A unified approach to the finite-horizon LQ regulator for discrete-time systems*

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Abstract

A closed-form expression parametrizing the solutions of the extended symplectic difference equation over a finite time interval is given under the mild assumption of modulus-controllability. This representation is expressed in terms of the strongly unmixed solution of a discrete ARE and of an algebraic Stein equation, and is exploited to derive a closed-form solution of a generalized version of the finite-horizon LQ problem.

1 Introduction

The extended symplectic system, arising in the solution of the discrete-time linear quadratic (LQ) regulator can be seen as the counterpart of the Hamiltonian system in the discrete time, and represents a set of necessary conditions for the optimality of a state trajectory and control law in several finite and infinite horizon optimization problems. The growing interest on the extended symplectic system is essentially due to the fact that it enables the conditions for optimality of LQ problems to be expressed in compact form even when the matrix weighting the control in the quadratic cost function and the system matrix are not invertible. On the contrary, other approaches aiming at deriving a direct discrete-time counterpart of the Hamiltonian differential equation by decoupling the control function

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from the state and costate in the Pontryagin equations *do* require the inversion of such matrices, see [12] and [10].
As such, in this paper we will establish the discrete-time version of the material presented in [5] in order to solve the finite-horizon linear quadratic optimal control problem with the most general form of affine constraints on the states at the end-points and with a perfomance index involving an arbitrary quadratic form of such states. However, since the theoretical and computational tools currently employed to study the properties of the extended symplectic system and the corresponding extended symplectic pencil are based on the general matrix pencil theory, the notions and techniques employed herein are considerably different from those exploited in [5].
As a first step, the set of solutions of the extended symplectic system over a finite-horizon time interval is parametrized in closed-form, leading to an expression involving a pair of n-dimensional deflating subspaces of the extended symplectic pencil. Under the weak assumption of modulus-controllability of the system – which is the discrete-time counterpart of the sign-controllability for continuous-time systems – these deflating subspaces can be expressed in a simple and easily computable form, which is particularly suitable for a software implementation.
The most important application of this expression is the finite-horizon LQ regulator in a framework which enables different kind of boundary conditions to be treated in closed-form, without resorting to the Riccati difference equation or reachability gramians for the computation of the optimal control function. Indeed, as for the continuous-time case, we will consider affine two-point boundary-value constraints on the state; moreover, since the initial state is not sharply assigned as in the standard version of the LQ problem, it can be quadratically penalized in the quadratic cost along with the terminal state, with respect to two assigned target states. In a manner similar to the continuous-time case, the boundary conditions can be imposed to the formula parametrizing the solutions of the extended symplectic system in order to determine the values of the parameter achieving optimality in finite terms, thus enabling the optimal state trajectory and the optimal control law to be analytically derived.

Nomenclature and Notation. Given a square matrix A, the symbol $\sigma(A)$ denotes the spectrum of A, and $\sigma^{-1}(A)$ stands for the set of reciprocal elements of $\sigma(A)$. More precisely, $\sigma^{-1}(A) \triangleq \{\lambda \in \mathbb{C} \,:\, \lambda^{-1} \in \sigma(A)\}$ if $\det A \neq 0$ and $\sigma^{-1}(A) \triangleq \{\lambda \in \mathbb{C} \,:\, \lambda^{-1} \in \sigma(A)\} \cup \{\infty\}$ if $\det A = 0$.
Given two matrices M and N of the same size, the matrix pencil $\lambda M - N$ [8, 6, 11] is said to be regular if M and N are square and $\det(\lambda M - N)$ is not identically zero. In this case, we denote by $\sigma(\lambda M - N)$ the set of *generalized eigenvalues* of $\lambda M - N$, i.e., $\sigma(\lambda M - N) \triangleq \{\lambda \in \mathbb{C} \,:\, \det(\lambda M - N) = 0\}$ if $\det(M) \neq 0$ and $\sigma(\lambda M - N) \triangleq \{\lambda \in \mathbb{C} \,:$

$\det(\lambda\, M-N)=0\} \cup \{\infty\}$ if $\det M=0$.

2 Statement of the problem

Consider the linear time-invariant discrete-time system governed by the state difference equation

$$x(t+1)=A\,x(t)+B\,u(t), \tag{1}$$

where, for all $t \in \mathbb{N}$, $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input, $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$. Let $N\in\mathbb{N}\setminus\{0\}$ be the length of the time horizon. Let $V_0, V_N\in\mathbb{R}^{s\times n}$ be such that $V \triangleq \left[\begin{array}{cc} V_0 & V_N \end{array}\right]$ is of full row rank and let $v\in\mathbb{R}^s$; consider the two-point boundary-value affine constraint

$$V_0\,x(0)+V_N\,x(N)=v. \tag{2}$$

In the case where $s=0$, the initial and terminal states are not constrained by (2).

Let $\Pi=\Pi^\top\geq 0$ be a square $(n+m)$-dimensional matrix partitioned as

$$\Pi=\left[\begin{array}{cc} Q & S \\ S^\top & R \end{array}\right],$$

with $Q\in\mathbb{R}^{n\times n}$, $S\in\mathbb{R}^{n\times m}$ and $R\in\mathbb{R}^{m\times m}$. Interestingly, here we do not assume the non-singularity of R, so that in the present context regular, singular and cheap problems are treated in a unified framework. We concisely denote by Σ the Popov triple (A,B,Π). Moreover, let

$$\Theta=\left[\begin{array}{cc} \Theta_1 & \Theta_2 \\ \Theta_2^\top & \Theta_3 \end{array}\right]=\Theta^\top\geq 0 \tag{3}$$

with $\Theta_1,\Theta_2,\Theta_3\in\mathbb{R}^{n\times n}$. Finally, consider the quadratic cost function

$$\begin{aligned} J(x,u) \triangleq& \sum_{t=0}^{N-1} \left[\begin{array}{cc} x^\top(t) & u^\top(t) \end{array}\right] \Pi \left[\begin{array}{c} x(t) \\ u(t) \end{array}\right] \\ &+\left[\begin{array}{cc} x^\top(0)-\theta_0^\top & x^\top(N)-\theta_N^\top \end{array}\right] \Theta \left[\begin{array}{c} x(0)-\theta_0 \\ x(N)-\theta_N \end{array}\right], \end{aligned} \tag{4}$$

where $\theta_0,\theta_N\in\mathbb{R}^n$. The optimal control problem dealt with herein can be stated as follows.

Problem 1 *Find a control function $u(t)$, $t\in\{0,\ldots,N-1\}$ and a state trajectory $x(t)$, $t\in\{0,\ldots,N\}$, minimizing the quadratic performance index $J(x,u)$ under the constraints (1-2).*

This formulation encompasses the standard case of assigned initial state and weighted terminal state, the fixed end-point case, where both the initial and terminal states are sharply assigned, and the point-to-point case, where the values at the end-points of an output $y(t) = C\,x(t)$ associated with (1) are assigned. In general, the existence of a state trajectory $x(t)$ satisfying the constraints (1-2) for some $u(t)$, $t \in \{0, \dots, N-1\}$, is not ensured, since we have not assumed reachability on (1). However, when a state and input functions satisfying (1-2) exist, the discrete counterpart of the Pontryagin equations presented in Lemma 1 in [5] takes the following form:

$$x(t+1) = A\,x(t) + B\,u(t), \tag{5}$$

$$V \begin{bmatrix} x(0) \\ x(N) \end{bmatrix} = v, \tag{6}$$

$$\lambda(t) = Q\,x(t) + A^\top \lambda(t+1) + S\,u(t), \tag{7}$$

$$\begin{bmatrix} -\lambda(0) \\ \lambda(N) \end{bmatrix} = \Theta \begin{bmatrix} x(0) - \theta_0 \\ x(N) - \theta_N \end{bmatrix} + V^\top \eta, \tag{8}$$

$$0 = S^\top x(t) + B^\top \lambda(t+1) + R\,u(t), \tag{9}$$

This set of equations represents a necessary and sufficient condition for an optimum, since the functional to be minimized is convex and all the constraints are expressed by linear algebraic equations. Hence we have the following.

Lemma 1 *If $u(t)$ and $x(t)$ are optimal for Problem 1, then $\lambda(t) \in \mathbb{R}^n$, $t \in \{0, \dots, N\}$ and $\eta \in \mathbb{R}^s$ exist such that $x(t)$, $\lambda(t)$, $u(t)$ and η satisfy the set of equations (5-9) for all $t \in \{0, \dots, N-1\}$. Conversely, if equations (5-9) admit solutions $x(t)$, $u(t)$, $\lambda(t)$, η, then $x(t)$, $u(t)$ minimize $J(x,u)$ subject to the constraints (1-2).*

The variables $\lambda(t)$ in (5-9) represent for all $t \in \{0, \dots, N\}$ the Lagrange multipliers associated with the constraint (1), [8, 6], while the variable $\eta \in \mathbb{R}^s$ is the Lagrange multiplier vector associated with the constraint (2).

3 The extended symplectic system

Unlike the continuous time, the *trasversality* condition (9) cannot be solved in $u(t)$ so as to lead to a set of $2n$ equations in the sole variables $x(t)$ and $\lambda(t)$, since here R may be singular. However, a very convenient form in which equations (5), (7) and (9) can be written is the descriptor form

$$F\,p(t+1) = G\,p(t) \qquad t \in \{0, \dots, T-1\}, \tag{10}$$

where

$$F \triangleq \begin{bmatrix} I & 0 & 0 \\ 0 & -A^\top & 0 \\ 0 & -B^\top & 0 \end{bmatrix}, \quad G \triangleq \begin{bmatrix} A & 0 & B \\ Q & -I & S \\ S^\top & 0 & R \end{bmatrix}, \quad p(t) \triangleq \begin{bmatrix} x(t) \\ \lambda(t) \\ u(t) \end{bmatrix}.$$

Equation (10) is called the extended symplectic system, and the associated matrix pencil $z\,F-G$ is known as the extended symplectic pencil, [8, 6]. If $z\,F-G$ is regular, equation (10) has $2n+m$ linearly independent solutions, spanning a $(2n+m)$-dimensional space, [3], [9]. Notice however that, since $u(t)$ is irrelevant both to the value of the cost function and to the satisfaction of the constraints (1-2), we may regard as equivalent two solutions $p(t)'$ and $p(t)''$ of (10) if they only differ for the value of $u(t)$. On the other hand, a solution $p(t)$ of (10) remains such when $u(t)$ is replaced with an arbitrary $\bar{u} \in \mathbb{R}^m$. Therefore, the dimension of the space of non-equivalent solutions of (10) equals $2n$. From now on, we will set as a representative element of each equivalence class of the solutions of (10) the vector $p(t)$ such that the corresponding $u(t)$ is zero.

In [4], it has been shown that under the assumption of reachability of the pair (A,B) a closed-form formula parametrizing the solutions of the extended symplectic system (10) can be exploited to derive expressions for the optimal control law and state trajectory, in terms of the stabilizing solution of a discrete-time algebraic Riccati equation and of the solution of an algebraic Stein equation, without the need of resorting to the iteration of the Riccati difference equation. The purpose of this work is that of extending the results presented in [4] in twofold directions: as already seen, here the problem statement is more general than that of [4]. Moreover, concerning the underlying assumptions, here we introduce a very weak form of controllability, which is the counterpart of sign-controllability for discrete-time systems, [1]:

Definition 1 *The pair (A,B) in (1) is said to be* modulus-controllable *if the set of uncontrollable eigenvalues of A does not contain pairs of elements in the form $(\lambda, \bar{\lambda}^{-1})$.*

It is easily seen that Assumption (A1) is weaker than that of reachability and even than that of stabilizability of the pair (A,B). Here we only assume that:

(A1) the pair $(A,\,B)$ is *modulus-controllable*;

(A2) the pencil $z\,F-G$ is regular and has no generalized eigenvalues on the unit circle.

4 Main results

Consider the discrete algebraic Riccati equation DARE(Σ)

$$P = A^\top P\,A - (A^\top P\,B + S)\,(R + B^\top P\,B)^{-1}(S^\top + B^\top P\,A) + Q.$$

To any solution $P = P^\top \in \mathbb{R}^{n \times n}$ of DARE(Σ) there corresponds the closed-loop matrix

$$A_P \triangleq A - B\,K_P, \quad K_P \triangleq (R + B^\top P\,B)^{-1}(S^\top + B^\top P\,A). \tag{11}$$

Definition 2 *The solution $P = P^\top$ of DARE(Σ) is said to be unmixed if the corresponding closed-loop matrix A_P is such that $\lambda, \bar{\lambda}^{-1} \in \sigma(A_P)$ implies $|\lambda| = 1$.*

The concept of unmixed solution of DARE(Σ) was first introduced in [2], while in [14] and [1] the existence of unmixed solutions of DARE(Σ) is discussed. However, in this paper, we need the following more stringent definition.

Definition 3 *The solution P of DARE(Σ) is strongly unmixed if the spectrum of the corresponding closed-loop matrix A_P is such that $\lambda \in \sigma(A_P)$ implies $\bar{\lambda}^{-1} \notin \sigma(A_P)$.*

Notice that, since A_P is a real matrix, this is equivalent to the fact that $\sigma(A_P)$ does not contain reciprocal pairs. Clearly, if P is unmixed and none of the eigenvalues of A_P lie in the unit circle, then P is strongly unmixed. Thus, under assumption (A2), if P is unmixed, then it is also strongly unmixed because, as shown e.g. in [6, p.175], $\sigma(A_P) \subseteq \sigma(zF - G)$.

The following lemma, that may be viewed as a corollary of Theorem 1.1. in [1], will be of crucial importance in the following.

Lemma 2 *Under assumptions (A1)-(A2), DARE(Σ) admits a strongly unmixed solution $P = P^\top$.*

Proof: The existence of an unmixed solution $P = P^\top$ is proved in [1, Theorem 1.1] under the assumption that the *Popov function*

$$\Psi(z) \triangleq \begin{bmatrix} B^\top (z^{-1} I - A^\top)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} (z\,I - A)^{-1} B \\ I \end{bmatrix}$$

is positive semidefinite for all the values $\bar{z}$ on the unit circle such that $\Psi(\bar{z})$ is well defined, and positive definite for at least one value of z on the unit circle. The first of such conditions is clearly satisfied in the present setting, since Π is assumed to be positive semidefinite. Concerning the second condition, it is shown in [6, Theorem 5.4.1] that it is a consequence of the regularity of $zF - G$. Hence, in view of [1, Theorem 1.1], an unmixed solution $P = P^\top$ of DARE(Σ) exists. As already observed, by virtue of assumption (A2) it is found that P is also strongly unmixed. ■

Lemma 3 *Let assumptions (A1)-(A2) hold. Let $P \in \mathbb{R}^{n \times n}$ be a solution of DARE(Σ), and let A_P be corresponding closed-loop matrix. The symmetric Stein equation*

$$A_P\,Y\,A_P^\top - Y + B\,(R + B^\top P\,B)^{-1} B^\top = 0 \tag{12}$$

admits a unique solution $Y = Y^\top \in \mathbb{R}^{n \times n}$ if and only if P is strongly unmixed.

Proof: Since P is strongly unmixed, for any $\lambda_1, \lambda_2 \in \sigma(A_P)$ we have $\lambda_1 \lambda_2 \neq 1$, hence (12) admits a unique solution (see e.g. [8, p.100], [6, p.10]). ■

In the following theorem it is shown how the set of representatives of the solutions of the extended symplectic system (10) can be parametrized in closed-form in terms of a $2n$-dimensional vector π.

Theorem 1 *Let assumptions (A1)-(A2) hold. Let P be a strongly unmixed solution of DARE(Σ), A_P be given by (11) and let Y be the corresponding solution of the Stein equation (12). The set of representatives of the solutions of the extended symplectic system (10) is parametrized in terms of $\pi \in \mathbb{R}^{2n}$ as*

$$p(t) = \begin{cases} \begin{bmatrix} V_1\, A_P^t & V_2\, (A_P^\top)^{T-t-1} \end{bmatrix} \pi & 0 \le t \le T-1 \\ \\ \begin{bmatrix} V_1'\, A_P^T & V_2' \end{bmatrix} \pi & t = N \end{cases} \tag{13}$$

where

$$V_1 \triangleq \begin{bmatrix} I \\ P \\ -K_P \end{bmatrix} \qquad \text{and} \qquad V_2 \triangleq \begin{bmatrix} Y\, A_P^\top \\ (P\,Y - I) A_P^\top \\ -K_\star \end{bmatrix} \tag{14}$$

with $K_\star \triangleq K_P\, Y\, A_P^\top - (R + B^\top P\, B)^{-1} B^\top$ and

$$V_1' \triangleq \begin{bmatrix} I \\ P \\ 0 \end{bmatrix} \qquad \text{and} \qquad V_2' \triangleq \begin{bmatrix} Y \\ P\,Y - I \\ 0 \end{bmatrix}$$

Proof: First, we prove that (13) satisfies the extended symplectic system (10). To this end, we show that $\mathcal{V}_1 \triangleq \operatorname{im} V_1$ and $\mathcal{V}_2 \triangleq \operatorname{im} V_2$ are deflating subspaces of the matrix pencil $zF - G$, and precisely that

$$F\, V_1\, A_P = G\, V_1, \tag{15}$$

$$F\, V_2 = G\, V_2\, A_P^\top. \tag{16}$$

Consider (15). By (11), $A_P = A - B\, K_P$. Hence, the first block-row equation of (15) holds. Moreover, DARE(Σ) can be written as $A^\top P\, A - P - (A^\top P\, B + S)\, K_P + Q = 0$, that leads to the second row-block equation of (15). Finally, it can be easily seen that by the definition of K_P the third block-row of (15) follows immediately. To prove (16), consider the Stein equation (12). From $A_P = A - B\, K_P$, it follows that $Y = A\, Y\, A_P^\top + B\, K_\star$. By postmultiplying the former by $A_P^\top$, we get the first block-row equation of (16). Now, by developing the products in the right-hand side of the identity $K_P^\top = (A^\top P\, B + S)\, (R + B^\top P\, B)^{-1}$ and by postmultiplying by $B^\top$ the expression thus obtained, it is found that $-A^\top P\, B\, (R+$

$B^\top P B)^{-1} B^\top = -K_P^\top B^\top + S\,(R + B^\top P B)^{-1} B^\top$. By adding and subtracting $A^\top$ in the right-hand side of the former and by using (12) in the left-hand side, we obtain $-A^\top P\,(Y - A_P\,Y\,A_P^\top) = -A^\top + A_P^\top + S(R + B^\top P B)^{-1} B^\top$. By (11), we get

$$
\begin{aligned}
-A^\top P\,Y + A^\top = {} & A^\top P\,B(R + B^\top P\,B)^{-1} B^\top P\,A\,Y\,A_P^\top - A^\top P\,A\,Y\,A_P^\top \\
& + A^\top PB(R + B^\top P\,B)^{-1} S^\top\,Y\,A_P^\top \\
& + A_P^\top + S(R + B^\top P\,B)^{-1} B^\top.
\end{aligned}
$$

By employing DARE(Σ), one can reduce the first three terms of the right side of the latter and, by postmultiplying the expression thus obtained by $A_P^\top$ and by taking into account the definitions of K_P and $K_\star$, the second block-row of (16) follows. Finally, consider the trivial identity $-(R + B^\top P\,B)(R + B^\top P\,B)^{-1} B^\top + B^\top = 0$. By using (12) we get $-B^\top P\,(Y - A_P\,Y\,A_P^\top) + B^\top - R\,(R + B^\top P\,B)^{-1} B^\top = 0$, that, by virtue of the third block-row of (15), leads to

$$
-B^\top P\,Y + B^\top = S^\top\,Y\,A_P^\top - R\,K_P\,Y\,A_P^\top + R\,(R + B^\top P\,B)^{-1} B^\top.
$$

By postmultiplying the former by $A_P^\top$ one obtains the third block-row equation of (16) in view of the definition of $K_\star$.
We are now ready to show that (13) are indeed solutions of the extended symplectic system (10) for all $\pi \in \mathbb{R}^{2n}$. First, let $t \in \{0, \ldots, T-2\}$. By substitution, it is seen that (13) satisfies (10) in view of (15-16). Let now $t = T-1$. Define $\bar{V}_1$ and $\bar{V}_2$ as in (14), except for their third block-row $m \times n$ submatrix, which is zero. Since the third block-column of F is zero, from (15) and (16) it follows that $F\,\bar{V}_1\,A_P = G\,V_1$ and $F\,\bar{V}_2 = G\,V_2\,A_P^\top$, so that, since $\bar{V}_2 = V_2' A_P^\top$ and $\bar{V}_1 = V_1'$, it follows that (13) satisfies (10) also for $t = T-1$, and hence for all $t \in \{0, \ldots, T-1\}$.

Conversely, we prove that all the solutions of the extended symplectic system (10) can be expressed by means of (13). To this purpose, we show that (13) has $2n$ linearly independent trajectories. In the first part of the proof we have shown that $\mathcal{V}_1$ and $\mathcal{V}_2$ are two deflating subspaces of $zF - G$, hence their intersection $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$ is such in view of Lemma 4. On the other hand, by (15) and (16) it follows that $\sigma(z\,F - G, \mathcal{V}_1) = \sigma(A_P)$ and $\sigma(z\,F - G, \mathcal{V}_2) = \sigma^{-1}(A_P)$, hence $\sigma(z\,F - G, \mathcal{V}_1)$ and $\sigma(z\,F - G, \mathcal{V}_2)$ are disjoint since P is strongly unmixed. Moreover, by virtue of Lemma 4, $\sigma(z\,F - G, \mathcal{V}) \subseteq \sigma(z\,F - G, \mathcal{V}_1) \cap \sigma(z\,F - G, \mathcal{V}_2)$. As a consequence, $\mathcal{V} = \{0\}$. Hence, for any given $\pi = \left[\begin{array}{cc} p^\top & q^\top \end{array}\right]$ with $p, q \in \mathbb{R}^n$, the two trajectories $V_1\,A_P^t\,p$ and $V_2\,(A_P^\top)^{T-t-1}\,q$ are linearly independent. Therefore, the dimension of the linear space of trajectories in (13) is given by the sum of the dimensions n_1 and n_2 of the subspaces of trajectories of (13) corresponding to $p = 0$ and to $q = 0$, respectively. By setting $t = 0$ and $t = T$, it is easily seen that $n_1 = n$ and $n_2 = n$, respectively. We may conclude that (13) admits $2n$ linearly independent solutions. Since the dimension of the linear space of non-equivalent solutions of (10) is exactly $2\,n$ as observed in Section 3, it follows that (13) parametrizes the complete set of solutions of the extended symplectic system (10) with $u(t)=0$. ■

Remark 1 *As in the continuous-time case, in the case where the pair (A,B) is stabilizable, in Theorem 1 we can choose as strongly unmixed solution of DARE(Σ) the stabilizing solution $P_+ = P_+^\top \geq 0$, such that the closed-loop matrix $A_{P_+} \triangleq A_+$ is strictly stable. In this case, the expressions of the optimal state and input functions are given in terms of powers of matrices, A_+ and $A_+^\top$, that are strictly stable in the overall time-interval, thus ensuring that such solution is numerically robust even for large time horizons.*

5 Derivation of the optimal solution

In Theorem 1 it has been shown that under assumptions (A1)-(A2) a strongly unmixed solution P of DARE(Σ) exists and yields an explicit formula parametrizing all the solutions of the extended symplectic system (10) in terms of π. In this section it is shown how, from this set of trajectories, one can select those, if any, that satisfy the boundary conditions. These trajectories are optimal for Problem 1, since they satisfy all the necessary and sufficient conditions for optimality (5-9), as pointed out in Section 2.
The following result provides the complete solution of Problem 1.

Theorem 2 *Let K_V be a basis matrix*[1] *of the null-space of V. Moreover, let $\widehat{P} \triangleq \mathrm{diag}\,(-P,P)$, $\theta \triangleq \begin{bmatrix} \theta_0 \\ \theta(t) \end{bmatrix}$, $L \triangleq \begin{bmatrix} I_n & Y\,(A_P^T)^\top \\ A_P^T & Y \end{bmatrix}$, $U \triangleq \begin{bmatrix} 0 & -(A_P^T)^\top \\ 0 & I_n \end{bmatrix}$ and*

$$N \triangleq \begin{bmatrix} V\,L \\ K_V^\top\,[\,(\widehat{P}-\Theta)\,L - U\,] \end{bmatrix}, \quad w \triangleq \begin{bmatrix} v \\ -K_V^\top\,\Theta\,\theta \end{bmatrix}. \tag{17}$$

Problem 1 admits solutions if and only if $w \in im\,N$.
If this is the case, let K_N be a basis matrix of the null-space of N, and define

$$\mathcal{P} \triangleq \{\pi = N^\dagger\,w + K_N\,\zeta \;:\; \zeta \text{ arbitrary}\,\}, \tag{18}$$

where $N^\dagger$ denotes the Moore-Penrose pseudoinverse of N. Then, the set of optimal solutions of Problem 1 is parametrized by

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} A_P^t & Y\,(A_P^\top)^{T-t} \\ -K_P A_P^t - K_\star\,(A_P^\top)^{T-t-1} \end{bmatrix} \pi, & 0 \leq t \leq T-1, \\[2ex] \begin{bmatrix} A_P^T & Y \\ 0 & 0 \end{bmatrix} \pi, & t = T, \end{cases} \tag{19}$$

for $\pi \in \mathcal{P}$.

[1] In the case when $ker\,V = \{0\}$, we consider K_V to be void.

Theorem 2 can be proved exactly in the same manner of its continuous-time counterpart Theorem 2 in [5], where however F is now defined as $F \triangleq A_P^T$. Notice that if $s = 2n$, the constraint represented by (2) completely assigns the initial and terminal states, and the second block-row of N and w disappear.

5.1 Optimal cost

The following proposition presents a simple formula for the computation of the optimal cost, as a quadratic form in the problem data. As observed in [5], such expression is very useful when dealing with more complex possibly parametric problems having the finite-horizon LQ as a subproblem.

Theorem 3 *Consider the matrices $\widehat{P}$, L, U, θ, N and w defined in Theorem 2, and let $\widehat{Y} \triangleq \operatorname{diag}(-Y, Y)$. Define*

$$\Upsilon \triangleq \begin{bmatrix} N^{\dagger\top} [L^\top (\Theta - \widehat{P})\, L + U^\top \widehat{Y} U]\, N^\dagger & -N^{\dagger\top} L^\top \Theta \\ -\Theta^\top L\, N^\dagger & \Theta \end{bmatrix}$$

and

$$G \triangleq \begin{bmatrix} I_s & 0 \\ 0 & -K_V^\top \Theta \\ 0 & I_{2n} \end{bmatrix}.$$

If Problem 1 has solutions, the optimal value $J^\star$ of the functional $J(x,u)$ is given by the quadratic form

$$J^\star = \begin{bmatrix} v^\top & \theta^\top \end{bmatrix} G^\top \Upsilon G \begin{bmatrix} v \\ \theta \end{bmatrix}. \tag{20}$$

Proof: By taking (7) and (9) into account, the value of the function

$$c\,(x,u) \triangleq \sum_{t=0}^{T-1} \begin{bmatrix} x^\top(t) & u^\top(t) \end{bmatrix} \Pi \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

corresponding to the optimal solution can be written as

$$\begin{aligned} c\,(x,u) &= \sum_{t=0}^{T-1} x^\top(t)\, Q\, x(t) + u^\top(t)\, R\, u(t) + 2\, x^\top(t)\, S\, u(t) \\ &= \sum_{t=0}^{T-1} x^\top(t) \Big(\lambda(t) - A^\top \lambda(t+1) - S\, u(t) \Big) \\ &\quad + u^\top(t) \Big(- S^\top x(t) - B^\top \lambda(t+1) \Big). \end{aligned}$$

Now, by virtue of (1), it follows that

$$\begin{aligned} c\left(x,u\right) &= \sum_{t=0}^{T-1} x^\top(t)\,\lambda(t) - x^\top(t)\,A^\top\,\lambda(t+1) - \left(x^\top(t+1) - x^\top(t)\,A^\top\right)\lambda(t+1) \\ &= \sum_{t=0}^{T-1} x^\top(t)\,\lambda(t) - x^\top(t+1)\,\lambda(t+1) = x_0^\top\,\lambda_0 - x^\top(t)\,\lambda(t). \end{aligned}$$

Such expression can be substituted in the optimal cost, thus leading to (20) as shown in Theorem 1 in [5]. ■

6 Existence and uniqueness of an optimum

In the discrete case, since the optimal control problem formulated in Section 2 involves a finite number of variables – precisely, $l = m{\cdot}T$ variables for the control plus n variables for the initial state – Problem 1 can be restated as a quadratic static optimization problem in these $l+n$ variables with linear constraints. Thus, it is clear that a solution to Problem 1 exists if and only if an admissible solution exists.

Theorem 4 *Problem 1 admits solutions if and only if a trajectory of (1) exists satisfying the constraint (2), i.e., if and only if $v \in \operatorname{im} V\,L$.*

Again, an equivalent condition for the existence of an optimum can be found which can be tested without the need of the computation of the DARE(Σ).

Corollary 1 *Let R_0 be a basis matrix of the reachable subspace from the origin. Problem 1 admits solutions if and only if $v \in \operatorname{im} V\,Z$, where*

$$Z \triangleq \begin{bmatrix} I_n & 0 \\ A^T & R_0 \end{bmatrix}$$

The proof of Corollary 1 follows with the obvious changes from the corresponding proof of Corollary 1 in [5]. The following theorem provides a necessary and sufficient condition for the uniqueness of the optimal solution depending only on the problem data; hence, it can be tested without the need of solving the algebraic Riccati equation.

Theorem 5

Proof: (If). We assume with no loss of generality that that $\begin{bmatrix} B & R \end{bmatrix}$ is full-column rank. Indeed, if $\begin{bmatrix} B & R \end{bmatrix}$ has non-trivial kernel, there exists a subspace $\mathcal{U}_0$ of the input space that does not influence the state dynamics and the cost function. Then, by performing a suitable (orthogonal) change of basis in the input space, we may eliminate $\mathcal{U}_0$ and obtain an equivalent problem for which this condition is satisfied.
Let $x'(t), u'(t)$ and $x''(t), u''(t)$ be two different solutions of Problem 1. They both satisfy

(5)-(9) for suitable $\lambda'(t)$, η', $\lambda''(t)$, η'', respectively. A straightforward computation over (5)-(9) shows that their difference $x(t) \triangleq x'(t) - x''(t), u(t) \triangleq u'(t) - u''(t)$ satisfies (5)-(9) with $\lambda(t) \triangleq \lambda'(t) - \lambda''(t)$, $\nu := \nu' - \nu''$, $\theta_0 = \theta_N = 0$, $v = 0$, hence it is the optimal solution of the optimal control problem consisting of the minimization of $J(x,u)$ with $\theta_0 = \theta_N = 0$, under the constraints (1) and $V \begin{bmatrix} x(0) \\ x(N) \end{bmatrix} = 0$. For this problem, the identically zero state and input functions are optimal, since they satisfy all the constraints and the corresponding cost is zero. Hence, $x(t)$ and $u(t)$ are not identically zero but the corresponding cost is zero. In other words, Problem 1 admits more than one solution if and only if there exists a non-identically zero solution with zero cost of the problem

$$\min_{x,u} J(x,u) = \sum_{t=0}^{N-1} \begin{bmatrix} x^\top(t) & u^\top(t) \end{bmatrix} \Pi \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} x^\top(0) & x^\top(N) \end{bmatrix} \Theta \begin{bmatrix} x(0) \\ x(N) \end{bmatrix},$$

subject to

$$x(t+1) = Ax(t) + Bu(t) \qquad t \in [0, N-1]$$

$$V \begin{bmatrix} x(0) \\ x(N) \end{bmatrix}$$

By the positive semidefiniteness of Π and Θ it follows that the solutions of this problem leading to zero cost must satisfy

$$\Pi \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad \forall t \in [0, N-1] \tag{21}$$

and

$$\Psi \begin{bmatrix} x(0) \\ x(N) \end{bmatrix} = 0, \qquad \text{where} \quad \Psi \triangleq \begin{bmatrix} V \\ \Theta \end{bmatrix}. \tag{22}$$

In view of the positive semidefiniteness of Π, it follows that $\ker S \supseteq \ker R$, or, equivalently, $\operatorname{im} S^\top \subseteq \operatorname{im} R$, so that from (21) it is easily found that

$$u(t) = -R^\dagger S^\top x(t) + R_\perp v(t), \tag{23}$$

where $R_\perp \triangleq R^\dagger R - I_m$ is a basis matrix of the null-space of R and $v(t)$ is arbitrary. Let $\tilde{A} := A + BR^\dagger S^\top$, $\tilde{B} := BR_\perp$, and $\tilde{Q} := Q - SR^\dagger S^\top$. Notice that $\tilde{B}$ is full column rank: in fact, the injectivity of $\begin{bmatrix} B \\ R \end{bmatrix}$ is equivalent to the condition $\ker B \cap \ker R = 0_m$, which in turn is equivalent to the fact that $\tilde{B} := B\, R_\perp$ is full column rank.

Now, we easily see that $x(t)$ must satisfy

$$\begin{cases} x(t+1) = \tilde{A}\, x(t) + \tilde{B}\, v(t) \\ \tilde{Q}\, x(t) = 0 \end{cases} \tag{24}$$

It is a well-known fact that the trajectories $x(t)$ solving (24) lie entirely on the output-nulling subspace $\tilde{\mathcal{V}}^\star$ of the triple $(\tilde{A}, \tilde{B}, \tilde{Q})$, i.e., on the largest $(\tilde{A}, \tilde{B})$-controlled invariant subspace contained in the null-space of $\tilde{Q}$. As a result, there exists more than one solution to Problem 1 if and only if there exists a non-identically zero solution to the problem

$$x(t) \in \tilde{\mathcal{V}}^\star \qquad \forall t \in [0, N-1]$$

and

$$\begin{bmatrix} x(0) \\ x(N) \end{bmatrix} \in \ker \Psi$$

Assume that such $x(t)$ does not exist. Let $\tilde{\mathcal{R}}^\star$ be the set of reachable states for the system $x(t+1) = \tilde{A}\, x(t) + \tilde{B}\, v(t)$ with the constraint that $x(t) \in \tilde{\mathcal{V}}^\star$ for all $t \in [0, N-1]$. Let $\bar{x} \in \tilde{\mathcal{R}}^\star$, then there exists a trajectory in $\tilde{\mathcal{V}}^\star$ with initial state $x(0) = 0$ with $x(n) = \bar{x}$ and $x(t) = 0$ for all $t \geq 2n$. Assuming $T > 2n$, this means in particular that $x(N) = x(0) = 0$, so that this state trajectory satisfies the constraint (22); this implies that the cost of the trajectory is zero since this trajectory lies on $\tilde{\mathcal{V}}^\star$ and hence $\bar{x} = 0$ (because we are assuming that the only zero cost trajectory is the zero trajectory). Thus $\tilde{\mathcal{R}}^\star = \{0\}$. Therefore, only free evolutions are possible in $\tilde{\mathcal{V}}^\star$. Let $\tilde{F}$ be a friend corresponding to $\tilde{\mathcal{V}}^\star$ so that $\hat{A} := \tilde{A} + \tilde{B}\tilde{F}$ is the corresponding system matrix. For all the trajectories of zero cost we have $x(0) \in \tilde{\mathcal{V}}^\star$ so that $x(N-1) = \hat{A}^{N-1} x(0)$ and $x(N) \in \hat{A}^N x(0) \mathrm{im}\, \tilde{B}$, equivalently, if the cost is zero we have $x(0) = V\, v_1$ and $x(N) = \hat{A}^N V\, v_1 + \tilde{B}\, v_2$ for some v_1 and v_2, where V is a matrix whose columns span $\tilde{\mathcal{V}}^\star$. Therefore, if the cost is zero we also have

$$\Psi \begin{bmatrix} V & 0 \\ A^\top V & \tilde{B} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

and since we are assuming that the only such trajectory is the zero trajectory (and since $\tilde{B}$ is full column rank), this implies

$$\ker \left(\Psi \begin{bmatrix} V & 0 \\ A^\top V & \tilde{B} \end{bmatrix} \right) = 0. \tag{25}$$

Conversely, the same arguments in the backward direction show that the if (25) and $\tilde{\mathcal{R}}^\star = \{0\}$ imply that the only zero cost trajectory is the zero trajectory.

In conclusion the solution is unique if and only if (25) and $\tilde{\mathcal{R}}^\star = \{0\}$ hold.

7 An illustrative example

Let $T = 8$ be the length of the time-horizon and consider the pair (A, B) given by

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix},$$

which is clearly sign-controllable (but not reachable nor stabilizable), and a Popov matrix Π in which

$$Q=\begin{bmatrix}1&0&0\\0&4&0\\0&0&0\end{bmatrix},\qquad S=\begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix},\qquad R=\begin{bmatrix}1&0\\0&0\end{bmatrix},$$

so that $\Pi=\Pi^\top\geq 0$. The extended symplectic pencil has no generalized eigenvalues on the unit circle, as can be checked by direct inspection. Let $\theta_0=\begin{bmatrix}3&3&1\end{bmatrix}^\top$ and $\theta(t)=\begin{bmatrix}-3&-3&1\end{bmatrix}^\top$ be the target states. Let Θ be partitioned as in (3), where $\Theta_1=\mathrm{diag}(1,4,0)$, $\Theta_2=0$ and $\Theta_3=\mathrm{diag}(4,4,4)$. Moreover, consider the following constraints on the extreme states:

$$\begin{aligned}x_0^1&=1, & x^1(t)&=-1\\ x_0^2&=1 & x^2(t)&=-1\end{aligned}$$
$$x_0^3-x^3(t)=2.$$

By Corollary 1, in this case Problem 1 admits solutions since

$$V\,Z=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\-1&0&-615&-1&0\\0&-1&1230&0&1\\0&0&-1023&0&0\end{bmatrix}$$

is clearly invertible. It is easily seen that the matrices

$$P=\begin{bmatrix}1&0&0\\0&3&0\\0&0&0\end{bmatrix}\quad\text{and}\quad Y=\begin{bmatrix}\frac{1}{2}&0&0\\0&\frac{1}{3}&0\\0&0&0\end{bmatrix}$$

are a strongly unmixed solution of DARE(Σ), and the corresponding solution of the Stein equation, respectively. First, we compute the parameter π by means of the formula $N\,\pi=w$ in Theorem 2. In this case N is invertible, so that $\pi=N^{-1}\,w$. Straightforward computations yield the following expression for π:

$$\pi^\top=\begin{bmatrix}1&1&-\frac{2}{2^T-1}&-2&-3&\frac{4(1-3\cdot 2^T)}{(2^T-1)^2}\end{bmatrix}$$

Now, by using (19) the expression of the optimal state trajectory and control law can be determined in closed form. For what concerns the optimal state we get

$$\begin{aligned}x_0^1&=\pi_1, & x^1(t)&=0, & 1\leq t<T,&& x^1(t)&=\tfrac{1}{2}\pi_4\\ x_0^2&=\pi_2, & x^2(t)&=0, & 1\leq t<T,&& x^2(t)&=\tfrac{1}{3}\pi_5\\ x_0^3&=\pi_3, & x^3(t)&=2^t\pi_3, & 1\leq t<T,&& x^3(t)&=2^T\pi_3\end{aligned}$$

The optimal control law is

$$
\begin{aligned}
u_0^1 &= -\pi_2, & u^1(t) &= 0, & 1 \le t < T-1,\\
u_0^2 &= -\tfrac{1}{2}\pi_1 - \tfrac{3}{2}\pi_3, & u^2(t) &= -\tfrac{3}{2}\cdot 2^t \pi_3, & 1 \le t < T-1,
\end{aligned}
$$

and

$$
\begin{aligned}
u_{T-1}^1 &= -\tfrac{1}{2}\,\pi_4 & u^1(t) &= 0\\
u_{T-1}^2 &= -\tfrac{3}{2}\cdot 2^{T-1}\pi_3 + \tfrac{1}{6}\,\pi_5, & u^2(t) &= 0.
\end{aligned}
$$

Finally, the optimal cost is computed in closed-form as

$$
\begin{aligned}
J^\star &= 61 + 2^{18}\,\pi_3^2 - 2^{11}\,\pi_3\\
&= 61 + \frac{2^{20}}{(2^T-1)^2} + \frac{2^{12}}{2^T-1} \simeq 93.1885.
\end{aligned}
$$

8 Conclusion

In this work we have extended the approach presented in [4] to the case when the pair (A, B) is modulus-controllable, for the solution of the finite-horizon LQ problem. Moreover, the problem formulation proposed here has been extended with respect to that characterizing the classic LQ regulator so as to yield, as particular cases, many LQ problems of interest in practice, such as the fixed end-point and the point-to-point LQ. The advantages of this approach are both theoretical, since closed-form solutions have been provided to manifold LQ problems that cannot be solved with the existing tools of LQ theory, and computational, since the methodology developed here relies only on the solutions of algebraic equations.

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Appendix

A subspace $\mathcal{V}$ is said to be *deflating* [8, 6, 11] for the regular matrix pencil $\lambda M - N$ if there exists a subspace $\mathcal{T}$ such that

$$\dim \mathcal{T} = \dim \mathcal{V}, \quad M\,\mathcal{V} \subseteq \mathcal{T}, \quad N\,\mathcal{V} \subseteq \mathcal{T}, \tag{26}$$

or, equivalently, if [8, Proposition 1.6.4]

$$\dim (M\,\mathcal{V} + N\,\mathcal{V}) = \dim \mathcal{V}. \tag{27}$$

Clearly, if (27) holds, $\mathcal{T} \triangleq M\,\mathcal{V} + N\,\mathcal{V}$ satisfies (26). The following well-known result [8, Theorem 1.6.2] is recalled for the readers' convenience.

Proposition 1 *Given a deflating subspace $\mathcal{V}$ of the regular pencil $\lambda M - N$ and a subspace $\mathcal{T}$ such that (26) holds, two non-singular matrices $W = \left[\begin{array}{cc} W' & W'' \end{array}\right]$ and $Z = \left[\begin{array}{cc} Z' & Z'' \end{array}\right]$ with $\operatorname{im} Z' = \mathcal{V}$ and $\operatorname{im} W' = \mathcal{T}$ exist such that*

$$W^{-1}(\lambda M - N)\, Z = \left[\begin{array}{cc} \lambda M_{11} - N_{11} & \lambda M_{12} - N_{12} \\ 0 & \lambda M_{22} - N_{22} \end{array}\right]. \tag{28}$$

The converse is true as well: if (28) holds for some invertible matrices $W = \left[\begin{array}{cc} W' & W'' \end{array}\right]$ and $Z = \left[\begin{array}{cc} Z' & Z'' \end{array}\right]$ with Z' and W' having the same number of columns, then $\mathcal{V} \triangleq \operatorname{im} Z'$ is deflating for $\lambda M - N$ and $\mathcal{T} \triangleq \operatorname{im} W'$ is such that (26) holds.

It can be shown [8, p.23] that $\sigma(\lambda M_{11} - N_{11})$ does not depend on the matrices W and Z, but only on $\mathcal{V}$. Hence, we can then use the symbol $\sigma(\lambda M - N, \mathcal{V})$ to denote $\sigma(\lambda M_{11} - N_{11})$. As a consequence of Proposition 1, we present the following lemma, that will be useful in the sequel.

Lemma 4 *Let $\mathcal{V}_1$, $\mathcal{V}_2$ be deflating subspaces for the regular pencil $\lambda M - N$. Then, the intersection $\mathcal{V} \triangleq \mathcal{V}_1 \cap \mathcal{V}_2$ is deflating for $\lambda M - N$ and $\sigma(\lambda M - N, \mathcal{V}) \subseteq \sigma(\lambda M - N, \mathcal{V}_1) \cap \sigma(\lambda M - N, \mathcal{V}_2)$.*

Proof: The fact that $\mathcal{V}$ is deflating for $\lambda M - N$ is a direct consequence of [8, Theorem 1.6.5]. Hence, a subspace $\mathcal{T}$ exists such that (26) holds. In particular we can choose $\mathcal{T} = M\,\mathcal{V} + N\,\mathcal{V}$ as already observed. Let $\mathcal{T}_1 \triangleq M\,\mathcal{V}_1 + N\,\mathcal{V}_1$ and $\mathcal{T}_2 \triangleq M\,\mathcal{V}_2 + N\,\mathcal{V}_2$ be the two subspaces associated with the two deflating subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. It is not difficult to check that $(M\,\mathcal{V} + N\,\mathcal{V}) \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$ so that $\mathcal{T} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$. Consider the invertible matrices $W = \left[\begin{array}{ccc} W' & W'' & W_1'' \end{array}\right]$ and $Z = \left[\begin{array}{ccc} Z' & Z'' & Z_1'' \end{array}\right]$ where W' is such that $\operatorname{im} W' = \mathcal{T}_1 \cap \mathcal{T}_2$, W'' is such that $\operatorname{im} \left[\begin{array}{cc} W' & W'' \end{array}\right] = \mathcal{T}_1$, W_1'' is such that W is non-singular, Z' is such that $\operatorname{im} Z' = \mathcal{V}_1 \cap \mathcal{V}_2$, Z'' is such that $\operatorname{im} \left[\begin{array}{cc} Z' & Z'' \end{array}\right] = \mathcal{V}_1$ and Z_1'' is such that Z is non-singular. Let $W^{-1} = \left[\begin{array}{ccc} Y_1^\top & Y_2^\top & Y_3^\top \end{array}\right]^\top$ be partitioned conformably with $W^\top$, so that $Y_2\,W' = 0$, $Y_3\,W' = 0$ and $Y_3\,W'' = 0$. With this construction, since $M\,\mathcal{V}_1 \subseteq \mathcal{T}_1$, $\operatorname{im} \left[\begin{array}{cc} M\,Z' & M\,Z'' \end{array}\right] = M\,\mathcal{V}_1 \subseteq \operatorname{im} \left[\begin{array}{cc} W' & W'' \end{array}\right]$, it follows that $Y_3 \left[\begin{array}{cc} M\,Z' & M\,Z'' \end{array}\right] = 0$. Similarly, $Y_3 \left[\begin{array}{cc} N\,Z' & N\,Z'' \end{array}\right] = 0$. Moreover, the inclusion $\operatorname{im} M\,Z' \subseteq \mathcal{T} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2 = \operatorname{im} W'$ together with $Y_2\,W' = 0$ implies that $Y_2\,M\,Z' = 0$. A similar argument shows that $Y_2\,N\,Z' = 0$. Thus

$$W^{-1}(\lambda M - N)\, Z = \left[\begin{array}{ccc} \lambda M_{11} - N_{11} & \lambda M_{12} - N_{12} & \lambda M_{13} - N_{13} \\ 0 & \lambda M_{22} - N_{22} & \lambda M_{23} - N_{23} \\ 0 & 0 & \lambda M_{33} - N_{33} \end{array}\right].$$

The particular structure of the former shows that $\sigma(\lambda M - N, \mathcal{V}) \subseteq \sigma(\lambda M - N, \mathcal{V}_1)$. By repeating the same procedure by taking W and Z such that $\operatorname{im} \left[\begin{array}{cc} W' & W'' \end{array}\right] = \mathcal{T}_2$ and

$\operatorname{im} \begin{bmatrix} Z' & Z'' \end{bmatrix} = \mathcal{V}_2$, it follows that $\sigma(\lambda M - N, \mathcal{V}) \subseteq \sigma(M, N, \mathcal{V}_2)$. Therefore, $\sigma(\lambda M - N, \mathcal{V}) \subseteq \sigma(\lambda M - N, \mathcal{V}_1) \cap \sigma(\lambda M - N, \mathcal{V}_2)$. ■