Asymptotic quotient observers for 2-D Fornasini Marchesini models

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Abstract— The concepts of conditioned-invariant, detectability and input-containing subspaces are developed within the context of observer design for 2-D Fornasini-Marchesini models in a general form. Specifically, a link is establised between these subspaces and the existence of so-called quotient observers, which estimate the local state modulo a conditioned invariant subspace. We also consider the synthesis of observers that are asymptotic in the sense that the estimation error (modulo a conditioned invariant subspace) tends to zero away from the boundary values.

I. INTRODUCTION

Conditioned invariant subspaces for 1-D systems were introduced by Basile and Marro in [1] as the dual of controlled invariant subspaces. The role of such subspaces in relation to the problem of estimation in the presence of unknown input signals was investigated by the same authors in [2]. An alternative definition of conditioned invariance was proposed by Willems in terms of the existence of certain observers [25]; also see the recent textbooks [3, Chapter 4] and [24, Chapter 5].

The purpose of this paper is to first extend the definition of conditioned invariance and input-containing subspaces given for 1-D systems in [1], to Fornasini-Marchesini models [7], [10] in the general form

$$
\begin{aligned}
x_{i+1,j+1} &= A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\
&\quad + B_0 u_{i,j} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \quad (1) \\
y_{i,j} &= C x_{i,j} + D u_{i,j},
\end{aligned}
$$

of Kurek [18]. Our approach to defining conditioned invariant subspaces is similar to that of Willems in that we ultimately seek an observer of the form[1]

$$
\begin{aligned}
\omega_{i+1,j+1} &= K_0 \omega_{i,j} + K_1 \omega_{i+1,j} + K_2 \omega_{i,j+1} \\
&\quad + L_0 y_{i,j} + L_1 y_{i+1,j} + L_2 y_{i,j+1}, \quad (2)
\end{aligned}
$$

so that the estimation error $e_{i,j} \triangleq Q x_{i,j} - \omega_{i,j}$, for some full row-rank matrix Q,[2] asymptotically approaches zero away from standard boundary conditions. To this end, we develop notions of *conditioned-invariant*, *detectability* and

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[1]This is different to the less general form considered in [22].

[2]That is, the error modulo the null space of Q.

input-containing subspaces, which turn out to be related to the existence of matrices Λ_i and Γ_i for which

$$
\begin{aligned}
&Q \begin{bmatrix} A_0 & A_1 & A_2 \end{bmatrix} - \begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_2 \end{bmatrix} \operatorname{diag}(C,C,C) \\
&= \begin{bmatrix} \Gamma_0 & \Gamma_1 & \Gamma_2 \end{bmatrix} \operatorname{diag}(Q,Q,Q).
\end{aligned}
$$

In this way, when there are no inputs (i.e. $u_{i,j} = 0$), with $L_i = \Lambda_i$ and $K_i = \Gamma_i$ the estimation error satisfies

$$
\begin{aligned}
e_{i+1,j+1} &= \Big(Q \begin{bmatrix} A_0 & A_1 & A_2 \end{bmatrix} \\
&\quad - \begin{bmatrix} L_0 & L_1 & L_2 \end{bmatrix} \operatorname{diag}(C,C,C) \Big) \begin{bmatrix} x_{i,j} \\ x_{i+1,j} \\ x_{i,j+1} \end{bmatrix} \\
&\quad - \begin{bmatrix} K_0 & K_1 & K_2 \end{bmatrix} \begin{bmatrix} \omega_{i,j} \\ \omega_{i+1,j} \\ \omega_{i,j+1} \end{bmatrix} \\
&= \Gamma_0 e_{i,j} + \Gamma_1 e_{i+1,j} + \Gamma_2 e_{i,j+1}.
\end{aligned}
$$

That is, the dynamics of the estimation error can be expressed as an autonomous FM model in Kurek form. It is interesting to note that, unlike the 1-D case, the required notions of conditioned invariance are not dual of the notion of controlled invariance developed in [6], [21]. This is because the obvious dual of (1) is not in the same form.

Notation. We denote the origin of $\mathbb{R}^n$ by $\mathbf{0}_n$. The image, kernel, transpose and Moore-Penrose inverse of a matrix M are denoted $\operatorname{im} M$, $\ker M$, $M^{\top}$ and $M^{\dagger}$, respectively. The $n \times m$ zero matrix is denoted by $0_{n \times m}$. We define $M_D \triangleq \operatorname{diag}(M,M,M)$, and, accordingly, given a subspace $\mathscr{J} \subseteq \mathbb{R}^n$, the symbol $\mathscr{J}_D$ denotes the subspace $\mathscr{J} \times \mathscr{J} \times \mathscr{J}$ of $\mathbb{R}^{3n}$, where $\times$ is the Cartesian product. Given the vector $\xi \in \mathbb{R}^n$, the symbol $\xi / \mathscr{J}$ denotes the canonical projection of ξ on the quotient space $\mathbb{R}^n / \mathscr{J}$. Finally, given a triple of matrices (M_0, M_1, M_2), we define $M_H \triangleq \begin{bmatrix} M_0 & M_1 & M_2 \end{bmatrix}$ and $M_V \triangleq \begin{bmatrix} M_0^{\top} & M_1^{\top} & M_2^{\top} \end{bmatrix}^{\top}$.

II. INVARIANT SUBSPACES FOR FM MODELS

We begin by considering the autonomous FM model

$$
x_{i+1,j+1} = A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1}. \quad (3)
$$

As *boundary conditions* for (3) we use $x_{i,j} = b_{i,j} \in \mathbb{R}^n$ for all $(i,j) \in \mathfrak{B}$ and some constants $b_{i,j} \in \mathbb{R}^n$, where $\mathfrak{B} \triangleq (\{0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{0\})$. [3]

A subspace $\mathscr{J}$ of $\mathbb{R}^n$ is said to be (A_0, A_1, A_2)-invariant if $\mathscr{J}$ is A_i-invariant for $i \in \{0,1,2\}$ in the usual 1-D sense; i.e., $A_i x \in \mathscr{J}$ for all $x \in \mathscr{J}$ and $i \in \{0,1,2\}$. The following provides geometric and matrix conditions for invariance.

[3]Other choices of $\mathfrak{B}$, for which a unique solution of (3) exists, are possible; see [11]. The results in this paper can be adapted to these cases.

Lemma 2.1: The following are equivalent:

1) $\mathscr{J}$ is (A_0, A_1, A_2)-invariant;
2) $A_H \mathscr{J}_D \subseteq \mathscr{J}$;
3) There exist $L_0, L_1, L_2 \in \mathbb{R}^{(n-r) \times (n-r)}$ such that $QA_i = L_i Q$ for $i \in \{0, 1, 2\}$, where $Q \in \mathbb{R}^{(n-r) \times n}$ is a full row-rank matrix such that $\ker Q = \mathscr{J}$, i.e., $QA_H = L_H Q_D$.

Proof: **1)** $\implies$ **2)** For $\xi_i \in \mathscr{J}$, $i \in \{0, 1, 2\}$, it follows that $A_i \xi_i \in \mathscr{J}$, and hence, $A_0 \xi_0 + A_1 \xi_1 + A_2 \xi_2 = A_H \begin{bmatrix} \xi_0^\top & \xi_1^\top & \xi_2^\top \end{bmatrix}^\top \in \mathscr{J}$.

2) $\implies$ **1)** Suppose there exist an $i \in \{0, 1, 2\}$ and an $\xi \in \mathscr{J}$ such that $A_i \xi \notin \mathscr{J}$; i.e., $\mathscr{J}$ is not (A_0, A_1, A_2)-invariant. Then $A_H \begin{bmatrix} \xi^\top & 0 & 0 \end{bmatrix}^\top \notin \mathscr{J}$, which contradicts **2)**.

2) $\Leftrightarrow$ **3)** Note that **2)** is equivalent to $\ker Q_D \subseteq \ker QA_H$, by which the result holds, since for any matrices $M \in \mathbb{R}^{p \times m}$ and $N \in \mathbb{R}^{q \times p}$, $\ker M \subseteq \ker N$ if and only if there exist an $L \in \mathbb{R}^{q \times m}$ such that $N = LM$. ∎

The following theorem is the 2-D counterpart of a well-known result (see [1]) concerning the decomposition of a 1-D system matrix with respect to an invariant subspace.

Theorem 2.1: The following are equivalent:

1) There exists an r-dimensional subspace $\mathscr{J} \subseteq \mathbb{R}^n$ that is (A_0, A_1, A_2)-invariant;
2) There exists a similarity transformation $S \in \mathbb{R}^{n \times n}$ such that for each $i \in \{0, 1, 2\}$

$$\hat{A}_i \triangleq S A_i S^{-1} = \begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ 0_{(n-r) \times r} & \hat{A}_i^{22} \end{bmatrix}. \quad (4)$$

Proof: In view of Lemma 2.1, the proof follows that of Theorem 2.1 in [21], via a similarity transformation T such that $T^{-1} = S$, where S is any non-singular matrix for which $QS^{-1} = \begin{bmatrix} 0_{(n-r) \times r} & I_{(n-r)} \end{bmatrix}$, where Q is a full row-rank matrix such that $\ker Q = \mathscr{V}$. In particular, with respect to the corresponding basis, the identities $QA_i = L_i A_i$ in **3)** of Lemma 2.1 can be expressed as

$$\begin{bmatrix} 0_{(n-r) \times r} & I_{(n-r)} \end{bmatrix} \begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ 0 & \hat{A}_i^{22} \end{bmatrix} = L_i \begin{bmatrix} 0_{(n-r) \times r} & I_{(n-r)} \end{bmatrix},$$

by which $L_i = \hat{A}_i^{22}$. ∎

A. Invariant Subspaces and Local-State Trajectories

Lemma 2.2: Consider an (A_0, A_1, A_2)-invariant subspace $\mathscr{J}$. A boundary condition $x_{i,j} = b_{i,j} \in \mathscr{J}$, for $(i, j) \in \mathfrak{B}$, gives rise to $x_{i,j} \in \mathscr{J}$ for all $i, j \geq 0$.

Proof: In the set of coordinates corresponding to the similarity transformation S in Theorem 2.1, it follows that

$$\begin{bmatrix} x'_{i+1,j+1} \\ x''_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} \hat{A}_0^{11} & \hat{A}_0^{12} \\ 0 & \hat{A}_0^{22} \end{bmatrix} \begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} + \begin{bmatrix} \hat{A}_1^{11} & \hat{A}_1^{12} \\ 0 & \hat{A}_1^{22} \end{bmatrix} \begin{bmatrix} x'_{i+1,j} \\ x''_{i+1,j} \end{bmatrix}$$
$$+ \begin{bmatrix} \hat{A}_2^{11} & \hat{A}_2^{12} \\ 0 & \hat{A}_2^{22} \end{bmatrix} \begin{bmatrix} x'_{i,j+1} \\ x''_{i,j+1} \end{bmatrix}. \quad (5)$$

Note that any boundary condition $x_{i,j} = b_{i,j} \in \mathscr{J}$ is such that $x''_{i,j} = 0$ for $(i, j) \in \mathfrak{B}$. Moreover, by (5), $x''_{i,j} = 0$ for all $i, j \geq 0$. Hence, $x_{i,j} \in \mathscr{J}$ for all $i, j \geq 0$. ∎

In the basis corresponding to S in the proof of Lemma 2.2, whereby $\begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} = S x_{i,j}$, the component $x'_{i,j}$ is the projection of the local state $x_{i,j}$ onto the invariant subspace $\mathscr{J}$, while $x''_{i,j}$ is the canonical projection on to the quotient space $\mathbb{R}^n / \mathscr{J}$.

B. Internal and External Stability of Invariant Subspaces

A necessary and sufficient condition for asymptotic stability of (3) – often said asymptotic stability of the triple (A_0, A_1, A_3) – is that $\forall (z_1, z_2) \in \mathfrak{P}$

$$\det(I_n - A_0 z_1 z_2 - A_1 z_2 - A_2 z_1) \neq 0, \quad (6)$$

where $\mathfrak{P} \triangleq \{(\zeta_1, \zeta_2) \in \mathbb{C} \times \mathbb{C} \mid |\zeta_1| < 1 \text{ and } |\zeta_2| < 1\}$; this is equivalent to $x_{i,j} \to 0$ as $i + j \to \infty$. Various, more computationally tractable, sufficient stability conditions have been proposed over the last two decades, in terms of Lyapunov equations and/or spectral radius conditions of certain matrices, see e.g. [14], [5]. In the very recent literature, new necessary and sufficient criteria have appeared for asymptotic stability in terms of conditions that can be checked in finite terms, see [27], [9]. For the sake of argument and clarity, however, the following simple sufficient condition for asymptotic stability, expressed in terms of an linear matrix inequality (LMI), will be used herein:

Lemma 2.3: ([14]) The triple (A_0, A_1, A_2) is asymptotically stable if three symmetric positive definite matrices P_0, P_1 and P_2 exist such that:

$$\text{diag}(P_0, P_1, P_2) - A_H^\top (P_0 + P_1 + P_2) A_H > 0. \quad (7)$$

We now show that stability of (3) can be "split" into two parts with respect to an invariant subspace $\mathscr{J} \subseteq \mathbb{R}^{n \times n}$. Expressing (3) in the set of coordinates corresponding to the similarity transformation S in Theorem 2.1,

$$\det(I_n - \hat{A}_0 z_1 z_2 - \hat{A}_1 z_2 - \hat{A}_2 z_1)$$
$$= \det(I - \hat{A}_0^{11} z_1 z_2 - \hat{A}_1^{11} z_2 - \hat{A}_2^{11} z_1)$$
$$\cdot \det(I - \hat{A}_0^{22} z_1 z_2 - \hat{A}_1^{22} z_2 - \hat{A}_2^{22} z_1).$$

It follows that (3) is asymptotically stable if and only if $(\hat{A}_0^{11}, \hat{A}_1^{11}, \hat{A}_2^{11})$ and $(\hat{A}_0^{22}, \hat{A}_1^{22}, \hat{A}_2^{22})$ are asymptotically stable.

Definition 2.1: The (A_0, A_1, A_2)-invariant subspace $\mathscr{J}$ is

- *internally stable* if the corresponding triple $(\hat{A}_0^{11}, \hat{A}_1^{11}, \hat{A}_2^{11})$ is asymptotically stable.
- *externally stable* if the corresponding triple $(\hat{A}_0^{22}, \hat{A}_1^{22}, \hat{A}_2^{22})$ is asymptotically stable.

Hence, (3) is asymptotically stable if and only if any invariant subspace is both internally and externally stable.

Corollary 2.1: Given a subspace $\mathscr{J}$ of $\mathbb{R}^n$, let $Q \in \mathbb{R}^{(n-r) \times n}$ be a full row-rank matrix such that $\ker Q = \mathscr{J}$. Then $\mathscr{J}$ is an externally stable (A_0, A_1, A_2)-invariant subspace if and only if an asymptotically stable triple (L_0, L_1, L_2) exists such that $QA_i = L_i Q$ for all $i \in \{0, 1, 2\}$.

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Proof: See last part of the proof of Theorem 2.1, whereby $\hat{A}_i^{22} = L_i$. ∎

III. CONDITIONED INVARIANT SUBSPACES

Now we focus on the definition of conditioned invariant subspaces for (1). Such subspaces are shown to be related to the existence of a so-called quotient observer as discussed in the introduction.

Definition 3.1: The subspace $\mathscr{S} \subseteq \mathbb{R}^n$ is conditioned invariant for (1) if $A_H(\mathscr{S}_D \cap \ker C_D) \subseteq \mathscr{S}$.

It is can be seen that the set of conditioned invariant subspaces is closed under subspace intersection. Its smallest element is $\mathbf{0}_n$, its largest element is $\mathbb{R}^n$.

Lemma 3.1: Let $\mathscr{S}$ be an s-dimensional subspace of $\mathbb{R}^n$, and let $Q \in \mathbb{R}^{(n-s) \times n}$ be such that $\ker Q = \mathscr{S}$ with Q of full row-rank. The following statements are equivalent:

1) the subspace $\mathscr{S}$ is conditioned invariant for (1);
2) there exist matrices $\Gamma = [\Gamma_0 \ \Gamma_1 \ \Gamma_2] \in \mathbb{R}^{(n-s) \times 3(n-s)}$ and $\Lambda = [\Lambda_0 \ \Lambda_1 \ \Lambda_2] \in \mathbb{R}^{(n-s) \times 3p}$ such that

$$QA_H = \Gamma Q_D + \Lambda C_D; \qquad (8)$$

3) there exist a matrix $G = [G_0 \ G_1 \ G_2] \in \mathbb{R}^{n \times 3p}$ such that

$$(A_H + GC_D)\mathscr{S}_D \subseteq \mathscr{S}. \qquad (9)$$

Proof: **1)** $\implies$ **2)**. Since $\mathscr{S}$ is such that $A_H(\mathscr{S}_D \cap \ker C_D) \subseteq \mathscr{S}$, it follows that $\ker \begin{bmatrix} Q_D \\ C_D \end{bmatrix} \subset \ker QA_H$ and as such, there exist $\Gamma \in \mathbb{R}^{(n-s) \times 3(n-s)}$ and $\Lambda \in \mathbb{R}^{(n-s) \times 3p}$ such that $QA_H = \Gamma Q_D + \Lambda C_D$; see Proof of Lemma 2.1. **2)** $\implies$ **3)**. Equation (9) follows from (8) with any G such that $\Lambda = -QG$. **3)** $\implies$ **1)**. This follows by defintion. ∎

Property **3)** in Lemma 3.1 means that $\mathscr{S}$ is conditioned invariant for (1) if and only if there exists an output-injection matrix $G = [G_0 \ G_1 \ G_2] \in \mathbb{R}^{n \times 3p}$ such that $\mathscr{S}$ is a $(A_0 + G_0 C, A_1 + G_1 C, A_2 + G_2 C)$-invariant subspace. Let Γ and Λ be such that (8) holds, which can be written as

$$QA_H = \begin{bmatrix} \Gamma & \Lambda \end{bmatrix} \begin{bmatrix} Q_D \\ C_D \end{bmatrix}, \qquad (10)$$

a linear equation which can be solve for Γ and Λ. Given a conditioned invariant subspace, the solutions of (10) are given by

$$\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = QA_H \begin{bmatrix} Q_D \\ C_D \end{bmatrix}^\dagger + KH, \qquad (11)$$

where the rows of H span the null-space of $\begin{bmatrix} Q_D^\top & C_D^\top \end{bmatrix}$ and K is an arbitrary matrix of suitable size. When $\begin{bmatrix} Q_D \\ C_D \end{bmatrix}$ is full-rank, matrix K has zero rows; i.e., the only solution of (10) is $[\Gamma \ \Lambda] = QA_H \begin{bmatrix} Q_D \\ C_D \end{bmatrix}^\dagger$. By (9), $\tilde{\Gamma}$ exists such that

$$Q(A_H + GC_D) = \tilde{\Gamma} Q_D. \qquad (12)$$

We now investigate the relation between the pairs (Γ, Λ) and $(G, \tilde{\Gamma})$ satisfying (10) and (12), respectively. First, notice that Given a pair $(G, \tilde{\Gamma})$ such that (12) holds, then (10) is satisfied with $\Gamma = \tilde{\Gamma}$ and $\Lambda = -QG$. Conversely,

given a pair of matrices (Γ, Λ) such that (10) holds, then (12) is satisfied with $\tilde{\Gamma} = \Gamma$ and with any G such that $\Lambda = -QG$. As such, no generality is lost by assuming $\tilde{\Gamma} = \Gamma$, and by representing the set of all friends of the conditioned invariant subspace $\mathscr{S}$ as the set of matrices $G \in \mathbb{R}^{n \times 3p}$ satisfying $\Lambda = -QG$, where $\Lambda \in \mathbb{R}^{(n-s) \times 3p}$ is any matrix for which another matrix $\Gamma \in \mathbb{R}^{(n-s) \times 3(n-s)}$ exists so that (10) holds. For any pair (Γ, Λ) such that (10) holds, the solutions of the linear equation $\Lambda = -QG$ are parameterised as

$$G = G_\Lambda + \tilde{G}, \qquad (13)$$

where $G_\Lambda \triangleq -Q^\top (QQ^\top)^{-1} \Lambda$ and $\tilde{G}$ is any $n \times 3p$ matrix such that $Q\tilde{G} = 0$, or, equivalently, such that $\operatorname{im} \tilde{G} \subseteq \ker Q$. The choice of $\tilde{G}$ affects the external stability of $\mathscr{S}$, but not the internal stability of $\mathscr{S}$. Similarly, G_Λ can affect the internal but not the external stability of $\mathscr{S}$. With reference to the proof of Corollary 2.1, note that with $S \triangleq \begin{bmatrix} S_c \\ Q \end{bmatrix}$, where the rows of S_c are linearly independent from those of Q, so that $QS^{-1} = [0 \ I]$, we have that for all $i \in \{0, 1, 2\}$

$$S(A_i + G_i C)S^{-1} = \begin{bmatrix} \Delta_i^{11}(\Lambda, \tilde{G}) & \Delta_i^{12}(\Lambda, \tilde{G}) \\ 0 & \Delta_i^{22}(\Lambda, \tilde{G}) \end{bmatrix}. \qquad (14)$$

Lemma 3.2: For all $i \in \{0, 1, 2\}$, the matrix $\Delta_i^{22}(\Lambda, \tilde{G})$ does not depend on $\tilde{G}$, and the matrix $\Delta_i^{11}(\Lambda, \tilde{G})$ does not depend on the particular Λ which satisfies (10) for some Γ.

Proof: First, let $\tilde{G}_1, \tilde{G}_2$ be such that $Q\tilde{G}_1 = 0$ and $Q\tilde{G}_2 = 0$. From (14) we find that

$$\begin{bmatrix} \Delta_i^{11}(\Lambda, \tilde{G}_1) - \Delta_i^{11}(\Lambda, \tilde{G}_2) & \Delta_i^{12}(\Lambda, \tilde{G}_1) - \Delta_i^{12}(\Lambda, \tilde{G}_2) \\ 0 & \Delta_i^{22}(\Lambda, \tilde{G}_1) - \Delta_i^{22}(\Lambda, \tilde{G}_2) \end{bmatrix}$$
$$= S(A_i + G_{\Lambda,i}C + \tilde{G}_1)S^{-1} - S(A_i + G_{\Lambda,i}C + \tilde{G}_2)S^{-1}$$
$$= \begin{bmatrix} S_c \\ Q \end{bmatrix} (\tilde{G}_1 - \tilde{G}_2)CS^{-1},$$

so that

$$Q(\tilde{G}_1 - \tilde{G}_2)C_i T^{-1}$$
$$= \begin{bmatrix} 0 & \Delta_i^{22}(\Lambda, \tilde{G}_1) - \Delta_i^{22}(\Lambda, \tilde{G}_2) \end{bmatrix} = 0,$$

since $Q\tilde{G}_1 = 0$ and $Q\tilde{G}_2 = 0$.

Now, from (14), it is follows that

$$S_c \left(A_i - Q^\top (QQ^\top)^{-1} \Lambda C + \tilde{G}C \right) \qquad (15)$$
$$= \Delta_i^{11}(\Lambda, \tilde{G})S_c + \Delta_i^{12}(\Lambda, \tilde{G})Q.$$

Let Λ_a and Λ_b such that $QA_H = \Gamma_\star Q_D + \Lambda_\star \hat{C}$, for $\star \in \{a, b\}$. The difference of these equations leads to $(\Lambda_a - \Lambda_b)C_D = -(\Gamma_a - \Gamma_b)Q_D$. By partitioning $(\Gamma_a - \Gamma_b)$ as $[\Xi_0 \ \Xi_1 \ \Xi_2]$, we get $(\Lambda_a - \Lambda_b)C_D = -\Xi_i Q$. Writing (15) with respect to Λ_a and Λ_b and by computing the difference

yields

$$S_c\left(-Q^\top(QQ^\top)^{-1}(\Lambda_a-\Lambda_b)C_D\right)$$
$$=\left(\Delta_i^{11}(\Lambda_a,\tilde{G})-\Delta_i^{11}(\Lambda_b,\tilde{G})\right)S_c$$
$$+\left(\Delta_i^{12}(\Lambda_a,\tilde{G})-\Delta_i^{12}(\Lambda_b,\tilde{G})\right)Q,$$

so that

$$S_c Q^\top(QQ^\top)^{-1}\Delta_i Q$$
$$=\left(\Delta_i^{11}(\Lambda_a,\tilde{G})-\Delta_i^{11}(\Lambda_b,\tilde{G})\right)S_c$$
$$+\left(\Delta_i^{12}(\Lambda_a,\tilde{G})-\Delta_i^{12}(\Lambda_b,\tilde{G})\right)Q.$$

Since Q and S_c have linearly independent rows, we find

$$S_c Q^\top(QQ^\top)^{-1}\Xi_i Q=\left(\Delta_i^{12}(\Lambda_a,\tilde{G})-\Delta_i^{12}(\Lambda_b,\tilde{G})\right)Q$$

and

$$\left(\Delta_i^{11}(\Lambda_a,\tilde{G})-\Delta_i^{11}(\Lambda_b,\tilde{G})\right)S_c=0,$$

the second yielding $\Delta_i^{11}(\Lambda_a,\tilde{G})=\Delta_i^{11}(\Lambda_b,\tilde{G})$ since S_c has linearly independent rows. ∎

Conditioned invariance is linked to the exisistence of 2-D quotient observers [22]. For an observer of the form (2) for (1) with $u_{i,j}=0$,[4] it follows that with $e_{i,j}:=Qx_{i,j}-\omega_{i,j}$,

$$e_{i+1,j+1}=(QA_H-L_HC_D)\begin{bmatrix}x_{i,j}\\x_{i+1,j}\\x_{i,j+1}\end{bmatrix}+K_H\begin{bmatrix}\omega_{i,j}\\\omega_{i+1,j}\\\omega_{i,j+1}\end{bmatrix}.\quad(16)$$

For $K_H=\Gamma$ and $L_H=\Lambda$, where (Λ,Γ) satisfy (10), this becomes

$$e_{i+1,j+1}=\Gamma Q_D\begin{bmatrix}x_{i,j}\\x_{i+1,j}\\x_{i,j+1}\end{bmatrix}+\Gamma\begin{bmatrix}\omega_{i,j}\\\omega_{i+1,j}\\\omega_{i,j+1}\end{bmatrix}$$
$$=\Gamma_0 e_{i,j}+\Gamma_1 e_{i+1,j}+\Gamma_2 e_{i,j+1},\quad(17)$$

so that with observer boundary conditions $\omega_{i,j}=x_{i,j}/\mathscr{S}$, for $(i,j)\in\mathfrak{B}$, it follows that $e_{i,j}=0$ for $(i,j)\in\mathfrak{B}$, and hence, all $(i,j)\in\mathbb{N}\times\mathbb{N}$ by (17). If $(\Gamma_0,\Gamma_1,\Gamma_2)$ is asymptotically stable, then the observer is said to be asymptotic in the sense that $e_{i,j}\to0$ as $i+j\to\infty$ for any boundary conditions. In view of Corollary 2.1, part 2), we are therefore interested in finding $G=\begin{bmatrix}G_0 & G_1 & G_2\end{bmatrix}$ such that $\mathscr{S}$ is an externally stable $(A_0+G_0C,A_1+G_1C,A_2+G_2C)$-invariant subspace; i.e., such that there exists an asymptotically stable triple $(\Gamma_0,\Gamma_1,\Gamma_2)$ for which $Q(A_H+GC_D)=\Gamma Q_D$. When such a G exists, $\mathscr{S}$ is called a *detectability* subspace.

For a given conditioned invariant $\mathscr{S}$, write (11) as

$$\begin{bmatrix}\Gamma_0 & \Gamma_1 & \Gamma_2 & \Lambda\end{bmatrix}=\begin{bmatrix}V_0 & V_1 & V_2 & V_3\end{bmatrix}+K\begin{bmatrix}H_0 & H_1 & H_2 & H_3\end{bmatrix},$$

where $\begin{bmatrix}V_0 & V_1 & V_2 & V_3\end{bmatrix}=QA_H\begin{bmatrix}Q_D\\C_D\end{bmatrix}^\dagger$ and the rows of $\begin{bmatrix}H_0 & H_1 & H_2 & H_3\end{bmatrix}$, partitioned comformably with $\begin{bmatrix}\Gamma_0 & \Gamma_1 & \Gamma_2 & \Lambda\end{bmatrix}$, span the kernel of $\begin{bmatrix}Q_D^\top & C_D^\top\end{bmatrix}$. If this null space is zero, i.e., if $\mathscr{S}_D+\ker C_D=\mathbb{R}^{3n}$, there is

[4]Recall that this is more general than the form considered in [22].

only one solution to (10), so that there are no degrees of freedom in the choice of the triple $(\Gamma_0,\Gamma_1,\Gamma_2)$. In this case, if $(\Gamma_0,\Gamma_1,\Gamma_2)=(V_0,V_1,V_2)$ is stable, then with the corresponding $\Lambda=\begin{bmatrix}\Lambda_0 & \Lambda_1 & \Lambda_2\end{bmatrix}=V_3$, the matrix $G_\Lambda\triangleq-Q^\top(QQ^\top)^{-1}\Lambda=\begin{bmatrix}G_{\Lambda,0} & G_{\Lambda,1} & G_{\Lambda,2}\end{bmatrix}$ is such that $\mathscr{S}$ is an externally stable $(A_0+G_{\Lambda,0}C,A_1+G_{\Lambda,1}C,A_2+G_{\Lambda,2}C)$-invariant subspace. On the other hand, if the triple $(\Gamma_0,\Gamma_1,\Gamma_2)=(V_0,V_1,V_2)$ is not asymptotically stable, the subspace $\mathscr{S}$ is not a detectability subspace.

Now, when $\mathscr{S}_D+\ker C_D\subset\mathbb{R}^{3n}$, the problem we need to solve is to find a matrix K such that the resulting triple $(\Gamma_0,\Gamma_1,\Gamma_2)=(V_0+KH_0,V_1+KH_1,V_2+KH_2)$ is asymptotically stable; the corresponding $\Lambda=\begin{bmatrix}\Lambda_0 & \Lambda_1 & \Lambda_2\end{bmatrix}=V_3+KH_3$, for which (Γ,Λ) is a solution of (8), is such that $G_\Lambda\triangleq-Q^\top(QQ^\top)^{-1}\Lambda$, yielding $Q(A_H+G_\Lambda C_D)=\Gamma Q_D$, so that $\mathscr{S}$ is an externally stable $(A_0+G_{\Lambda,0}C,A_1+G_{\Lambda,1}C,A_2+G_{\Lambda,2}C)$-invariant subspace. Towards characterising a subset of such matrices K, we can rewrite the sufficient condition for asymptotic stability in Lemma 2.3 for the triple $(\Gamma_0,\Gamma_1,\Gamma_2)$ as shown below

$$\begin{bmatrix}\Phi & 0 & 0\\0 & \Psi & 0\\0 & 0 & \Theta-\Phi-\Psi\end{bmatrix}-\begin{bmatrix}\Gamma_0^\top\\\Gamma_1^\top\\\Gamma_2^\top\end{bmatrix}\Theta\begin{bmatrix}\Gamma_0 & \Gamma_1 & \Gamma_2\end{bmatrix}>0,$$

for some $\Phi\triangleq P_0>0$, $\Psi\triangleq P_1>0$ and $\Theta\triangleq P_0+P_1+P_2>0$. Standard manipulation and $\Gamma_i=V_i+KH_i$, for $i=0,1,2$, yield the equivalent condition

$$\begin{bmatrix}\Phi & 0 & 0 & \tilde{V}_i^\top\\0 & \Psi & 0 & \tilde{V}_1^\top\\0 & 0 & \Theta-\Phi-\Psi & \tilde{V}_2^\top\\\tilde{V}_0 & \tilde{V}_1 & \tilde{V}_2 & \Theta\end{bmatrix}>0\quad(18)$$

for some $\Phi>0$, $\Psi>0$, $\Theta>0$ and Π of suitable dimensions, where $\tilde{V}_i\triangleq\Theta V_i+\Pi H_i$ and $\Pi=\Theta K$.

Theorem 3.1: Let $\mathscr{S}$ be a conditioned invariant subspace for (1), $\begin{bmatrix}V_0 & V_1 & V_2 & V_3\end{bmatrix}=QA_H\begin{bmatrix}Q_D\\C_D\end{bmatrix}^\dagger$ and $\begin{bmatrix}H_0 & H_1 & H_2 & H_3\end{bmatrix}$ be such that its rows are a basis for the kernel of $\begin{bmatrix}Q_D^\top & C_D^\top\end{bmatrix}$. The subspace $\mathscr{S}$ is a detectability subspace if there exist $\Phi=\Phi^\top>0$, $\Psi=\Psi^\top>0$, $\Theta=\Theta^\top>0$ and Π of suitable dimensions such that (18) holds.

Moreover, given a quadruple (Θ,Φ,Ψ,Π) in the convex set defined by (18), a matrix K for which the triple $(\Gamma_0,\Gamma_1,\Gamma_2)$ is asymptotically stable is given by $K=\Theta^{-1}\Pi^\top$.

IV. INPUT-CONTAINING SUBSPACES

Now we turn our attention to input-containing subspaces, which are particular types of conditioned invariant subspaces useful in the context of various filtering/estimation problems, like unknown-input observation [22].

Definition 4.1: We define a *input-containing subspace* $\mathscr{S}$ for (1) as a subspace of $\mathbb{R}^n$ such that

$$\begin{bmatrix}A_H & B_H\end{bmatrix}\left((\mathscr{S}_D\times\mathbb{R}^{3m})\cap\ker\begin{bmatrix}C_D & D_D\end{bmatrix}\right)\subseteq\mathscr{S}.$$

The set of input-containing subspaces for (1) is denoted by the symbol $\mathfrak{S}_0$. The intersection of two input-containing subspaces is input-containing. It follows that the set $\mathfrak{S}_0$ is closed under subspace intersection. The same is not true for subspace addition. The intersection of all the input-containing subspaces of Σ is the smallest input-containing subspace of Σ, and is usually denoted by $\mathscr{S}^\star$.

Lemma 4.1: Given the s-dimensional subspace $\mathscr{S}$ of $\mathbb{R}^n$, let $Q \in \mathbb{R}^{(n-s) \times n}$ be such that $\ker Q = \mathscr{S}$ with Q of full row-rank. The following statements are equivalent:

1) the subspace $\mathscr{S}$ is input-containing for (1);
2) two matrices $\Gamma \in \mathbb{R}^{(n-s) \times 3(n-s)}$ and $\Lambda \in \mathbb{R}^{(n-s) \times 3p}$ exist such that

$$Q \begin{bmatrix} A_H & B_H \end{bmatrix} = \Gamma \begin{bmatrix} Q_D & 0 \end{bmatrix} + \Lambda \begin{bmatrix} C_D & D_D \end{bmatrix}; \quad (19)$$

3) a matrix $G \in \mathbb{R}^{n \times 3p}$ exists such that

$$\begin{bmatrix} A_H + G C_D & B_H + G D_D \end{bmatrix} \left(\mathscr{S}_D \times \mathbb{R}^{3m} \right) \subseteq \mathscr{S} \quad (20)$$

Proof: The result follows in the same way as the result in Lemma 3.1. ∎

As before, given an input-containing subspace it is not difficult to see that there exists a quotient observer of the form (2) for (1) in the presence of unkown inputs (possibly non-zero). In particular, it follows that

$$e_{i+1,j+1} = Q A_H \begin{bmatrix} x_{i,j} \\ x_{i+1,j} \\ x_{i,j+1} \end{bmatrix} + Q B_H \begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i,j+1} \end{bmatrix} - \Gamma \begin{bmatrix} \omega_{i,j} \\ \omega_{i+1,j} \\ \omega_{i,j+1} \end{bmatrix}$$

$$- Q G C_D \begin{bmatrix} x_{i,j} \\ x_{i+1,j} \\ x_{i,j+1} \end{bmatrix} - Q G D_D \begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i,j+1} \end{bmatrix}$$

$$= Q \begin{bmatrix} A_H + G C_D & B_H + G D_D \end{bmatrix} \begin{bmatrix} x_{i,j} \\ x_{i+1,j} \\ x_{i,j+1} \\ u_{i,j} \\ u_{i+1,j} \\ u_{i,j+1} \end{bmatrix} - \Gamma \hat{\omega}(i,j)$$

$$= \Gamma_0 e_{i,j} + \Gamma_1 e_{i+1,j} + \Gamma_2 e_{i,j+1},$$

where (20) has been used. Moreover if the input-containing subspace is a detectability subspace then the observer is asymptotic.

The following is an algorithm for computing the smallest input-containing subspace $\mathscr{S}^\star$.

Algorithm 4.1: The sequence of subspaces $(\mathscr{S}^i)_{i \in \mathbb{N}}$ described by the recurrence

$$\mathscr{S}^0 = 0_n$$
$$\mathscr{S}^i = \begin{bmatrix} A_H & B_H \end{bmatrix} \left((\mathscr{S}_D^{i-1} \times \mathbb{R}^{3m}) \cap \ker \begin{bmatrix} C_D & D_D \end{bmatrix} \right),$$

for $i > 0$, is monotonically non-increasing. An integer $k \leq n - 1$ exists such that $\mathscr{S}^{k+1} = \mathscr{S}^k$. For such k, the identity $\mathscr{S}^\star = \mathscr{S}^k$ holds.

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