x)\}$, then there exists constants $Y_n, n \geq 1$, such that

$$y_0(x) = \sum_{n=1}^\infty Y_n \sin(\alpha_n^* x).$$

Taking $Y_n = T_0 A_n$ ensures that (18) with $\alpha_n = \alpha_n^*$ also satisfies the initial conditions (3b), and is therefore the unique solution of (3). Since α^* is the first positive solution of equation (17), it follows from constraints (19b) and (19c) that for each $n \geq 1$,

$$c - (\alpha_n^*)^2 \leq c - (\alpha^*)^2 \leq -\varepsilon < 0.$$

This shows that all eigenvalues are negative. ■

Theorem 2.1 requires that the initial function $y_0(x)$ be contained within the linear span of sinusoidal functions $\sin(\alpha_n^* x)$, where each α_n^* is a solution of equation (17) corresponding to Θ^*. The good thing about this condition is that it can be verified numerically by solving the following optimization problem: choose span coefficients $Y_n, 1 \leq n \leq N$, to minimize

$$J = \int_0^1 \left| y_0(x) - \sum_{n=1}^N Y_n \sin(\alpha_n^* x) \right|^2 dx, \tag{20}$$

where N is a sufficiently large integer and each α_n^* is a solution of equation (17) corresponding to the optimal solution of Problem 2.2. If the optimal cost value for this optimization problem is sufficiently small, then the span condition in Theorem 2.1 is likely to be satisfied, and therefore closed-loop stability is expected. Based on our computational experience, the span condition in Theorem 2.1 is usually satisfied. In fact, as we show in [12], the solutions α_n^* of (17) converge to the integer multiples of π. Thus, it is reasonable to expect that the linear span of $\{\sin(\alpha_n^* x)\}$ is "approximately" the same as the linear span of $\{\sin(n\pi x)\}$, which is known to be a basis for the space of continuous functions defined on $[0,1]$.

III. NUMERICAL COMPUTATION

Problem 2.2 is an optimal parameter selection problem with decision parameters θ_1, θ_2 and α. In principle, such problems can be solved as nonlinear optimization problems using sequential quadratic programming or other nonlinear optimization methods. However, to do this, we need the gradients of the cost functional (7) and the constraint functions (19) with respect to the decision parameters.

Since the constraint functions in (19) are explicit functions of the decision variables, their gradients are easily derived using elementary differentiation. The cost functional (7), on the other hand, is an implicit function of Θ because it depends on the state trajectory $y(x,t)$. Thus, computing the gradient of (7) is a non-trivial task. We now develop a computational method, analogous to the costate method in the optimal control of ordinary differential equations [7], [8], [14], for computing this gradient.

We define the following costate PDE system:

$$\begin{cases} v_t(x,t)+v_{xx}(x,t)+cv(x,t) \\ \qquad + y(x,t;\Theta)-k(x;\Theta)v_x(1,t)=0, & (21a)\\ v(0,t)=v(1,t)=0, & (21b)\\ v(x,T)=0. & (21c)\end{cases}$$

Let $v(x,t;\Theta)$ denote the solution of the costate PDE system (21) corresponding to the parameter vector Θ. Then we have the following theorem.

Theorem 3.1: The gradient of the cost functional (7) is given by

$$\nabla_{\theta_1} g_0(\Theta) = -\int_0^T\int_0^1 x v_x(1,t)y(x,t)dxdt+\frac{1}{3}\theta_1+\frac{1}{4}\theta_2,$$
$$\nabla_{\theta_2} g_0(\Theta) = -\int_0^T\int_0^1 x^2 v_x(1,t)y(x,t)dxdt+\frac{1}{4}\theta_1+\frac{1}{5}\theta_2,$$

where $y(x,t)=y(x,t;\Theta)$ and $v_x(x,t)=v_x(x,t;\Theta)$.

Proof: For simplicity, we write $y(x,t;\Theta)$ as $y(x,t)$ and $k(x;\Theta)$ as $k(x)$. Let $\psi(x,t)$ be an arbitrary function satisfying

$$\psi(x,T)=0, \quad \psi(0,t)=\psi(1,t)=0. \tag{22}$$

Then we can rewrite the cost functional (7) in augmented form as follows:

$$g_0(\Theta)=\frac{1}{2}\int_0^T\int_0^1 y^2(x,t)dxdt+\frac{1}{2}\int_0^1 k^2(x)dx + \int_0^T\int_0^1 \psi(x,t)\big\{-y_t(x,t)+y_{xx}(x,t)+cy(x,t)\big\}dxdt. \tag{23}$$

Using integration by parts and applying conditions (3b), (3c) and (22), we can simplify the augmented cost functional (23) to obtain

$$\begin{aligned} g_0(\Theta) = {} & \frac{1}{2}\int_0^T\int_0^1 y^2(x,t)dxdt+\frac{1}{2}\int_0^1 k^2(x)dx \\ & + \int_0^T\int_0^1 \big\{\psi_t(x,t)+\psi_{xx}(x,t)+c\psi(x,t)\big\}y(x,t)dxdt \\ & + \int_0^1 \psi(x,0)y_0(x)dx-\int_0^T \psi_x(1,t)y(1,t)dt. \end{aligned}$$

Now, consider a perturbation $\delta\rho$ in the parameter vector Θ, where δ is a constant of sufficiently small magnitude and ρ is an arbitrary vector. The corresponding perturbation in the state is,

$$y_\delta(x,t)=y(x,t)+\delta\langle\nabla_\Theta y(x,t),\rho\rangle+\mathscr{O}(\delta^2), \tag{24}$$

and the perturbation in the feedback kernel is,

$$k_\delta(x)=k(x)+\delta\langle\nabla_\Theta k(x),\rho\rangle+\mathscr{O}(\delta^2), \tag{25}$$

where $\mathscr{O}(\delta^2)$ denotes omitted second-order terms such that $\delta^{-1}\mathscr{O}(\delta^2)\to 0$ as $\delta\to 0$. For notational simplicity, we define $\eta(x,t)=\langle\nabla_\Theta y(x,t),\rho\rangle$. Obviously, $\eta(x,0)=0$, because the initial profile $y_0(x)$ is independent of the parameter vector Θ. Based on (24) and (25), the perturbed augmented cost functional takes the following form:

$$\begin{aligned} g_0(\Theta+\delta\rho) = {} & \frac{1}{2}\int_0^T\int_0^1 \big\{y(x,t)+\delta\eta(x,t)\big\}^2 dxdt \\ & + \int_0^T\int_0^1 \big\{\psi_t(x,t)+\psi_{xx}(x,t)\big\}\big\{y(x,t)+\delta\eta(x,t)\big\}dxdt \\ & + \int_0^T\int_0^1 c\psi(x,t)\big\{y(x,t)+\delta\eta(x,t)\big\}dxdt \\ & - \int_0^T \psi_x(1,t)\left[\int_0^1 k(x)\big\{y(x,t)+\delta\eta(x,t)\big\}dx\right]dt \\ & - \int_0^T \psi_x(1,t)\left[\int_0^1 \delta\langle\nabla_\Theta k(x),\rho\rangle y(x,t)dx\right]dt \\ & + \frac{1}{2}\int_0^1 \big\{k(x)+\delta\langle\nabla_\Theta k(x),\rho\rangle\big\}^2 dx \\ & + \int_0^1 \psi(x,0)y_0(x)dx+\mathscr{O}(\delta^2). \end{aligned} \tag{26}$$

Taking the derivative of (26) with respect to δ and setting $\delta=0$ gives

$$\begin{aligned} \langle\nabla_\Theta g_0(\Theta),\rho\rangle = {} & \left.\frac{dg_0(\Theta+\delta\rho)}{d\delta}\right|_{\delta=0} \\ = {} & \int_0^T\int_0^1 \big\{y(x,t)+\psi_t(x,t)+\psi_{xx}(x,t)+c\psi(x,t)\big\}\eta(x,t)dxdt \\ & - \int_0^T\int_0^1 \psi_x(1,t)k(x)\eta(x,t)dxdt \\ & - \int_0^T\int_0^1 \psi_x(1,t)\langle\nabla_\Theta k(x),\rho\rangle y(x,t)dxdt \\ & + \int_0^1 k(x)\langle\nabla_\Theta k(x),\rho\rangle dx. \end{aligned}$$

Since the perturbation ρ was selected arbitrarily, the theorem follows immediately by setting $\psi(x,t)=v(x,t;\Theta)$ and taking

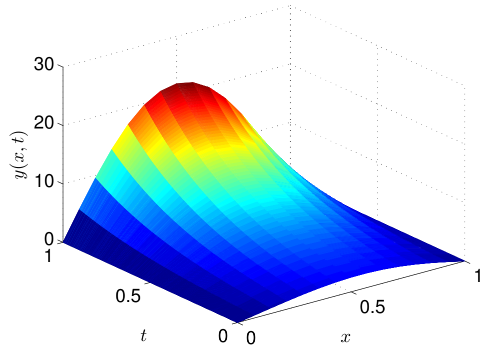

Fig. 2. Uncontrolled open-loop response.

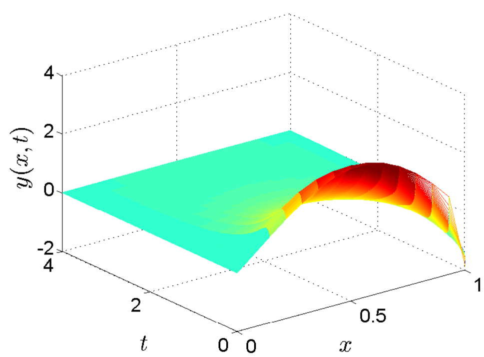

Fig. 3. Optimal closed-loop response.

ρ to be the standard unit basis vectors in $\mathbb{R}^2$. This completes the proof. ■

By combining the gradient formulas in Theorem 3.1 with a standard gradient-based optimization method (such as sequential quadratic programming), Problem 2.2 can be solved efficiently.

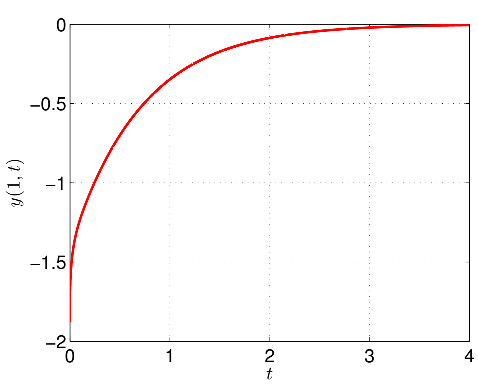

Fig. 4. Optimal boundary control.

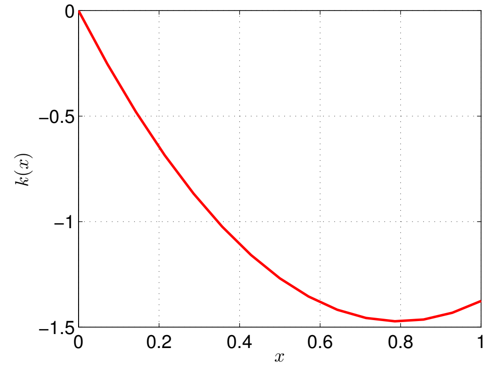

Fig. 5. Optimal kernel function.

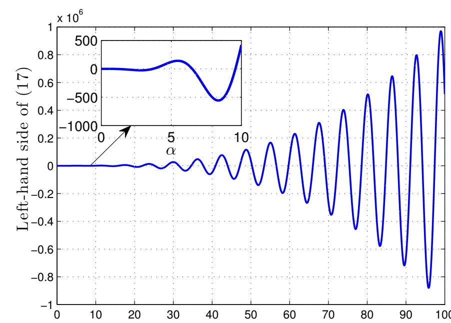

Fig. 6. Equation (17) has an infinite number of positive solutions.

IV. NUMERICAL EXAMPLE

We consider Problem 2.2 over a time horizon of $[0,T] = [0,4]$. To solve the problem, we wrote a MATLAB program that combines the FMINCON optimization function with the gradient formulas in Theorem 3.1. The state system (3) and the costate system (21) are solved numerically using the finite difference method (with 14 spatial intervals and 5000 temporal intervals). All numerical simulations were performed on a desktop computer with the following configuration: Intel Core i7-2600 3.40GHz CPU, 4.00GB RAM, 64-bit Windows 7 Operating System.

Consider the uncontrolled version of (3) in which $u(t) = 0$. In this case, the exact solution is [5]

$$y(x,t) = 2\sum_{n=1}^{\infty} C_n \mathrm{e}^{(c-n^2\pi^2)t} \sin(n\pi x)dx, \tag{27}$$

where C_n are the Fourier coefficients defined by

$$C_n = \int_0^1 y_0(x)\sin(n\pi x)dx.$$

The eigenvalues of (27) are $c - n^2\pi^2$, $n = 1,2,\ldots$ The largest eigenvalue is therefore $c - \pi^2$, which indicates that system (3) with $u(t) = 0$ is unstable for $c > \pi^2 \approx 9.8696$.

We choose $c = 12$ and $y_0(x) = (2+x)\sin(\pi x)$. The corresponding uncontrolled open-loop response (see equation (27)) is shown in Fig. 2. As we can see from Fig. 2, the state

TABLE I
SOLUTIONS OF EQUATION (17) AND CORRESPONDING SPAN COEFFICIENTS.

n	α_n^*	α_n^*/π	Y_n
1	3.6934	1.0801	−0.4211
2	6.4961	2.0678	1.5226
3	9.5794	3.0492	1.4629
4	12.6751	4.0346	−0.7636
5	15.7975	5.0285	0.4483
6	18.9223	6.0231	−0.3458
7	22.0544	7.0201	0.2681
8	25.1874	8.0173	0.9085
9	28.3233	9.0155	1.0726
10	31.4597	10.0139	0.9415
11	34.5975	11.0127	1.0418
12	37.7356	12.0116	0.9709
13	40.8745	13.0107	1.0156
14	44.0136	14.0099	0.9968
15	47.1532	15.0093	0.9908

of the uncontrolled system grows as time increases. For the feedback kernel optimization, we suppose that the lower and upper bounds for the optimization parameters are $a_i = -10$ and $b_i = 10$, respectively. We also choose $\varepsilon = 1$ in (19b). Starting from the initial guess $(\theta_1, \theta_2, \alpha) = (-1, 2, 0)$, our program terminates after 29 iterations and 21.0904 seconds. The optimal cost value is $g_0 = 1.2092$ and the optimal solution of Problem 2.2 is $(\theta_1^*, \theta_2^*, \alpha^*) = (-3.6977, 2.3220, 3.6934)$.

The spatial-temporal response of the controlled plant corresponding to (θ_1^*, θ_2^*) is shown in Fig. 3. The figure clearly shows that the controlled system (3) with optimized parameters (θ_1^*, θ_2^*) is stable. The corresponding optimal boundary control and kernel function are shown in Fig. 4 and Fig. 5, respectively.

Recall from Theorem 2.1 that closed-loop stability is guaranteed if $\alpha^* = 3.6934$ is the first positive solution of equation (17) and the initial function $y_0(x)$ is contained within the linear span of $\{\sin(\alpha_n^* x)\}$, where each α_n^* is a solution of equation (17) corresponding to (θ_1^*, θ_2^*). By viewing a plot of the left-hand side of equation (17), it can be easily verified that α^* is indeed the first positive solution; see Fig. 6. To verify the linear span condition, we use FMINCON in MATLAB to minimize (20) for $N = 20$. The first 15 positive solutions of (17) corresponding to the optimal parameters $\theta_1^* = -3.6977$ and $\theta_2^* = 2.3220$ are given in Table I. The optimal span coefficients that minimize (20) are also given. The optimal value of J in (20) is 7.7387×10^{-14}, which indicates that the span condition holds. Note also from Table I that α_n^*/π converges to an integer as $n \to \infty$.

V. CONCLUSIONS

In this paper, we have introduced a new gradient-based optimization approach for boundary stabilization of parabolic PDE systems. Our new approach involves expressing the boundary controller as an integral state feedback in which a kernel function needs to be designed judiciously. We do not determine the feedback kernel by solving Riccati-type or Klein-Gorden-type PDEs; instead, we approximate the feedback kernel by a quadratic function and then optimize the quadratic's coefficients using dynamic optimization techniques. This preliminary work has also raised several issues that require further investigation: (i) Can the proposed kernel optimization approach be applied to other classes of PDE plant models (i.e., 2D or 3D domains)? (ii) Is it possible to develop methods for minimizing cost functional (7) over an infinite time horizon? These issues will be explored in future work.

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