

# Initial Soil Springs Stiffness for laterally loaded Piles

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## ABSTRACTS

Total number of 180 finite element models is created. Soil is modelled by three dimensional elastic isotropic brick elements with cylindrical shaft cavity at the centre. Pile is modelled with ordinary Euler-Bernoulli beam elements. Different connectivity scenarios between the soil and the pile surface are examined. Elastic properties of the soil are changed for each FE model within wide range of possible values. A globally covering database is created for free - head pile stiffness from FE analysis results. Lateral springs stiffness are calculated by equating pile head stiffness from FE models to predicted value from beam on elastic support theory. Spring's stiffness is correlated to mechanical properties of the soil, shaft diameter, and pile flexibility factor using curve fitting techniques. It is found that spring stiffness is dependent to shaft diameter. It is also found that relative connectivity between the soil and pile's skin, highly affects the spring's stiffness. Comparison to work by other researchers is made.

*Keywords:* Pile lateral stiffness, lateral springs stiffness, soil pile interaction, Lateral springs stiffness

## 1. INTRODUCTION

Beam on Lateral foundation is one of the earliest methods used in the study soil-pile interaction. The well known "p-y" method (e.g. Reese and Van Impe 2001) is also based on beam on lateral springs, reflecting effects of soil nonlinearity in the springs stiffness.

Modelling the pile as a beam on lateral springs is a cost effective technique to analyse lateral soil pile interaction. Linearly elastic spring stiffness may be successfully used where lateral deflection of the system is intentionally limited to small values. For example, it is essential for their serviceability to restrict vibration amplitude of gas compressors on piled foundations. This results in very low strain in soil, which justifies using an elastic approach.

Seismic load assessment is another example of using an elastic approach: most seismic codes of practice require an estimation of natural frequency of the structure. Free vibration frequency of the pile-structure system is normally determined via Eigen value analysis, Ritz vector analysis, etc. All of these types of analysis are based on linearly elastic models.

The stiffness of lateral springs has been subject of research for the past few decades. Vesic(1961) provided expression for modulus of sub grade reaction. Bowles (1997) proposed a modified version of Vesic's (1961) relationship to use in the analysis of laterally loaded piles. Poulos and Davis (1980) provided non-dimensional curves for single piles based on numerical integration of Mindlin (1936) solution on a thin rectangular vertical strip. Randolph (1981) conducted a parametric finite element study and proposed expressions for top deflection of a pile under lateral load and bending moment. Carter (1984) and Ling (1988) reported full scale test results which concluded in diameter-dependent lateral spring stiffness.

In this paper springs stiffness are calculated by curve fitting on three dimensional FE results.

## 2. BACKGROUND

Governing differential equation of a beam on lateral springs is given for a beam along vertical "z" axis, which bends in "yz" plane. Lateral springs constant is arbitrarily written as a fraction of soil shear modulus:

$$k_s = Gk_k \quad (1)$$

$$E_p I_p \frac{d^4 Y(z)}{dz^4} + Gk_k Y(z) = 0 \quad (2)$$

Where

G= Shear modulus of soil

$k_s$ = Lateral spring constant

$k_k$  = Non dimensional lateral spring constant

H=Soil layer's thickness = pile embedded length

$E_p$ =Young modulus of elasticity of pile  
 $I_p$  = Pile second moment of area  
 General solution of equation (2) can be written as:

$$Y(z) = \frac{F_0}{2EI\beta^3} [A \sin(\beta z) \cosh(\beta z) + B \sin(\beta z) \sinh(\beta z) + C \cos(\beta z) \sinh(\beta z) + D \cos(\beta z) \cosh(\beta z)] \quad (3)$$

Where “A,B,C and D” are integration constants, “ $F_0$ ” is the lateral load on pile top in “y” direction and:

$$\beta = \sqrt[4]{\frac{k_s}{4E_p I_p}} = \sqrt[4]{\frac{Gk_k}{4E_p I_p}} \quad (4)$$

Ratio of horizontal force “ $F_0$ ” to top translation “ $Y(0)$ ”, is called free head lateral stiffness “ $K_h$ ”. Constants “A” to “D” are derived by applying free head conditions on pile top and fixed conditions on pile tip. For a pile with a finite length “H”:

$$K_h = \frac{2}{3} (\beta H)^3 \frac{\cosh(2\beta H) + \cos(2\beta H) + 2}{\sinh(2\beta H) - \sin(2\beta H)} \left( \frac{3E_p I_p}{H^3} \right) \quad (5)$$

Where, “ $K_h$ ” is the pile free head stiffness. Equation (5) is written as a fraction of a free standing cantilever, i.e. “ $3E_p I_p / H^3$ ”. It is easy to prove that “ $K_h$ ” approaches the stiffness of a free standing cantilever for very soft soils (i.e.  $\beta H \rightarrow 0$ ).

If the pile is infinitely long, solution to differential equation (2) becomes:

$$Y(z) = \frac{F_0}{2EI\beta^3} e^{-\beta z} \cos(\beta z) \quad (6)$$

The pile free head stiffness for an infinitely long pile is written as:

$$K_h = 2E_p I_p \beta^3 \quad (7)$$

### 3. METHODOLOGY

Beam on lateral springs is a single parameter model. The main challenge on using this model is to determine springs stiffness as a function of soil and pile mechanical and/ or geometrical characteristics. Spring stiffness may be determined by direct solution of elasto-static differential equations. Due to the complexity of the problem, numerical methods are used to solve these equations and results are presented as graphs and/or tables. Alternatively, if pile head stiffness is known (e.g. by full scale testing); the springs stiffness can be derived from equation (5). Full scale tests may reflect variations of soil characteristic parameters with depth and effect of nonlinearity of soil.

In the present study, we used finite element analysis for deriving pile head stiffness. This gave us the advantage of eliminating all un-assessable effects that may be included in a full scale test.

Pile was modelled as ordinary beam elements along vertical axis “z”. We modelled a layer of soil with elastic-homogenous, brick elements. Thickness of the soil layer was ten metres and it was extended 29.5 metres from each side of the pile. Examination showed that changing far end restraint conditions of the modelled soil does not significantly affect the calculated head stiffness of the pile. We then concluded that the extension of the soil was adequately large. Only one quarter of the space was modelled and symmetry boundary conditions were applied to the artificial boundaries. Soil elements were fully restrained at the bottom to resemble rigid bedrock. The excavated (or displaced) soil was modelled as a cylindrical cavity. By changing diameter of this cavity, while all other parameters were kept constant, shaft diameter effect on stiffness was investigated. A total number of 71,060 brick elements were used. A typical Young’s modulus of elasticity of 25000 MPa was chosen for the pile. Pile was fully connected to soil at horizontal surfaces. To investigate the effects of slippage between the pile skin and the soil; another set of models were created. In the new set of models soil and pile were constraints together only in horizontal directions. A horizontal force in “y” direction was applied to the pile top. Figure 1 shows configuration of finite element models.

Throughout 50 sets of models with cavity diameter of 1000mm, soil characteristics were varied such that a range of Poisson's ratio from 0 to 0.48 and a range of modulus ratio ( $E_p/G$ ) from 100 to 100,000 were covered.

In order to investigate the effect of shaft diameter, six sets of FE models were created with cavity diameters 100, 400, 600, 800, 1500 and 2000 millimetres. Poisson's ratio for each model was set to zero while modulus ratios were varied in the range of 100 to 100,000.

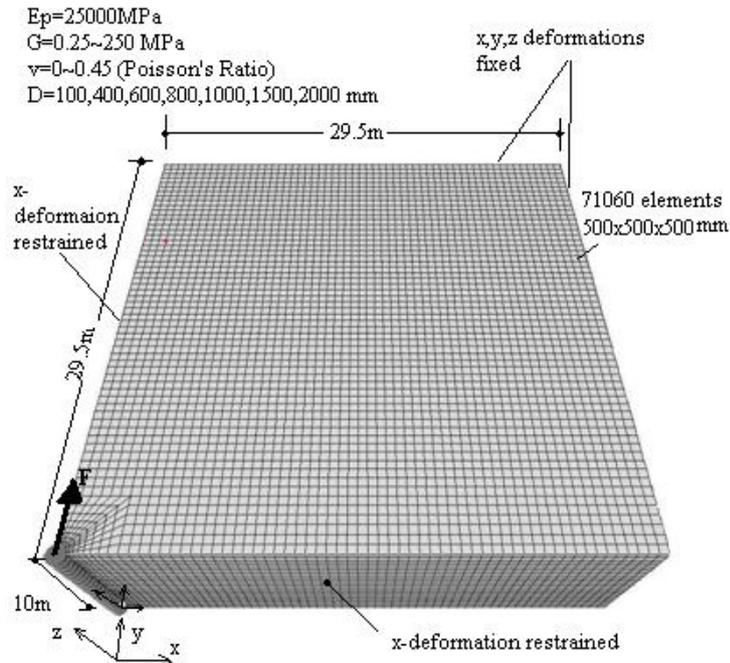


Figure 1- 3D finite element model of 1/4 soil half space and pile in the corner

Pile second moment of inertia can also be considered as an effective factor, independent to the shaft diameter. In order to investigate its effect, pile second moment of inertia was varied in the models with cavity diameter of 1 metre. The changes were irrespective to the shaft diameter.

The effect of slippage between the pile skin and the soil was also investigated.

Results of analyses are shown in Figures 2a and 2b. Figure 2a shows variations of pile stiffness with shaft diameter, when pile second moment of inertia is constant. It can be seen that a linear relationship exist between the pile stiffness and the shaft diameter. Dependence of pile stiffness and shaft diameter is more critical for stiffer soil. It loses its importance for soft soils. Figure 2b shows the same quality as Figure 2a does, this time with the pile second moment of inertia varying correspondingly with the shaft diameter.

#### 4. SPRING STIFFNESS

Curve fitting methods are used to derive simple relationship for non-dimensional spring stiffness " $k_k$ ". The spring stiffness is considered as "reference stiffness" which is then scaled for the effects of Poisson's ratio, flexural rigidity " $E_p I_p$ " and shaft diameter. The "reference stiffness" is the spring stiffness derived for a pile with 1m shaft diameter and for zero Poisson's ratio.

$$k_s = Gk_0 S_\nu S_I S_D \quad (8)$$

Where:

$k_0$  = Reference stiffness for the shaft diameter " $D_{Ref}=1m$ " and zero Poisson's ratio

$S_\nu$  = Poisson's ratio scale factor.

$S_I$  = Flexural rigidity scale factor

$S_D$  = Shaft diameter scale factor.

Figure 3a includes curves of " $k_0$ " for a fully connected model as well as that for a model in which pile and soil are vertically disconnected at the pile cavity. According to the figure, noticeable dependency between the reference stiffness and connectivity conditions exists. The following relationships are

proposed for the case of full pile-soil connectivity (upper limit) and partial pile-soil connectivity (lower limit):

$$(k_0)_u = 2.86 + 20.15 \left( \frac{G}{E_p} \right)^{0.41} \quad (9)$$

$$(k_0)_l = 2.65 + 9.10 \left( \frac{G}{E_p} \right)^{0.40} \quad (10)$$

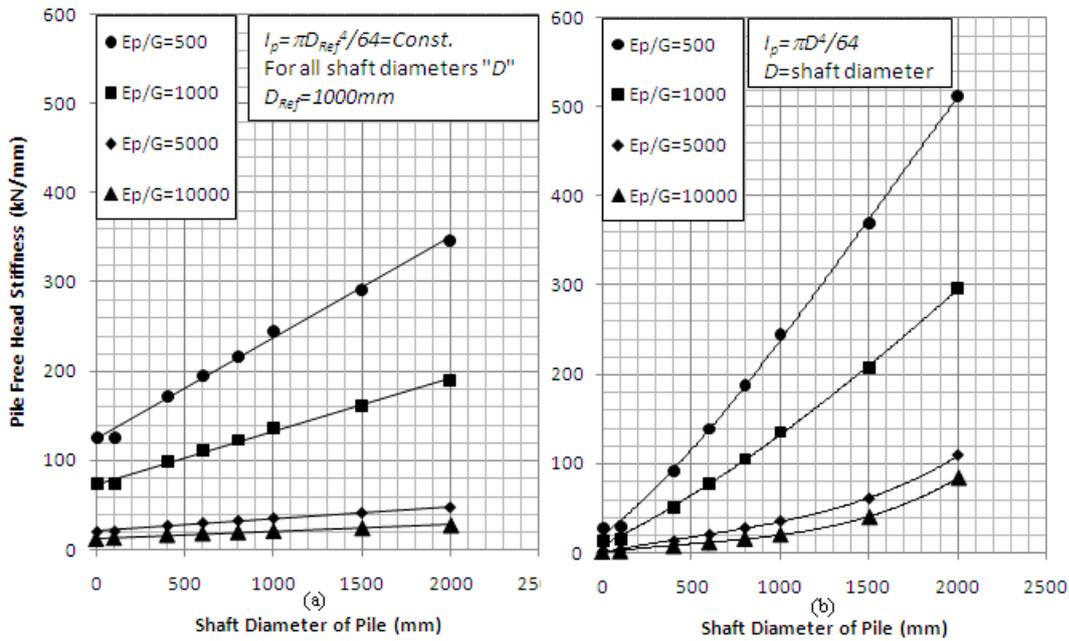


Figure 2. Variations of pile free head stiffness with shaft diameter:  
a) Pile second moment of inertia is (artificially) kept constant for all different shaft diameters in FE models.  
b) Pile second moment of inertia is equal to that of a solid circular (non-tubular) section fit in the shaft cavity.

Scaling factors are obtained from the rest of the database:

$$S_v = 1 + \left( \frac{G}{50E_p} \right)^{0.0678} \frac{\nu}{1-\nu} \quad (11)$$

$$S_D = 0.5(1 + D/D_{Ref}) \quad (12)$$

$$S_I = \left( \frac{E_{Ref} I_{Ref}}{E_p I_p} \right)^\gamma \quad (13)$$

Where “ $I_{Ref}$ ” is the second moment of inertia of a circle 1 m diameter and “ $E_{Ref}$ ” = 25000MPa. “ $\gamma$ ” is the second moment of inertia of the pile and “ $\square$ ” is a power that varies from 0.15 for very stiff soils to 0.036 for very soft soils:

$$\gamma = 0.2256 - 0.017 \ln(E_p/G) \quad (14)$$

Since the above equations are based on curve fitting results, some degree of deviation from accurate data may be experienced. Investigation showed that the average error of using formulas (8) to (14) is about 11%. Maximum error of 20% may occur for very small diameter piles in very soft soils. In a more practical range of data, the relative error of using these formulas is about 6%.

## 5. COMPARISON WITH OTHER SOLUTIONS

Vesic (1961) proposed a relationship for spring stiffness of a beam on an elastic foundation:

$$(k_s)_{Vesic} = \frac{0.65 E_s}{1-\nu^2} \sqrt[12]{\frac{E_s B^4}{E_p I_p}} = G \frac{1.377}{1-\nu} \sqrt[12]{\frac{(1+\nu) G B^4}{E_p I_p}} \quad (15)$$

“ $E_s$ ” is Young modulus of elasticity of the soil and “ $B$ ” is the width of the beam. Bowles (1997) recommended using twice the value of the equation (15) while using pile’s diameter in place of beam’s width.

Randolph (1981) conducted a finite element study, very similar to the present study. He used triangular-linear strain elements to model the soil. From this point of view, the present study is more accurate, because general three dimensional brick elements are used. Randolph proposed expressions for deflection and rotation at pile's top under the action of a lateral force and a bending moment. Using Randolph's expression for zero bending moment, pile free head stiffness " $K_h$ " can be derived as follows, after some algebraic manipulation:

$$(K_h)_{Randolph} = 2DG \sqrt[7]{\left(1 + \frac{3}{4}\nu\right)^6} \sqrt[7]{\frac{E_p}{G} \frac{I_p}{\pi D^4 / 64}} \quad (16)$$

Substituting equation (16) in equation (7) results in the spring stiffness:

$$(k_s)_{Randolph} = 2.731G \left(1 + \frac{3}{4}\nu\right)^{\frac{8}{7}} \left(\frac{GD^4}{E_p I_p}\right)^{\frac{1}{7}} \quad (17)$$

Novak and El Sharnoubi (1983) summarized results of dynamic continuum mechanics solution for pile dynamic stiffness into tabulated values. Their formulas are presented here with slight changes in notations:

$$K_{HH} = (\pi f_u / 8) D E_p, \quad K_{MM} = (\pi f_\psi / 32) D^3 E_p, \quad K_{HM} = (\pi f_{c1} / 16) D^2 E_p \quad (18)$$

Where  $K_{HH}$ ,  $K_{MM}$  and  $K_{HM}$  are translational, rotational and coupled stiffness, respectively.  $f_u$ ,  $f_\psi$  and  $f_{c1}$  are dimensionless tabulated parameters. Free head pile stiffness can be derived from the following relationship (Pender, Carter et al. 2007):

$$(K_h)_{Novak} = \frac{K_{HH}K_{MM} - K_{HM}^2}{K_{MM} - eK_{HM}} = \frac{\pi}{8} \left(f_u - \frac{f_{c1}^2}{f_\psi}\right) D E_p \quad (19)$$

Where " $e=M/F_0$ " is the ratio of bending moment to shear force on pile top. Substituting equation (19) in equation (7) results in spring's stiffness.

$$(k_s)_{Novak} \cong 0.114G \left(\frac{E_p}{G}\right) \left(f_u - \frac{f_{c1}^2}{f_\psi}\right)^{4/3} \left(\frac{D^4}{I_p}\right)^{1/3} \quad (20)$$

Poulos and Davis (1980) provided general continuum solutions for floating piles. Their solution for a pile under the action of a horizontal force (zero bending moment) at its top can be written as:

$$(K_h)_{Poulos} = \frac{E_s H}{I_{\rho H}} = 2(1+\nu)G \frac{H}{I_{\rho H}} \quad (21)$$

Where " $H$ " is the pile length and " $E_s$ " is the soil elastic modulus. Values for " $I_{\rho H}$ " are given in graphs for different values of " $H/D$ ". Springs stiffness can be calculated from equation (21), using equations (2) and (7):

$$(k_s)_{Poulos} \cong 4G \left(\frac{1+\nu}{I_{\rho H}}\right)^{\frac{4}{3}} \sqrt[3]{\frac{GH^4}{E_p I_p}} \quad (22)$$

Comparison among the results of the present study and solutions of Vesic (1961), Randolph (1981), Poulos and Davis (1980) and Novak and El Sharnoubi (1983) is made in Figure 3.

Novak and El Sharnoubi's solution together with Randolph's value show correlation with Vesic (1961). They also show spring stiffness lower in value compared to those of Poulos and Davis. The reason might become clear; noting that underlying theory of Novak and El Sharnoubi's solution is based on a plane strain soil. Randolph also used triangular linear strain elements in his study. Poulos and Davis's solution is closer to the results of the present solution. The difference may come from different sources, including the approximate nature of Poulos and Davis solution, the thin rectangular shape they used for the pile and that the curves for floating pile is used to calculate the spring stiffness in Figure 3a. An important feature of Poulos and Davis (1980) solution is its dependency to the pile diameter. Figure 3b shows values of equation (12) in comparison with the factors obtained from the curve on the a.m. reference. There is good agreement between the results of the present studies and those calculated from the Poulos and Davis's solution, keeping in mind that their pile had a thin rectangular cross section, as opposed to the circular cross section in the present study.

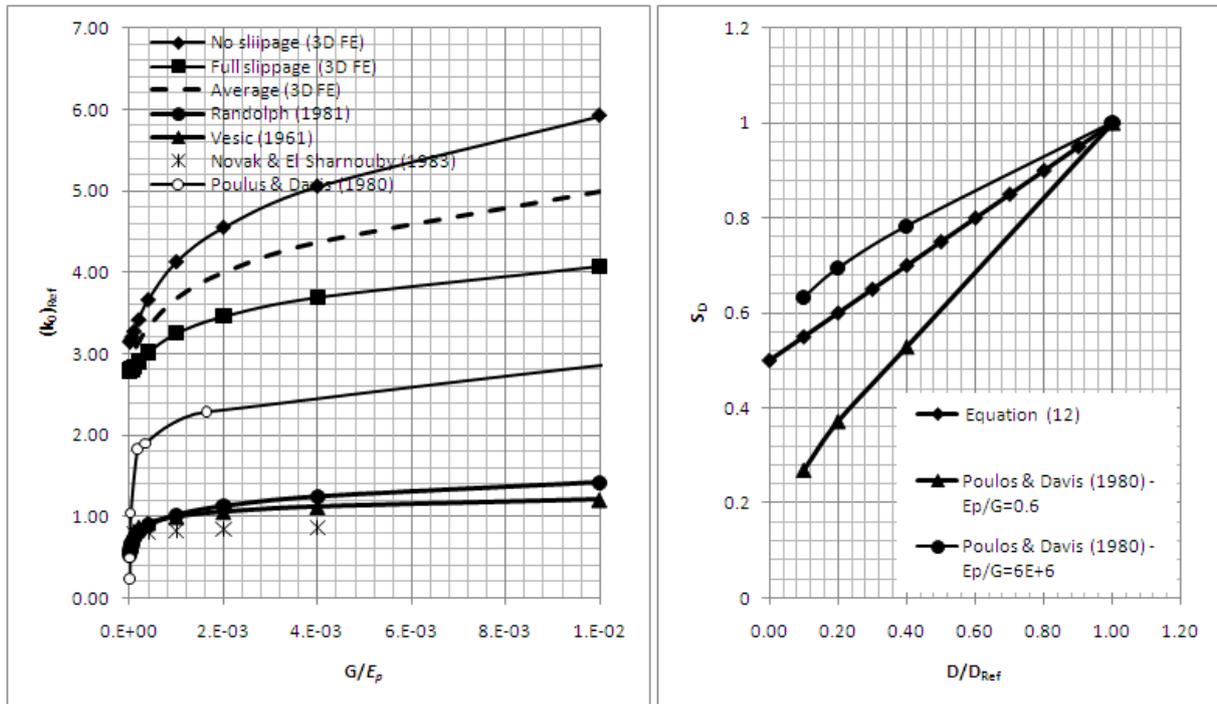


Figure 3a- Comparison among different solutions

3b- Diameter scaling factor

## 6. CONCLUSION

Three dimensional finite element analyses show higher values for initial spring stiffness than those based on analyses on two dimensional models. Dependence of spring's stiffness with pile diameter is an important outcome of the present study. This phenomenon, which is in agreement with the works of other researchers, yet needs more attention and can be a subject for further studies. Scaling factors introduced in this study may find particular application in extrapolation of pile test results to new designs, where designed pile diameter might be different from that of the test pile.

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