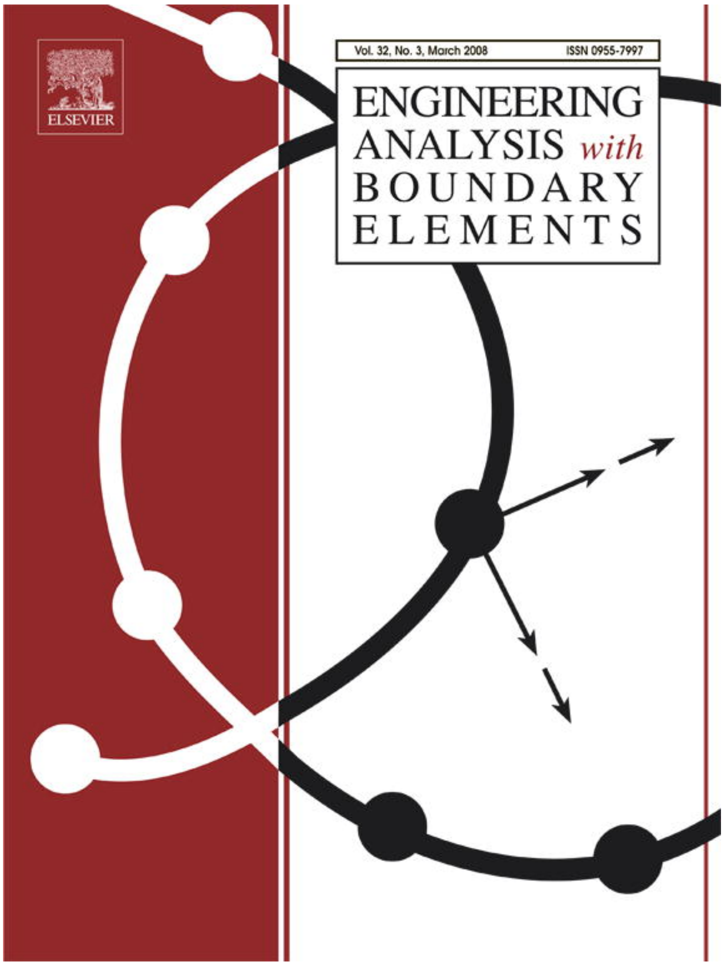
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Transient image principles for anisotropic diffusion with basic concepts and 2D linear analytical solutions

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Abstract

In this paper, based on the traditional method of images, new image principles for two-dimensional (2D) transient diffusion problems in limited anisotropic and inhomogeneous media are systematically developed. Using these principles for the first time, Green's functions for transient diffusion (heat conduction) phenomena for a bimaterial, quartmaterial and a two-layer spaces with limited anisotropic and inhomogeneous media are presented. The solutions are not only very useful for construction of the inverse and more complex direct solutions in many domains of applied science but also are the fundamentals for solutions of nonlinear systems which are linear convertible.

Keywords: Transient image principles; Method of images; Anisotropy; Fundamental solution; Green's function; Transient diffusion; Mathematical physics computation

1. Introduction

The clarity of the expression of physical concepts generated by the method of images, has prompted its wide use in mathematical physics and its applications. For example, in the theory and application of exploration geophysics, the method plays an important role, as indicated in the books by Keller and Frischknecht [1] and Eskola [2]. In electromagnetic theory, the books authored by Stratton [3] and Lindell [4] employed the method of images to obtain solutions for electromagnetic fields. In seismology, the method of images is used to determine the reflected ray travel times from seismic reflectors as demonstrated in the book by Cordier [5]. A huge number of articles in journal publications use the method of images to construct solutions for specific problems in applied and engineering related mathematics. For example, the most recent applications of this method were most likely given by Tadeu and Simões [6], Enders and Clark [7] and Tadeu and Simões [8].

In recent decades, the method of images has been extended from isotropic media to anisotropic media. It appears that the first publication using the method of images for anisotropic media was produced in 1974 by Asten [9], who limited anisotropy to the case with one of the principal axes parallel to the half-space surface. In 1988, Dellinger and Muir [10] gave a physical model for an anisotropic half-space image and Uren [11], in 1989, showed how to locate the image point in elliptically anisotropic media for seismic shear waves. In 1993, Lindell et al. [12] discussed the static image principle for anisotropic half-space problems with perfect electric and magnetic conductor boundaries. In 1998, Li and Uren [13] showed mathematically how to locate the anisotropic reflection images in arbitrarily oriented anisotropic media. Li and Uren [14,15] then extended the concepts of the method of images for isotropic media to reflection and transmission ($R\&T$) in anisotropic media. Li and Stagnitti [16] extended the method to a case where there was a right-angled interface between anisotropic media. However, except for totally reflecting interfaces and totally absorbing interfaces, all of the above-mentioned applications of the method of images are used to satisfy the

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boundary conditions of the relevant elliptical partial differential equations. To the present time, when solving transient diffusion or heat conduction problems, only the reflection method of images [17] has been used for obtaining solutions for a half-space with a total reflection boundary. In this paper, using the existing method of images, we will build up principles for the use of the method of images for transient sources in limited structures in anisotropic and inhomogeneous diffusion media. Here, Green's functions for anisotropic bimaterial and quartmaterial and two-layer anisotropic spaces are obtained for the first time.

Finding Green's functions for certain physical phenomena is one of the fundamental issues in many mathematical physics computations. Fundamental solutions for transient diffusion in two-dimensional (2D) or higher dimensional isotropic media have been known for decades and have been discussed in many textbooks, such as those by Morse and Feshbach [18], Hill and Dewynne [19] and Evans [20]. However, it appears that the fundamental solution for transient diffusion in homogeneous anisotropic whole space was first published by Chang et al. [21]. An extension of this solution to half-space anisotropic media was first reported by Chang [22] using the Fourier transformation method. Unfortunately, due to the choice of the mathematical expressions in Chang's paper, the transient solution was so complex that it is only applicable to the case of an infinite slab with simple boundary conditions.

In recent years, a number of published papers [23–26] tackled heat conduction and diffusion problems, by finding the exact analytical solutions for media with geometrically simple boundaries. These publications only addressed steady-state problems. In a very recent study by Kuo and Chen [27], a transient Green's function for anisotropic conduction was given, in which an exponentially graded conductivity tensor was used to develop Green's function in whole and half-spaces.

This present paper is organized as follows. First, the governing equation for anisotropic diffusion from a point source function is transformed into the canonical form by coordinate rotation, translation and stretching. Then the fundamental solution for anisotropic diffusion is given. The transient image principle (TIP) for total reflecting and absorbing boundaries is introduced in Section 3. Green's function for transient diffusion in a half-space is then given. In Section 4, based on the definition of bimaterial and the application of the TIP for boundaries with continuous boundary conditions, Green's function for a bimaterial is presented. This is followed by Section 5, where we introduce new concepts, such as front medium (FM), back media (BM) and diffusion instance (DI), for the solutions given in Sections 6 and 7. In Section 6, the TIP for multiple reflection and transmission boundaries is demonstrated for a two-layer space. New Green's functions are given for the parallel layered case. In Section 7, a new concept called the reverse image source (RIS) for a partially reflecting and transmitting boundary is introduced, followed by new Green's functions for the quartmaterial domain. Section 8 gives two numerical examples. Finally, Section 9 discusses several generalizations and limitations of the solutions developed in this paper.

2. Fundamental solution

The fundamental solution for transient diffusion in an unbounded anisotropic medium was given by Chang et al. [21]. In this present paper, a different method is employed to develop the same solution using a clearer and mathematically simpler mathematical effort.

Transient diffusion due to a source $S(t,x,y,x_p,y_p)$ in a 2D whole space with an anisotropic diffusive medium is governed by the following nonhomogeneous PDE,

$$\frac{\partial u}{\partial t} - \nabla \cdot D \cdot \nabla u = S(t,x,y,x_p,y_p). \tag{1}$$

Here $t>0$ and $(x,y,x_p,y_p) \in U$, where $U \subset \mathbb{R}^2$ is open. The unknown is $u : \bar{U} \times [0,\infty) \to \mathbb{R}$, and $u = u(t,x,y)$ is the diffusion field function, such as temperature or chemical concentration in a layered solid. D is the diffusivity tensor which is symmetric and positive definite. $S : U \times [0,\infty) \to \mathbb{R}$ is a source function. If a point source exhibits an instantaneous spike at time 0, the source function can be given by,

$$S(t,x,y,x_p,y_p) := \delta(t)\delta(x-x_p)\delta(y-y_p), \quad (x,y,x_p,y_p) \in \mathbb{R}, \quad t \geqslant 0, \tag{2}$$

where δ represents the delta function, $P(x_p,y_p), (x_p,y_p) \in \mathbb{R}$, is the source location. δ is called the primary point source function. However, if the amplitude of a point source is arbitrarily transient, the delta function for time is replaced by $I(t)$. If $I(t) = c$, where c is a constant, then the function is called the primary static source function.

The fundamental solution, of PDE (1) with source function (2) is subject to the radiation condition (boundary condition at infinity, $u(t,x,y)|_{x\to\pm\infty, y\to\pm\infty} = 0$), and an initial value condition, $(u(t,x,y)|_{t=0} = 0)$.

Definition 1 (Fundamental solution). The function

$$u(t,x,y) := \begin{cases} 0 & (x,y) \in \mathbb{R}, \quad t<0, \\ \dfrac{\sqrt{\det[\sigma]}}{4\pi t} \mathrm{e}^{-\eta^2/4t} & (x,y) \in \mathbb{R}, \quad t>0, \end{cases} \tag{3}$$

is called the fundamental solution of the anisotropic diffusion governed by PDE (1) with source function (2), where σ is the inverse matrix of diffusivity tensor D. η is called anisotropic weighted diffusion time lapse (AWDTL) given by

$$\eta := \sqrt{(x-x_p, y-y_p) \cdot \sigma \cdot (x-x_p, y-y_p)}. \tag{4}$$

Note that AWDTL means the time lapse in the diffusion process between the source point $P(x_p,y_p)$ and the observation point (x, y), and that $u(t, x, y)$ is singular at $t = 0$ when the point $(x, y) = (x_p, y_p)$.

Theorem 1 (Canonical form of the governing equation). *If the governing equation* (1) *with source function* (2) *has the following canonical form*:

$$\frac{\partial u(t,\eta)}{\partial t} - \nabla^2 u(t,\eta) = \sqrt{\det[\sigma]}\delta(t)\delta(\eta) \quad (\eta \in \mathbb{R},\ t \geqslant 0), \tag{5}$$

its fundamental solution is the function given by (3) *in which the* η *is defined by* (4).

Proof. To prove this theorem, we need to (i) justify that Eq. (5) is the canonical form of the governing equation (1); (ii) validate the fundamental solution with its point source function at any point in $\mathbb{R}^3$ in the time domain $t > 0$; (iii) show that the radiation boundary condition is satisfied; and (iv) show that the zero initial conditions are also satisfied.

(i) Apply the following standard rotation and translation coordinate transformations for (1),

$$\xi_1 = \alpha_{11}(x - x_p) + \alpha_{12}(y - y_p),$$
$$\xi_2 = \alpha_{21}(x - x_p) + \alpha_{22}(y - y_p), \tag{6}$$

and let the term with $\frac{\partial^2 u}{\partial \xi_1 \partial \xi_2}$ in the left-hand side of (1) be zero. Then (1) with source function (2) becomes

$$\frac{\partial u(t,\xi_1,\xi_2)}{\partial t} - \left(\lambda_1 \frac{\partial^2 u(t,\xi_1,\xi_2)}{\partial \xi_1^2} + \lambda_2 \frac{\partial^2 u(t,\xi_1,\xi_2)}{\partial \xi_2^2}\right) = \delta(t)\delta(\xi_1)\delta(\xi_2) \quad (\xi_1,\xi_2) \in \mathbb{R},\ t > 0, \tag{7}$$

where λ_1 and λ_2 are eigenvalues of the tensor D, and $D = \alpha_{k,i}\lambda_k\alpha_{j,k}$. Now introduce two new variables which stretch the coordinate system in the ξ_1 and ξ_2 directions

$$\xi_1 = \eta_1\sqrt{\lambda_1}, \quad \xi_2 = \eta_2\sqrt{\lambda_2}. \tag{8}$$

Then (7) becomes (see p. 309 [28])

$$\frac{\partial u(t,\eta_1,\eta_2)}{\partial t} - \left(\frac{\partial^2 u(t,\eta_1,\eta_2)}{\partial \eta_1^2} + \frac{\partial^2 u(t,\eta_1,\eta_2)}{\partial \eta_2^2}\right) = \frac{1}{\sqrt{\lambda_1\lambda_2}}\delta(t)\delta(\eta_1)\delta(\eta_2) \quad (\eta_1,\eta_2) \in \mathbb{R},\ t > 0. \tag{9}$$

Using the property of matrix symmetry, the relation $\frac{1}{\lambda_1\lambda_2} = \det[\sigma] = \det[D^{-1}]$ is obtained. It is obvious that (9) and (5) are the same thing, the solution for the two being [29]

$$u(t,\eta_1,\eta_2) = \begin{cases} 0 & (\eta_1,\eta_2) \in \mathbb{R},\ t < 0, \\ \dfrac{\sqrt{\det(\sigma)}}{4\pi t} e^{-(\eta_1^2+\eta_2^2)/4t} & (\eta_1,\eta_2) \in \mathbb{R},\ t > 0. \end{cases} \tag{10}$$

We see that (10) is the same as (3) with $\eta^2 = \eta_1^2 + \eta_2^2$. From (6) and (8), it follows from noting $\sigma_{i,j} = \alpha_{k,i}\lambda_k^{-1}\alpha_{j,k}$ that

$$|\eta| = \sqrt{\sigma_{11}(x - x_p)^2 + \sigma_{22}(y - y_p)^2 + 2\sigma_{12}(x - x_p)(y - y_p)}. \tag{11}$$

Thus, (4) follows readily.

(ii) We consider the source point $(x = x_p, y = y_p)$, and the surrounding field (any point except the source point in the medium) separately. In the surrounding field, the right side of (1) is zero. Substitute (10) for u in the left-hand side of (1), and we see that the left side of (1) is also zero when $t \geqslant 0$. Thus, (10) is valid in the field of surrounding the source point. At the point (x_p, y_p), when $t \geqslant 0$, we start from (5), which is obtained by coordinate rotation, translation and stretching of (1). Consider a small circular region G of radius $\varepsilon \ll 1$ around the point $(0, 0)$ in the coordinate system (η_1, η_2). It is the point (x_p, y_p) in (x, y) coordinate system. Because of the circular symmetry in the 2D domain, (5) can be written in the form

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u}{\partial \eta}\right) = \sqrt{\det[\sigma]}\delta(\eta)\delta(t) \quad (\eta \in \mathbb{R}, t \geqslant 0). \tag{12}$$

Integrate (12) over the circle G and the time domain $T = (0, \tau)$. This gives

$$\int\int_G\int_T (-\Delta u + u_t)\,dt\,dS = \sqrt{\det[\sigma]}. \tag{13}$$

The first term of the left-hand side of (13) can be computed using Gauss' theorem, producing

$$\int\int_G div\,\mathbf{F}\,dS = \int_C (\mathbf{F}\cdot n)\,dc, \tag{14}$$

where $\mathbf{F}$ is a 2D vector field in the planar domain G having area element dS, and boundary C with arc element dc, and outward unit normal $\mathbf{n}$. The integral (14) is computed within the interior of a circle of radius ε centered at the origin (in (η_1, η_2) coordinate system). Thus,

$$\int\int_{G_\varepsilon} -\Delta u\,dS = \int_C -u_\eta\,dc. \tag{15}$$

Using the solution (10), by setting $\mathbf{F} = \nabla u$ within (15) and letting $dc \approx r\,d\theta$ in the circle of radius ε, we obtain

$$\int_C -u_\eta\,dc = \int \frac{\sqrt{\det[\sigma]}\,e^{-\eta^2/4t}\eta}{8\pi t^2}\,dc = \int_0^{2\pi} \frac{\sqrt{\det[\sigma]}\,e^{-\varepsilon^2/4t}\varepsilon^2}{8\pi t^2}\,d\theta = \frac{\sqrt{\det[\sigma]}\,e^{-\varepsilon^2/4t}\varepsilon^2}{4t^2}. \tag{16}$$

When we integrate (16) for t on $(0,\tau)$, the first term of the left-hand side of (13) becomes

$$\int\int_G\int_T -\Delta u\,\mathrm{d}t\,\mathrm{d}S = \sqrt{\det[\sigma]}\,\mathrm{e}^{-\varepsilon^2/4\tau}, \tag{17}$$

$$\lim_{\varepsilon\to 0}\sqrt{\det[\sigma]}\,\mathrm{e}^{-\varepsilon^2/4\tau} = \sqrt{\det[\sigma]}. \tag{18}$$

Since $0<\varepsilon<1$

$$\lim_{\tau\to 0}\sqrt{\det[\sigma]}\,\mathrm{e}^{-\varepsilon^2/4\tau} = 0. \tag{19}$$

Substituting (10) for u in the second term of the left-hand side of (13), we have

$$\int\int_G\int_T u_t\,\mathrm{d}t\,\mathrm{d}S = \int\int_G\int_T \frac{\sqrt{\det[\sigma]}\,\mathrm{e}^{-\eta^2/4t}\eta^2}{16\pi t^3} - \frac{\sqrt{\det[\sigma]}\,\mathrm{e}^{-\eta^2/4t}}{4\pi t^2}\,\mathrm{d}t\,\mathrm{d}S. \tag{20}$$

Let $\mathrm{d}S \approx \frac{1}{2}\eta^2\,\mathrm{d}\theta$ in the circle of radius ε with t on $(0,\tau)$. Then,

$$\begin{aligned}\int_0^\tau\int_0^\pi &\frac{\sqrt{\det[\sigma]}\,\mathrm{e}^{-\eta^2/4t}\eta^2(-4t+\eta^2)}{32\pi t^3}\,\mathrm{d}\theta\,\mathrm{d}t\\ &= \int_0^\tau \frac{\sqrt{\det[\sigma]}\,\mathrm{e}^{-\eta^2/4t}\varepsilon^2(-4t+\varepsilon^2)}{18t^3}\,\mathrm{d}t\\ &= \left[\frac{2\det[\sigma]\,\mathrm{e}^{-\varepsilon^2/4t}\varepsilon^2}{9t}\right]_{t=0}^{t=\tau}.\end{aligned} \tag{21}$$

Since $0<\varepsilon<1$, we have

$$\lim_{\tau\to 0}\frac{2\det[\sigma]\,\mathrm{e}^{-\varepsilon^2/4\tau}\varepsilon^2}{9\tau} = \lim_{t\to 0}\frac{2\det[\sigma]\,\mathrm{e}^{-\varepsilon^2/4t}\varepsilon^2}{9t} = 0. \tag{22}$$

Thus, (21) is zero, the governing equation (1) with the source function (2) is satisfied.

When the observation point (x,y) goes to infinity ($x\to\infty$ and/or $y\to\infty$), the AWDTL must be infinite. One can clearly see that the solution (10) goes to zero if the AWDTL goes to infinity. It is obvious that when $t\to 0$ the solution (10) is zero. Thus, conditions (iii) and (iv) are justified. □

3. TIPs for half-space

Consider a half-space, where the diffusion governing equation (1) with the source function (2) is subject to the following first or second boundary condition on the surface given by (23) or (24):

$$u(t,x,y)|_{y=f(x)} = 0 \quad (x\in\mathbb{R}, t>0), \tag{23}$$

or

$$[\nabla_u(t,x,y)\cdot\sigma\cdot\nabla(y-f(x))]_{y=f(x)} \equiv 0 \quad (x,y\in\mathbb{R}, t>0), \tag{24}$$

where $y=f(x)$ is the 2D surface function of the half-space and $\mathbf{n}$ is the normal to the surface $y=f(x)$. Also, the radiation boundary conditions $u(t,x,y)|_{x\to\pm\infty,y\to\infty}=0$, and initial condition $u(t,x,y)|_{t\to 0}=0$, have to be satisfied, where $(x,y)\in\mathbb{R}, t>0$.

When the first boundary condition (23) is imposed on the surface of the half-space, it is then called a total absorption boundary or total absorber for the transient diffusion field. However, when in the same space, a transient diffusion field u satisfies the second boundary condition (24), the interface is called a total diffusion reflection boundary, total diffusion reflector or diffusion barrier.

Definition 2 (Reflection or absorption image). Suppose a point $P(x_p,y_p)$ and a straight line $y=bx+c$ are located in an anisotropic diffusion medium with inverse diffusivity tensor σ, and that a point $R(x_r,y_r)$ can be found such that the following identical relation

$$|\eta_p|_{y=bx+c} \equiv |\eta_r|_{y=bx+c}, \tag{25}$$

is satisfied, where η_p is the AWDTL between the point P and the field observation point (x,y), and η_r is the AWDTL between the point R and the same field point (x,y). Then, the point R is called either a reflection or an absorption image (RAI) of P, subject to the surface boundary condition on the straight line $y=bx+c$. If P is a primary diffusion source point, then R is called the RAI source point of P.

Lemma 1 (The location of RAI point). *Consider an anisotropic diffusion medium with the inverse diffusivity tensor* $\sigma = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}$. *Suppose that a diffusion point source is located at the point* $P(x_p,y_p)$. *Then, the location of its RAI point created by reflection at the interface* $y=bx+c$ *is given by*

$$\begin{aligned}x_r &= \frac{x_p(\sigma_{11}-b^2\sigma_{22}) - 2(c-y_p)(b\sigma_{22}+\sigma_{12})}{\sigma_{11}+b(2\sigma_{12}+b\sigma_{22})},\\ y_r &= \frac{2(c+bx_p)(\sigma_{11}+b\sigma_{12}) + y_p(b^2\sigma_{22}-\sigma_{11})}{\sigma_{11}+b(2\sigma_{12}+b\sigma_{22})}.\end{aligned} \tag{26}$$

Proof. Considering $\sigma = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}$, $P(x_p,y_p)$ and $R(x_r,y_r)$, and noting (11), the following two AWDTLs are obtained:

$$\begin{aligned}\eta_p &= \sqrt{r_p\cdot\sigma\cdot r_p},\\ \eta_r &= \sqrt{r_r\cdot\sigma\cdot r_r},\end{aligned} \tag{27}$$

where $r_p=(x-x_p, y-y_p)$ and $r_r=(x-x_r, y-y_r)$. Substituting (27) for η_p and η_r in (25), noting $y=bx+c$, the following equations are obtained:

$$\begin{aligned}&(x_p-x_r)(x_p+x_r)\sigma_{11} + (-2cy_p+y_p^2+2cy_r-y_r^2)\sigma_{22}\\ &\quad - 2(c(x_p-x_r) - x_py_p + x_ry_r)\sigma_{12} = 0,\\ &(x_p-x_r)\sigma_{11} + b(y_p-y_r)\sigma_{22}\\ &\quad + (b(x_p-x_r)+y_p-y_r)\sigma_{12} = 0.\end{aligned} \tag{28}$$

Solving (28) for x_r and y_r, we obtain (26). □

Remarks. (i) From (26), it can be seen that the RAI location is complex and affected by every parameter of the point source, medium tensor and interface line system. However, if the interface is a horizontal line $y = c$, (26) becomes

$$x_r = \frac{2(-c + y_p)\sigma_{12}}{\sigma_{11}} + x_p,$$
$$y_r = 2c - y_p, \qquad (29)$$

where x_r is affected by the distance from the source point to the interface $y = c$ and ratio $\frac{\sigma_{12}}{\sigma_{11}}$. In this case, y_r is independent of the diffusivity tensor. Moreover, if the diffusivity tensor is a diagonal matrix in which $\sigma_{12} = 0$, (29) becomes

$$x_r = x_p,$$
$$y_r = 2c - y_p. \qquad (30)$$

We see that the RAI location is independent of the tensor of the medium containing the source. It depends only on the position of the source with respect to the interface. (ii) The position of the RAI created by the interface shown in Fig. 1 is always in the second medium.

Theorem 2 (Green′s functions for half − space with totally reflecting or totally absorbing). *Consider a point source with source function* (2) *located in an anisotropic half-space, with the surface boundary defined by* $y = bx + c$, *and the inverse diffusive tensor* $\sigma = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}$ *which is symmetric and positive definite in the half-space. If the surface is a total reflector and the diffusion governing PDE is given by* (1) *with source function* (2), *then Green's function for the half-space is*

$$u(t, x, y, x_p, y_p) := \frac{\sqrt{\det[\sigma]}}{4\pi t}\left(e^{-\eta_p^2/4t} + e^{-\eta_r^2/4t}\right)$$
$$(x, y, x_p, y_p) \in \mathbb{R}, \quad t \geqslant 0. \qquad (31)$$

If the surface is a total absorber with the same governing PDE (1) *and source function* (2), *Green's function for the half-space is*

$$u(t, x, y, x_p, y_p) := \frac{\sqrt{\det[\sigma]}}{4\pi t}\left(e^{-\eta_p^2/4t} - e^{-\eta_r^2/4t}\right)$$
$$(x, y, x_p, y_p) \in \mathbb{R}, \quad t \geqslant 0. \qquad (32)$$

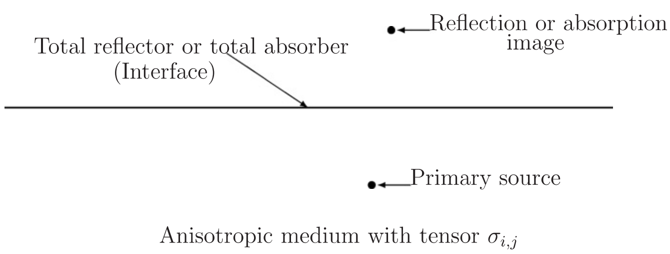

Fig. 1. Anisotropic half-space with source and its image.

Here, η_p *and* η_r *are given in* (27) *in which the source point is given by* (26), (29) *or* (30) *depending on the interface and tensor conditions.*

Proof. To prove this theorem, we need to justify the solution to (i) the governing equation (1), (ii) the boundary condition (24) and (iii) the initial condition ($u(t, x, y)|_{t=0} = 0$) are satisfied.

(i) From Theorem (1), we can see that the governing equation (1) is satisfied. This is because the solutions given by (31) and (32) are actually the superposition of two fundamental solutions with primary and reflection and transmission image (RTI) sources, respectively.

(ii) For convenience, let $y = c$. Then, the surface boundary conditions become

$$u(t, x, y)|_{y=c} = 0 \quad (x \in \mathbb{R}, \quad t > 0), \qquad (33)$$

$$[\nabla_u(t, x, y) \cdot \sigma \cdot \nabla(y - c)]_{y=c} \equiv 0 \quad (x, y \in \mathbb{R}, t > 0). \qquad (34)$$

Noting (27) and (29) then substituting (31) and (32), respectively, for u in the first and the second equations of (34), we can see that the boundary conditions are satisfied. When $x \to \pm\infty$ and/or $y \to \infty$, η_p and η_r goes to infinity and when (31) and (32) go to zero we see that the radiation boundary condition is also satisfied.

(iii) It is obvious that when $t \to 0$, the solutions (31) and (32) go to zero. □

4. TIPs for bimaterial

Both reflection and transmission (*R&T*) occur when a diffusion field encounters a boundary across which temperature and heat flux are continuous. If a diffusion field is initiated by a point source, the boundary with continuous boundary conditions causes the creation of a reflection image and a transmission image of the source, so that its boundary conditions are satisfied. In this section, we will examine this issue.

Suppose that a diffusion field encountering an interface satisfies the following two identical conditions,

$$u_1(t, x, y)|_{y=bx+c} \equiv u_2(t, x, y)|_{y=bx+c} \qquad (35)$$

and

$$[\nabla u_1(t, x, y) \cdot \sigma_1 \cdot \nabla(y - (bx + c))]_{y=bx+c}$$
$$\equiv [\nabla u_2(t, x, y) \cdot \sigma_2 \cdot \nabla(y - (bx + c))]_{y=(bx+c)}, \qquad (36)$$

then this interface is called a boundary with continuous boundary conditions to the diffusion field. These two conditions are called continuous boundary conditions subject to the governing equation (1), where u_1 and u_2 are respectively the diffusion field functions (temperature) in the first and second media, and $y = bx + c$ is the equation to the interface between these two media. If a space is composed of two anisotropic media bonded along

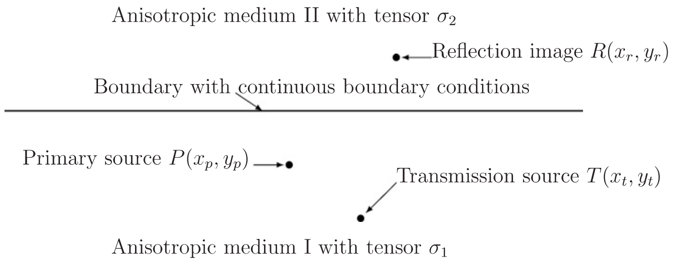

Fig. 2. Bimaterial space with source and its images.

$y = bx + c$ in a 2D space, the anisotropic tensors for the two half-spaces are σ_1 and σ_2 and the bonded interface is a boundary with continuous boundary conditions. This physical model is called a bimaterial space as shown in Fig. 2.

Definition 3 (RTIs). Suppose a bimaterial is described by the two anisotropic inverse diffusivity tensors σ_1 and σ_2 with limitation $(\sigma_1)_{11} = (\sigma_2)_{11}$, a point $P(x_p, y_p)$ $(x_p, y_p \in \mathbb{R})$ which is located in the first medium of the bimaterial space, and a straight line $y = bx + c$ which is the bimaterial interface with the continuous boundary conditions. Further, suppose that we can find two points $R(x_r, y_r)$ and $T(x_t, y_t)$ $(x_r, y_r, x_t, y_t \in \mathbb{R})$ which satisfy the following identical relations:

$$|\eta_p|_{y=bx+c} \equiv |\eta_r|_{y=bx+c} \tag{37}$$

and

$$|\eta_p|_{y=bx+c} \equiv |\eta_t|_{y=bx+c}. \tag{38}$$

Then, $R(x_r, y_r)$ is called the reflection image point of $P(x_p, y_p)$ subject to the boundary function $y = bx + c$, while $T(x_t, y_t)$ is called the transmission image point of $P(x_p, y_p)$ subject to the same boundary function. Here,

$$\begin{cases} \eta_p = \sqrt{r_p \cdot \sigma_1 \cdot r_p}, \\ \eta_r = \sqrt{r_r \cdot \sigma_1 \cdot r_r}, \\ \eta_t = \sqrt{r_t \cdot \sigma_2 \cdot r_t} \end{cases} \tag{39}$$

are AWDTL from points $P(x_p, y_p), R(x_r, y_r)$ and $T(x_t, y_t)$, respectively, to the field observation point (x, y), where $r_p = (x - x_p, y - y_p)$, $r_r = (x - x_r, y - y_r)$ and $r_t = (x - x_t, y - y_t)$. If $P(x_p, y_p)$ is a primary source point, then $R(x_r, y_r)$ and $T(x_t, y_t)$ are called the reflection and transmission image (RTI) source points of $P(x_p, y_p)$, respectively.

Lemma 2 (RTI location). *In a bimaterial anisotropic diffusion space with inverse diffusivity tensors σ_1 and σ_2 in the first and second media respectively, if a point source $P(x_p, y_p)$ is located in the first medium, the locations of its reflection $R(x_r, y_r)$ and transmission $T(x_t, y_t)$ image source points subject to the interface $y = c$ which is a boundary with continuous boundary conditions, are given by*

$$x_r = 2\alpha_1(y_p - c) + x_p,$$
$$y_r = 2c - y_p \tag{40}$$

and

$$x_t = \beta_{12}(y_p - c) + x_p,$$
$$y_t = \gamma_{12}(y_p - c) + c, \tag{41}$$

where

$$\alpha_1 = \frac{(\sigma_1)_{12}}{(\sigma_1)_{11}},$$

$$\beta_{12} = \frac{(\sigma_1)_{12}}{(\sigma_1)_{11}} - \frac{(\sigma_2)_{12}\sqrt{-(\sigma_1)_{12}^2 + (\sigma_1)_{11}(\sigma_1)_{22}}}{(\sigma_1)_{11}\sqrt{-(\sigma_2)_{12}^2 + (\sigma_1)_{11}(\sigma_2)_{22}}},$$

$$\gamma_{12} = \frac{\sqrt{-(\sigma_1)_{12}^2 + (\sigma_1)_{11}(\sigma_1)_{22}}}{\sqrt{-(\sigma_2)_{12}^2 + (\sigma_1)_{11}(\sigma_2)_{22}}}.$$

Here, in α_1, β_{12} and γ_{12} the subscripts "1" *and* "2" *represent the first and second media respectively, and* "12" *represents a transition from the first medium to second medium.*

Proof. The proof for the location of the reflection image (40) is the same as what is given in Lemma 1. Hence, we will now only give the proof for the transmission image location. Considering $P(x_p, y_p)$ and $T(x_t, y_t)$, where for convenience $y = c$, and substituting the expression of η_p and η_t given by (39) into the identity relation (38), we obtain

$$\begin{cases} 2x_p(\sigma_1)_{11} - 2x_t(\sigma_1)_{11} - 2h(\sigma_1)_{12} + 2y_p(\sigma_1)_{12} \\ \quad +2h(\sigma_2)_{12} - 2y_t(\sigma_2)_{12} = 0, \\ -x_p^2(\sigma_1)_{11} + x_t^2(\sigma_1)_{11} + 2hx_p(\sigma_1)_{12} - 2x_p y_p(\sigma_1)_{12} \\ \quad -h^2(\sigma_1)_{22} + 2hy_p(\sigma_1)_{22} - y_p^2(\sigma_1)_{22} - 2hx_t(\sigma_2)_{12} \\ \quad +2x_p y_p(\sigma_2)_{12} + h^2(\sigma_2)_{22} - 2hy_t(\sigma_2)_{22} + y_t^2(\sigma_2)_{22} = 0. \end{cases} \tag{42}$$

Solving (42) for x_t and y_t, gives (41). $\square$

Remarks. From the above two sections, we find that: (i) for a total reflector, only one image of the source is generated. However, at a conductive boundary, there are two image sources produced, the reflection and transmission images; (ii) the coordinates of the reflection image are dependent on the location of the primary point source, the reflection boundary function and the diffusivity tensor of the medium containing the source. However, if the inverse diffusivity tensor is a diagonal matrix and the boundary with continuous boundary conditions is a horizontal interface, the coordinates of the image of the source are only dependent on the position of the source point.

Theorem 3 (Green's function for bimaterial). *Consider a bimaterial space with two inverse diffusivity tensors, $\sigma_1 = \begin{bmatrix} (\sigma_1)_{11} & (\sigma_1)_{12} \\ (\sigma_1)_{12} & (\sigma_1)_{22} \end{bmatrix}$ and $\sigma_2 = \begin{bmatrix} (\sigma_2)_{11} & (\sigma_2)_{12} \\ (\sigma_2)_{12} & (\sigma_2)_{22} \end{bmatrix}$, respectively, with the condition that in the first and second half-space media $(\sigma_1)_{11} = (\sigma_2)_{11}$. A boundary with continuous boundary conditions is located at $y = c$, and a primary point source with source function* (2) *is located in the first anisotropic* *medium. The diffusion governing equation* (1) *is subject to initial conditions*:

$$u_1(t,x,y)|_{t\to 0}=0,$$
$$u_2(t,x,y)|_{t\to 0}=0, \tag{43}$$

continuous boundary conditions:

$$u_1(t,x,y)|_{y=c}=u_2(t,x,y)|_{y=c}, \tag{44}$$

$$[\nabla u_1(t,x,y)\cdot\sigma_1\cdot\nabla(y-c)]_{y=c}\equiv[\nabla u_2(t,x,y)\cdot\sigma_2\cdot\nabla(y-c)]_{y=c}, \tag{45}$$

and radiation boundary conditions:

$$u_1(t,x,y)|_{x\to\pm\infty, y\to\infty}=0,$$
$$u_2(t,x,y)|_{x\to\pm\infty, y\to-\infty}=0.$$

Green's functions for this bimaterial space are

$$u_1(t,x,y,x_p,y_p){:=}\frac{\sqrt{\det[\sigma_1]}}{4\pi t}(\mathrm{e}^{-\eta_p^2/4t}+\theta_{12}\,\mathrm{e}^{-\eta_r^2/4t}),$$
$$u_2(t,x,y,x_p,y_p){:=}\vartheta_{12}\frac{\sqrt{\det[\sigma_2]}}{4\pi t}\,\mathrm{e}^{-\eta_t^2/4t},$$
$$(x,y,x_p,y_p)\in\mathbb{R},\quad t>0, \tag{46}$$

where u_1 *and* u_2 *are the diffusion fields in the first and second media respectively*, θ_{12} *and* ϑ_{12} *are called reflection and transmission coefficients from medium I to medium II respectively, which are given by*

$$\theta_{12}=\frac{\sqrt{\det[\sigma_2]}-\sqrt{\det[\sigma_1]}}{\sqrt{\det[\sigma_2]}+\sqrt{\det[\sigma_1]}},$$
$$\vartheta_{12}=\frac{2\sqrt{\det[\sigma_1]}}{\sqrt{\det[\sigma_2]}+\sqrt{\det[\sigma_1]}}, \tag{47}$$

while η_p, η_r *and* η_t *are given in* (39). *We refer to the medium containing the source as the source medium.*

Proof. The proof is similar to the proof given for Theorem 2. We also need to demonstrate that (i) the governing equation (1) with source function (2) is satisfied; (ii) the initial conditions (43) are satisfied; and (iii) the boundary conditions (44) and (45) are satisfied. However, the proof for (i) and (ii) are basically the same as that given for Theorem 2. It suffices to justify that boundary conditions (44) and (45) are applicable.

Substituting the coordinates of the reflection point given in (40) for (x_r, y_r) into η_r given in (39), the coordinates of the transmission points given in (41) for (x_t, y_t) into η_t given in (39), and then substituting (39) into (46) for η_p^2, η_r^2 and η_t^2, we obtain a full expression for the solution to the bimaterial diffusion problem. Substituting this full expression (46) for u_1 and u_2 in boundary condition (44), yields,

$$\sqrt{-(\sigma_1)_{12}^2+(\sigma_1)_{11}(\sigma_1)_{22}}+\theta_{12}\sqrt{-(\sigma_1)_{12}^2+(\sigma_1)_{11}(\sigma_1)_{22}}$$
$$-\vartheta_{12}\sqrt{-(\sigma_2)_{12}^2+(\sigma_1)_{11}(\sigma_2)_{22}}\equiv 0. \tag{48}$$

Then, by substituting the same full expression (46) for u_1 and u_2 in the boundary condition (45), the following relation is obtained:

$$2(-1+\theta_{12}+\vartheta_{12})\equiv 0. \tag{49}$$

To solve (48) and (49) for θ_{12} and ϑ_{12}, we obtain the solution (47). □

Remarks. (i) From the above two sections, it can be seen that the reflection coefficient θ_{12} depends on the type of reflection boundary conditions. When the boundary is a total reflector, θ_{12} is equal to 1 (31). When the boundary is a total absorber, θ_{12} is equal to -1 (32). However, if the boundary is a boundary with continuous boundary conditions where partial reflection and transmission occur, the value of θ_{12} is given by the first expression of (47). (ii) ϑ_{12} represents the transmission coefficient. (iii) Both coefficients depend only on the bimaterial diffusivity tensors and are independent of the source point locations.

5. Medium formation and DI for 2D space

Developing an analytical solution for 2D space is very difficult when the geometrical structure of the interface within inhomogeneous media is complex. Imposing limitations on the media, however, may allow analytical solutions to be derived. The TIP method given in this paper can be used to construct analytical solutions for more complex media than in the introduction given in the first section. However, this method is directly dependent on how the geometrical structure of the interface formation in a space is described mathematically. With different mathematical descriptions of the same medium formation, we obtain different mathematical expressions for the solution. Concerning this property of the analytical solutions, a suitable description of the media is briefly introduced by the following two definitions and the simple space given in Fig. 3. In this example, the space in Fig. 3 consists of four 2D rectangular media where the continuous boundary conditions apply on the interfaces between the media. We assign "NM" to represent the Nth medium number and "NB" to represent the Nth boundary number. It is clear that the space has 4 media and 12 boundaries with continuous boundary conditions.

Definition 4 (Front medium and back media). If a medium is bound by simple connected straight lines and is acted on by a source, it is called a front medium (FM) with respect to the source. Simple connected boundaries refer to those boundaries which consist of straight lines such that for any arbitrary straight line starting from the source and

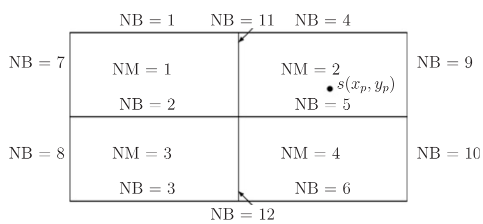

Fig. 3. A space with four 2D rectangles.

passing into an adjoining medium, only one boundary intersection occurs. In other words, when a geometrical source of illumination is located at the position of the source, and every boundary is fully illuminated by it, then these boundaries are simply connected. All other directly joining media which are illuminated by such a geometrical source of illumination are called back media (BM).

Remark. In the case of boundaries that extend to infinity surrounding a space, this definition of BM is still valid under the assumption that these infinite boundaries will meet at infinity. It can be easily derived that a reflection image's FM is its primary source's FM, and that the transmission image's FM is the BM of the primary source subject to the concerning boundary. As we can see in Fig. 3, if medium 2 is acted on by source $s(x_s, y_s)$ boundary segments (NB $= 11, 5, 9, 4$) are illuminated, so medium 2 is called FM and media 0 (represents the medium surrounding the four 2D square media), 1 and 4 are called BM of the source. If medium 1 were to contain source $s(x_s, y_s)$ the FM is medium 1 and the BM are media 0, 2 and 3.

Definition 5 (Diffusion instance). A matrix characterizing a source is called a diffusion instance (DI). It must have seven elements {Amp, SLC, SLN, NMM, NBM, BIM, BMM}. Here Amp denotes amplitude of the source; SLC denotes the source location coordinates; SLN is a NM that represents the medium in which the source is located; NMM is a matrix that consists of the NM representing the medium that has been acted on by the source; NBM is a matrix consisting of the NBs which refer to the boundaries that produce the source; BIM is a matrix consisting of NBs, which correspond to the boundaries that are illuminated by the source; and BMM is a matrix which contains the NMs that refer to the BM of the illuminated boundaries by the source.

Remarks. (i) DI is an abstraction drawn from the physical process of a source in a diffusion medium, leading to mathematical expressions for the solution of multi-media diffusion problems with a point source. Consider the application of the definition to the space given in Fig. 3. Amp is the source amplitude, of the primary source, which is given by the source function in the governing PDE. For the RTI source, the amplitude is given by the R&T coefficients that are presented in the following two sections. For the primary source, the SLC are given by the location of the source as (x_s, y_s). For RTIs, the reflection and transmitted image locations are determined by the source location such as (40) and (41). If the SLC are known, the SLN is defined automatically. Always, for a primary source, NMM has one element which is the SLN. However, for RTI sources NMM has more than one element, for example, if the source s is an RTI source in Fig. 3, NMM could be a matrix, (1, 4), which means the source causes diffusion into media 1 and 4 at the same time. We can see that if the point source s in Fig. 3 is an RTI, then this RTI can be produced by multiple boundaries. For example, if s in Fig. 3 is a reflection image source which can be produced by boundary 4 and 7, then the NBM is a matrix, (4,7). It is obvious that BIM is a matrix which contains all FM's BNs when the s is primary source, however, when a source is not given primarily its BIM's elements are dependent on the R&T process of the space concerned. In the given space as shown in Fig. 3, if the s is a transmission image source located in medium 2 the matrix BIM could be (5,6,10,12) and is not (11,1,7,2). As we can see from the definition that BMM is a companion matrix of BIM, when the BIMs is (6,8,11) the source's BMM is $(1, 0, 0)$ in Fig. 3. (ii) From the definition, we can see that DI can represent all kinds of sources, such as primary, reflection and transmission image sources. So from now on, there is no difference between the terms "source" and "DI". (iii) If DI refers to a primary source, we assign pDI to represent it. Sometimes we also use rDI and tDI to represent, respectively, reflection and transmission image sources.

6. TIPs for two-layer space

Consider a two-layer space that has two parallel interfaces, where the first is a total reflector located at $y = 0$ and the second is a boundary with continuous boundary conditions located at $y = c$ as shown in Fig. 4. Let the reflector NB $= 1$ and for the boundary with continuous boundary conditions NB $= 2$, and the NM for the top layer as 1 and for the lower medium infinite extent as 2. In this section we will deal with the TIPs in this two layer-space composed anisotropic media. Green's function will be given and the new concepts of reflection and transmission streams (RTS) and the functions of the streams will be introduced.

Definition 6 (R & T streams, stream matrix and stream function). An RTS is a set of consecutive DIs obtained from a pair of interfaces. The DI's SLC increasingly away from a pair of interfaces by the same amount with each previous DI, and where the amplitude changes in the same proportion each time, each with the same NMM and the same NBM. The stream matrix (SM) contains the streams for the whole concerning space.

The following function is called the *R&T* stream function,

$$sf[t,x,y,i,j,k] := \frac{\sqrt{\det[\sigma_i]}}{4\pi t} \sum_{l=1}^{n_k} c_{jkl}\, \mathrm{e}^{-\mathscr{H}^2_{ijkl}/4t},$$

$$(x,y) \in \mathbb{R}, \quad (i,j,k) \in \mathbb{Z}, \quad t > 0, \tag{50}$$

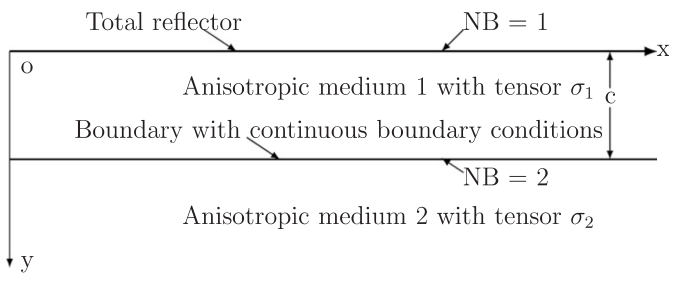

Fig. 4. Two-layer anisotropic space.

where i refers to NM; c_{jkl} is Amp of the lth DI in the kth stream of the jth stream matrix, n_k is the length of the kth stream which can be finite or infinite; and $\mathscr{H}^2_{ijkl}$ is given by,

$$\mathscr{H}^2_{ijkl} := \{x - x_s^{jkl}, y - y_s^{jkl}\} \cdot \sigma_i \cdot \{x - x_s^{jkl}, y - y_s^{jkl}\},$$
$$(x, y) \in \mathbb{R}, \ (i, j, k, l) \in \mathbb{Z}. \tag{51}$$

Here x_s^{jkl} and y_s^{jkl} are SLCs for the lth DI in the kth stream of the jth SM.

Lemma 3 (RTSs for a two − layer space). *In a two-layer space such as that given in Fig. 4, when the source is located in the first layer, the RTSs are given in the space as follows*:

$$s_{11} = \{\theta^i_{12}, \{x_p + 2\alpha_1(y_p + ic), -y_p - 2ic\}, -1, \{1\}, \{1\}, \{2, 1\}, \{2, 0\}\} \quad (i = 0, 1, 2, 3, \ldots), \tag{52}$$

$$s_{12} = \{\theta^i_{12}, \{x_p - 2i\alpha_1 c, y_p + 2ic\}, 2, \{1\}, \{2\}, \{1, 2\}, \{0, 2\}\} \quad (i = 1, 2, 3, \ldots), \tag{53}$$

$$s_{13} = \{\theta^i_{12}\vartheta_{12}, \{x_p + 2(ci + y_p)\alpha_1 - (c + 2ci + y_p)\beta_{12}, c - (c + 2ci + y_p)\gamma_{12}\}, -2, \{2\}, \{2\}, \{2\}, \{1\}\} \quad (i = 0, 1, 2, 3, \ldots), \tag{54}$$

$$s_{14} = \{\theta^i_{12}, \{x_p + 2ic\alpha_1, y_p - 2ic\}, -1, \{1\}, \{1\}, \{2, 1\}, \{2, 0\}\} \quad (i = 1, 2, 3, \ldots), \tag{55}$$

$$s_{15} = \{\theta^i_{12}, \{x_p + 2(y_p - ic)\alpha_1, -y_p + 2ic\}, 2, \{1\}, \{2\}, \{1, 2\}, \{0, 2\}\} \quad (i = 1, 2, 3, \ldots), \tag{56}$$

$$s_{16} = \{\theta^i_{12}\vartheta_{12}, \{x_p + 2ci\alpha_1 + (-c - 2ci + y_p)\beta_{12}, c + (-c(1 + 2i) + y_p)\gamma_{12}\}, -2, \{2\}, \{2\}, \{2\}, \{1\}\} \quad (i = 0, 1, 2, 3, \ldots), \tag{57}$$

$$pDI = \{1, \{x_p, y_p\}, 1, \{1\}, \{0\}, \{1, 2\}, \{0, 2\}\}. \tag{58}$$

When the source is located in the second layer, there are three streams:

$$s_{21} = \{\theta^i_{12}\vartheta_{21}, \{x_p + (-c + y_p)\beta_{21} + 2\alpha_1(c + ci + (-c + y_p)\gamma_{21}), - c(1 + 2i) + (c - y_p)\gamma_{21}\}, -1, \{1\}, \{1\}, \{2, 1\}, \{2, 0\}\} \quad (i = 0, 1, 2, 3, \ldots), \tag{59}$$

$$s_{22} = \{\theta^i_{12}, \vartheta_{21}, \{x_p - 2ci\alpha_1 + (-c + y_p)\beta_{21}, c + 2ci + (-c + y_p)\gamma_{21}\}, 2, \{1\}, \{2\}, \{1, 2\}, \{0, 2\}\} \quad (i = 0, 1, 2, 3, \ldots), \tag{60}$$

$$s_{23} = \{\theta^i_{12}\vartheta_{12}\vartheta_{21}, \{x_p + (-c + y_p)\beta_{21} + \beta_{12}(-2c(1 + i) + (c - y_p)\gamma_{21}) + 2\alpha_1(c(1 + i) + (-c + y_p)\gamma_{21}), c + \gamma_{12}(-2c(1 + i) + (c - y_p)\gamma_{21})\}, -2, \{2\}, \{2\}, \{2\}, \{1\}\} \quad (i = 0, 1, 2, 3, \ldots), \tag{61}$$

with two additional DIs:

$$pDI = \{1, \{x_p, y_p\}, 2, \{2\}, \{0\}, \{2\}, \{1\}\}, \tag{62}$$

$$rDI = \{\theta_{21}, \{2\alpha_2(y_p - c) + x_p, 2c - y_p\}, - 2, \{2\}, \{2\}, \{2\}, \{1\}\}, \tag{63}$$

where

$$\gamma_{21} = 1/\gamma_{12},$$
$$\theta_{21} = \frac{\sqrt{\det[\sigma_1]} - \sqrt{\det[\sigma_2]}}{\sqrt{\det[\sigma_2]} + \sqrt{\det[\sigma_1]}},$$
$$\vartheta_{21} = \frac{2\sqrt{\det[\sigma_2]}}{\sqrt{\det[\sigma_2]} + \sqrt{\det[\sigma_1]}},$$
$$\beta_{21} = \frac{(\sigma_2)_{12}}{(\sigma_2)_{11}} - \frac{(\sigma_1)_{12}\sqrt{-(\sigma_2)^2_{12} + (\sigma_2)_{11}(\sigma_2)_{22}}}{(\sigma_2)_{11}\sqrt{-(\sigma_1)^2_{12} + (\sigma_2)_{11}(\sigma_1)_{22}}}.$$

Note. For calculating the diffusion field, the pDI given by (58) is included in the stream s_{12} in which the first term is a pDI and i starts from 0 instead of 1.

Proof. Because some of this proof was given in the previous sections, we only outline the proof here. When the point source is located in the first medium, the diffusion field encounters the upper total reflector and the lower boundary with continuous boundary conditions in Fig. 4. Considering only the upper boundary effect as the first step of the RTS, the point source $P(x_p, y_p)$ initiates reflection streams (52), (53) and (54) as indicated by R_i, r_i and T_i of Fig. 5, respectively. Applying the half-space reflection theorem, Theorem 2, and reflection image point (29) for the total reflector at the first reflection we have the first (where $i = 0$) DI in stream (52), see R_0 in Fig. 5. Then, with this reflection image as a source and applying (40), (41) and (47) for the lower boundary, the first DIs in streams (53) (where $i = 1$) and (54) (where $i = 0$) are given. These are also indicated by r_1 and T_0 in Fig. 5. Further, using the first DI of (53) as a new source and applying Theorem 2 and reflection image point (29) again for the upper reflector, we obtain the second DI of stream (52) (where $i = 1$) which is indicated by R_1 in Fig. 5. Then, by using this reflection image as a source and applying (40), (41) and (47) for lower boundary again, the second DIs in streams (53) (where $i = 2$) and (54) (when $i = 1$) are obtained. At this point we use the second DI of (53) as a source $P(x_p, y_p)$ and generate

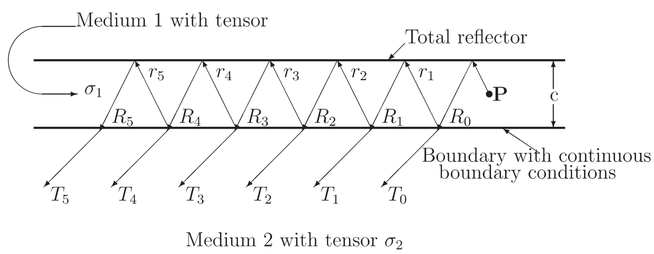

Fig. 5. RTS for upper reflection with source location in medium 1.

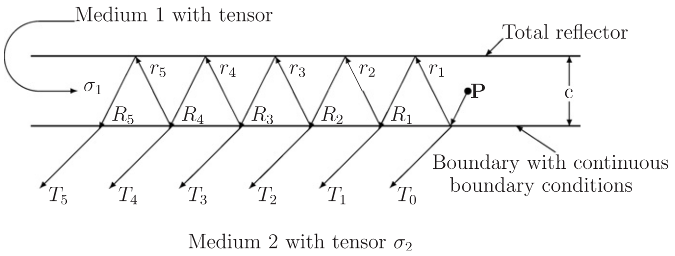

Fig. 6. RTS for down reflection with source location in medium 1.

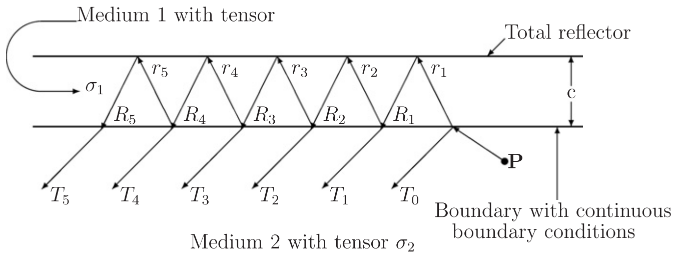

Fig. 7. RTS with source location medium 2.

the upper total reflection image (29), lower reflection (40) and transmission (41) images, and the coefficient correction relation (47) repetitively. Hence we obtain the whole streams (52), (53) and (54).

In Fig. 6 we consider the lower boundary effect at the first step in the reflection stream, the point source $P(x_p, y_p)$ initiates reflection streams (55), (56) and (57) as indicated, respectively, by r_i, R_i and T_i. It is exactly the same as the above process for streams (52)–(54), except for the first step, where we apply (40), (41) and (47) for the lower boundary instead.

When the point source is located in the second medium, the diffusion field acts to the lower boundary first, as shown in r_i, R_i and T_i of Fig. 7. This boundary then produces a reflection image given by (63) and a transmission image as shown by the first DI in (60) (where $i = 0$). We use this transmission image as a point source in the first medium to repeat the exact same process as for streams (52)–(54). Then, we derive streams (59)–(61). □

Remark. In TIP applications, the use of the *R&T* streams is a very effective way to resolve the huge number of *R&T* images which originate from complex domains and higher dimensional diffusion problems. We can treat streams with huge numbers of DIs as a point source in a diffusion domain with complex boundaries. Unfortunately, based on the definition of the streams given in Definition 6, there are still a few DIs that cannot be included in the *R&T* streams, such as those defined by (62) and (63). These DIs are normally the main contributors, such as pDI and the first rDIs or tDIs, to the concerned diffusion field when compared with others included in the *R&T* streams. From Definition 6 we see that when the primary source is located in medium 1, the SM is:

$$sm1 = \{s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}\}. \tag{64}$$

On the other hand, when the primary source is located in medium 2, the SM is:

$$sm2 = \{s_{21}, s_{22}, s_{23}\}. \tag{65}$$

Theorem 4 (Green′s function for two – layer space). *Consider a two-layer space with two inverse diffusivity tensors* $\sigma_1 = \begin{bmatrix} (\sigma_1)_{11} & (\sigma_1)_{12} \\ (\sigma_1)_{12} & (\sigma_1)_{22} \end{bmatrix}$ *and* $\sigma_2 = \begin{bmatrix} (\sigma_1)_{11} & (\sigma_2)_{12} \\ (\sigma_2)_{12} & (\sigma_2)_{22} \end{bmatrix}$ *in first and second layers, respectively. Then it has a total reflector at* $y = 0$ *and a boundary with continuous boundary conditions located at* $y = c$. *When* u_1 *and* u_2 *are, respectively, diffusion fields in first and second layered media, the diffusion governing equation* (1) *with source function* (2) *is subject to the following initial conditions*:

$$u_1(t,x,y)|_{t\to 0} = 0,$$
$$u_2(t,x,y)|_{t\to 0} = 0,$$

top total reflection boundary condition:

$$[\nabla u_1(t,x,y)\cdot\sigma_1\cdot\nabla(y-c)]_{y=0} \equiv 0, \tag{66}$$

continuous boundary conditions:

$$u_1(t,x,y)|_{y=c} \equiv u_2(t,x,y)|_{y=c}, \tag{67}$$

$$[\nabla u_1(t,x,y)\cdot\sigma_1\cdot\nabla(y-c)]_{y=c} \equiv [\nabla u_2(t,x,y)\cdot\sigma_2\cdot\nabla(y-c)]_{y=c}, \tag{68}$$

and radiation boundary conditions:

$$u_1(t,x,y)|_{x\to\pm\infty} = 0,$$
$$u_2(t,x,y)|_{x\to\pm\infty, y\to+\infty} = 0.$$

When a point source with function (2) *is located in the first anisotropic medium, Green's functions for this two-layer space are*

$$u_1(t,x,y) := \sum_{k=1}^{4} sf[t,x,y,1,1,k] \quad (x,y)\in\mathbb{R}, \ t>0 \tag{69}$$

$$u_2(t,x,y) := \sum_{k=1}^{2} sf[t,x,y,2,1,k] \quad (x,y)\in\mathbb{R}, \ t>0, \tag{70}$$

where the function sf is definite in (50). *Eq.* (69) *is the solution of the diffusion field in layer* 1, *where the respective streams for the stream functions are* (52), (53), (55) *and* (56). *Eq.* (70) *is the solution of the diffusion field in layer* 2, *where the corresponding streams for the stream functions are* (54) *and* (57).

When a point source with source function (2) *is located in the second anisotropic medium, Green's functions for this two-layer space are*

$$u_1(t,x,y) := \sum_{k=1}^{2} sf[t,x,y,1,2,k] \quad (x,y)\in\mathbb{R}, \ t>0, \tag{71}$$

$$u_2(t,x,y) := sf[t,x,y,2,2,3] + \frac{\sqrt{\det[\sigma_1]}}{4\pi t}\left(\mathrm{e}^{-\eta_p^2/4t} + \theta_{21}\,\mathrm{e}^{-\eta_r^2/4t}\right)$$
$$(x,y)\in\mathbb{R}, \ t>0, \tag{72}$$

where (71) *is the solution for the diffusion field in layer* 1, *in which the corresponding streams for the stream functions are* (59) *and* (60). *Function* (72) *is the solution of the diffusion field in layer* 2, *the corresponding stream for the stream function is* (61), *and the two single DIs which contribute to the last term of* (72) *are the primary source* (62) *and the first reflection image source* (63).

Proof. Similar to those given in previous sections, the following three aspects need to be confirmed. (i) Satisfying the governing equation. From (1), we see that at any point of the medium the governing equation must be zero, except for the primary source point $P(x_p, y_p)$. As we can see from the RTSs given in Lemma 3 this condition has been satisfied. (ii) The boundary conditions are naturally satisfied. This is because *R&T* processes are initiated by satisfying the boundary conditions given in (66)–(68) and the RTSs are produced by repetitively satisfying the boundary conditions of $y = 0$ and $y = c$. (iii) The initial condition for the solution of a point source diffusion has been already justified in Section 3. Green's function for the two-layer diffusion space is a superposition of all the contributions of the DIs to the diffusion field given in streams (52)–(63). □

7. TIPs for quartmaterial

Up to now in this paper we have considered whole, half-space, bimaterial and two-layer space anisotropic transient diffusion problems. However, for more complex boundary problems, the present principles of *R&T* image for anisotropic diffusion may not be sufficient. For example, in the last section, we considered two parallel-boundaries diffusion problems, in which the reflection images of the first boundary are always treated as a primary sources for the second boundary. If the two boundaries are not parallel, then what is the *R&T* process? And how are the DIs organized in RTSs and further in SMs? In this section, we use a quartmaterial space as shown in Fig. 8 which is quite simple, but is sufficient in presenting a new concept, called reverse images, that can be employed to easily obtain solutions for more complex boundary value problems.

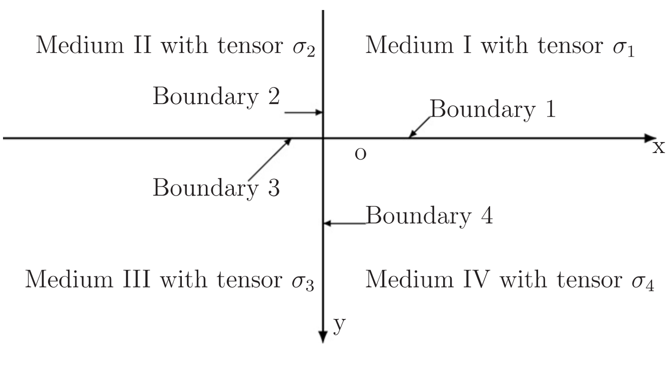

Fig. 8. Quartmaterial anisotropic space

The quartmaterial is composed of four anisotropic media bonded along two straight and perpendicular lines (for convenience, we use $x = 0$ and $y = 0$) in a 2D space, and the bonded four interfaces are boundaries with continuous boundary conditions. The four anisotropic tensors for the quartmaterial are $\sigma_1 = \begin{bmatrix} (\sigma_1)_{11} & (\sigma_1)_{12} \\ (\sigma_1)_{12} & (\sigma_1)_{22} \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} (\sigma_2)_{11} & (\sigma_2)_{12} \\ (\sigma_2)_{12} & (\sigma_2)_{22} \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} (\sigma_3)_{11} & (\sigma_3)_{12} \\ (\sigma_3)_{32} & (\sigma_3)_{32} \end{bmatrix}$ and $\sigma_4 = \begin{bmatrix} (\sigma_4)_{11} & (\sigma_4)_{12} \\ (\sigma_4)_{12} & (\sigma_4)_{22} \end{bmatrix}$. The four media and four boundaries are numbered as shown in Fig. 8.

Definition 7 (Reverse image source). If a DI is in a space, whose SLN is not in BMM, then we name this image source a reverse image source (RIS), with the exception of pDI. In other words, if a source acts on a medium but it cannot illuminate the boundaries of the medium, then this image source is an RIS of the medium.

Remarks. (i) As we have already discussed in previous sections, *R&T* images are determined by identities of AWDTL from a given boundary and an existing primary source. However, a reverse image is basically a "fake" primary source that is determined based on a given image (reflection or transmission) source subject to an alternative boundary. (ii) The concept of the RIS is an important component in tackling diffusion problems in a complex space with multi-media and multi-boundaries. To conveniently show this concept here, we use four simple diffusion anisotropies in a quartmaterial as given in the following theorem.

Theorem 5 (DIs in a quartmaterial space). *A quartmaterial space has four inverse diffusivity tensors* $\sigma_1 = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} \sigma_{33} & 0 \\ 0 & \sigma_{22} \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} \sigma_{33} & 0 \\ 0 & \sigma_{44} \end{bmatrix}$ *and* $\sigma_4 = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{44} \end{bmatrix}$. *If a primary source* $P(x_p, y_p)$ *is located in the first medium of a quartmaterial as shown in Fig.* 8, *then all the DIs for medium* 1 *through to medium* 4 *are given as follows*:

$$DI_{11} := \{1, \{x_p, y_p\}, 1, \{1\}, \{0\}, \{1, 2\}, \{4, 2\}\}, \tag{73}$$

$$DI_{12} := \left\{ \frac{\sqrt{\sigma_{44}} - \sqrt{\sigma_{22}}}{\sqrt{\sigma_{22}} + \sqrt{\sigma_{44}}}, \{x_p, -y_p\}, 4, \{1\}, \{1\}, \{1, 2\}, \{4, 2\} \right\}, \tag{74}$$

$$DI_{13} := \left\{ \frac{\sqrt{\sigma_{33}} - \sqrt{\sigma_{11}}}{\sqrt{\sigma_{11}} + \sqrt{\sigma_{33}}}, \{-x_p, y_p\}, 2, \{1\}, \{2\}, \{2, 1\}, \{2, 4\} \right\}, \tag{75}$$

$$DI_{14} := \left\{ \frac{(\sqrt{\sigma_{11}} - \sqrt{\sigma_{33}})(\sqrt{\sigma_{22}} - \sqrt{\sigma_{44}})}{(\sqrt{\sigma_{11}} + \sqrt{\sigma_{33}})(\sqrt{\sigma_{22}} + \sqrt{\sigma_{44}})}, \{-x_p, -y_p\}, 3, \{1\}, \{1, 2\}, \{2, 1\}, \{2, 4\} \right\}, \tag{76}$$

$$DI_{21} := \left\{ \frac{2\sqrt{\sigma_{11}}}{\sqrt{\sigma_{11}} + \sqrt{\sigma_{33}}}, \left\{ \frac{x_p\sqrt{\sigma_{11}}}{\sqrt{\sigma_{33}}}, y_p \right\}, 1, \{2\}, \{2\}, \{2,3\}, \{1,3\} \right\}, \tag{77}$$

$$DI_{22} := \left\{ \frac{-2\sqrt{\sigma_{11}}(\sqrt{\sigma_{22}} - \sqrt{\sigma_{44}})}{(\sqrt{\sigma_{11}} + \sqrt{\sigma_{33}})(\sqrt{\sigma_{22}} + \sqrt{\sigma_{44}})}, \left\{ \frac{x_p\sqrt{\sigma_{11}}}{\sqrt{\sigma_{33}}}, -y_p \right\}, 4, \{2\}, \{2,3\}, \{3,2\}, \{3,1\} \right\}, \tag{78}$$

$$DI_{31} := \left\{ \frac{4\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}}{(\sqrt{\sigma_{11}} + \sqrt{\sigma_{33}})(\sqrt{\sigma_{22}} + \sqrt{\sigma_{44}})}, \left\{ \frac{x_p\sqrt{\sigma_{11}}}{\sqrt{\sigma_{33}}}, \frac{y_p\sqrt{\sigma_{22}}}{\sqrt{\sigma_{44}}} \right\}, 1, \{3\}, \{4,3\}, \{3,4\}, \{2,4\} \right\}, \tag{79}$$

$$DI_{41} := \left\{ \frac{2\sqrt{\sigma_{22}}}{\sqrt{\sigma_{22}} + \sqrt{\sigma_{44}}}, \left\{ x_p, \frac{y_p\sqrt{\sigma_{22}}}{\sqrt{\sigma_{44}}} \right\}, 1, \{4\}, \{1\}, \{1,4\}, \{1,3\} \right\}, \tag{80}$$

$$DI_{42} := \left\{ -\frac{2\sqrt{\sigma_{22}}(\sqrt{\sigma_{11}} - \sqrt{\sigma_{33}})}{(\sqrt{\sigma_{11}} + \sqrt{\sigma_{33}})(\sqrt{\sigma_{22}} + \sqrt{\sigma_{44}})}, \left\{ -x_p, \frac{y_p\sqrt{\sigma_{22}}}{\sqrt{\sigma_{44}}} \right\}, 2, \{4\}, \{1,4\}, \{4,1\}, \{3,1\} \right\}. \tag{81}$$

Proof. It is known from previous sections that *R&T* images are produced by boundaries with continuous boundary conditions, with their locations determined by identities of anisotropic weighted distances on the boundary. For this quartmaterial space, the following 16 identities must be satisfied:

$$|\eta_p^{(1)}|_{y=0} \equiv |\eta_r^{(1)}|_{y=0}, \quad |\eta_p^{(1)}|_{y=0} \equiv |\eta_t^{(4)}|_{y=0}, \tag{82}$$

$$|\eta_p^{(1)}|_{x=0} \equiv |\eta_r^{(1)}|_{x=0}, \quad |\eta_p^{(1)}|_{x=0} \equiv |\eta_t^{(2)}|_{x=0}, \tag{83}$$

$$|\eta_p^{(2)}|_{y=0} \equiv |\eta_r^{(2)}|_{y=0}, \quad |\eta_p^{(2)}|_{y=0} \equiv |\eta_t^{(3)}|_{y=0}, \tag{84}$$

$$|\eta_p^{(2)}|_{x=0} \equiv |\eta_r^{(2)}|_{x=0}, \quad |\eta_p^{(2)}|_{x=0} \equiv |\eta_t^{(1)}|_{x=0}, \tag{85}$$

$$|\eta_p^{(3)}|_{y=0} \equiv |\eta_r^{(3)}|_{y=0}, \quad |\eta_p^{(3)}|_{y=0} \equiv |\eta_t^{(2)}|_{y=0}, \tag{86}$$

$$|\eta_p^{(3)}|_{x=0} \equiv |\eta_r^{(3)}|_{x=0}, \quad |\eta_p^{(3)}|_{x=0} \equiv |\eta_t^{(4)}|_{x=0}, \tag{87}$$

$$|\eta_p^{(4)}|_{y=0} \equiv |\eta_r^{(4)}|_{y=0}, \quad |\eta_p^{(4)}|_{y=0} \equiv |\eta_t^{(1)}|_{y=0}, \tag{88}$$

$$|\eta_p^{(4)}|_{x=0} \equiv |\eta_r^{(4)}|_{x=0}, \quad |\eta_p^{(4)}|_{x=0} \equiv |\eta_t^{(3)}|_{x=0}, \tag{89}$$

where
$\eta_p^{(1)} = \sqrt{r_p \cdot \sigma_1 \cdot r_p}, \eta_p^{(2)} = \sqrt{r_p \cdot \sigma_2 \cdot r_p}, \eta_p^{(3)} = \sqrt{r_p \cdot \sigma_3 \cdot r_p}, \eta_p^{(4)} = \sqrt{r_p \cdot \sigma_4 \cdot r_p}$,
$\eta_r^{(1)} = \sqrt{r_r \cdot \sigma_1 \cdot r_r}, \eta_r^{(2)} = \sqrt{r_r \cdot \sigma_2 \cdot r_r}, \eta_r^{(3)} = \sqrt{r_r \cdot \sigma_3 \cdot r_r}, \eta_r^{(4)} = \sqrt{r_r \cdot \sigma_4 \cdot r_r}$,
$\eta_t^{(1)} = \sqrt{r_t \cdot \sigma_1 \cdot r_t}, \eta_t^{(2)} = \sqrt{r_t \cdot \sigma_2 \cdot r_t}, \eta_t^{(3)} = \sqrt{r_t \cdot \sigma_3 \cdot r_t}, \eta_t^{(4)} = \sqrt{r_t \cdot \sigma_4 \cdot r_t}$,
and where $r_p = \{x_p, y_p\}, r_r = \{x_r, y_r\}$ and $r_t = \{x_t, y_t\}$. Each DI's contribution to the diffusion field of the space, u_{mn}, must satisfy the following corresponding continuous boundary conditions given that m represents a medium number and n represents a DI number in the medium m:

$$u_{1,i}(t,x,y)|_{y=0} \equiv u_{4,j}(t,x,y)|_{y=0}, \tag{90}$$

$$[\nabla u_{1,i}(t,x,y) \cdot \sigma_1 \cdot \nabla y]_{y=0} \equiv [\nabla u_{4,j}(t,x,y) \cdot \sigma_4 \cdot \nabla y]_{y=0}, \tag{91}$$

$$u_{1,i}(t,x,y)|_{x=0} \equiv u_{2,j}(t,x,y)|_{x=0}, \tag{92}$$

$$[\nabla u_{1,i}(t,x,y) \cdot \sigma_1 \cdot \nabla x]_{x=0} \equiv [\nabla u_{2,j}(t,x,y) \cdot \sigma_2 \cdot \nabla x]_{x=0}, \tag{93}$$

$$u_{2,i}(t,x,y)|_{y=0} \equiv u_{3,j}(t,x,y)|_{y=0}, \tag{94}$$

$$[\nabla u_{2,i}(t,x,y) \cdot \sigma_2 \cdot \nabla y]_{y=0} \equiv [\nabla u_{3,j}(t,x,y) \cdot \sigma_3 \cdot \nabla y]_{y=0}, \tag{95}$$

$$u_{3,i}(t,x,y)|_{x=0} \equiv u_{4,j}(t,x,y)|_{x=0}, \tag{96}$$

$$[\nabla u_{3,i}(t,x,y) \cdot \sigma_3 \cdot \nabla x]_{x=0} \equiv [\nabla u_{4,j}(t,x,y) \cdot \sigma_4 \cdot \nabla x]_{x=0}, \tag{97}$$

where i and j are the DI numbers for the corresponding medium.

As we can see, DI_{11} is a pDI. Because the primary source has an amplitude of one unit and is located in the first medium, the elements AMP, SLC, SLN and NMM of DI_{11} are 1, (x_p, y_p), 1 and $\{1\}$, respectively. According to the definition, it is easy to obtain $NBM = \{0\}$, $BIM = \{1,2\}$ and $BMM = \{4,2\}$.

Using identities given by (82), the SLCs of DI_{12} and DI_{41} are obtained. Using continuous boundary conditions (90) and (91), amplitudes in DI_{12} and DI_{41} are obtained. DI_{12} is the reflection DI of DI_{11}, so its NMM is the same as DI_{11}, however DI_{41} is the transmission DI of DI_{11}, its $NMM = \{4\}$ which is BM of DI_{11} subject to boundary 1. It is obvious that the SLN for DI_{12} is 4, and for DI_{41} is 1. Because DI_{12} and DI_{41} are, respectively, the reflection and transmission on boundary 1, NBM in DI_{12} and DI_{41} are the same as $\{1\}$. Due to DI_{12} being the reflection of DI_{11} with its $SLN = 4$, the BIM must be $\{1,2\}$. Because DI_{41} is the transmission of DI_{11}, its BIM must be $\{1,4\}$. The definition for BMM is subject to BIM, so the values for BMM in DI_{12} and DI_{41} are $\{4,2\}$ and $\{1,3\}$, respectively. Following this analysis process exactly and using identities given by (83) and boundary conditions (92) and (93), the DI pair, DI_{13} and DI_{21}, can be easily confirmed.

From Definition 7, we see that $DI_{14}, DI_{22}, DI_{31}$ and DI_{42} are RISs and they are produced by two different methods, each with a two-step *R&T* process. The first method for DI_{14}, is reflecting DI_{11} through boundary 1 and then reflecting again by boundary 2. The second method is reflecting DI_{11} through boundary 2 and then reflecting again by boundary 1. In these two methods, the reflection identities (82) and (83), and continuous boundary conditions (90)–(93) are applied, and the elements of each DI are set out accordingly based on the definition of DI. Similarly,

DI_{31} is a two-way and two-step transmission, DI_{22} and DI_{42} are two-way reflections and transmissions. All the remaining identities and boundary conditions are applied, and the values of the elements in the DIs are confirmed. □

Theorem 6 (Green′s function for a quartmaterial space). *Consider a quartmaterial space as shown in Fig. 8 has four inverse diffusivity tensors* $\sigma_1 = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} \sigma_{33} & 0 \\ 0 & \sigma_{22} \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} \sigma_{33} & 0 \\ 0 & \sigma_{44} \end{bmatrix}$ and $\sigma_4 = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{44} \end{bmatrix}$ *in each of the four media. A point source* $P(x_p, y_p)$ *is located in the first medium and the two straight perpendicular separating lines are* $x = 0$ *and* $y = 0$ *as shown in Fig. 8. If diffusion fields for these four media are* u_1, u_2, u_3 and u_4 *respectively, then the diffusion governing equation* (1) *with source function* (2) *is subject to the continuous boundary conditions* (90)–(97), *as well as the following radiation boundary conditions*:

$$u_1(t,x,y)|_{x\to\infty, y\to\infty} = 0, \tag{98}$$

$$u_2(t,x,y)|_{x\to-\infty, y\to\infty} = 0, \tag{99}$$

$$u_3(t,x,y)|_{x\to-\infty, y\to-\infty} = 0, \tag{100}$$

$$u_4(t,x,y)|_{x\to\infty, y\to-\infty} = 0, \tag{101}$$

and the following initial conditions:

$$u_1(t,x,y)|_{t\to 0} = 0, \tag{102}$$

$$u_2(t,x,y)|_{t\to 0} = 0, \tag{103}$$

$$u_3(t,x,y)|_{t\to 0} = 0, \tag{104}$$

$$u_4(t,x,y)|_{t\to 0} = 0. \tag{105}$$

Green's functions for medium 1 *through to medium* 4 *are given as follows*:

$$u_1[t,x,y] := \frac{\sqrt{\det[\sigma_1]}}{4\pi t} \sum_{k=1}^{4} c_{1k}\, e^{-\mathscr{H}_{1k}^2/4t} \quad (x,y) \in \mathbb{R}, \quad t > 0, \tag{106}$$

$$u_2[t,x,y] := \frac{\sqrt{\det[\sigma_2]}}{4\pi t} \sum_{k=1}^{2} c_{2k}\, e^{-\mathscr{H}_{2k}^2/4t} \quad (x,y) \in \mathbb{R}, \quad t > 0, \tag{107}$$

$$u_3[t,x,y] := \frac{\sqrt{\det[\sigma_3]}}{4\pi t} c_{31}\, e^{-\frac{\mathscr{H}_{31}^2}{4t}} \quad (x,y) \in \mathbb{R}, \quad t > 0, \tag{108}$$

$$u_4[t,x,y] := \frac{\sqrt{\det[\sigma_4]}}{4\pi t} \sum_{k=1}^{2} c_{4k}\, e^{-\mathscr{H}_{4k}^2/4t} \quad (x,y) \in \mathbb{R}, \quad t > 0, \tag{109}$$

where c_{ik} *is Amp of the kth DI in the ith medium, and* $\mathscr{H}_{ik}^2$ *is given by*

$$\mathscr{H}_{ik}^2 := \{x - x_s^{ik}, y - y_s^{ik}\} \cdot \sigma_i \cdot \{x - x_s^{ik}, y - y_s^{ik}\}, \tag{110}$$

where x_s^{ik} *and* y_s^{ik} *are SLCs for the kth DI in the ith medium.*

The proof for this solution is similar to those given in previous sections, and is not repeated here.

Remarks. We see that the RIS plays a very important role in a quartmaterial space, where all the two-step and two-way *R&T*s are coincided in a medium. This coincided RIS can satisfy all boundary conditions for a medium. We also can see that these RISs satisfy the boundary conditions of one medium, but are located in another, for example, DI_{22} satisfies boundary conditions for medium 2 while its location is in medium 4.

8. Numerical examples

Green's functions developed in Sections 4 and 5 were implemented using Mathematica 5.1 on a personal

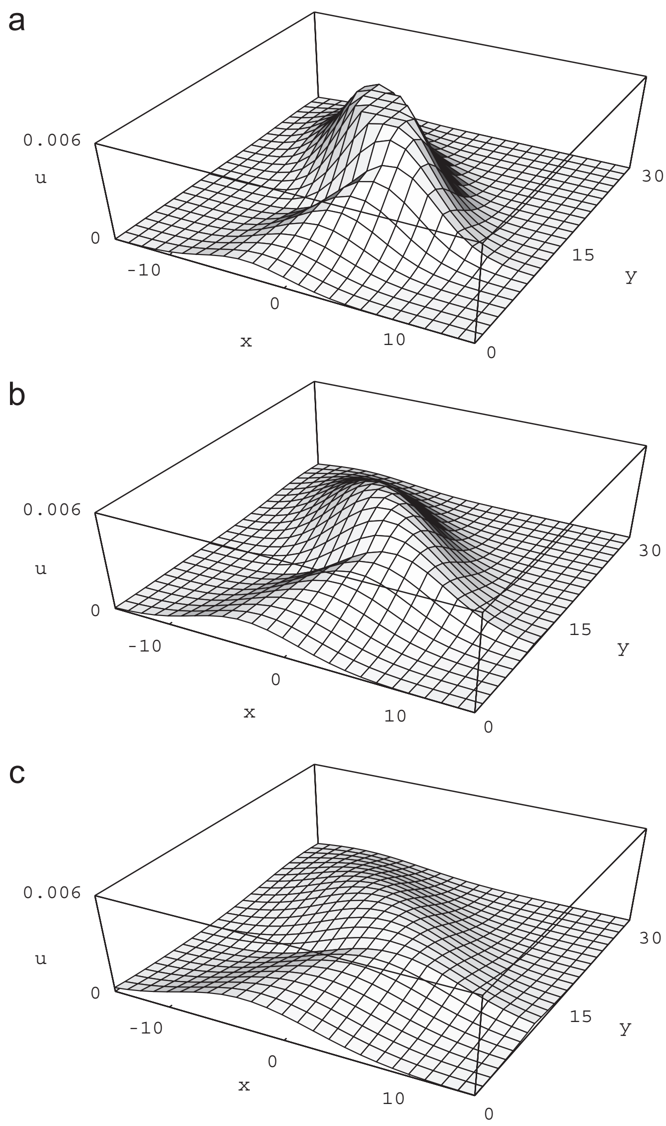

Fig. 9. Diffusion in a two-layer space initiated by a transient source located in the second layer at position (0, 15.0 m). The boundary between the two layers is at $y = 10.0$ m: (a) h 10, t 2; (b) h 10, t 3 and (c) h 10, t 5.

computer. Accurate numerical calculations of a diffusion field in two dimensions were obtained within minutes, once the relevant equations were established.

Figs. 9(a), (b) and (c) show diffusion field (temperature) distributions in a 2D two-layer space at times $t = 2$, 3 and 5 s, respectively, arising from a point transient source (delta function). This source is located in the second layer at (0, 15.0 m) with the boundary with continuous boundary conditions at $y = 10.0\,\mathrm{m}$, and where the inverse diffusive tensors (W/mK) are $\sigma_1 = \begin{bmatrix} 0.3 & -0.2 \\ -0.2 & 0.21 \end{bmatrix}$ and $\sigma_2 = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.25 \end{bmatrix}$ in the first and second media, respectively.

Figs. 10(a), (b) and (c) show the diffusion field (temperature) distributions at times $t = 2, 3$ and 5 s. A point source is located in the first layer at $(0, 14.9\,\mathrm{m})$ with the boundary with continuous boundary conditions at $y = 15.0\,\mathrm{m}$. The inverse diffusive tensors are $\sigma_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.25 \end{bmatrix}$ and $\sigma_2 = \begin{bmatrix} 0.3 & -0.2 \\ -0.2 & 0.21 \end{bmatrix}$ in first and second media, respectively. We can now clearly see the anisotropic effects on the diffusion fields.

Figs. 11 and 12 show the diffusion field distributions in a 2D quartmaterial space where a point source located in the first medium at point $(1.0, 1.0\,\mathrm{m})$, with the boundaries with continuous boundary conditions $x = 0$ and $y = 0$. The inverse diffusive tensors are $\sigma_1 = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} \frac{5}{8} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} \frac{5}{8} & 0 \\ 0 & \frac{1}{16} \end{bmatrix}$ and $\sigma_4 = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{16} \end{bmatrix}$. Fig. 11(a)is a diffusion field distribution with a combination of inverse diffusive tensors $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ in the first to fourth media, respectively. Fig. 11(b) is a diffusion field distribution when the inverse diffusive tensors from medium 1 through to medium 4 are $\sigma_2, \sigma_3, \sigma_4$ and σ_1, respectively. Fig. 12(a) is a diffusion field distribution where the four media inverse diffusive tensors are given in the first to fourth media by $\sigma_3, \sigma_4, \sigma_1$ and σ_2, respectively. Fig. 12(b) is a diffusion field distribution with a combination of inverse diffusive tensors $\{\sigma_4, \sigma_1, \sigma_2, \sigma_3\}$ in the first to fourth media, respectively. We can clearly see that the different combinations of the source location and the inverse diffusive tensors have very

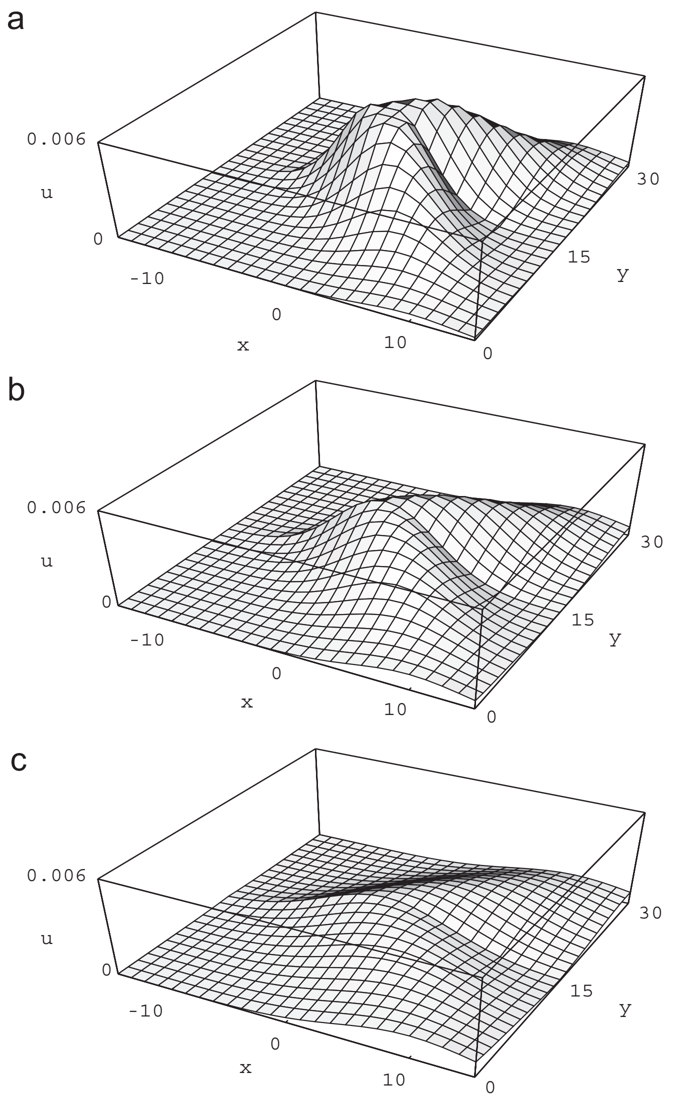

Fig. 10. Diffusion in a two-layer space initiated by a transient source located in the second layer at position $(0, 14.9\,\mathrm{m})$. The boundary between the two layers is at $y = 15.0\,\mathrm{m}$: (a) $h = 15$, $t = 2$; (b) $h = 15$, $t = 3$ and (c) $h = 15$, $t = 5$.

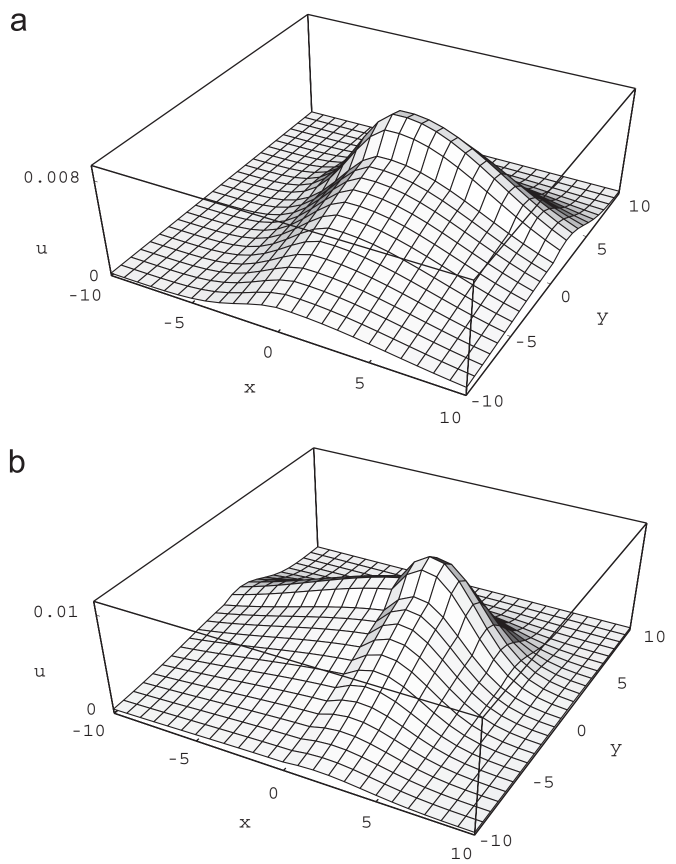

Fig. 11. Diffusion in a quartmaterial space initiated by a transient source located at position $(1.0, 1.0\,\mathrm{m})$ with a 2 s time lapse and tensor combinations 1 and 2: (a) with tensor combination 1, 2, 3, 4 and (b) with tensor combinaiton 2, 3, 4, 1.

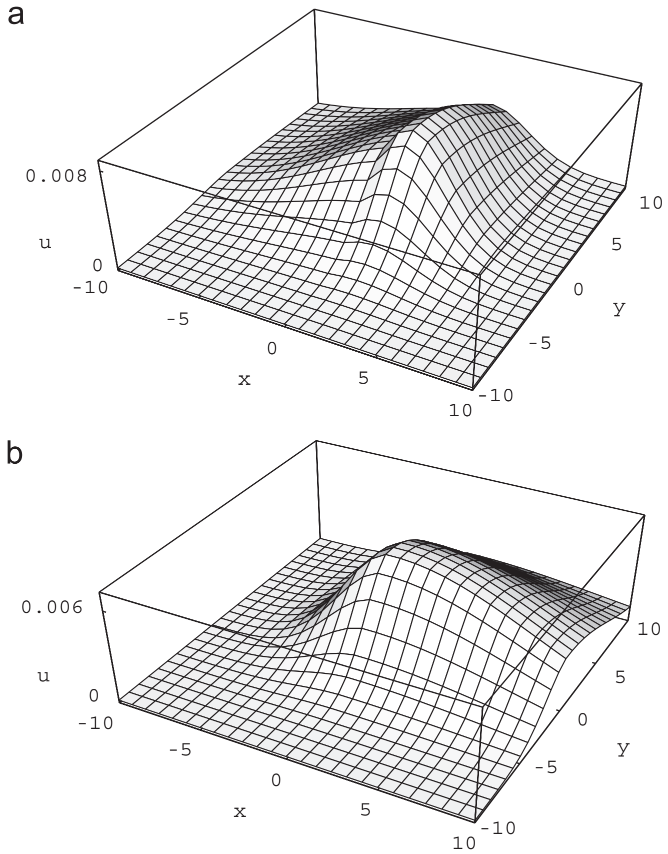

Fig. 12. Diffusion in a quartmaterial space initiated by a transient source located at position (1.0, 1.0 m) with a 2 s time lapse and tensor combinations 3 and 4: (a) with tensor combination 3, 4, 1, 2 and (b) with tensor combinaiton 4, 1, 2, 3.

different distributions to the diffusion field in the quartmaterial space.

9. Conclusion

In this paper, the following new concepts were introduced: (i) diffusion instances and their physical interpretations; (ii) reflection and transmission streams; and (iii) reverse images. They are the fundamental bases for the theory of transient image principles used in the computation of mathematical physics with multi-media and multi-boundaries.

The static image method applies to elliptical PDEs, while the transient image method applies to hyperbolic PDEs. For solutions in a space with boundaries with continuous boundary conditions, the static image method primarily deals with isotropic problems, and then may be extended to anisotropic cases, whereas the transient image method deals with limited anisotropic cases first and then enables the solution of the isotropic problems to be done in the future.

By imposing limits on the anisotropy of media, solutions to transient diffusion become possible. This does enable more general solutions to be obtained which would otherwise not be obtainable.

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