Continuously controlled options: derivatives with added flexibility*

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Abstract

The paper introduces special options such that the holder selects dynamically a continuous time process controlling the distribution of the payments (benefits) over time. For instance, the holder can select dynamically the quantity of a commodity purchased or sold by a fixed price given constraints on the cumulative quantity. In a modification of the Asian option, the control process can represent the averaging kernel describing the distribution of the purchases. The pricing of these options requires to solve special stochastic control problems with constraints for the cumulative control similar to a knapsack problem. Some existence results and pricing rules are obtained via modifications of parabolic Bellman equations.

Key words: exotic options, controlled options, continuous time market, stochastic control, HJB equation, knapsack problem

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1 Introduction

There are many different types of options and financial derivatives: European, American, Asian, Bermudian, Israeli, Russian, and Parisian options, swing options, reload options, shout options, and multiple rescindable options; see, e.g., Bender and Schoenmakers [1] , Bender [2], Briys *et al* [3], Carmona and Dayanik [4], Carmona and Touzi [5], Dai and Kwok [6], Dokuchaev [9], Kifer [13], Kramkov *et al* [15], Kyprianou [14], Meinshausen and Hambly [17], Peskir [19]. Pricing of exotic options require special methods.

Typically, new type of options are developed with the purpose to offer some additional flexibility for an option holder and take into account some particular features of business models. As a next step in this direction, the paper suggests a family of options that allows the holder to select dynamically continuous time processes that controlling the distribution of the payments (benefits) over time. The control processes are assumed to be adapted to the current flow of information. We call the new options *controlled options*. There is a similarity with passport options introduced in Hyer *et al* [11] as a generalization of the American option (see also Delbaen and Yor [7], Kampen [12], Nagayama [18]). The passport options allow the holder to select investment strategies for an account; the writer guarantees protection from the losses. The difference with the control options introduced in this paper is that the holder of passport options selects a portfolio strategy, and the holders of the controlled options select the weight functions controlling the distribution of the payments over time.

The controlled options may have applications in commodities and energy trading. In a modification of the Asian option, a control process $u(t)$ may represent the averaging kernel (i.e., the weight of the integral) describing the distribution of the purchases. In another example, a non-negative control process $u(t)$ can describe the purchase dynamic such that $u(t)dt$ is the quantity of some commodity purchased during a vanishingly small time interval $[t, t + dt]$, given that the cumulative quantity is fixed,i.e., $\int_0^T u(t)dt = 1$, where T is the terminal time. This is a limit case of multi-exercise option studied in [1]-[6], [9], [17], where the distribution of exercise times approaching a continuous distribution. Therefore, controlled options can be used also as an auxiliary tool to study the multi-exercise options. Analysis of these controlled option is more straightforward in some aspects since optimal stopping problem is excluded and replaced by an optimal stochastic control problem. These and other examples of controlled options are studied below.

To avoid some problems with replicability and hedging, we used a relaxed modification of the classical fair price that we called unimprovable price. In this framework, we were able to establish the general martingale pricing formula; the pricing requires to solve of a stochastic control problem with constraints on cumulative control that reminds a continuous time knapsack problem. Some existence results and pricing rules are obtained in Markov diffusion setting based on dynamic programming and various modifications of problems with constraints on cumulative control, optimal stopping problems, and degenerate parabolic Bellman equations, or Hamilton-Jacobi-Bellman (HJB) equations.

The paper is organized as follows. In Section 2, two classes of controlled options are introduced:

(i) options where an adapted control process $u(t)$ is selected such that $\int_0^T u(t)dt = 1$; and

(ii) options where an adapted process $u(t)$ is selected without restrictions on the cumulative value $\int_0^T u(t)dt$, and where the payoff is defined by the normalized weight $v(t) = u(t)\left(\int_0^T u(s)ds\right)^{-1}$ which is not an adapted process.

Some motivation for this setting is given in Section 2. In Section 3, the market model and the definition of unimprovable price is introduced, and the general martingale pricing formula is presented. In Section 4, the pricing is discussed for the case (i); and analog of Merton's "no early exercise" theorem is given. In Section 5, the pricing is discussed for the case (ii). In Section 6, non-arbitrage property and superrepication is discussed. The proofs are given in Appendix.

An earlier version of this paper was web-published in 2010 [10] and presented in Quantitative Methods in Finance conference in Sydney, Australia, 2011.

2 Controlled options: definition and examples

Consider a risky asset (stock, commodity, a unit of energy) with the price $S(t)$, where $t \in [0, T]$, for a given $T > 0$. Consider an option with the payoff that depends on $S(\cdot)$ and on a control process $u(\cdot)$ that is selected by an option holder from a certain class of admissible controls. We call the corresponding options controlled options. Clearly, an American option is a special case of controlled options, where the exercise time is selected. Two new classes of controlled options are suggested and discussed below. We assume the option holder selects control processes $u(t)$

that has to be adapted to the filtration $\mathcal{F}_t$ generated by the observable process $S(t)$. The process $u(t)$ is assumed to be observable by the option writer.

For simplicity, we assume that all options give the right on the corresponding payoff in cash rather than the right to buy or sell stock or commodities.

Let $\mathcal{U}$ be the class of processes $u(t)$ that are progressively measurable with respect to the filtration $\mathcal{F}_t$ and such that $u(t) \in [d_0, d_1]$, where d_0 and d_1 are given, $0 \le d_0 < d_1 < +\infty$.

Let $\mathcal{U}_1$ be the class of processes $u \in \mathcal{U}$ such that

$$\int_0^T u(t)dt \le 1. \tag{2.1}$$

To ensure that this class in non-empty, we assume that $d_0 T < 1$.

Let $g : \mathbf{R} \to \mathbf{R}$ and $f : \mathbf{R} \times [0,T] \to \mathbf{R}$ be some given functions.

We consider two different classes of options with payoff at time T. First, we consider options with the payoff

$$F_u = g\left(\int_0^T u(t) f(S(t), t)dt\right), \tag{2.2}$$

where $u(t)$ is selected in $\mathcal{U}_1$. The process $u(t)$ can be called weight process.

Second, we consider options with the payoff

$$F_u = g\left(\int_0^T v(t) f(S(t), t)dt\right), \tag{2.3}$$

where

$$v(t) = \frac{u(t)}{\int_0^T u(s)ds}, \tag{2.4}$$

and where $u \in \mathcal{U}$. Similarly to the control process for option (2.2), $\int_0^T v(t)dt = 1$ if $u(\cdot) \neq 0$. The case when $d_0 = 0$ and $u \equiv 0$ is not excluded. In this case, the payoff is defined to be

$$F_u = g\left(f(S(T), T)\right), \tag{2.5}$$

This is the limit of payoff (2.3) for $u(t) \equiv \varepsilon \mathbb{I}_{\{t \ge T - \varepsilon\}}$ as $\varepsilon \to 0$. In this case, $v(t)$ can be interpreted as the delta function with the mass concentrated at $t = T$.

The difference with option (2.2) is that the process $v(t)$ is not assumed to be adapted to the filtration $\mathcal{F}_t$, i.e., $v \notin \mathcal{U}_1$. In this setting, the current selection of the value of $u(t)$ is recorded, and the payoff occurs at terminal time T. The process $u(t)$ can be called weight process again, and $v(t)$ can be called normalized weight process.

It can be noted that the options with payoffs (2.3) give more flexibility to the holder than the related options (2.2). First, the selection of $u(t)$ is obviously more restricted for options (2.2) than for options (2.3): the option holder has to obey restriction (2.1) on the total amount of cumulated $u(t)$. Second, options (2.3) provide some opportunities to correct past decisions. Assume that the option holder selects $u(t)$ with the purpose to maximize the payoff. For the holder of options (2.3), the effect of past unfortunate decisions can be smoothed by selecting larger $u(t)$ in the future. Similarly, the relative weight of the past good decisions can be enlarged by selecting small future values of $u(t)$. This opportunity is absent for options (2.2).

The option with payoff (2.2) with $f(x,t) \equiv x$ represents a generalization of the Asian option where the weight $u(t)$ is selected by the holder. The option with payoff (2.3) represents another useful generalization of the Asian option. Consider, for instance, a customer in the energy market who consumes time variable and random quantity $u(t)dt$ of energy per time period $(t, t+dt)$, with the price $S(t)$ for a unit. The cumulated number of units consumed up to time T is $\bar{u} = \int_0^T u(t)dt$; it is unknown at times $t < T$. To hedge against the price rise, the customer would purchase a portfolio of M call options; each option gives the right to purchase one energy unit for the price K. To minimize the impact of price fluctuations, the Asian options are commonly used. These options can be described as the options with the payoff $(\bar{S} - K)^+$, where $\bar{S} = T^{-1} \int_0^T S(t)dt$. However, for accounting and tax purposes, the average price of energy for a particular customer has to be calculated as $\bar{S}_u = \bar{u}^{-1} \int_0^T u(t)S(t)dt$ rather than $\bar{S}$. Therefore, more certainty in financial and tax situation can be achieved if one uses the portfolio of M options with the payoff $(\bar{S}_u - K)^+$ defined by the consumption of the particular customer. This is a special case of option (2.3). Since $\bar{u}$ is random and unknown, options (2.2) cannot be used for this model.

Another example of possible applications of options (2.3) could be based on a fact that Australian Taxation Office uses the volume weighted average price (VWAP) to determine the market value of the share.

Special cases and possible modifications

Useful special cases can be covered with $g(x) = x$, $g(x) = (x-K)^+$, $g(x) = (K-x)^+$, $g(x) = \min(K, x)$, where $K > 0$, and with

$$f(x,t) = x, \quad f(x,t) = (x-K)^+, \quad f(x,t) = (K-x)^+, \tag{2.6}$$

or

$$f(x,t) = e^{r(T-t)}(x-K)^+, \quad f(x,t) = e^{r(T-t)}(K-x)^+, \tag{2.7}$$

where $r > 0$ is the annual bank interest rate. We denote $x^+ \triangleq \max(0, x)$.

Functions (2.6) can be used for options such that the payments are made at current times $t \in [0,T]$. Functions (2.7) can be used for options such that the payments is made at the terminal time T. This model takes into account accumulation of interest up to time T on any payoff.

The option with $g(x) \equiv x$ represents a limit version of the multi-exercise options, when the distribution of exercise time approaches a continuous distribution. The restriction $|u(t)| \leq \text{const}$ represents the continuous analog of the requirement that exercise times for the multi-exercise options must be on some distance from each other. These options can be used, for instance, for energy trading with $u(t)dt$ representing the quantity of energy purchased during the time interval $[t, t+dt]$ for the fixed price K when the market price is above K.

A possible modification payoff (2.2) is the payoff

$$F_u = \int_0^T u(t) f(S(t), t) dt + \left(1 - \int_0^T u(t) dt\right) f(S(T), T).$$

In this case, the unused $u(t)$ are accumulated and used at the terminal time.

An alternative selection of the payoff (2.3) for $u(t) \equiv 0$ is $F_u = g\left(\frac{1}{T}\int_0^T S(t)dt\right)$, i.e., as the limit of the payoff (2.3) for $u(t) \equiv \varepsilon$ as $\varepsilon \to 0$.

3 Market model

We investigate pricing of the options described above for the simplest case of Black-Scholes model, i.e, for a complete continuous time diffusion market model with constant volatility. We consider the model of a securities market consisting of a risk free bond or bank account with the price $B(t)$ and a risky stock with the price $S(t)$, $t \in [0,T]$.

We assume that the prices of the stocks evolves as

$$dS(t) = S(t)\left(a(t)dt + \sigma dw(t)\right), \tag{3.1}$$

where $a(t)$ is the appreciation rate, $\sigma > 0$ is the volatility coefficient.

In (3.1), $w(\cdot)$ is a standard Wiener process on a given standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is a set of elementary events, $\mathcal{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is a probability measure.

The price of the bond evolves as

$$B(t) = e^{rt}B(0). \tag{3.2}$$

We assume that $\sigma > 0$, $r \geq 0$, $B(0) > 0$, and $S(0) > 0$, are given constants.

Let $\mathcal{F}_t$ be the filtration generated by $w(t)$ and augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$. For simplicity, we assume that $a(t)$ is a bounded process progressively measurable with respect to $\mathcal{F}_t$. In this case, $\mathcal{F}_t$ is also the filtration generated by $S(t)$.

Let $\mathbf{P}_*$ be the probability measure such that the process $e^{-rt}S(t)$ is a martingale under $\mathbf{P}_*$. By the assumptions on (a, σ, r), this measure exists and is unique. Let $\mathbf{E}_*$ be the corresponding expectation. Under the risk neutral measure $\mathbf{P}_*$,

$$w_*(t) \triangleq w(t) + \int_0^t \sigma^{-1}[a(s) - r]ds$$

is a Wiener process, and the process $\widetilde{S}(t)$ is a martingale, since $d\widetilde{S}(t) = \sigma dw_*(t)$.

Admissible portfolio strategies

Let $X(0) > 0$ be the initial wealth at time $t = 0$ and let $X(t)$ be the wealth at time $t > 0$. We assume that

$$X(t) = \beta(t)B(t) + \gamma(t)S(t). \tag{3.3}$$

Here $\beta(t)$ is the quantity of the bond portfolio, $\gamma(t)$ is the quantity of the stock portfolio, $t \geq 0$. The pair $(\beta(\cdot), \gamma(\cdot))$ describes the state of the bond-stocks securities portfolio at time t. Each of these pairs is called a strategy.

The process $\widetilde{X}(t) \triangleq e^{-rt}X(t)$ is said to be the discounted wealth, and the process $\widetilde{S}(t) \triangleq e^{-rt}S(t)$ is said to be the discounted stock price.

Definition 3.1 *A pair $(\beta(\cdot),\gamma(\cdot))$ is said to be an admissible self-financing strategy if $\beta(t)$ and $\gamma(t)$ are random processes which are progressively measurable with respect to the filtration $\mathcal{F}_t$ and such that there exists a sequence of Markov times $\{T_k\}_{k=1}^{+\infty}$ (with respect to the filtration $\mathcal{F}_t$) such that $T_k \to T-0$ a.s. and*

$$\mathbf{E}\int_0^{T_k} \left(\beta(t)^2 B(t)^2 + S(t)^2\gamma(t)^2\right) dt < +\infty \qquad \forall k = 1,2,\ldots$$

and

$$dX(t) = \beta(t)dB(t) + \gamma(t)dS(t). \tag{3.4}$$

It is well known that (3.4) is equivalent to

$$d\widetilde{X}(t) = \gamma(t)d\widetilde{S}(t). \tag{3.5}$$

(See, e.g., [8], p.83). It follows that $\widetilde{X}(t)$ is a martingale with respect to the probability measure $\mathbf{P}_*$.

Let $X(0)$ be an initial wealth, and let $\widetilde{X}(t)$ be the discounted wealth generated by an admissible self-financing strategy $(\beta(\cdot),\gamma(\cdot))$. For any Markov time τ such that $\tau \in [0,T]$, we have

$$\mathbf{E}_*\widetilde{X}(\tau) = X(0) + \mathbf{E}_*\int_0^{\tau} \gamma(t)d\widetilde{S}(t) = X(0) + \mathbf{E}_*\int_0^{\tau} \gamma(t)\widetilde{S}(t)^{-1}dw_*(t) = X(0).$$

The unimprovable price of an option

Let us consider a controlled option with the payoff $F_u = \mathbf{F}(u(\cdot),S(\cdot))$, where $\mathbf{F}: U \times C(0,T)$ is a measurable mapping such that $\sup_{u\in\mathcal{U}} \mathbf{E}_*|F_u|^2 < +\infty$. Here U is the set of all admissible controls $u(\cdot)$; we will be using $U = \mathcal{U}_1$ and $U = \mathcal{U}$.

Definition 3.2 *An unimprovable price of an option is any price c such that*

- *The option writer cannot ensure fulfillment of option obligations at terminal time T using the wealth raised from the initial wealth $X(0) < c$ with self-financing strategies.*

- *A rational option buyer wouldn't buy an option for a higher price than c.*

The following theorem is formulated for the case of constant r. However, this theorem holds for any model where the risk-neutral measure $\mathbf{P}_*$ exists and is unique; the extension on the case of time variable $r = r(t)$ is straightforward.

Theorem 3.1 *The unimprovable price c_F of an option with the payoff F_u is unique and can be found as*

$$c_F = e^{-rT} \sup_{u \in U} \mathbf{E}_* F_u.$$

Proofs are given in the Appendix.

4 Pricing of options with adapted weight

We assume that $S(t)$ and $\mathcal{F}_t$ are such as described in Section 3. Consider an option with payoff (2.2), where $f(x,t) : (0,+\infty) \times [0,T] \to \mathbf{R}$ and $g(x) : (0,+\infty) \to \mathbf{R}$ are given continuous non-negative functions such that $|f(x,t)| + |g(x)| \le \mathrm{const}\,(|x|+1)$ and $|\partial f(x,t)/\partial x| + |dg(x)/dx| \le \mathrm{const}$. In addition, we assume that the function $g(x)$ is non-decreasing. The function $u(t)$ is the control process that is selected by the option holder.

By Theorem 3.1, the unimprovable price of this option is

$$c_F = e^{-rT} \sup_{u \in \mathcal{U}_1} \mathbf{E}_* F_u. \tag{4.1}$$

Lemma 4.1 *Assume that the function g is concave on $(0,+\infty)$. In this case, an optimal control for problem (4.1) exists in $\mathcal{U}_1$.*

Up to the end of this section, we discuss solution of optimal control problem (4.1). Note that this problem reminds the so-called knapsack problem, due to restriction on $u \in \mathcal{U}_1$.

4.1 Equivalent problems

Let us consider optimal stopping problem

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E}_* g(x(\tau)) \quad \text{over} \quad u(\cdot) \in \mathcal{U} \\ \text{subject to} \quad & dx(t) = u(t) f(e^{rt}\widetilde{S}(t), t)dt, \\ & dy(t) = u(t)dt, \\ & d\widetilde{S}(t) = \sigma \widetilde{S}(t) dw_*(t), \end{aligned} \tag{4.2}$$

where $\tau = T \wedge \inf\{t \in [0,T] :\ y(t) \ge 1\}$.

Lemma 4.2 *Problems (4.1) and (4.2) are equivalent in the following sense:*

$$\sup_{u\in\mathcal{U}_1} \mathbf{E}_* F_u = \sup_{u\in\mathcal{U}} \mathbf{E}_* g(x(\tau)),$$

where $x(t)$, $y(t)$, and τ are defined as in (4.2) given $u(\cdot)$ and given that $x(0)=0$, $y(0)=0$, and $S(0)=S_0$.

The state process for problem (4.2) is $(x(t), y(t), \widetilde{S}(t))$, and the corresponding matrix of the diffusion coefficients is degenerate. In addition, this problem involves the first exit from a domain with a boundary. This makes it difficult to use the classical methods of solution. Hence it will be more convenient to use an alternative model that does not feature a first exit time. Similarly to Lemma 4.2, it can be shown that problem (4.2) is equivalent to the optimal stochastic control problem

$$\begin{aligned}
&\text{Maximize} && \mathbf{E}_* g(x(T)) \quad \text{over} \quad u(\cdot)\in\mathcal{U} \\
&\text{subject to} && dx(t) = \mathbb{I}_{\{y(t)<1\}} u(t) f(e^{rt}\widetilde{S}(t), t)dt, \\
& && dy(t) = u(t)dt, \\
& && d\widetilde{S}(t) = \sigma \widetilde{S}(t) dw_*(t).
\end{aligned} \tag{4.3}$$

More precisely, we have that $c_F = e^{-rT} \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* g(x(T))$, where $x(t)$ is defined by (4.3) given that $x(0)=0$, $y(0)=0$, and $S(0)=S_0$.

4.2 Pricing via dynamic programming

The state equation for problem (4.3) has a discontinuous drift coefficient for $x(t)$. To remove this feature, we approximate the problem as the following.

Let functions $f_\varepsilon : (0,+\infty)\times[0,T] \to \mathbf{R}$ be selected such that the following holds.

(i) The functions $\widehat{f}_\varepsilon(z,t) \triangleq f_\varepsilon(e^z, t)$ are bounded and continuously differentiable in $(z,t)\in \mathbf{R}\times(0,T)$. The corresponding derivatives are bounded uniformly in $\varepsilon>0$.

(ii) $f_\varepsilon(x,t) \le f(x,t)+\varepsilon$ for all x,t.

(iii) $f_\varepsilon(x,t) \to f(x,t)$ as $\varepsilon\to 0$ for all x,t.

Let functions $g_\varepsilon(x) : \mathbf{R}\to\mathbf{R}$ and $\xi_\varepsilon(y) : \mathbf{R}\to[0,1]$ be selected such that the following holds.

(i) The functions ξ_ε are non-increasing, continuously differentiable, and such that $\xi_\varepsilon(y) = 1$ for $y < 1 - \varepsilon$, and $\xi_\varepsilon(y) = 0$ for $y > 1 - \varepsilon + \varepsilon^2$.

(ii) The functions $g_\varepsilon(x)$ are bounded and twice differentiable. The corresponding first order derivatives are bounded uniformly in $\varepsilon > 0$, and

$$g_\varepsilon(x) \to g(x) \quad \text{as} \quad \varepsilon \to 0, \qquad g_\varepsilon(x) \le g(x) + \varepsilon \quad \text{for all} \quad x.$$

The corresponding functions f_ε and g_ε can be obtained via convolutions of functions $\min(f(x,t), \varepsilon^{-1})$ and $\min(g(x), \varepsilon^{-1})$ with appropriate smooth convolution kernels such as $\bar{k}_\varepsilon(x) = \varepsilon^{-1}\bar{k}(x/\varepsilon)$, where $\bar{k}(x)$ is the density for the standard normal distribution, or with some other appropriate smoothing kernel. The corresponding functions ξ_ε can be obtained via convolutions with smooth enough kernels with finite support that is vanishing as $\varepsilon \to 0$ (see, e.g., Krylov [16], pp. 48–49).

Consider the following stochastic control problem:

$$\begin{aligned}
\text{Maximize} \qquad & \mathbf{E}_* g_\varepsilon(x(T)) \quad \text{over} \quad u(\cdot) \in \mathcal{U} \\
\text{subject to} \qquad & dx(t) = u(t)\xi_\varepsilon(y(t)) f_\varepsilon(e^{rt}\widetilde{S}(t), t)dt, \\
& dy(t) = u(t)dt, \\
& d\widetilde{S}(t) = \sigma \widetilde{S}(t) dw_*(t).
\end{aligned} \tag{4.4}$$

Consider the corresponding value function

$$J_\varepsilon(x, y, s, t) \triangleq \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* \Big\{ g_\varepsilon\left(x(T)\right) \Big| x(t) = x,\ y(t) = y,\ \widetilde{S}(t) = s \Big\}. \tag{4.5}$$

Let $D = \mathbf{R}^3 \times [0, T]$. Let $\mathcal{X}$ be the class of continuous functions $v(x, y, z, t) : D \to \mathbf{R}$ such that there exists $c > 0$ such that $|v(x, y, z, t)| \le c(|x| + |y| + |z| + 1)$ for all $(x, y, z, t) \in D$. Let $\mathcal{X}_1$ be the class of functions $v \in \mathcal{X}$ such that v'_x, v'_y, and v'_z belong to $\mathcal{X}$. Let $\mathcal{X}_2$ be the class of functions $v \in \mathcal{X}_1$ such that v'_t and v''_{zz} belong to $\mathcal{X}$.

Theorem 4.1 *(i) The unimprovable option price can be found as*

$$c_F = \lim_{\varepsilon\to 0} e^{-rT} J_\varepsilon(0, 0, S(0), 0). \tag{4.6}$$

(ii) The value function $J = J_\varepsilon$ satisfies the Bellman equation

$$\begin{aligned}
& J'_t + \max_{u\in[d_0,d_1]} \{J'_x u \xi_\varepsilon f_\varepsilon + J'_y u\} + \tfrac{1}{2} J''_{ss}\sigma^2 s^2 = 0, \\
& J(x, y, s, T) = g_\varepsilon(x).
\end{aligned} \tag{4.7}$$

The Bellman equation has unique solution in the class of functions $J = J_\varepsilon(x, y, s, t)$ such that $J_\varepsilon(x, y, s, t) = V_\varepsilon(x, y, \log s, t)$ for some function $V_\varepsilon \in \mathcal{X}_2$. The Bellman equation holds as an equality that is satisfied for a.e. $(x, y, s, t) \in \mathbf{R}^2 \times (0, +\infty) \times [0, T]$.

4.3 Case of linear g

The dimension of the Bellman equation can be reduced for the case when $g(x) \equiv x$. Similarly to Lemma 4.2, c_F can be found via solution of stochastic control problem

$$\begin{aligned}
&\text{Maximize} && \mathbf{E}_* \int_0^\tau u(t) f(e^{rt}\widetilde{S}(t), t)dt \quad \text{over} \quad u(\cdot) \in \mathcal{U} \\
&\text{subject to} && dy(t) = u(t)dt, \\
& && d\widetilde{S}(t) = \sigma \widetilde{S}(t) dw_*(t),
\end{aligned} \tag{4.8}$$

where $\tau = T \wedge \inf\{t \in [0, T] : \ y(t) \geq 1\}$. Consider the corresponding value function

$$\bar{J}(y, s, t) \triangleq \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* \Big\{ \int_t^{\tau_u^{x,t}} u(s) f(S(s), s) ds \Big| y(t) = y, \ \widetilde{S}(t) = s \Big\}. \tag{4.9}$$

Here

$$\tau_u^{y,s} = T \wedge \inf\{\theta \in [t, T] : \ y + \int_s^\theta u(q)dq \geq 1\}.$$

The unimprovable option price is $c_F = e^{-rT}\bar{J}(0, S(0), 0)$. The Bellman equation satisfied formally by $\bar{J}$ is

$$\begin{aligned}
&\bar{J}_t + \max_{u\in[d_0,d_1)} \{\bar{J}'_y u + u f(s, t)\} + \tfrac{1}{2}\bar{J}''_{ss}\sigma^2 s^2 = 0, \\
&\bar{J}(1, s, T) = 0, \qquad \bar{J}(y, s, T) = 0.
\end{aligned} \tag{4.10}$$

The Bellman equation holds for $x > 0$, $y < 1$, $s > 0$, $t < T$. However, to derive this equation and to prove the Verification Theorem, we need to overcome again some technical difficulties arising from the presence of the boundary and from the fact that the diffusion matrix in the state equation is degenerate. Instead, we suggest to use an alternative stochastic control problem

$$\begin{aligned}
&\text{Maximize} && \mathbf{E}_* \int_0^T \mathbb{I}_{\{y(t)\leq 1\}} u(t) f(e^{rt}\widetilde{S}(t), t)dt \quad \text{over} \quad u(\cdot) \in \mathcal{U} \\
&\text{subject to} && dy(t) = u(t)dt, \\
& && d\widetilde{S}(t) = \sigma \widetilde{S}(t) dw_*(t).
\end{aligned} \tag{4.11}$$

This problem does not involve the first exit time. The Bellman equation satisfied formally by its value function $J = J(y, s, t)$ is

$$
\begin{aligned}
&J_t + \max_{u\in[d_0,d_1]} \{J'_y u + \mathbb{I}_{\{y\le 1\}} uf\} + \tfrac{1}{2} J''_{ss}\sigma^2 s^2 = 0,\\
&J(y, s, T) = 0.
\end{aligned}
$$

Since the state equation for problem (4.11) is degenerate again, we will use an equation with more regular coefficients as an approximation.

Let functions f_ε and ξ_ε be such as described above. Consider a stochastic control problem

$$
\begin{aligned}
&\text{Maximize} \quad && \mathbf{E}_* \int_0^T u(t)\xi_\varepsilon(y(t)) f_\varepsilon(e^{rt}\widetilde{S}(t), t)dt \quad \text{over} \quad u(\cdot) \in \mathcal{U}\\
&\text{subject to} && dy(t) = u(t)dt,\\
& && d\widetilde{S}(t) = \sigma \widetilde{S}(t) dw_*(t). \qquad (4.12)
\end{aligned}
$$

Consider the corresponding value function

$$
J_\varepsilon(y, s, t) \triangleq \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* \Big\{ \int_t^T u(t)\xi_\varepsilon(y(t)) f_\varepsilon(e^{rt}\widetilde{S}(t), t)dt \Big| y(t) = y,\ \widetilde{S}(t) = s \Big\}. \qquad (4.13)
$$

Let $D' = \mathbf{R}^2 \times [0, T]$. Let $\mathcal{Y}$ be the class of functions $v(y, z, t) : D' \to \mathbf{R}$ such that v is continuous and there exists $c > 0$ such that $|v(y, z, t)| \le c(|y| + |z| + 1)$ for all $(y, z, t) \in D'$. Let $\mathcal{Y}_1$ be the class of functions $v \in \mathcal{Y}$ such that v'_y and v'_z belong to $\mathcal{Y}$. Let $\mathcal{Y}_2$ be the class of functions $v \in \mathcal{Y}_1$ such that v'_t and v''_{zz} both belong to $\mathcal{Y}$.

Theorem 4.2 *(i) The unimprovable option price can be found as*

$$
c_F = \lim_{\varepsilon\to 0} e^{-rT} J_\varepsilon(0, S(0), 0). \qquad (4.14)
$$

(ii) The value function $J = J_\varepsilon(y, s, t)$ for problem (4.11) satisfies the Bellman equation

$$
\begin{aligned}
&J_t + \max_{u\in[d_0,d_1]} \{J'_y u + u\xi_\varepsilon f_\varepsilon\} + \tfrac{1}{2} J''_{ss}\sigma^2 s^2 = 0,\\
&J(y, s, T) = 0. \qquad (4.15)
\end{aligned}
$$

The Bellman equation has unique solution in the class of functions $J = J_\varepsilon(x, y, s, t)$ such that $J_\varepsilon(y, s, t) = V_\varepsilon(y, \log s, t)$ for some function $V_\varepsilon \in \mathcal{Y}_2$. The Bellman equation holds as an equality that is satisfied for a.e. $(y, s, t) \in \mathbf{R} \times (0, +\infty) \times [0, T]$.

4.4 An analog of the Merton's "no early exercise" theorem

In this section, we consider the case when $g(x) \equiv x$ and $d_0 = 0$. We consider an option with the payoff at time T

$$F_u = \int_0^T u(t) f(S(t), t) dt,$$

where $f : (0, +\infty) \times [0, T] \to \mathbf{R}$ is a given function such that $|f(x,t)| \le \text{const}\,(1 + |x|)$ and $f(x,t) \ge 0$. Here $u(t)$ is the control process that is selected by the option holder. The set $\mathcal{U}_1$ of admissible processes $u(t)$ consists of the processes that are adapted to $\mathcal{F}_t$ and such that

$$u(t) \in [0, d_1], \quad \int_0^T u(t) dt \le 1,$$

where $d_1 \in (0, +\infty)$.

If $d_1 T \le 1$ then the optimal solution is $u \equiv d_1$. Hence we assume that $d_1 T > 1$.

This option represents the limit version of multi-exercise options when the distribution of exercise times approaching a continuous distribution. This model can be used, for instance, for energy trading with $u(t)dt$ representing the quantity of energy purchased during the time interval $[t, t + dt]$ for the fixed price K when the market price is above K. The total amount $\int_0^T u(t)dt$ of energy that can be purchased is limited per option.

Merton's no early exercise theorem states that the American and European options with the same parameters have the same price and that an early exercise is not rational. The following theorem represents an extension of this theorem on the case of the controlled options.

Theorem 4.3 *Let $f(S(t), t) = e^{r(T-t)} h(S(t))$, where the function $h(x)$ is convex and non-linear in $x > 0$, and such that at least one of the following conditions holds:*

(i) the function $\alpha^{-1} h(\alpha x)$ is non-decreasing in $\alpha \in (0, 1]$; or

(ii) $r = 0$.

Then $\sup_{u(\cdot) \in \mathcal{U}_1} \mathbf{E}_* F_u$ is achieved for the control process

$$\widehat{u}(t) = \begin{cases} d_1, & t \ge T - 1/d_1 \\ 0, & t < T - 1/d_1, \end{cases}$$

and the unimprovable price of the option is

$$e^{-rT}\mathbf{E}_* \int_0^T \widehat{u}(t) f(S(t),t)dt = \frac{e^{-rT}}{d_1}\mathbf{E}_* \int_{T-1/d_1}^T f(S(t),t)dt.$$

Remark 4.1 The function $h(x) = (x-K)^+$ is such that assumption (i) of Theorem 4.3 is satisfied; this function corresponds to the call option with continuously distributed payoff time. On the other hand, assumption (i) of Theorem 4.3 is not satisfied for $h(x) = (K-x)^+$ that corresponds to the put option. The pricing for this case with $r > 0$ is an interesting problem. A solution could be a useful approximation of the classical optimal stopping pricing rule for the American options. For the controlled options with admissible controls such that $u(t) \in [0, d_1]$, the limit case when $d_1 \to +\infty$ leads to optimal stopping and a Stefan problem with unknown boundary. The solution for a large finite d_1 can be used as an approximation that is easier to find since it does not involve a Stefan problem. We leave it for future research.

5 Pricing for non-adapted normalized weight $v(t)$

Consider an option with payoff (2.3), where functions $f(x,t)$ and $g(x)$ have the same properties as in Section 4.

We don't exclude the case when $d_0 = 0$ and $u \equiv 0$. In this case, the payoff is assumed to be defined as (2.5).

For $\varepsilon > 0$, let functions $f_\varepsilon(x,t) : (0,+\infty) \to \mathbf{R}$ be such as described in Section 4.

Let $\mathrm{f}_\varepsilon(u,x,t) = h_\varepsilon(u,t) f_\varepsilon(x,t)$, where $h_\varepsilon(u,t) = (1-\psi_\varepsilon(t))u + d_1\psi_\varepsilon(t)$, and where $\psi_\varepsilon(t) : \mathbf{R} \to [0,1]$ is a continuously differentiable non-decreasing function such that $\psi_\varepsilon(t) = 0$ for $t < T - \varepsilon$, $\psi_\varepsilon(t) = d_1$, $t > T - \varepsilon + \varepsilon^2$.

Let functions $\mathrm{g}_\varepsilon(x,y) : \mathbf{R}^2 \to \mathbf{R}$ be selected such that the following holds.

(i) $\mathrm{g}_\varepsilon(x,y) \le g(x/y) + \varepsilon$ for all $y \neq 0, x, t$.

(ii) The functions $\mathrm{g}_\varepsilon(x,y)$ are bounded and twice differentiable in (x,y). The corresponding first order derivatives are bounded uniformly in ε.

(iii) If $y \neq 0$ then $\mathrm{g}_\varepsilon(x,y) \to g(x/y)$ as $\varepsilon \to 0$ for all x.

(iv) If, for some $c \in \mathbf{R}$, we have that $\varepsilon \to 0$, $y \to 0$, $x/y \to c$, then $\mathrm{g}_\varepsilon(x,y) \to g(c)$.

Consider optimal stochastic control problem

$$\begin{aligned}
\text{Maximize} \quad & \mathbf{E}_* \mathrm{g}_\varepsilon(x(T), y(T)) \quad \text{over} \quad u(\cdot) \in \mathcal{U} \\
\text{subject to} \quad & dx(t) = \mathrm{f}_\varepsilon(u(t), e^{rt}\widetilde{S}(t), t)dt, \\
& dy(t) = h_\varepsilon(u(t), t)dt, \\
& d\widetilde{S}(t) = \sigma \widetilde{S}(t) dw_*(t).
\end{aligned} \tag{5.1}$$

Consider the corresponding value function

$$J_\varepsilon(x, y, s, t) \triangleq \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* \Big\{ \mathrm{g}_\varepsilon\,(x(T), y(T)) \,\Big|\, x(t) = x,\ y(t) = y,\ \widetilde{S}(t) = s \Big\}. \tag{5.2}$$

Let $\mathcal{X}_2$ be the space introduced in Section 4.

Theorem 5.1 *(i) The unimprovable option price can be found as*

$$c_F = \lim_{\varepsilon\to 0} e^{-rT} J_\varepsilon(0, 0, S(0), 0). \tag{5.3}$$

(ii) The value function $J = J_\varepsilon$ satisfies the Bellman equation

$$\begin{aligned}
& J_t + \max_{u\in[d_0,d_1]} \{J'_x \mathrm{f}_\varepsilon + J'_y h_\varepsilon\} + \tfrac{1}{2} J''_{ss} \sigma^2 s^2 = 0, \\
& J(x, y, s, T) = \mathrm{g}_\varepsilon(x, y).
\end{aligned} \tag{5.4}$$

The Bellman equation has unique solution in the class of functions $J = J_\varepsilon(x, y, s, t)$ such that $J_\varepsilon(x, y, s, t) = V_\varepsilon(x, y, \log s, t)$ for some function $V_\varepsilon \in \mathcal{X}_2$. The Bellman equation holds as an equality that is satisfied for a.e. $(x, y, s, t) \in \mathbf{R} \times \mathbf{R} \times (0, +\infty) \times [0, T]$.

6 On superreplication and non-arbitrage for writer

Usually, the fair price of an option is established as the minimal initial wealth that allows superreplication. This property implies that any higher price leads to arbitrage opportunities. It is shown below that a similar property holds for the modified approximating problems.

Theorem 6.1 *(i) Under the assumptions of Theorem 4.1, the value $X(0) = J_\varepsilon(0, 0, S(0), 0)$ is the minimal initial wealth such that, for any $u \in \mathcal{U}$, there exists a self-financing strategy such that the corresponding wealth $X(t)$ is such that*

$$X(T) \geq g_\varepsilon(x(T)) \quad a.s., \tag{6.1}$$

where $x(\cdot)$ is defined by $u(\cdot)$ according to (4.4).

(ii) *Under the assumptions of Theorem 4.2, the value $X(0) = J_\varepsilon(0, S(0), 0)$ is the minimal initial wealth such that, for any $u \in \mathcal{U}$, there exists a self-financing strategy such that the corresponding wealth $X(t)$ is such that*

$$X(T) \geq \int_0^T u(t)\xi_\varepsilon(y(t))f_\varepsilon(S(t), t)dt \quad a.s.,$$

where $x(\cdot)$ is defined by $u(\cdot)$ according to (4.12).

(iii) *Under the assumptions of Theorem 5.1, the value $X(0) = J_\varepsilon(0, 0, S(0), 0)$ is the minimal initial wealth such that, for any $u \in \mathcal{U}$, there exists a self-financing strategy such that the corresponding wealth $X(t)$ is such that*

$$X(T) \geq \mathrm{g}_\varepsilon(x(T), y(T)) \quad a.s.,$$

where $x(\cdot)$ and $y(t)$ are defined by $u(\cdot)$ according to (5.1).

Corollary 6.1 *Any price larger than c_F ensures an arbitrage opportunities for the option writer. This holds for the options with payoff (2.2) and (2.3) under the assumptions of Section 4 and Section 5 respectively.*

It follows from Theorems 4.1, 4.2, 5.1, 6.1, and Corollary 6.1, that the unimprovable price c_F ensures for the option writer possibility of approximate superreplication of the payoff such that the error $\inf_u (X(T) - F_u)^+$ can be made vanishingly small. So far, it is unclear yet if thus error can be made zero.

Appendix: Proofs

Proof of Theorem 3.1. It is known that the discounted wealth is a martingale under $\mathbf{P}_*$ for any admissible strategy. For a given u, the ability to fulfill the option obligations means that $X(T) \geq F_u$ and $\widetilde{X}(T) \geq e^{-rT} F_u$. Hence $X(0) = \mathbf{E}_* \widetilde{X}(T) \geq e^{-rT} \mathbf{E}_* F_u$. It follows that

$$c_F \geq e^{-rT} \sup_{u \in U} \mathbf{E}_* F_u.$$

Further, suppose that there exists $\varepsilon > 0$ such that, for all $u(\cdot) \in U$,

$$c_F \geq e^{-rT}\mathbf{E}_* F_u + \varepsilon.$$

In this case, for any strategy $u(\cdot) \in U$, the claim F_u can be replicated with the initial wealth $X_0 \leq c_F - \varepsilon$. Therefore, any potential option buyer could save $\varepsilon > 0$ quantity of cash if she select to replicate the payoff F_u with some self-financing strategy. Therefore, a rational option buyer would not buy an option for the price c_F. This completes the proof. $\square$

Proof of Lemma 4.1. Let $\mathcal{H}$ be the Hilbert space formed as the completion of the set of all square integrable and adapted processes in the norm of $L_2([0,T]\times\Omega)$. The set $\mathcal{U}_1$ is a convex and closed (and, therefore, weakly closed) subset of $\mathcal{H}$. Hence $\mathcal{U}_1$ is compact in the weak topology of $\mathcal{H}$.

Consider the mapping $\Phi : \mathcal{U} \to \mathbf{R}$ such that $\Phi(u) = \mathbf{E}_* F_u$.

Let $\{u_j\} \subset \mathcal{H}$ be a sequence such that

$$\Phi(u_j) \to \sup_{u\in\mathcal{H}} \Phi(u) \quad \text{as} \quad j \to +\infty. \tag{A.1}$$

There exists a subsequence $\{u_k\}$ and $\bar{u} \in \mathcal{U}_1$ such that $u_k \to \bar{u} \in \mathcal{H}$ weakly in $\mathcal{H}$ as $k \to +\infty$. By Mazur's Theorem, there exists a sequence of integer numbers $k = k_i \to +\infty$ such that there exist sets of real numbers $\{a_{mk}\}_{m=1}^{k} \subset [0,1]$ such that $\sum_{m=1}^{k} a_{mk} = 1$ and that

$$\widetilde{u}_k \triangleq \sum_{m=1}^{k} a_{mk} u_m \to \bar{u} \quad \text{in} \quad \mathcal{H}. \tag{A.2}$$

(See, e.g., Theorem 5.1.2 from Yosida [20]). In addition, there exists a subsequence $\{\widehat{u}_m\}$ of this sequence such that $\widehat{u}_m \to \bar{u}$ a.e. as $k \to +\infty$.

Consider mappings $G(u,y) : \mathcal{U}_1 \times C([0,T] \to \mathbf{R})$ and $I(u,y) : \mathcal{U}_1 \times C([0,T] \to \mathbf{R})$ such that $I(u,y) = \int_0^T u(t) f(y(t),t)dt$ and $G(u,y) = g(I(u,y))$. Clearly, $G(\widehat{u}_k, S(\cdot)) \to G(\bar{u}, S(\cdot))$ a.s. as $k \to +\infty$. In addition,

$$\begin{aligned} &|I(u,S)| \leq T \max_{t\in[0,T]} |f(S(t),t)| \max u(t) \leq \text{const } \cdot d_1 T \max_{t\in[0,T]} (S(t)+1), \\ &|G(u,S)| \leq \text{const}\, I(u,S(\cdot)). \end{aligned}$$

By the Lebesgue's Dominated Convergence Theorem, it follows that $\Phi(\widehat{u}_k) \to \Phi(\bar{u})$.

By the linearity of I, we have that $I(\widehat{u}_k, S(\cdot)) = \sum_{m=1}^{k} a_{mk} I(u_m, S(\cdot))$. By the concavity of g, it follows that

$$g(I(\widehat{u}_k, S(\cdot))) \geq g\left(\sum_{m=1}^{k} a_{mk} I(u_m, S(\cdot))\right).$$

Hence

$$\Phi(\widehat{u}_k) \geq \sum_{m=1}^{k} a_{mk}\Phi(u_m) \to \sup_{u\in\mathcal{U}_1} \Phi(u) \quad \text{as} \quad k \to +\infty.$$

Hence $\Phi(\bar{u}) = \sup_{u\in\mathcal{H}} \Phi(u)$, i.e., $\bar{u}$ is a optimal control. $\square$

Proof of Lemma 4.2. For any $u \in \mathcal{U}_1$, we have that $F_u = g(x(\tau))$, where $x(\cdot)$ is defined by (4.2) with any $\widetilde{u} \in \mathcal{U}$ such that $\widetilde{u}(t)\mathbb{I}_{\{t\leq\tau\}} = u(t)$, given that $x(0) = 0$, $y(0) = 0$, and $S(0) = S_0$. On the other hand, for any $u \in \mathcal{U}$, we have that $g(x(\tau)) = F_{\widehat{u}}$, where $x(\cdot)$ is defined by (4.2) given that $x(0) = 0$, $y(0) = 0$, and $S(0) = S_0$, where $\widetilde{u}(t) = u(t)\mathbb{I}_{\{t\leq\tau\}}$. In addition, we have that $\widehat{u} \in \mathcal{U}_1$. This completes the proof. $\square$

Proof of Theorem 4.1. Let us prove statement (i). By Lemma 4.1, $c_F = e^{-rT}\mathbf{E}_* F_{\widehat{u}}$ for some $\widehat{u} \in \mathcal{U}_1$. Let $\widehat{y}(t) = \int_0^t \widehat{u}(s)ds$. Let $\widehat{\phi}_\varepsilon(u, x, y, s) = u\xi_\varepsilon(y) f_\varepsilon(x, t)$ and

$$F_u^\varepsilon = g_\varepsilon\left(\int_0^T \widehat{\phi}_\varepsilon(u(t), y(t), S(t), t)dt\right), \tag{A.3}$$

where $y(\cdot)$ is defined by (4.4) given that $x(0) = 0$, $y(0) = 0$, and $S(0) = S_0$.

By the assumptions on $f_\varepsilon, g_\varepsilon$, and ξ_ε, it follows that $\widehat{\phi}_\varepsilon(u, x, y, s) \leq uf(x, t) + u\varepsilon$. Hence

$$e^{-rT} J_\varepsilon(0, 0, S(0), 0) \leq c_F + \varepsilon d_1 T \sup_{x,\varepsilon} \left|\frac{dg_\varepsilon}{dx}(x)\right|$$

for any $\varepsilon > 0$. On the other hand,

$$J_\varepsilon(0, 0, S(0), 0) \geq \mathbf{E}_* g_\varepsilon\left(\int_0^T \widehat{\phi}_\varepsilon(\widehat{u}(t), \widehat{y}(t), S(t), t)dt\right).$$

By the Lebesgue's Dominated Convergence Theorem and by the assumptions on $g_\varepsilon, f_\varepsilon$, we obtain that

$$F_{\widehat{u}}^\varepsilon = g_\varepsilon\left(\int_0^T \widehat{\phi}_\varepsilon(\widehat{u}(t), \widehat{y}(t), S(t), t)dt\right) \to g\left(\int_0^T \widehat{u}(t)\mathbb{I}_{\{\widehat{y}(t)<1\}} f(S(t), t)dt\right) = F_{\widehat{u}} \quad \text{a.s. as} \quad \varepsilon \to 0.$$

By the Lebesgue's Dominated Convergence Theorem again, statement (i) follows.

Let us prove statement (ii). Using the Itô formula, we obtain that the change of variables $R(t) = \ln \widetilde{S}(t)$ transfers problem (4.4) into the problem

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E}_* g_\varepsilon(x(T)) \quad \text{over} \quad u(\cdot) \in \mathcal{U} \\ \text{subject to} \quad & dx(t) = \widehat{\phi}_\varepsilon(u(t), y(t), e^{R(t)}, t)dt, \\ & dy(t) = u(t)dt, \\ & dR(t) = -\frac{\sigma^2}{2}dt + \sigma dw_*(t). \end{aligned}$$

Consider the corresponding value function

$$V(x, y, z, t) \triangleq \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* \Big\{ g_\varepsilon\,(x(T)) \,\Big|\, x(t) = x,\ y(t) = y,\ R(t) = z \Big\}.$$

The coefficients of this problem are such that the assumptions of Theorem 4.1.4 and Theorem 4.4.3 from Krylov [16], pp.167 and 192, are satisfied.

By Theorem 4.1.4 from [16], p. 167, the function V satisfies the corresponding parabolic Bellman equation that has unique solution in $\mathcal{X}_1$. The Bellman equation holds in the generalized sense, i.e., as an equality of the distributions. This equation includes only one partial derivative of the second order, V''_{zz}, which is presented with the coefficient $\sigma^2/2 > 0$. By Theorem 4.4.3 from [16], p.192, the derivative $V'_t(x, y, z, t)$ belongs to $\mathcal{X}$. It follows that $V''_{zz} \in \mathcal{X}$. The Bellman equation for $J(x, y, s, t) = V(x, y, \log s, t)$ is defined by the equation for V with the corresponding change of variables. Then the proof of statement (ii) and Theorem 4.1 follows. $\square$.

Proof of Theorem 4.2 is similar to the proof of Theorem 4.1.

Proof of Theorem 4.3. By Lemma 4.1, it follows that there exists an optimal control $u \in \mathcal{U}_1$. Let $u(\cdot) \neq \widehat{u}(\cdot)$ be a process in $\mathcal{U}_1$. To prove the theorem, it suffices to show that $\Phi(u) \leq \Phi(\widehat{u})$, where the mapping $\Phi : \mathcal{U} \to \mathbf{R}$ is such that $\Phi(u) = \mathbf{E}_* F_u$. Suppose that Theorem 4.3 does not hold. In this case, there exist $u \in \mathcal{U}_1$ and $M > 0$ such that

$$\Phi(u) = \Phi(\widehat{u}) + M. \tag{A.4}$$

Let $\mathcal{U}_{BB}$ be the set of all pathwise right continuous processes $u \in \mathcal{U}$ such that $u(t) \in \{0, d_1\}$ a.e. (i.e., it is the set of all bang-bang controls).

Let us construct a sequence $u_i \in \mathcal{U}_{BB}$ according to the following procedure. We select $t_k =$

kT/i, $k=0,1,\dots,i$, and $u_i(t)|_{t\in[0,t_1)}=0$. Further, we select

$$u_i(t)|_{t\in[t_k,t_{k+1})}=0 \quad \text{if} \quad y_i(t_k)\ge y(t_k),$$
$$u_i(t)|_{t\in[\theta_k,t_1)}=d_1 \quad \text{if} \quad y_i(t_k)< y(t_k),$$

where $y(t)=\int_0^t u(s)ds$ and $y_i(t)=\int_0^t u_i(s)ds$. We have that $y_i(t)\to y(t)$ for any t and $u_i\to u$ weakly in $L_2(0,T)$ as $i\to+\infty$. It follows that

$$\Phi(u_i)\to\Phi(u), \quad \int_0^t u_i(s)ds\to\int_0^t u(s)ds \quad \forall t\in[0,T] \qquad \text{as} \quad i\to+\infty. \tag{A.5}$$

Let $\widehat{T}=T-1/d_1$, and let $\widehat{T}_i=\widehat{k}/i$, where $\widehat{k}=\min\{k:\ k/i\ge\widehat{T}\}$.

Consider random sets $J_0=\{k:\ u_i(t_k)=d_1,\quad t_k<\widehat{T}_i\}$ and $J_1=\{m:\ u_i(t_k)=0,\quad t_m\ge\widehat{T}_i\}$. Let $N_0=N_0(i)$ be the number of elements in J_0, and let $N_1=N_1(i)$ be the number of elements in J_1. We have that $J_0=\{\tau_1,\dots,\tau_{N_0}\}$ and $J_1=\{\theta_1,\dots,\theta_{N_1}\}$, where τ_n and θ_n are Markov times with respect to $\mathcal{F}_t$ such that $\tau_n\le\tau_{n+1}$ and $\theta_n\le\theta_{n+1}$ for all n.

Further, let $v_i\in\mathcal{U}_{BB}$ be selected such that $v_i(t)=0$ if $t<\widehat{T}_i$, $v_i(t)=d_1$ for $t\ge\widehat{T}_i$. Let $N=\min(N_0,N_1)$.

Let $\mathcal{T}_i=\{t\in[0,T_1):\ u_i(t)=d_1\}$ and $\widehat{\mathcal{T}}_i=\{t\in[T_1,T]:\ u_i(t)=0\}$. We have that

$$\begin{aligned}\Phi(u_i)-\Phi(v_i)&=\mathbf{E}_*\Big[\int_{\mathcal{T}_i}d_1f(S(t),t)dt-\int_{\widehat{\mathcal{T}}_i}d_1f(S(t),t)dt\Big]\\ &=d_1\mathbf{E}_*\Big[\sum_{n=1}^{N_0}\int_{\tau_k}^{\tau_k+\varepsilon}f(S(t),t)dt-\sum_{m=1}^{N_1}\int_{\theta_m}^{\theta_m+\varepsilon}f(S(t),t)dt\Big]\\ &=d_1\mathbf{E}_*\Big[\sum_{n=1}^{N}\int_{\tau_k}^{\tau_k+\varepsilon}f(S(t),t)dt-\sum_{m=1}^{N}\int_{\theta_m}^{\theta_m+\varepsilon}f(S(t),t)dt\Big]+R_i,\end{aligned} \tag{A.6}$$

where $R_i=R_{1i}-R_{2i}$, $\varepsilon=t_{k+1}-t_k$,

$$R_{1i}=d_1\mathbf{E}_*\mathbb{I}_{\{N_0>N\}}\sum_{n=N}^{N_0}\int_{\tau_n}^{\tau_n+\varepsilon}f(S(t),t)dt, \quad R_{2i}=d_1\mathbf{E}_*\mathbb{I}_{\{N_1>N\}}\sum_{n=N}^{N_1}\int_{\theta_m}^{\theta_m+\varepsilon}f(S(t),t)dt.$$

Let ρ_0 and ρ_1 be Markov times with respect to $\mathcal{F}_t$ such that $0\le\rho_0\le\rho_1\le T$. Let us show that

$$f(S(\rho_0),\rho_0)\le\mathbf{E}_*\{f(S(\rho_1),\rho_1)|\mathcal{F}_{\rho_0}\}. \tag{A.7}$$

We use the approach from [8], p.132. We have that $\widetilde{S}(t)$ is a martingale under $\mathbf{P}_*$. By the convexity of $g(\cdot)$ and the Jensen's inequality, it follows that

$$\mathbf{E}_*\{h(S(\rho_1))|\mathcal{F}_{\rho_0}\}=\mathbf{E}_*\{h(e^{r\rho_1}\widetilde{S}(\rho_1))|\mathcal{F}_{\rho_0}\}\ge h(e^{r\rho_1}\widetilde{S}(\rho_0))=h(e^{r(\rho_1-\rho_0)}S(\rho_0)).$$

By the properties of h, we have that

$$e^{-r(\rho_1-\rho_0)}h(e^{r(\rho_1-\rho_0)}S(\rho_0)) \ge h(S(\rho_0)).$$

Hence $h(e^{r(\rho_1-\rho_0)}S(\rho_0)) \ge e^{r(\rho_1-\rho_0)}h(S(\rho_0))$ and

$$\mathbf{E}_*\{h(S(\rho_1))|\mathcal{F}_{\rho_0}\} \ge h(e^{r(\rho_1-\rho_0)}S(\rho_0)) \ge e^{r(\rho_1-\rho_0)}h(S(\rho_0)).$$

Hence

$$\mathbf{E}_*\{f(S(\rho_1),\rho_1)|\mathcal{F}_{\rho_0}\} = e^{r(T-\rho_1)}\mathbf{E}_*\{h(S(\rho_1))|\mathcal{F}_{\rho_0}\} \ge e^{r(T-\rho_1)}e^{r(\rho_1-\rho_0)}h(S(\rho_0)) = f(S(\rho_0),\rho_0).$$

It follows that (A.7) holds.

Further, it follows from (A.6) and (A.7) that

$$\Phi(u_i)-\Phi(v_i) \le d_1\mathbf{E}_*\sum_{n=1}^{N}\Big[\int_{\tau_n}^{\tau_n+\varepsilon} f(S(t),t)dt - \varepsilon f(S(\theta_n),\theta_n)\Big] + R_i.$$

Hence

$$\Phi(u_i)-\Phi(v_i) \le d_1\mathbf{E}_*\sum_{n=1}^{N}\int_0^{\varepsilon}[f(S(\tau_n+q),\tau_n+q)dq - f(S(\theta_n),\theta_n)]dt + R_i$$

$$\le d_1\mathbf{E}_*\sum_{n=1}^{N}\int_0^{\varepsilon}[f(S(\tau_n+q),\tau_n+q) - \mathbf{E}\{f(S(\theta_n),\theta_n)|\mathcal{F}_{\tau_n+q}\}]dq + R_i \le R_i. \qquad \text{(A.8)}$$

The last inequality follows from (A.7) again.

Since $u\in\mathcal{U}_1$, it follows from the definition of u_i and (A.5) that

$$\int_0^{\widehat{T}} u_i(t)dt - \int_0^{\widehat{T}}(d_1-u_i(t))dt \to 0 \quad \text{as} \quad i\to+\infty.$$

Clearly, $\widehat{T}_i \to \widehat{T}$ as $i\to+\infty$. Hence

$$\int_0^{\widehat{T}_i} u_i(t)dt - \int_0^{\widehat{T}_i}(d_1-u_i(t))dt \to 0 \quad \text{as} \quad i\to+\infty. \qquad \text{(A.9)}$$

By the definition of N_1 and N_0, it follows from (A.9) that $\varepsilon(N_1-N_0)\to 0$ as $i\to+\infty$ and

$$R_i \to 0 \quad \text{as} \quad i\to+\infty. \qquad \text{(A.10)}$$

Since $\widehat{T}_i\to\widehat{T}$ as $i\to+\infty$, it follows that

$$\Phi(v_i)\to\Phi(\widehat{u}) \quad \text{as} \quad i\to+\infty. \qquad \text{(A.11)}$$

By (A.5), (A.8), (A.10)-(A.11), a large i can be selected such that $\Phi(u_i) - \Phi(u) \le M/4$, $\Phi(v_i) - \Phi(\widehat{u}) \le M/4$, and $R_i < M/4$. We have that $\Phi(u_i) - \Phi(v_i) \le R_i$ and

$$\Phi(u) - \Phi(\widehat{u}) = \Phi(u) - \Phi(u_i) + \Phi(u_i) - \Phi(v_i) + \Phi(v_i) - \Phi(\widehat{u}) \le 3M/4.$$

This contradicts to (A.4). This completes the proof of Theorem 4.3. □

Proof of Theorem 5.1. Let $\{u_i\} \subset \mathcal{U}$ be a sequence such that

$$\mathbf{E}_* F_{u_i} \to \sup_{u\in\mathcal{U}} \mathbf{E}_* F_u \quad \text{as} \quad i \to +\infty. \tag{A.12}$$

Let F_u^ε be defined as

$$F_u^\varepsilon = \mathrm{g}_\varepsilon \left(\int_0^T \mathrm{f}_\varepsilon(u(t)S(t),t)dt, \int_0^T h_\varepsilon(u(t),t)dt \right). \tag{A.13}$$

By the definitions,

$$J_\varepsilon(0,0,S(0),0) = \sup_{u\in\mathcal{U}} \mathbf{E}_* F_u^\varepsilon \ge \mathbf{E}_* F_{u_i}^\varepsilon$$

for all i. Similarly to the proof of Theorem 4.1, we obtain from the properties of $(\mathrm{f}_\varepsilon, \mathrm{g}_\varepsilon)$ that $\sup_{u\in\mathcal{U}} \mathbf{E}_* F_u^\varepsilon \le e^{rT} c_F + \mathrm{const} \cdot \varepsilon$, where $c_F = \sup_{u\in\mathcal{U}} \mathbf{E}_* F_u$. Hence

$$e^{-rT} J_\varepsilon(0,0,S(0),0) \le c_F + \mathrm{const} \cdot \varepsilon \quad \forall \varepsilon > 0. \tag{A.14}$$

Let i be fixed. If $\int_0^T u_i(t)dt > 0$ then, by the Lebesgue's Dominated Convergence Theorem and by the assumptions on g_ε and f_e,

$$\mathbf{E}_* F_{u_i}^\varepsilon = \mathrm{g}_\varepsilon \left(\int_0^T \mathrm{f}_\varepsilon(u_i(t), S(t), t)dt, \int_0^T h_\varepsilon(u_i(t),t)dt \right) \to g\left(\frac{\int_0^T u_i(t) f(S(t),t)dt}{\int_0^T u_i(t)dt} \right) = F_{u_i} \quad \text{a.s. as} \quad \varepsilon \to 0.$$

If $\int_0^T u_i(t)dt = 0$ then, by assumption (2.5) and by the assumptions on g_ε and f_ε,

$$\begin{aligned} \mathbf{E}_* F_{u_i}^\varepsilon &= \mathrm{g}_\varepsilon \left(\int_0^T \mathrm{f}_\varepsilon(u_i(t), S(t), t)dt, \int_0^T h_\varepsilon(u_i(t),t)dt \right) \\ &= \mathrm{g}_\varepsilon \Bigg(\int_{T-\varepsilon}^{T-\varepsilon+\varepsilon^2} h_\varepsilon(u_i(t),t) f_\varepsilon(S(t),t)dt + \int_{T-\varepsilon+\varepsilon^2}^T d_1 f_\varepsilon(S(t),t)dt, \int_{T-\varepsilon}^{T-\varepsilon+\varepsilon^2} h_\varepsilon(u_i(t),t)dt \\ &\qquad + \int_{T-\varepsilon+\varepsilon^2}^T d_1 dt \Bigg) \\ &= \mathrm{g}_\varepsilon \left(O(\varepsilon^2) + \int_{T-\varepsilon+\varepsilon^2}^T d_1 f_\varepsilon(S(t),t)dt,\, O(\varepsilon^2) + (\varepsilon - \varepsilon^2) d_1 \right) \to g(f(S(T),T)) = F_{u_i} \\ &\qquad \text{a.s. as} \quad \varepsilon \to 0. \end{aligned}$$

By the Lebesgue's Dominated Convergence Theorem and by the assumptions on g_ε and f_ε, it follows that

$$\mathbf{E}_*^\varepsilon F_{u_i} \to \mathbf{E}_* F_{u_i} \quad \text{as} \quad \varepsilon \to 0. \tag{A.15}$$

We now in the position to prove statement (i). By (A.14), it suffices to show that, for any $\delta > 0$, there exists $\varepsilon_* > 0$ such that $J_\varepsilon(0,0,S(0),0) \geq c_F - \delta$ for $\varepsilon \leq \varepsilon_*$. Let i be such that $\mathbf{E}_* F_{u_i} \geq \sup_{u\in\mathcal{U}} \mathbf{E}_* F_u - \delta/2$. By (A.15), there exists $\varepsilon_* = \varepsilon_*(\delta, i) > 0$ such that $\mathbf{E}_* F_{u_i}^\varepsilon \geq \mathbf{E}_* F_{u_i} - \delta/2$ for all $\varepsilon \leq \varepsilon_*$. Hence $J_\varepsilon(0,0,S(0),0) \geq \mathbf{E}_* F_{u_i} - \delta/2$ and $J_\varepsilon(0,0,S(0),0) \geq \sup_{u\in\mathcal{U}} \mathbf{E}_* F_u - \delta$ for these ε. Then statement (i) follows.

Let us prove statement (ii). Using the Itô formula, we obtain that the change of variables $R(t) = \ln \widetilde{S}(t)$ transfers problem (5.1) into the problem

$$\begin{aligned} &\text{Maximize} && \mathbf{E}_* \mathrm{g}_\varepsilon(x(T), y(T)) \quad \text{over} \quad u(\cdot) \in \mathcal{U} && \text{(A.16)} \\ &\text{subject to} && dx(t) = \mathrm{f}_\varepsilon(u(t), e^{rt+R(t)}, t)dt, \\ &&& dy(t) = h_\varepsilon(u(t), t)dt, \\ &&& dR(t) = -\frac{\sigma^2}{2}dt + \sigma dw_*(t). && \text{(A.17)} \end{aligned}$$

Consider the corresponding value function

$$V(x,y,z,t) \triangleq \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* \Big\{ \mathrm{g}_\varepsilon\left(x(T), y(T)\right) \Big| x(t) = x,\, y(t) = y,\, R(t) = z \Big\}. \tag{A.18}$$

Note that the coefficients of this problem are such that the assumptions of Theorem 4.1.4 and Theorem 4.4.3 from [16], pp.167,192, are satisfied. The remaining part of the proof repeats the proof of Theorem 4.1(ii). This completes the proof of Theorem 5.1. □

Proof of Theorem 6.1. Let us prove statement (i). Let $J = J_\varepsilon$ be the solution of Bellman equation (4.7). Let $\widehat{u} \in \mathcal{U}$ be such that $\alpha(t, u(\cdot)) \geq 0$ a.s. for a.e. t for all $u(\cdot) \in \mathcal{U}$, where

$$\alpha(t, u(\cdot)) \quad \triangleq \quad (\widehat{u}(t) - u(t)) \left[J_x'(x(t), y(t), \widetilde{S}(t), t)\xi_\varepsilon(y(t)) f_\varepsilon(S(t), t) + J_y'(x(t), y(t), \widetilde{S}(t), t) \right].$$

In other words, $\widehat{u}(t)$ is the process that delivers the maximum in the Bellman equation. Since the interval $[d_0, d_1]$ is compact, this process exists.

Let $u \in \mathcal{U}$, set

$$\widetilde{X}(t) = e^{-rT}\Big(J(x(t), y(t), \widetilde{S}(t), t) + \int_0^t \alpha(s, u(\cdot))ds \Big), \tag{A.19}$$

where $x(t)$ and $y(t)$ are defined by $u(t)$ according to (4.4).

By Itô formula applied to the process $\widetilde{X}(t)$ defined by (A.19), we have that

$$d\widetilde{X}(t) = e^{-rT}[J'_t(x(t),y(t),\widetilde{S}(t),t) + J'_x(x(t),y(t),\widetilde{S}(t),t)u(t)\xi_\varepsilon(y(t))f_\varepsilon(S(t),t) \\ +J'_y(x(t),y(t),\widetilde{S}(t),t)u(t) + \tfrac{1}{2}J''_{ss}(x(t),y(t),\widetilde{S}(t),t)\sigma^2\widetilde{S}(t)^2 + \alpha(t,u(\cdot)]dt \\ +e^{-rT}J'_s(x(t),y(t),\widetilde{S}(t),t)\sigma\widetilde{S}(t)dw_*(t).$$

By the definition of the process $\alpha(t,u(\cdot))$, this equation can be rewritten as

$$d\widetilde{X}(t) = e^{-rT}[J'_t(x(t),y(t),\widetilde{S}(t),t) + J'_x(x(t),y(t),\widetilde{S}(t),t)\widehat{u}(t)\xi_\varepsilon(y(t))f_\varepsilon(S(t),t) \\ +e^{-rT}J'_y(x(t),y(t),\widetilde{S}(t),t)\widehat{u}(t) + \tfrac{1}{2}J''_{ss}(x(t),y(t),\widetilde{S}(t),t)\sigma^2\widetilde{S}(t)^2]dt \\ +e^{-rT}J'_s(x(t),y(t),\widetilde{S}(t),t)\sigma\widetilde{S}(t)dw_*(t).$$

The selection of the function $J = J_\varepsilon$ ensures that the last equation can be rewritten as

$$d\widetilde{X}(t) = e^{-rT}J'_s(x(t),y(t),\widetilde{S}(t),t)\sigma\widetilde{S}(t)dw_*(t) = e^{-rT}J'_s(x(t),y(t),\widetilde{S}(t),t)d\widetilde{S}(t).$$

It follows that $\widetilde{X}(t)$ is the discounted wealth for the self-financing strategy such that the quantity of stock shares at time t is $e^{-rT}J'_s(x(t),y(t),\widetilde{S}(t),t)$ and that the initial wealth is $X(0) = e^{-rT}J(0,0,S(0),0)$. Since $\alpha(t,u(\cdot)) \geq 0$, it follows from (A.19) that (6.1) holds.

Suppose that there exists some other initial wealth $\bar{X}(0)$ such that

$$\bar{X}(0) < X(0) = e^{-rT}J_\varepsilon(0,0,S(0),0) = e^{-rT}\sup_{u(\cdot)\in\mathcal{U}}\mathbf{E}_*g_\varepsilon(x(T)) \tag{A.20}$$

and such that, for any $u \in \mathcal{U}$, there exists an admissible strategy such that, for the corresponding discounted wealth $\bar{X}(t)$,

$$\bar{X}(T) \geq e^{-rT}g_\varepsilon(x(T)) \quad \text{a.s.}. \tag{A.21}$$

Hence $\bar{X}(0) < e^{-rT}\sup_{u(\cdot)\in\mathcal{U}}\mathbf{E}_*g_\varepsilon(x(T))$. Therefore, there exists $\bar{u} \in \mathcal{U}$ such that $\bar{X}(0) < e^{-rT}\mathbf{E}_*g_\varepsilon(x(T))$, where $x(t)$ is defined by this $\bar{u}$ according to (4.4). By (A.21), $\mathbf{E}_*\bar{X}(T) = \bar{X}(0) \geq e^{-rT}\mathbf{E}_*g_\varepsilon(x(T))$. It follows that (A.20) does not hold. This completes the proof of Theorem 6.1(i). The proof for statements (ii)-(iii) is similar and will be omitted. □

Proof of Corollary 6.1. Let the payoffs F_u^ε be defined under the assumptions of Theorem 6.1(i)-(iii) respectively. In particular, F_u^ε is defined by (A.3) under the assumptions of Theorem 6.1(i), and F_u^ε is defined by (A.13) under the assumptions Theorem 6.1(iii). Let $\delta > 0$ be given.

It follows from the definitions that $F_u \le F_u^\varepsilon + c\varepsilon$ for some constant $c > 0$ that is independent from ε. It follows from (4.6) and Theorem 6.1(i), from (4.14) and Theorem 6.1(ii), and from (5.3) and Theorem 6.1(iii), that the initial wealth $X_\varepsilon(0) = e^{-rT} \sup_u \mathbf{E}_* F_u^\varepsilon + \delta$ ensures superreplication of the claim $F_u^\varepsilon + e^{rT}\delta$ for any u, i.e., $X_\varepsilon(T) \ge \mathbf{E}_* F_u^\varepsilon + e^{rT}\delta$ for an appropriate choice of strategies for the option writer. Therefore, $X_\varepsilon(T) \ge e^{-rT} \sup_u \mathbf{E}_* F_u - c\varepsilon + e^{rT}\delta$. This leads to an arbitrage if $\varepsilon < e^{rT}\delta/c$. □

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