

110 NEW RESULTS IN SINGULAR LINEAR QUADRATIC OPTIMAL CONTROL

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Abstract: This paper focuses on the singular infinite-horizon linear quadratic (LQ) optimal control problem for continuous-time systems. In particular, we are interested in the stabilising impulse-free solutions to this problem that can be expressed as a static state feedback. In particular we establish a link between the geometric properties of the so-called Hamiltonian system associated with the optimal control problem at hand and the so-called proper deflating subspaces of the Hamiltonian matrix pencil.

Key words: Singular LQ optimal control; Hamiltonian system; Hamiltonian matrix pencil; Deflating subspaces.

1 INTRODUCTION

This paper introduces new results on the singular linear quadratic optimal control problem for continuous-time systems. It is a well known fact that when the matrix penalising the control in the performance index to be minimised – traditionally denoted by R – is positive definite, the optimal control can be expressed as a static state-feedback, whose gain depends on the solution of a standard algebraic Riccati equation. This equation involves the inverse of matrix R . When this matrix is singular, the optimal control is guaranteed to exist for any initial condition only if the set of allowed inputs is extended to include distributions (Dirac delta and its derivatives in the sense of distributions). In this case, the standard Riccati equation is not defined, and the problem has been solved in the literature mainly by resorting to a geometric approach, see e.g. Willems *et al.* (1986); Hautus and Silverman (1983); Saberi and Sannuti (1987).

A different perspective was established in Prattichizzo *et al.* (2008), where the main focus of the geometric analysis was the Hamiltonian system. Indeed, the cornerstone of that paper was the interpretation of the LQ regulator as an output nulling problem referred to the Hamiltonian system. In particular, by writing the conditions for optimality in the form of the Hamiltonian system, whose output has to be maintained identically equal to zero, the singular LQ problem reduces to finding a state feedback such that the state-costate trajectory entirely lies on the largest stabilisability subspace of the Hamiltonian system. The analysis carried out in that paper was restricted to the *cheap* LQ problem, i.e., the one in which the matrices weighting the control in the objective function are zero.

In recent years, another important tool aimed at characterising the solutions of the so-called *generalised continuous algebraic Riccati equation* has been introduced in the literature: the Hamiltonian matrix pencil (sometimes also referred to as “extended Hamiltonian pencil” in analogy with the extended symplectic pencil of the discrete time counterpart), see van Dooren (1983); Weiss (1994); Ionescu and Oară (1996); Ionescu *et al.* (1996). The aim of this paper is to establish a link between the approach taken in Prattichizzo *et al.* (2008) based on the Hamiltonian system with that based on the Hamiltonian pencil.

Indeed, the latter is nothing more than the Rosenbrock matrix pencil associated with the Hamiltonian system, see Ionescu *et al.* (1996), p. 86. In this paper, we show that there exists a simple correspondence between the proper right deflating subspaces of the Hamiltonian matrix pencil with the output-nulling subspaces of the Hamiltonian system and that a dual correspondence exists between the proper left deflating subspaces of the Hamiltonian matrix pencil with the input-containing subspaces of the Hamiltonian system. This very simple observation is crucial as it can immediately lead to the derivation of simple conditions under which the singular LQ problem admits an impulse-free solution expressed in terms of a stabilising state feedback for any initial condition. To that end, the approach taken in Weiss (1994), which provides conditions under which an n -dimensional stable and proper right deflating subspace of the Hamiltonian matrix pencil exists, is exploited in this paper to show that under the same conditions the projection of the largest stabilisability subspace of the Hamiltonian system on the state space of the original system coincides with the state space itself. Therefore for any initial condition we can find an optimal stabilising state feedback optimal control by suitably manipulating the friend associated with such stabilisability subspace.

2 SINGULAR LQ PROBLEM

Consider the linear time-invariant (LTI) state differential equation with initial condition

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad (2.1)$$

where, for all $t \geq 0$, the vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ represent the state and the control input, respectively, and A, B are real constant matrices of proper sizes, i.e., $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Let $Q \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$ and $R \in \mathbb{R}^{m \times m}$ be such that

$$\Pi \stackrel{\text{def}}{=} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \Pi^\top \geq 0. \quad (2.2)$$

In view of (2.2), matrix Π can be factored as

$$\Pi = \begin{bmatrix} C^\top & D^\top \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} \quad \text{where } Q = C^\top C, \quad S = C^\top D \quad \text{and } R = D^\top D. \quad (2.3)$$

Using the nomenclature of Ionescu *et al.* (1996), matrix Π is referred to as *Popov matrix*. Notice that here we do not require R to be positive definite. We denote by Σ a quadruple (A, B, C, D) where C and D are such that (2.3) holds.

The singular LQ problem we consider in this paper can be stated as follows.

Problem 6 *Determine under which conditions for all $x_0 \in \mathbb{R}^n$ the input $u(t)$ that minimises the performance index*

$$J(x, u) = \int_0^\infty \begin{bmatrix} x^\top(t) & u^\top(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

is impulse-free and can be expressed as a static state feedback $u(t) = Fx(t)$, with the additional requirement that $A + BF$ be asymptotically stable.

In general, it is well known that Problem 6 is guaranteed to be solvable for any initial condition $x_0 \in \mathbb{R}^n$ only if the set of allowed inputs is extended to include distributions (Dirac delta and its distributional derivatives). In this case, the standard continuous algebraic Riccati equation is not defined, and the problem has been solved in the literature mainly by resorting to a geometric approach, see e.g. Willems *et al.* (1986); Hautus and Silverman (1983); Saberi and Sannuti (1987). Here we are interested in the impulse-free solutions, with the requirement of asymptotic stability of the closed loop. The approach based on the Hamiltonian system hinges on the fact that if $u(t)$ and $x(t)$ are optimal for Problem 6, then a costate function $\lambda(t) \in \mathbb{R}^n$, exists such that $x(t)$, $\lambda(t)$ and $u(t)$ satisfy for all $t \geq 0$ the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (2.4)$$

$$\dot{\lambda}(t) = -Qx(t) - Su(t) - A^\top \lambda(t), \quad (2.5)$$

$$Ru(t) + S^\top x(t) + B^\top \lambda(t) = 0, \quad (2.6)$$

$$x(0) = x_0. \quad (2.7)$$

We now introduce some fundamental objects associated with the classical LQ optimal control problem.

3 THE HAMILTONIAN SYSTEM AND THE HAMILTONIAN MATRIX PENCIL

Recall that the Hamiltonian system associated with the Popov triple Σ is an LTI system defined by the equations

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ -Q & -A^\top \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} B \\ -S \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} S^\top & B^\top \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + R u(t), \end{aligned} \tag{3.1}$$

where the variable $\lambda(t)$ is the *costate*. We define $\widehat{A} \stackrel{\text{def}}{=} \begin{bmatrix} A & 0 \\ -Q & -A^\top \end{bmatrix}$, $\widehat{B} \stackrel{\text{def}}{=} \begin{bmatrix} B \\ -S \end{bmatrix}$, $\widehat{C} \stackrel{\text{def}}{=} [S^\top \quad B^\top]$ and $\widehat{D} \stackrel{\text{def}}{=} R$. The Hamiltonian system (3.1) is identified with the quadruple $\widehat{\Sigma} \stackrel{\text{def}}{=} (\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$. The Hamiltonian system is a fundamental tool in the solution of continuous-time differential and algebraic Riccati equations, and it has strong relations with the corresponding optimal control problem. Indeed, comparing (3.1) with (2.4)-(2.6), it emerges that the optimal control and state trajectory satisfy the Hamiltonian system with an identically zero output for a suitable costate function λ . This consideration is crucial, and recasts the optimal control problem into an output-nulling problem for the Hamiltonian system (3.1). On the other hand, the trajectories of $\widehat{\Sigma}$ that yield an identically zero output are those and only those for which the state-costate vector $\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$ lies entirely on an output-nulling subspace¹ of $\widehat{\Sigma}$. Since in addition we have a stability requirement on the closed-loop system, $\begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}$ must belong to the largest stabilisability subspace² of the Hamiltonian system, herein denoted by $\widehat{\mathcal{V}}_g^*$, see also Prattichizzo *et al.* (2008). Thus, we have the following result.

Theorem 3.1 *Let $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ be a basis matrix of $\widehat{\mathcal{V}}_g^*$. Given $x_0 \in \mathbb{R}^n$, a state and input functions $x(t)$ and $u(t)$ exist satisfying (2.4-2.7) if and only if $x_0 \in \text{im } V_1$.*

Corollary 3.1 *A state and input functions $x(t)$ and $u(t)$ satisfy (2.4-2.6) for any $x_0 \in \mathbb{R}^n$ if and only if $\text{im } V_1 = \mathbb{R}^n$.*

Another important tool that is often introduced in the theory of continuous-time Riccati equations is the so-called Hamiltonian matrix pencil, which coincides with the Rosenbrock matrix pencil associated with the Hamiltonian system (3.1). More explicitly, the Hamiltonian matrix pencil is $N - sM$, where the matrices N and M are defined as

$$M \triangleq \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad N \triangleq \begin{bmatrix} A & 0 & B \\ -Q & -A^\top & -S \\ S^\top & B^\top & R \end{bmatrix}.$$

It follows that the invariant zeros of the Hamiltonian system are the generalised eigenvalues of the Hamiltonian pencil. The Hamiltonian pencil has been used by several authors to characterise the stabilising solution of the generalised algebraic Riccati equation, see van Dooren (1983); Weiss (1994); Ionescu and Oară (1996).

Theorem 3.2 *Given a proper right deflating subspace of the Hamiltonian matrix pencil $N - sM$ spanned by the matrix $V = [V_1^\top \quad V_2^\top \quad V_3^\top]^\top$ partitioned conformably with M and N , the columns of $V = [V_1^\top \quad V_2^\top]^\top$ are a basis of an output-nulling subspace of (3.1). Conversely, given an output-nulling subspace of the Hamiltonian system (3.1) spanned by the basis matrix $V = [V_1^\top \quad V_2^\top]^\top$ partitioned conformably with (3.1), and given two matrices Ξ and Ω such that (3.5) holds, the matrix $V = [V_1^\top \quad V_2^\top \quad \Omega^\top]^\top$ spans a proper right deflating subspace of $N - sM$.*

Proof: Consider the basis matrix $V = [V_1^\top \quad V_2^\top \quad V_3^\top]^\top$ of a proper right deflating subspace of the Hamiltonian matrix pencil $N - sM$. By definition, a matrix Φ exists such that $MV\Phi = NV$ where MV is injective, so that

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \Phi = \begin{bmatrix} A & 0 & B \\ -Q & -A^\top & -S \\ S^\top & B^\top & R \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{3.2}$$

We now notice that equation (3.2) can be equivalently written as

$$\begin{bmatrix} A & 0 \\ -Q & -A^\top \\ S^\top & B^\top \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ 0 \end{bmatrix} \Phi - \begin{bmatrix} B \\ -S \\ R \end{bmatrix} V_3, \quad (3.3)$$

which says that $\text{im} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ is output-nulling for the Hamiltonian system by virtue of the equivalence of (3.4) and (3.5), see the Notes at the end of the paper. Conversely, if $\text{im} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ is output-nulling for the Hamiltonian system, two matrices Ξ and Ω exist such that (3.5) holds with \widehat{A} , \widehat{B} , \widehat{C} and \widehat{D} in place of A , B , C , D , which means that (3.2) holds with $\Phi = \Xi$ and $V_3 = \Omega$. Hence, (3.2) holds under the same substitutions, which means that $V = [V_1^\top \ V_2^\top \ \Omega^\top]^\top$ is a right deflating subspace of the Hamiltonian matrix pencil $N - sM$. It is proper since $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ is of full column rank. ■

The dual of Theorem 3.2 is as follows.

Theorem 3.3 *Given a proper left deflating subspace $\ker W$ of the Hamiltonian matrix pencil $N - sM$, where W is partitioned as $[W_1 \ W_2 \ W_3]$ conformably with M and N , then $\ker [W_1 \ W_2]$ is an input-containing³ subspace of (3.1). Conversely, given an input-containing subspace of (3.1) equal to $\ker W$ where $W = [W_1^\top \ W_2^\top]^\top$ is full row-rank and partitioned conformably with (3.1), and given two matrices Γ and Λ such that (3.7) holds, the null-space of the matrix $W = [W_1 \ W_2 \ \Lambda]$ is a proper left deflating subspace of the Hamiltonian matrix pencil $N - sM$.*

Proof: By definition, a left deflating subspace $\ker W$ is such that there exists Ψ for which $\Psi W M = W N$ with $W M$ full row-rank. The proof follows by dualising the proof of Theorem 3.2, by noticing the equivalence of

$$[W_1 \ W_2] \begin{bmatrix} A & 0 & B \\ -Q & -A^\top & -S \end{bmatrix} = \Gamma [W_1 \ W_2 \ 0] + \Lambda [S^\top \ B^\top \ R]$$

with $[W_1 \ W_2 \ W_3] N = \Psi [W_1 \ W_2 \ W_3] \text{diag}\{I_n, I_n, 0\}$ under the substitutions $\Gamma = \Psi$ and $\Lambda = -W_3$. ■

The following result, which relates the fundamental subspaces of the Hamiltonian system $\widehat{\Sigma}$ with those of the original system Σ , generalises Lemma 4.3 in Prattichizzo *et al.* (2008) in two directions. First, we are not assuming that D is zero (i.e., that the LQ problem is *cheap*). Second, Σ is not necessarily left invertible.

Theorem 3.4 *Let \mathcal{R}^* , \mathcal{V}^* and \mathcal{S}^* denote the largest controllability subspace, the largest output-nulling and the smallest input containing subspaces of Σ , respectively. Moreover, let $\widehat{\mathcal{R}}^*$, $\widehat{\mathcal{V}}^*$ and $\widehat{\mathcal{S}}^*$ denote the same subspaces referred to the Hamiltonian system $\widehat{\Sigma}$. The following identities hold:*

$$\begin{aligned} \dim \widehat{\mathcal{R}}^* &= \dim \mathcal{R}^* \\ \dim \widehat{\mathcal{S}}^* &= 2 \dim \mathcal{S}^* - \dim \mathcal{R}^* \\ \dim \widehat{\mathcal{V}}^* &= 2n - 2 \dim \mathcal{S}^* + \dim \mathcal{R}^* \end{aligned}$$

The following result can be found in Weiss (1994).

Theorem 3.5 *Let the pair (A, B) be stabilisable and let the Hamiltonian system (3.1) be devoid of invariant zeros on the imaginary axis. Then, the Hamiltonian pencil $N - sM$ has an n -dimensional stable proper deflating subspace. Let this deflating subspace be spanned by the matrix $V = [V_1^\top \ V_2^\top \ V_3^\top]^\top$ partitioned conformably with M and N . Then, V_1 is invertible.*

Proof: This statement follows from Weiss (1994), p. 679, by observing that the absence of generalised eigenvalues on the imaginary axis of the Hamiltonian pencil is equivalent to the absence of invariant zeros on the imaginary axis of the Hamiltonian system. This, in view of the stabilisability of the pair (A, B) , in turn corresponds to the absence of unobservable eigenvalues of the pair (A, C) , where C is any matrix which, together with a matrix D , factorises the Popov matrix as in (2.3). ■

Corollary 3.2 *Consider the factorisation (2.3), and let $\text{rank } G(j\omega) = \text{rank } D$ for every $\omega \in \mathbb{R}$. Let the pair (A, B) be stabilisable and let the Hamiltonian system (3.1) be devoid of invariant zeros on the imaginary axis. Then, the largest stabilisability subspace $\widehat{\mathcal{V}}_g^*$ of the Hamiltonian system has dimension n .*

Proof: This result follows directly from Theorem 3.5 and Theorem 3.2. ■

Theorem 3.6 *Let $\text{rank } G(j\omega) = \text{rank } D$ for every $\omega \in \mathbb{R}$. Let the pair (A, B) be stabilisable and let the Hamiltonian system (3.1) be devoid of invariant zeros on the imaginary axis. For all $x_0 \in \mathbb{R}^n$, a control exists such that (2.4-2.7) hold and the state converges to zero, and such control can be expressed as a static state feedback.*

Proof: Under the assumptions considered, the state-costate trajectory lies on $\widehat{\mathcal{V}}_g^*$, whose dimension is equal to n . Since from Theorem 3.5 a basis matrix for $\widehat{\mathcal{V}}_g^*$ is given by the columns of $\begin{bmatrix} V_1^\top & V_2^\top \end{bmatrix}^\top$ with V_1 square and invertible, it follows that the projection of $\widehat{\mathcal{V}}_g^*$ on the state space \mathbb{R}^n coincides with \mathbb{R}^n . For any $x_0 \in \mathbb{R}^n$, there exists $\lambda_0 \in \mathbb{R}^n$ such that $\begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} \in \widehat{\mathcal{V}}_g^*$. The corresponding control can be expressed as a feedback of the sole state $x(t)$. Let $\widehat{F} = \begin{bmatrix} F_x & F_\lambda \end{bmatrix}$ be a friend of $\widehat{\mathcal{V}}_g^*$ that assigns stable closed-loop eigenvalues of $\widehat{A} + \widehat{B}\widehat{F}$ restricted to $\widehat{\mathcal{V}}_g^*$. The control can therefore be expressed as a static feedback of the state-costate vector using \widehat{F} , i.e. $u(t) = \begin{bmatrix} F_x & F_\lambda \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$, where for all $t \geq 0$ we have $\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \in \widehat{\mathcal{V}}_g^*$, so that we can write $\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \alpha(t)$ for a suitable function $\alpha(t)$. Thus, the state $x(t)$ identically lies on the projection of $\widehat{\mathcal{V}}_g^*$ on the state space, which is spanned by V_1 . Hence

$$u(t) = \begin{bmatrix} F_x & F_\lambda \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \alpha(t) = \begin{bmatrix} F_x & F_\lambda \end{bmatrix} \begin{bmatrix} I_n \\ V_2 V_1^{-1} \end{bmatrix} V_1 \alpha(t) = (F_x + F_\lambda V_2 V_1^{-1}) x(t),$$

i.e., we have expressed the control as a state feedback of the sole state $x(t)$. Moreover, since \widehat{F} drives the state-costate trajectory to the origin as $t \rightarrow \infty$, then $F_x + F_\lambda V_2 V_1^{-1}$ drives the state to the origin as $t \rightarrow \infty$. ■

Example 3.1 Consider the Popov triple characterised by the matrices

$$A = \begin{bmatrix} -9 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 9 \end{bmatrix}, \quad Q = \begin{bmatrix} 18 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R = 0.$$

This system is stabilisable (the uncontrollable eigenvalue is equal to -9). Moreover, from the factorisation $\Pi = \begin{bmatrix} C^\top \\ D^\top \end{bmatrix} [C \ D]$ with $C = \begin{bmatrix} 3\sqrt{2} & 0 \end{bmatrix}$ and $D = 0$, since $G(s) = C(sI - A)^{-1}B + D \equiv 0$, we find that indeed $\text{rank } G(j\omega) = \text{rank } D = 0$ for all $\omega \in \mathbb{R}$. A direct check shows that the zeros of the Hamiltonian system are $\{9, -9\}$. The largest stabilisability subspace of the Hamiltonian system is $\widehat{\mathcal{V}}_g^* = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

A friend \widehat{F} of $\widehat{\mathcal{V}}_g^*$ that assigns the eigenvalue -9 with double multiplicity to the spectrum of $\widehat{A} + \widehat{B}\widehat{F}$ restricted to $\widehat{\mathcal{V}}_g^*$ is $\widehat{F} = \begin{bmatrix} 0 & -1 & 0 & 0 \end{bmatrix}$. The feedback matrix $F = F_x + F_\lambda V_2 V_1^{-1} = \begin{bmatrix} 0 & -1 \end{bmatrix}$ assigns the spectrum $\sigma(A + BF) = \{-9\}$ with double multiplicity. However, since this system is not left invertible, as the largest controllability subspace⁴ is $\mathcal{R}^* = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can find the gains by changing the spectrum of $A + BF$ restricted to \mathcal{R}^* . Let us compute a friend \widehat{F} of $\widehat{\mathcal{V}}_g^*$ which assigns the eigenvalues $\{-3, -9\}$ to the spectrum of $(\widehat{A} + \widehat{B}\widehat{F} | \widehat{\mathcal{V}}_g^*)$. We get $\widehat{F} = \begin{bmatrix} 0 & -1/3 & 0 & 0 \end{bmatrix}$. Then, $F = F_x + F_\lambda V_2 V_1^{-1} = \begin{bmatrix} 0 & -1/3 \end{bmatrix}$ assigns the spectrum $\sigma(A + BF) = \{-3, -9\}$.

Notes

1. We recall that, given a quadruple (A, B, C, D) , an output-nulling subspace is a subspace \mathcal{V} of the state-space satisfying the inclusion

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \oplus \{0\}) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{3.4}$$

which is equivalent to the existence of a matrix F (referred to as a *friend* of \mathcal{V}) such that $(A + BF)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(C + DF)$, see Trentelman *et al.* (2001), p. 160. Notice that in view of (3.4), we have that given a subspace \mathcal{V} and a basis matrix V (i.e., a matrix whose columns are linearly independent and that span \mathcal{V}), then \mathcal{V} is output-nulling if and only if two matrices Ξ and Ω exist such that

$$\begin{bmatrix} A \\ C \end{bmatrix} V = \begin{bmatrix} V \\ 0 \end{bmatrix} \Xi + \begin{bmatrix} B \\ D \end{bmatrix} \Omega, \tag{3.5}$$

see Ntogramatzidis (2007); Ntogramatzidis (2008). Since the set of output-nulling subspaces of a given quadruple is closed under subspace addition, there exists a largest output-nulling subspace – denoted by \mathcal{V}^* – which represents the set of all initial states for which a control can be found that maintains the output at zero.

2. Given a quadruple (A, B, C, D) , an output-nulling subspace \mathcal{V} is referred to as a *stabilisability subspace* if a friend F exists such that the map $A + BF$ restricted to \mathcal{V} is asymptotically stable. When the pair (A, B) is stabilisable, then also the map induced by $A + BF$ in the quotient space \mathbb{R}^n/\mathcal{V} is stable, so that F exists such that the spectrum of $A + BF$ is stable. The set of stabilisability subspaces is closed under addition, and its largest element is denoted by \mathcal{V}_g^* .

3. Given a quadruple (A, B, C, D) , an input-containing subspace is a subspace \mathcal{S} of the state-space satisfying the inclusion

$$\begin{bmatrix} A & B \end{bmatrix} ((\mathcal{S} \oplus \mathbb{R}^m) \cap \ker \begin{bmatrix} C & D \end{bmatrix}) \subseteq \mathcal{S}, \quad (3.6)$$

Trentelman *et al.* (2001), p. 185. In view of (3.4), given a subspace \mathcal{S} and a full row-rank matrix W such that $\ker W = \mathcal{S}$, then \mathcal{S} is input containing if and only if two matrices Γ and Λ exist such that

$$W \begin{bmatrix} A & B \end{bmatrix} = \Gamma \begin{bmatrix} W & 0 \end{bmatrix} + \Lambda \begin{bmatrix} C & D \end{bmatrix}, \quad (3.7)$$

see Ntogramatzidis (2007); Ntogramatzidis (2008). Since the set of input-containing subspaces is closed under subspace intersection, there exists a smallest input-containing subspace indicated by \mathcal{S}^* .

4. The largest controllability subspace of a quadruple (A, B, C, D) can be computed as the intersection $\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^*$, Trentelman *et al.* (2001), Theorem 8.22. This subspace represents the states that can be driven to the origin by maintaining the output identically equal to zero. If $\begin{bmatrix} B \\ D \end{bmatrix}$ is full column-rank, the quadruple (A, B, C, D) is left invertible if and only if $\mathcal{R}^* = \{0\}$.

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