On the Foundation of the Popular Ratio Test for GNSS Ambiguity Resolution

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BIOGRAPHY

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ABSTRACT

Integer carrier phase ambiguity resolution is the key to fast and high-precision global navigation satellite system (GNSS) positioning and navigation. It is the process of resolving the unknown cycle ambiguities of the double-differenced carrier phase data as integers. For the problem of estimating the ambiguities as integers a rigorous theory is available. The user can choose from a whole class of integer estimators, from which integer least-squares is known to perform best in the sense that no other integer estimator exists which will have a higher success rate.

Next to the integer estimation step, also the integer validation plays a crucial role in the process of ambiguity resolution. Various validation procedures have been proposed in the literature. One of the earliest and most popular ways of validating the integer ambiguity solution is to make use of the so-called Ratio Test.

In this contribution we will study the properties and underlying concept of the popular Ratio Test. This will be done in two parts. First we will criticize some of the properties and underlying principles which have been assigned in the literature to the Ratio Test. Despite this criticism however, we will show that the Ratio Test itself is still an important, albeit not optimal, candidate for validating the integer solution. That is, we will also show that the procedure underlying the Ratio Test can indeed be given a firm theoretical footing. This is made possible by the recently introduced theory of Integer Aperture Inference. The necessary ingredients of this theory will be briefly described. It will also be shown that one can do better than the Ratio Test. The optimal test will be given and the difference between the optimal test and the Ratio Test will be discussed and illustrated.

1 INTRODUCTION

Integer carrier phase ambiguity resolution is the key to fast and high-precision global navigation satellite system (GNSS) positioning and navigation. It applies to a great variety of current and future models of GPS, modernized GPS and Galileo, with applications in surveying, navigation, geodesy and geophysics. These models may differ greatly in complexity and diversity. They range from single-baseline models used for kinematic positioning to multi-baseline models used as a tool for studying geodynamic phenomena. The models may or may not have the relative receiver-satellite geometry included. They may also be discriminated as to whether the slave receiver(s) is stationary or in motion, or whether or not the differential atmospheric delays (ionosphere and troposphere) are included as unknowns. An overview of these models can be found in textbooks like (Hofmann-Wellenhof et al., 2001; Leick, 2003; Parkinson and Spilker, 1996; Strang and Borre, 1997; Teunissen and Kleusberg, 1998).

Any linear(ized) GNSS model can be cast in the following system of linearized observation equations:

$$E\{y\} = Aa + Bb\ ,\ a \in \mathbb{Z}^n,\ b \in \mathbb{R}^p \tag{1}$$

with $E\{.\}$ the mathematical expectation operator, y the m-vector of observables, a the n-vector of unknown integer parameters and b the p-vector of unknown real-valued parameters. The data vector y will then usually consist of the 'observed minus computed' single- or multi-frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range, while the entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

The procedure for solving the above GNSS model can be divided conceptually into three steps. In the first step one simply discards the integer constraints $a \in \mathbb{Z}^n$ and performs a standard adjustment. As a result one obtains the so-called 'float' solution $\hat{a}$ and $\hat{b}$. This solution is real-valued. Then in the second step the float solution $\hat{a}$ is further adjusted so as to take in some pre-defined way the integerness of the ambiguities into account. This gives

$$\check{a} = S(\hat{a}) \tag{2}$$

in which S is an n-dimensional mapping that in some way takes the integerness of the ambiguities into account. This estimator is then used in the final step to adjust the float estimator $\hat{b}$. As a result one obtains the so-called 'fixed' estimator of b as

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \check{a}) \tag{3}$$

in which $Q_{\hat{a}}$ denotes the vc-matrix of $\hat{a}$ and $Q_{\hat{b}\hat{a}}$ denotes the covariance matrix of $\hat{b}$ and $\hat{a}$.

The above three-step procedure is still ambiguous in the sense that it leaves room for choosing the n-dimensional map S. Different choices for S will lead to different ambiguity estimators and thus also to different baseline estimators $\check{b}$. One can therefore now think of constructing a family of maps S with certain desirable properties.

Next to the integer estimation step, also integer validation plays a crucial role in the process of ambiguity resolution. After all, even when one uses an optimal, or close to optimal, integer ambiguity estimator, one can still come up with an unacceptable integer solution. Unfortunately, however, there does not yet exist a rigorous probabilistic theory for the validation of the integer ambiguities. Various validation procedures have been proposed in the literature. Some seem to have a good performance, others however can be shown to perform poorly, while still others perform poor in some cases while good in other cases. One of the earliest and most popular ways of validating the integer ambiguity solution is to make use of the so-called Ratio Test. The test statistic of the Ratio Test is defined as the ratio of the squared norm of the 'second-best' ambiguity residual vector and the squared norm of the 'best' ambiguity residual vector. The computed integer ambiguity solution is then rejected in favor of the float solution when this ratio is below a certain user-defined threshold.

In this contribution we will study the properties and underlying concept of the popular Ratio Test. This will be done in two parts. First we will criticize some of the properties and underlying principles which have been assigned in the literature to the Ratio Test. This criticism will be substantiated by means of counter examples. Topics to which this criticism applies are, for example, the use of the classical theory of hypothesis testing for 'deriving' the Ratio Test, the probability distribution which is often assigned to the test statistic of the Ratio Test, and the lack of being able to give a rigorous overall probabilistic evaluation of the combination of integer estimation and integer validation when using the Ratio Test. Despite this criticism however, we will show that the Ratio Test itself is still an important, albeit not optimal, candidate for validating the integer solution. That is, we will also show that the procedure underlying the Ratio Test can indeed be given a firm theoretical footing, but one that differs significantly from the ones described in the literature. For the practical user of the Ratio Test this has the important consequence that the Ratio Test has to be evaluated differently as thought so far.

The firm theoretical basis that can be given to the Ratio Test is made possible by the recently introduced theory of Integer Aperture Inference (Teunissen, 2003b; Teunissen, 2004). The necessary ingredients of this theory will be briefly described. It will be shown that the procedure underlying the Ratio Test is a member from the class of integer aperture estimators. This allows us (i) to quantify and qualify the acceptance region of the Ratio Test, (ii) to give an exact and overall probabilistic evaluation of the combined integer estimation and validation solution when using the Ratio Test, and (iii) to show the user how he/she needs to compute the critical value of the Ratio Test. However, it will also be shown that one can do better than the Ratio Test. The optimal test will be given and the difference between the optimal test and the Ratio Test will be discussed and illustrated. Having the practical user in mind, we also present the concrete procedure together with the implementation steps.

The outline of this contribution is as follows. First, the theory of integer estimation is reviewed. In section 3 the properties and concept of the Ratio Test are described and criticized. Next it will be shown that a firm theoretical basis for the Ratio Test can be given by the introduction of the class of Integer Aperture estimators in section 5. The optimal test will be presented in section 4. Finally, section 7 presents some examples in order to illustrate the principles and properties of the Ratio Test and the optimal test.

2 INTEGER ESTIMATION

In section 1 the procedure for solving the GNSS model was outlined. The second step was the integer estimation of the ambiguities. The space of integers, $\mathbb{Z}^n$, is of a discrete nature, which implies that the map S in (2) must be a many-to-one map, and not one-to-one. In other words, different real-valued ambiguity vectors a will be mapped to the same integer vector. Therefore, a subset $S_z \subset \mathbb{R}^n$ can be assigned to each integer vector $z \in \mathbb{Z}^n$:

$$S_z = \{x \in \mathbb{R}^n \mid \quad z = S(x)\}, \quad z \in \mathbb{Z}^n \tag{4}$$

This subset S_z contains all real-valued float ambiguity vectors that will be mapped to the same integer vector z, and it is called the pull-in region of z (Jonkman, 1998; Teunissen, 1998).

The integer estimator is completely defined by the pull-in region S_z, so that it is possible to define different classes of integer estimators by imposing various conditions on the pull-in regions. An integer estimator, $\check{a}$, is said to be admissible when its pull-in region, S_z, satisfies:

$$
\begin{aligned}
&(i) \quad \bigcup_{z \in Z^n} S_z = \mathbb{R}^n \\
&(ii) \quad \text{Int}(S_u) \cap \text{Int}(S_z) = \emptyset, \quad \forall u, z \in \mathbb{Z}^n, u \neq z \qquad (5) \\
&(iii) \quad S_z = z + S_0, \qquad \forall z \in \mathbb{Z}^n
\end{aligned}
$$

where 'Int' denotes the interior of the subset.

In (Teunissen, 1998) the motivation for this definition is given. The first condition states that the collection of all pull-in regions should cover the complete space $\mathbb{R}^n$, so that indeed all real-valued vectors will be mapped to an integer vector. The second condition states that the pull-in regions should be disjunct, i.e. there should be no overlap between the pull-in regions, so that the float solution is mapped to just one integer vector. Finally, the third condition is that of translational invariance, which means that if the float solution is perturbed by $z \in \mathbb{Z}^n$, the corresponding integer solution is perturbed by the same amount: $S(\hat{a}+z) = S(\hat{a})+z$. This allows one to apply the integer remove-restore technique: $S(\hat{a}-z)+z = S(\hat{a})$.

Examples of integer estimators that belong to the class of admissible integer estimators are integer rounding, integer bootstrapping, and integer least-squares (ILS). The latter is shown to be optimal, cf.(Teunissen, 1999), which means that the probability of correct integer estimation is maximized. The integer least-squares estimator is defined as:

$$\check{a} = \arg \min_{z \in \mathbb{Z}^n} \|\hat{a} - z\|^2_{Q_{\hat{a}}} \tag{6}$$

with the squared norm $\| \cdot \|^2_Q = (\cdot)^T Q^{-1} (\cdot)$, and where $\check{a} \in \mathbb{Z}^n$ is the fixed ILS ambiguity solution. The pull-in region that belongs to the integer z follows as:

$$S_z = \left\{ x \in \mathbb{R}^n \mid \|x - z\|^2_{Q_{\hat{a}}} \leq \|x - u\|^2_{Q_{\hat{a}}}, \forall u \in \mathbb{Z}^n \right\} \tag{7}$$

If we use:

$$
\begin{aligned}
&\|x - z\|^2_{Q_{\hat{a}}} \leq \|x - u\|^2_{Q_{\hat{a}}} \iff \\
&\qquad (u - z)^T Q^{-1}_{\hat{a}} (x - z) \leq \frac{1}{2} \|u - z\|^2_{Q_{\hat{a}}}, \forall u \in \mathbb{Z}^n
\end{aligned}
$$

it follows that

$$S_z = \bigcap_{c \in \mathbb{Z}^n} \{ x \in \mathbb{R}^n \mid \ |c^T Q^{-1}_{\hat{a}} (x - z)| \leq \frac{1}{2} \|c\|^2_{Q_{\hat{a}}} \} \tag{8}$$

The ILS pull-in regions are thus constructed as intersecting half-spaces, which are bounded by the plane orthogonal to $(u - z)$, $u \in \mathbb{Z}^n$ and passing through $\frac{1}{2}(u + z)$. It can be shown that at the most $2^n - 1$ pairs of such half-spaces are needed for the construction.

The fixed solution should only be used if there is enough confidence in this solution. Therefore, the probability that the ambiguities are fixed to the correct integers, the success rate, is a very important measure. It is given by:

$$P_{s,ILS} = P(\check{a} = a) = \int_{S_a} f_{\hat{a}}(x) dx \tag{9}$$

where $P(\check{a} = a)$ is the probability that $\check{a} = a$, and $f_{\hat{a}}(x)$ is the probability density function of the float ambiguities, for which in practice the normal (Gaussian) distribution is used.

3 INTEGER VALIDATION

A parameter resolution theory cannot be considered complete without rigorous measures for validating the parameter solution. In the classical theory of linear estimation, the vc-matrices provide sufficient information on the precision of the estimated parameters. The reason is that a linear model applied to normally distributed (Gaussian) data, provides linear estimators that are also normally distributed, and the peakedness of the multivariate normal distribution is completely captured by the vc-matrix.

Unfortunately, this relatively simple approach cannot be applied in case integer parameters are involved in the estimation process, since the integer estimators do not have a Gaussian distribution, even if the model is linear and the data are normally distributed. Instead of the vc-matrices, the parameter distribution itself has to be used in order to obtain the appropriate measures that can be used to validate the integer parameter solution.

In the past, the problem of non-Gaussian parameter distributions was circumvented by simply ignoring the randomness of the fixed ambiguities. Several testing procedures for the validation of the fixed solution have been proposed using this approach. An overview of these procedures and their pitfalls was given in (Verhagen, 2004).

Discrimination tests are used in order to compare the likelihood of the fixed solution to that of another set of integers. Of course it is known that the likelihood of the fixed solution $\check{a}$ is always larger than the likelihood of any other integer vector in case it was obtained with integer least-squares estimation. However, if the likelihood of $\check{a}$ is not sufficiently larger than the likelihood of $\check{a}'$, the two solutions cannot be discriminated with enough confidence.

3.1 The Ratio Test

A very popular discrimination test is the one introduced by Euler and Schaffrin (1990). It is given by:

$$\text{Accept } \check{a} \text{ iff:} \quad \frac{\|\hat{a} - \check{a}_2\|^2_{Q_{\hat{a}}}}{\|\hat{a} - \check{a}\|^2_{Q_{\hat{a}}}} = \frac{R_2}{R_1} \geq c \tag{10}$$

where the notation R_i is used for the squared norm of ambiguity residuals of the best ($i = 1$), and second-best ($i = 2$) integer solution, $\check{a}$ and $\check{a}_2$ respectively, as measured by the squared norm of the ambiguity residual vector.

It is derived by applying the classical theory of hypothesis testing. Three hypotheses are considered:

$$H_0: \ a = \hat{a}, \quad H_1: \ a = \check{a} \quad H_2: \ a = \check{a}_2 \tag{11}$$

In order to determine the critical value c it was assumed that:

$$\frac{(m-n-p)R_i}{n\|\hat{e}\|^2_{Q_y}} \sim F(n, m-n-p, \lambda_i),\ i = 1,2 \quad (12)$$

with $\hat{e}$ the residuals of the float solution, and $F(n, m-n-p, \lambda_i)$ denotes the non-central F-distribution with n and $m-n-p$ degrees of freedom and non-centrality parameter λ_i. Hence, from eq.(12) follows that it is assumed that $\|\hat{e}\|^2_{Q_y}$ and R_i are independent. In principle, however, this assumption is not allowed. Firstly, because the entries of $\check{a}$, $\check{a}_2$ and $\hat{e}$ depend on the same vector y. If y changes, also $\check{a}$ and $\check{a}_2$ will change. Secondly, since the vector y is assumed to be random, also the fixed ambiguities obtained with integer least-squares estimation will be stochastic.

Another problem with this approach is that the determination of the critical value is not straightforward. Euler and Schaffrin (1990) used test computations, from which followed that a critical value between 5 and 10 should be chosen depending on the degrees of freedom.

The conclusion is that it is not possible to use the classical theory of hypothesis testing for deriving the Ratio Test. For that purpose the probabilistic characteristics of the parameters should be taken into account, which is not possible here.

Other approaches have also been proposed in literature, all based on choosing a fixed critical value for test (10).

Wei and Schwarz (1995) propose to use the test with a critical value of $c = 2$. They do not claim that there is a theoretical justification for using this test. Moreover, they acknowledge that it is risky to apply the test to an individual epoch in the case of poor satellite geometry or noisy observations. Therefore, it is proposed to apply the test to all epochs in a time period of 30-60 seconds, and only accept the integer solution if the same solution is accepted for all epochs. It is mentioned that the test is conservative.

Han and Rizos (1996) showed that good results can be obtained with the Ratio Test with a critical value of $c = 1.5$ if one can have confidence in the correctness of the stochastic model. For their experiments this was accomplished by applying satellite elevation-dependent weighting, with the weights determined for the receiver at hand.

A final point of criticism is that the combined integer estimation and validation solution lacks an overall probabilistic evaluation.

Despite the criticism, integer validation based on the Ratio Test often works satisfactorily. The reason is that the stochasticity of $\check{a}$ may indeed be neglected if there is sufficient probability mass located at one integer grid point of $\mathbb{Z}^n$, that is if the success rate is very close to one. Then an empirically determined fixed critical value can be expected to give good results. Therefore in many software packages a fixed value for the ratio is used, e.g. $c = 3$ (Leick, 2003).

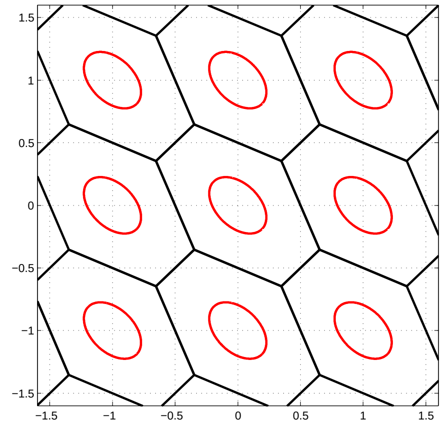

Figure 1: Two-dimensional example of aperture pull-in regions (red), together with the ILS pull-in regions (black).

4 INTEGER APERTURE ESTIMATION

4.1 The integer aperture estimator

In practice, a user will decide not to use the fixed solution if either the probability of failure is too high, or if the discrimination test is not passed. This gives rise to the thought that it might be interesting to use an ambiguity estimator defined such that three situations are distinguished: *success* if the ambiguity is fixed correctly, *failure* if the ambiguity is fixed incorrectly, and *undecided* if the float solution is maintained. This can be accomplished by dropping the condition that there are no gaps between the pull-in regions, so that the only conditions on the pull-in regions are that they should be disjunct and translational invariant. Then integer estimators can be determined that somehow regulate the probability of each of the three situations mentioned above.

The new class of ambiguity estimators was introduced in (Teunissen, 2003a; Teunissen, 2003c), and is called the class of Integer Aperture (IA) estimators. It is defined as:

$$\begin{array}{ll} (i) & \bigcup_{z\in\mathbb{Z}^n} \Omega_z = \Omega \\ (ii) & \text{Int}(\Omega_u) \cap \text{Int}(\Omega_z) = \emptyset,\ \forall u, z \in \mathbb{Z}^n, u \neq z \\ (iii) & \Omega_z = z + \Omega_0, \qquad \forall z \in \mathbb{Z}^n \end{array} \quad (13)$$

$\Omega \subset \mathbb{R}^n$ is called the aperture space. From (i) follows that this space is built up of the Ω_z, which will be referred to as aperture pull-in regions. Conditions (ii) and (iii) state that these aperture pull-in regions must be disjunct and translational invariant.

Figure 1 shows a two-dimensional example of aperture pull-in regions that fulfill the conditions in (13), together with the ILS pull-in regions fulfilling the conditions in (5).

The integer aperture estimator, $\bar{a}$, is now given by:

$$\bar{a} = \sum_{z\in\mathbb{Z}^n} z\omega_z(\hat{a}) + \hat{a}(1 - \sum_{z\in\mathbb{Z}^n} \omega_z(\hat{a})) \quad (14)$$

with the indicator function $\omega_z(x)$ defined as:

$$\omega_z(x) = \begin{cases} 1 & \text{if } x \in \Omega_z \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

So, when $\hat{a} \in \Omega$ the ambiguity will be fixed using one of the admissible integer estimators, otherwise the float solution is maintained. This means that indeed the following three cases can be distinguished:

$\hat{a} \in \Omega_a$	success: correct integer estimation
$\hat{a} \in \Omega \setminus \Omega_a$	failure: incorrect integer estimation
$\hat{a} \notin \Omega$	undecided: ambiguity not fixed to an integer

The corresponding probabilities of success (s), failure (f) and undecidedness (u) are given by:

$$\begin{aligned} P_s &= P(\bar{a} = a) = \int_{\Omega_a} f_{\hat{a}}(x)dx \\ P_f &= \sum_{z \in \mathbb{Z}^n \setminus \{a\}} \int_{\Omega_z} f_{\hat{a}}(x)dx = \int_{\Omega_0} f_{\check{\epsilon}}(x)dx - \int_{\Omega_a} f_{\hat{a}}(x)dx \\ P_u &= 1 - P_s - P_f = 1 - \int_{\Omega_0} f_{\check{\epsilon}}(x)dx \end{aligned} \tag{16}$$

The first two probabilities are referred to as success rate and fail rate respectively. Note the difference with the ILS success rate given in eq.(9), where the integration is over the ILS pull-in region $S_a \supset \Omega_a$. The expression for the fail rate is obtained by using the probability density function of the ambiguity residuals $\check{\epsilon} = \hat{a} - \check{a}$:

$$f_{\check{\epsilon}}(x) = \sum_{z \in \mathbb{Z}^n} f_{\hat{a}}(x + z)dx s_0(x) \tag{17}$$

with $s_0(x) = 1$ if $x \in S_0$ and $s_0(x) = 0$ otherwise, cf.(Teunissen, 2002; Verhagen and Teunissen, 2004).

4.2 Fixed fail rate approach

As mentioned in the beginning of this section, for a user it is especially important that the probability of failure, the fail rate, is below a certain limit. The approach of integer aperture estimation allows us now to choose a threshold for the fail rate, and then determine the *size* of the aperture pull-in regions such that indeed the fail rate will be equal to or below this threshold. So, applying this approach means that implicitely the ambiguity estimate is validated using a sound criterion. However, there are still several options left with respect to the choice of the *shape* of the aperture pull-in regions.

It is very important to note that Integer Aperture estimation with a fixed fail rate is an *overall* approach of integer estimation and validation, and allows for an exact and overall probabilistic evaluation of the solution. With the traditional approaches, e.g. the Ratio Test applied with a fixed critical value, this is not possible. Two important probabilistic measures are the fail rate, which will never exceed the user-defined threshold, and the probability that the fixed solution is correct, $\check{a} = a$, if the integer aperture solution is fixed, $\bar{a} = z$:

$$\begin{aligned} P_{s|\bar{a}=z} &= P(\check{a} = a | \bar{a} = z) \\ &= \frac{P(\check{a} = a, \bar{a} = z)}{P(\bar{a} = z)} \\ &= \frac{P_s}{P_s + P_f} \end{aligned} \tag{18}$$

where the expressions for the success and fail rates in eqs.(9) and (16) are used. Note that if $P_f << P_s$:

$$P_{s|\bar{a}=z} \approx 1 \tag{19}$$

This is an important result, since it means that with IA estimation it is not only guaranteed that the fail rate will be below a given threshold, but also the probability that the integer solution is correct will be close to one if IA estimation results in a fix and $P_f << P_s$. If P_s is small, this is not necessarily the case, but then the probability that the solution will be fixed is also very small, since this probability equals $P_s + P_f$.

5 THEORETICAL FOUNDATION FOR RATIO TEST

Despite the criticism on the Ratio Test given in section 3 it is possible to give a firm theoretical basis for this test. This is made possible by the theory of Integer Aperture Inference as presented in the previous section.

First, it will be shown that the procedure underlying the Ratio Test is a member from the class of integer aperture estimators. In the sequel the inverse of the test statistic of eq.(10) will be used, so that the test becomes:

$$\text{Accept } \check{a} \text{ iff: } \quad \frac{R_1}{R_2} \leq \mu, \quad 0 < \mu \leq 1 \tag{20}$$

The critical value is denoted as μ.

It can now be shown that the Ratio Test is an IA estimator. The acceptance region or aperture space is given as:

$$\Omega = \{x \in \mathbb{R}^n \mid \|x - \check{x}\|^2_{Q_{\hat{a}}} \leq \mu \|x - \check{x}_2\|^2_{Q_{\hat{a}}}, \, 0 < \mu \leq 1\} \tag{21}$$

with $\check{x}$ and $\check{x}_2$ the best and second-best ILS estimator of x. Let $\Omega_z = \Omega \cap S_z$, i.e. Ω_z is the intersection of Ω with the ILS pull-in region as defined in (7). Then all conditions of (13) are fulfilled, since:

$$\begin{aligned} \Omega_0 &= \{x \in \mathbb{R}^n \mid \|x\|^2_{Q_{\hat{a}}} \leq \mu \|x - z\|^2_{Q_{\hat{a}}}, \, \forall z \in \mathbb{Z}^n \setminus \{0\}\} \\ \Omega_z &= \Omega_0 + z, \, \forall z \in \mathbb{Z}^n \\ \Omega &= \bigcup_{z \in \mathbb{Z}^n} \Omega_z \end{aligned} \tag{22}$$

The proof was given in (Teunissen, 2003b).

The acceptance region of the Ratio Test consists thus of an infinite number of regions, each one of which is an integer translated copy of $\Omega_0 \subset S_0$. The acceptance region plays the role of the aperture space, and μ plays the role of aperture parameter since it controls the size of the aperture pull-in regions.

It has now been shown that indeed there is a theoretical basis for the Ratio Test, since the Ratio Test is an integer aperture estimator and there is a sound criterion available for choosing the critical value, or aperture parameter, by means of the fixed fail rate approach described in section 4.2. This implies that the acceptance region is qualified and quantified by means of eq.(21). Moreover, an exact and overall probabilistic evaluation of the combined estimation and validation solution is available

6 OPTIMAL INTEGER APERTURE ESTIMATION

The approach of integer aperture estimation with a fixed fail rate has two important advantages. The first is that IA estimation can always be applied, independent of the precision, since the user does not have to be afraid that the fail rate is too high. The second advantage is that for the first time sound theoretical criteria are available for the validation of the estimates. For that purpose, the Ratio Test can be used. However, it will now be shown that also an *optimal* integer aperture (OIA) estimator exists.

As with integer estimation, the optimality property would be to maximize the success rate, but in this case for a fixed fail rate. So, the optimization problem is to determine the aperture space which fulfills:

$$\max_{\Omega_0 \subset S_0} P_s \quad \text{subject to:} \quad P_f = \beta \tag{23}$$

where β is a chosen fixed value for the fail rate. The solution of the optimization problem is given by, cf.(Teunissen, 2003c; Teunissen, 2004):

$$\Omega_0 = \{x \in S_0 \mid \sum_{z \in \mathbb{Z}^n} f_{\hat{a}}(x+z) \leq \mu f_{\hat{a}}(x+a)\} \tag{24}$$

The best choice for S_0 is the ILS pull-in region. The reason is that $P_s + P_f$ is independent of S_0, but P_s is not. Therefore, any choice of S_0 which makes P_s larger, automatically makes P_f smaller. The best choice for S_0 follows from:

$$\max_{S_0} \int_{\Omega \cap S_a} f_{\hat{a}}(x)dx \quad \text{subject to} \quad \Omega = \Omega + z, \forall z \in \mathbb{Z}^n$$

as the ILS pull-in region, see (Teunissen, 1999).

From eq.(24) the optimal test follows as:

$$\text{Accept } \check{a} \text{ iff:} \quad \frac{f_{\check{\epsilon}}(\hat{a}-\check{a})}{f_{\hat{a}}(\hat{a}-\check{a})} \leq \mu \tag{25}$$

where we used eq.(17).

Compare this result with the Ratio Test in eq.(20). Both test statistics are defined as a ratio. In the case of the Ratio Test, it only depends on $\|\hat{a}-\check{a}\|^2_{Q_{\hat{a}}}$ and $\|\hat{a}-\check{a}_2\|^2_{Q_{\hat{a}}}$, whereas from eq.(24) it follows that the optimal test statistic depends on all $\|\hat{a}-z\|^2_{Q_{\hat{a}}}$, $z \in \mathbb{Z}^n$ if it is assumed that the float solution is normally distributed.

7 EVALUATION OF THE TESTS

7.1 Performance in the 2-D case

In order to illustrate the principle and the properties of the Ratio Test and optimal test, examples will be given here. These examples are also used to illustrate the differences between the tests.

Simulations were carried out to generate 500,000 samples of the float range and ambiguities. The following vc-matrices $Q_{\hat{a}}$ are used:

$$Q_1 = \begin{bmatrix} 0.0577 & -0.0242 \\ -0.0242 & 0.0564 \end{bmatrix}, \; Q_2 = 1.5Q_1, \; Q_3 = 6Q_1$$

$$Q_4 = Q_2 + \begin{bmatrix} 0.8 & 0 \\ 0 & 0 \end{bmatrix}$$

The first three vc-matrices correspond to a dual-frequency GPS model for one satellite-receiver pair, where the undifferenced standard deviations of the code and phase observables are varied.

The first step is to use a random generator to generate n independent samples from the univariate standard normal distribution $N(0,1)$, and then collect these in a vector x. This vector is transformed by means of $\hat{a} = Gx$, with G equal to the Cholesky factor of $Q_{\hat{a}} = GG^T$. The result is a sample $\hat{a}$ from $N(0, Q_{\hat{a}})$, and this sample is used as input for integer least-squares estimation and the validation test. Three outcomes can be distinguished:

success (s) : $\check{a} = 0$ and is accepted;
failure (f) : $\check{a} \neq 0$ and is accepted;
undecided (u) : estimate is rejected.

This process can be repeated an N number of times, and one can count how many times each of the outcomes is obtained, say N_s, N_f and N_u times. The approximations of the success rate and fail rate follow then as:

$$P_s = \frac{N_s}{N}, \quad P_f = \frac{N_f}{N}$$

In order to get good approximations, the number of samples N must be chosen sufficiently large, see (Teunissen, 1998).

In figures 2-5 the following parameters are plotted as function of the fail rate:

P_s : success rates
P_{id} : percentage of Ratio Test and Optimal IA solutions identical to each other
μ : aperture parameters

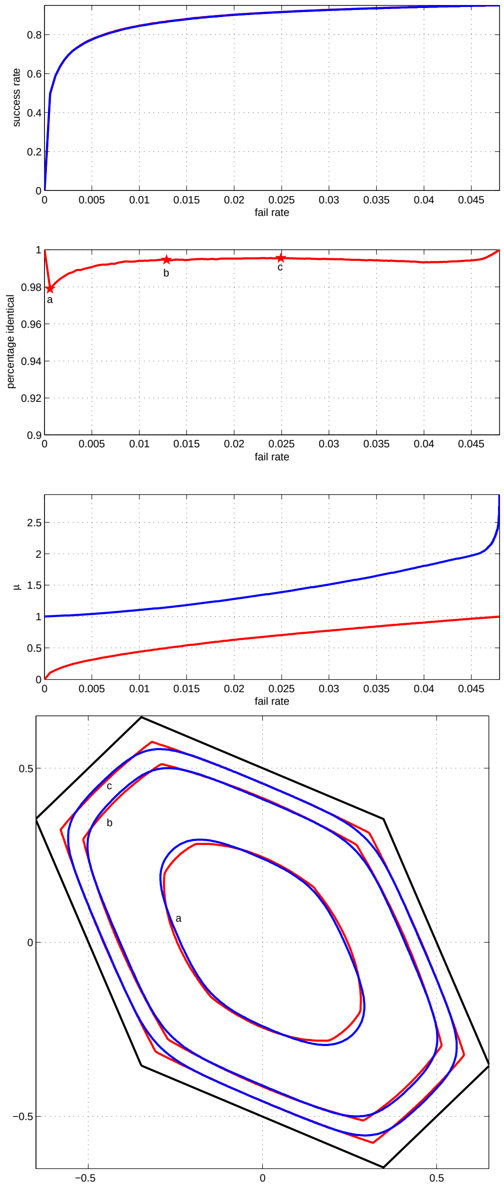

Figure 2: Q_1. Top to bottom: Success rate as function of fail rate; Percentage of solutions identical to OIA solution; μ as function of the fail rate; Aperture pull-in regions for: a) the μ that results in the smallest percentage of solutions identical to the optimal test; b) $\mu_R = 0.5$; c) $P_f = 0.025$. Blue: optimal test. Red: Ratio Test.

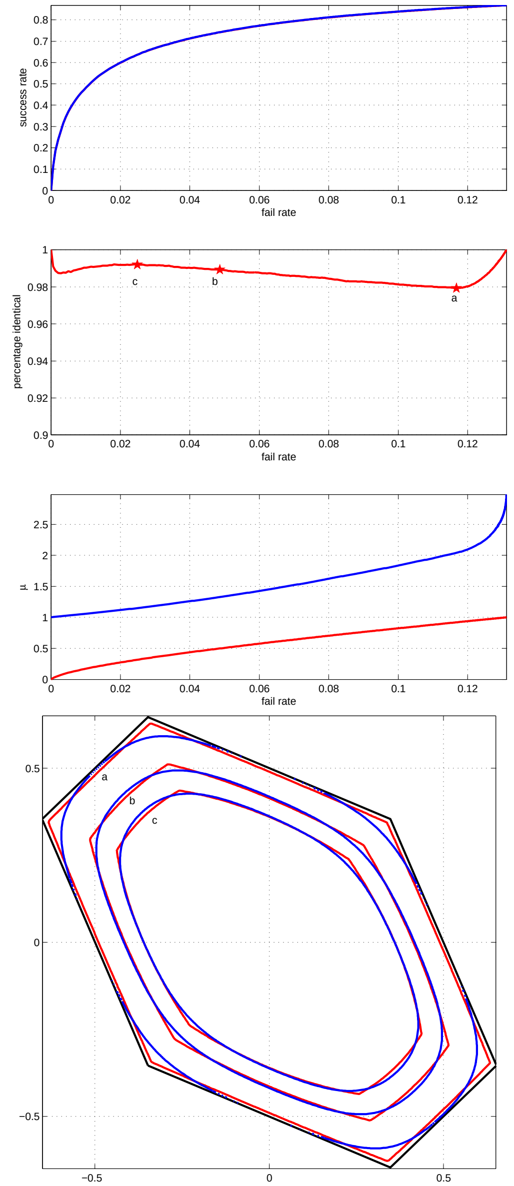

Figure 3: Q_2. Top to bottom: Success rate as function of fail rate; Percentage of solutions identical to OIA solution; μ as function of the fail rate; Aperture pull-in regions for: a) the μ that results in the smallest percentage of solutions identical to the optimal test; b) $\mu_R = 0.5$; c) $P_f = 0.025$. Blue: optimal test. Red: Ratio Test.

Note that the maximum fail rate and success rate are equal to those obtained with integer least-squares.

Obviously, the differences between the success rates of the Ratio Test and the optimal IA estimator are very small. P_{id} is very high if the precision is good, which is the case for Q_1 and Q_2. For Q_4 the Ratio Test IA estimator clearly performs somewhat poorer, since a difference between the success rates of both estimators can be seen and P_{id} is much lower than with the other vc-matrices.

In order to explain the differences in the performance of the Ratio Test and the optimal test, three aperture pull-in regions for both tests are plotted for each vc-matrix. They correspond to the following three situations:

a : μ chosen such that for both tests the fail rate is obtained for which P_{id} is minimum

b : Ratio Test with fixed critical value of $\mu = 0.5$, μ for optimal test chosen such that fail rates of both tests are equal

c : Ratio Test and Optimal Test both with fixed fail rate of $P_f = 0.025$

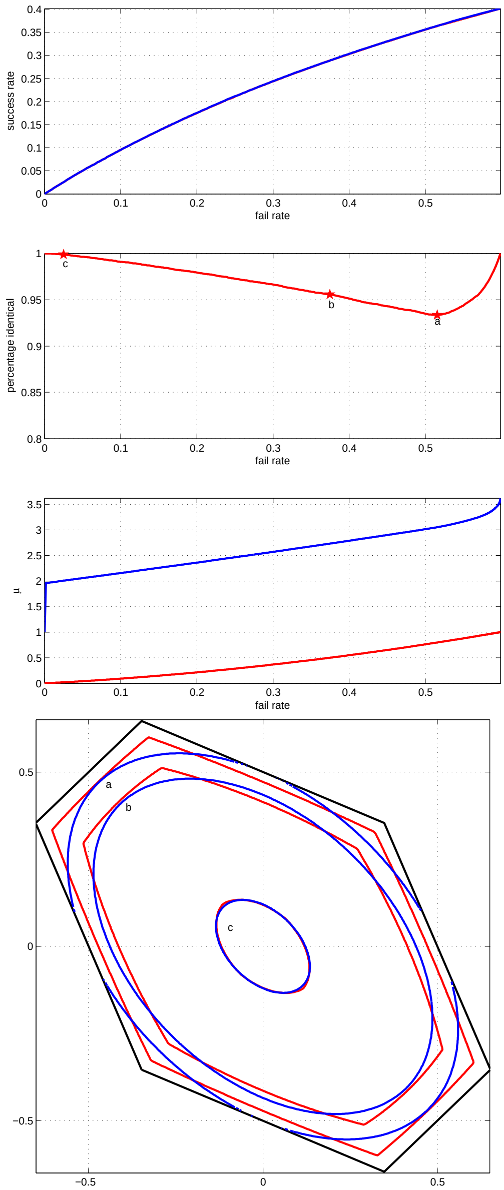

Figure 4: Q_3. Top to bottom: Success rate as function of fail rate; Percentage of solutions identical to OIA solution; μ as function of the fail rate; Aperture pull-in regions for: a) the μ that results in the smallest percentage of solutions identical to the optimal test; b) $\mu_R = 0.5$; c) $P_f = 0.025$. Blue: optimal test. Red: Ratio Test.

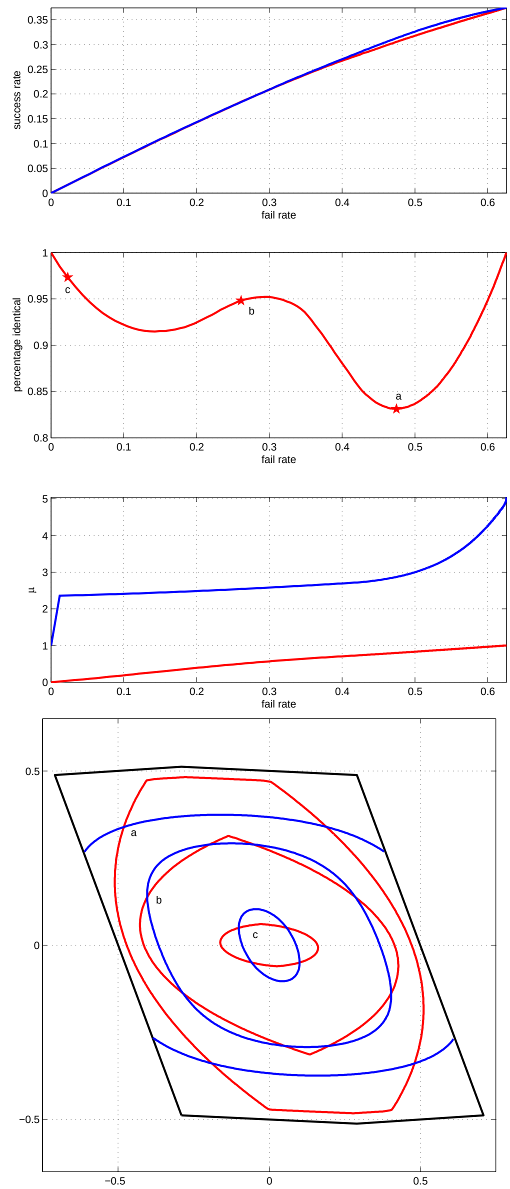

Figure 5: Q_4. Top to bottom: Success rate as function of fail rate; Percentage of solutions identical to OIA solution; μ as function of the fail rate; Aperture pull-in regions for: a) the μ that results in the smallest percentage of solutions identical to the optimal test; b) $\mu_R = 0.5$; c) $P_f = 0.025$. Blue: optimal test. Red: Ratio Test.

The aperture pull-in regions are shown in the bottom panels of figures 2-5. The corresponding fail rates and P_{id} are depicted with the stars in the graphs second from top.

It follows that the first situation, smallest P_{id} for the same fail rate, in general occurs when the OIA pull-in region just touches the ILS pull-in region, and thus the aperture parameter is large. The reason is that for a large aperture parameter the shape of the Ratio Test pull-in region starts to resemble the shape of the ILS pull-in region and is larger in the direction of the corner points of the ILS pull-in regions where little probability mass is located. The OIA pull-in region, on the other hand, touches the ILS pull-in regions there where more probability mass is located in S_a, since $f_{\hat{a}}(x)$ is elliptically shaped. Only for vc-matrix Q_1 the smallest P_{id} is not obtained for a large aperture pull-in region, because the precision is so high that $f_{\hat{a}}(x)$ is peaked and the probability mass close to the boundary of the ILS pull-in region is very low.

If the aperture pull-in regions are chosen even larger than shown in figures 2-5, the OIA pull-in region is cut off by the ILS pull-in region. So, if μ is increased the OIA pull-in region cannot expand in all directions, and therefore the fail rate will not change much. This explains why for high fail rates μ suddenly starts to increase much faster, as can be seen in the graphs showing μ as function of the fail rate.

The aperture pull-in regions of the Ratio Test and optimal test are especially different in the direction of the vertices of the ILS pull-in region. That is because in those directions there are two integers, $\check{a}_2$ and $\check{a}_3$ with the same distance to $\hat{a}$, but only $\|\hat{a} - \check{a}_2\|^2_{Q_{\hat{a}}}$ is considered by the Ratio Test, whereas with the optimal test the likelihood of all integers is considered, see section 6.

In the graphs showing P_{id} as function of the fail rate, it can be seen that using a fixed critical value for the Ratio Test (b) results in very different and sometimes large fail rates, depending on the vc-matrix. The reason is that the size of the aperture pull-in regions is not adjusted if the precision changes. This can be seen for Q_1, Q_2, and Q_3 which are scaled versions of each other: the Ratio Test aperture pull-in regions corresponding to situation b are all equal.

The fixed fail rate approach works much better than using a fixed critical value for the Ratio Test, because the size of the aperture pull-in region is nicely adjusted in case of a fixed fail rate, as can be seen by comparing the results corresponding to situation c for Q_1, Q_2, and Q_3. In practice the choice of a fixed μ often works satisfactorily, but that is because either the value is chosen very conservative, cf. (Han, 1997), or it is required that the ILS success rate is close to one before an attempt to fix the ambiguities is made.

7.2 Performance in the geometry-based case

The performance of the Ratio Test and optimal test was evaluated and compared for the simple 2-D case in the preceding section, since it is then possible to analyze and illustrate the results by looking at the pull-in regions. The question is now whether or not the conclusions from these examples are also valid for the higher-dimensional, geometry-based GNSS models in which users will be interested. Therefore, also simulations are used for several geometry-based models. The GPS constellation is based on the Yuma almanac for GPS week 184 and a cut-off elevation of 15^o. Undifferenced standard deviations of $\sigma_p = 30$cm and $\sigma_\phi = 3$mm are used for both frequencies. The GPS model is set up for a single epoch for two different locations and times. In one case 4 satellites are visible and a medium baseline length is considered; in the other case 6 satellites are visible and a longer baseline is considered. For the following epochs it is simply assumed that

$$Q_k = \frac{1}{k} Q_1$$

with k the epoch number. This relation is valid when the satellite geometry is not changed.

The aperture parameters, success rates and fail rates as function of the number of epochs are determined using simulations, similarly as in section 7.1. The following approaches are considered:

- Optimal IA estimation, fixed fail rate $P_f = 0.005$;
- Ratio Test IA estimation, fixed fail rate $P_f = 0.005$;
- Ratio Test, fixed critical value $\mu = \frac{1}{3}$;
- Ratio Test, fixed critical value $\mu = \frac{1}{2}$ **and** fixed solution is only accepted if $P_{s,ILS} \geq 0.99$;
- fixed solution is only accepted if $P_{s,ILS} \geq 0.995$.

If the ILS fail rate is smaller than 0.005, all solutions are accepted with the first two IA estimators, i.e. $\Omega_0 = S_{0,ILS}$. Note that the last approach also guarantees that the fail rate will never exceed 0.005.

For (near) real-time applications it is important that the time to first fix is as short as possible. The probability that a fix is made at a certain epoch equals:

$$P_{fix} = P_s + P_f \tag{26}$$

Figures 6 and 7 show P_{fix} and $P_{s|\bar{a}=z}$ obtained with the 5 approaches as function of the number of epochs. Obviously, IA estimation with a fixed fail rate result in the highest P_{fix}. Again, it follows that the performance of the Ratio Test IA estimator is close to optimal only with the fixed fail rate approach.

Applying the Ratio Test with a fixed critical value of $1/3$ (green line) may give comparable probabilities P_{fix} in the first epochs. Note that if the success rate is higher than the one obtained with the fixed fail rate, this implies that the corresponding fail rate is higher than 0.005. In later epochs the fixed critical value is far too conservative, so that the time to first fix may be unnecessarily long.

For the first epoch the probability $P_{s|\bar{a}=z}$ is always low for the examples considered here. The reason is that the success rate P_s is very low, and thus the approximation (19) is not valid. For later epochs $P_{s|\bar{a}=z}$ becomes reasonably high with all approaches, and is highest for the fixed critical value approach. Note, however, that this is only the case because this approach is too conservative; P_{fix} is very low in the first epochs, so that if the solution is fixed it can be expected to be correct since the aperture space, and thus also the fail rate, is very small.

With the two approaches where it is required that the ILS success rate exceeds a certain limit (black and light blue lines) no attempt is made to fix the solution in the first epochs. As soon as a fix is made, the probability $P_{s|\bar{a}=z}$ is high.

With the optimal test and the Ratio Test with a fixed fail rate there is a very high probability that the solution will be fixed much faster than with the other approaches, and the Ratio Test performs close to optimal. At the same time it is guaranteed that the fail rate will not exceed the user-defined threshold. This is not the case if the Ratio Test with a fixed critical value is used. It can be concluded that the performance of the Ratio Test with a fixed fail rate is much

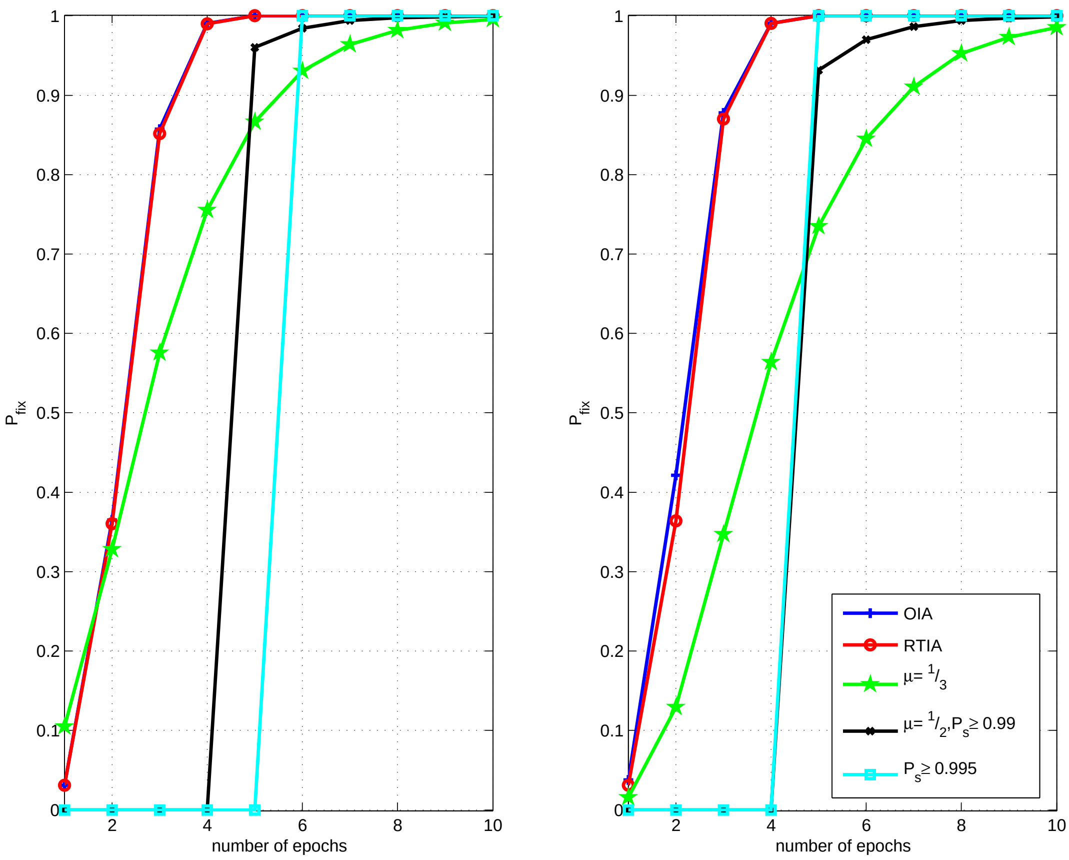

Figure 6: Probabilities that solution will be fixed as function of the number of epochs with 5 different tests. Left: 4 visible satellites, Right: 6 visible satellites.

better than with a fixed critical value, especially when more epochs of data become available. The reason is that using a fixed critical value implies that the aperture pull-in region Ω_0 does not change as function of the number of epochs. Recall that the success rate is given by:

$$P_s = \int_{\Omega_0} f_{\hat{a}}(x + a)dx$$

So, if more epochs of data become available, the success rate only increases because $f_{\hat{a}}(x+a)$ becomes more peaked due to the improved precision. Using a fixed fail rate, on the other hand, means that Ω_0 becomes larger when the precision improves, and thus the success rate increases more than with the fixed critical value approach.

8 IMPLEMENTATION ASPECTS

A problem not discussed so far is how to determine the aperture parameter given the fixed fail rate. It will be clear from the definition of the fail rate in eq.(16) and the aperture pull-in region Ω_0 in eqs.(21) and (24) that there is no closed-form expression in order to compute this parameter. The following approach can be used for Ratio Test Integer Aperture estimation.

1. choose fixed fail rate: $P_f = \beta$
2. collect observations and apply least-squares adjustment: $\hat{a}$, $Q_{\hat{a}}$;
3. apply integer least-squares estimation: $\check{a}$, $\check{\epsilon}$, $\|\hat{a} - \check{a}\|^2_{Q_{\hat{a}}}$, $\|\hat{a} - \check{a}_2\|^2_{Q_{\hat{a}}}$;
4. determine $\mu' = \frac{\|\hat{a}-\check{a}\|^2_{Q_{\hat{a}}}}{\|\hat{a}-\check{a}_2\|^2_{Q_{\hat{a}}}}$,
   This means that $\hat{a} - \check{a}$ lies on the boundary of the aperture pull-in region Ω'_0 determined by μ';
5. generate N samples of float ambiguities $\hat{x}_i \sim N(0, Q_{\hat{a}})$;
6. for each sample determine the ILS fixed solution $\check{x}_i$ and $\mu_i = \frac{\|\hat{x}_i-\check{x}_i\|^2_{Q_{\hat{a}}}}{\|\hat{x}_i-\check{x}_{i,2}\|^2_{Q_{\hat{a}}}}$;
7. count number of samples N_f that are fixed incorrectly but fall in Ω'_0: $\mu_i \leq \mu'$ and $\check{x}_i \neq 0$;
8. compute fail rate with μ' as critical value: $P_f(\mu') = \frac{N_f}{N}$;
9. if $P_f(\mu') \leq \beta$: continue with step 10, otherwise use $\hat{a}$;

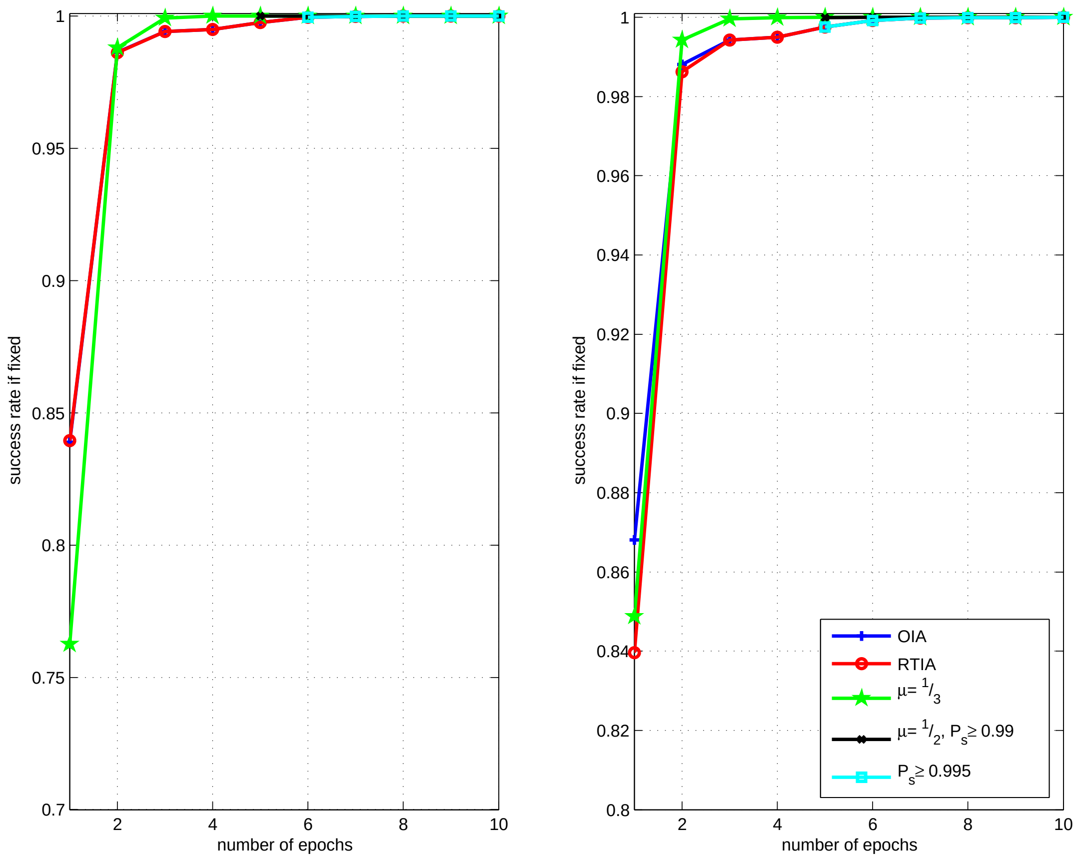

Figure 7: Success rates conditioned on $\bar{a} = z$ as function of the number of epochs with 5 different tests. Left: 4 visible satellites, Right: 6 visible satellites.

10. count number of samples N_s that are correctly fixed and fall in Ω'_0: $\mu_i \leq \mu'$ and $\check{x}_i = 0$
11. compute success rate: $P_s(\mu') = \frac{N_s}{N}$ and $P_{s|\bar{a}=z}$;
12. if $P_{s|\bar{a}=z} \approx 1$: **accept** $\check{a}$.

This approach is allowed since P_f is monotonically increasing with μ, and therefore:

$$P_f(\mu') \leq \beta = P_f(\mu_\beta) \iff \mu' = \frac{\|\hat{a} - \check{a}\|^2_{Q_{\hat{a}}}}{\|\hat{a} - \check{a}_2\|^2_{Q_{\hat{a}}}} \leq \mu_\beta \tag{27}$$

and the latter inequality is the Ratio Test IA estimation criterion which determines the aperture space.

The LAMBDA (Least-squares AMBiguity Decorrelation Adjustment) method can be used for efficient determination of the ILS solutions in steps 3 and 6, see e.g. (De Jonge and Tiberius, 1996; Teunissen, 1993; Teunissen, 1995).

In order to get a good approximation of the fail rate in step 8 the number of samples must typically be chosen large, $N > 100,000$. But in order to reduce the computation time, which is important in real-time applications, it has been investigated how much the results would differ with N much smaller so that the computation time becomes acceptable. It followed that already with a few thousand samples the results are quite good – same decision as with N much larger in more than 99% of the cases –, and in any case better than with the traditional approaches.

9 CONCLUDING REMARKS

In this contribution an overall approach is presented for the problem of integer estimation and validation. This approach for the first time includes an exact and overall probabilistic evaluation of the solution

This is made possible by the introduction of the theory of Integer Aperture Inference, on which the theoretical justification of the popular Ratio Test is based, but the theory also allows for the definition of an optimal test.

Furthermore, rigorous measures are available for deciding whether or not to use the fixed solution by using the fixed fail rate approach. Simulations indicate that the Ratio Test gives a close to optimal performance, which is another justification of its popularity. However, this justification is only valid when the fixed fail rate approach is used, but not if the classical approach of using a fixed critical value is used.

With this theory available, there is no need anymore to make incorrect assumptions on the distribution of the parameters; until now it was simply assumed in the literature that the fixed ambiguities are deterministic, which is only a valid assumption when the precision is high. The Ratio Test as it is currently used in practice is generally applied with a fixed critical value, but this value will in general be either too conservative, or only valid when the precision is high.

Using the fixed fail rate approach implies that the critical value depends on the model (and thus the precision) at hand. This means that the time to first fix will be shorter, and at the same time it is guaranteed that the probability of incorrect fixing is below a user-defined threshold.

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