A Computational Method for a Class of Non-standard Time Optimal Control Problems Involving Multiple Time Horizons

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Abstract. In this paper, we consider a class of non-standard time optimal control problems involving a dynamical system consisting of multiple subsystems evolving over different time horizons. Different subsystems are required to reach their respective target sets at different termination times. The goal is to minimize the maximum of these termination times. By introducing a discrete variable to represent the system termination ordering, we reformulate this problem as a discrete optimization problem. A discrete filled function method is developed to solve this discrete optimization problem. For illustration, a numerical example is solved.

Key Words. Optimal control problem, multiple time horizons, discrete filled function method, time scaling transformation, control

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parametrization enhancing technique.

1. Introduction

Time optimal control problems arise frequently in practical applications. For this reason, they have been extensively studied in the literature by both mathematicians and engineers alike. Some relevant references are [1], [2], [3] and [4]. In these works, the state of the system is required to travel from a given initial state to a target in minimum time. Some papers in the literature (e.g. see [5] and [6]) have also considered minimum-time problems involving multiple targets. Nevertheless, in all of these papers, the dynamical system is defined on a single time horizon and terminates when the desired target is reached.

In this paper, we consider a class of time optimal control problems involving a dynamical system composed of M subsystems. Each of these subsystems is required to reach a desired target. These desired targets can be different for different subsystems. Furthermore, these subsystems reach their respective targets at different times. In other words, each of these M subsystems are defined on a different time horizon. Each time horizon starts from zero and ends at some unspecified final time. Our objective is to find a control such that the maximum of these final times is minimized. This class of time optimal control problems has many practical applications. Examples include situations where a large project consisting of several sub-projects is to be completed in minimum time. Due to the complex nature of these problems, it is not possible to derive an analytical solution. Thus, it is unavoidable to rely on numerical methods for solving these problems.

The difficulty with this type of problem is that the cost function is non-differentiable. Furthermore, because the chronological order of system termination times is not known *a priori*, existing methods cannot be used to solve such a problem directly. These included the time scaling transform, which is introduced and referred to as the control parametrization enhancing transformation in [2], although this approach has been successfully used for solving many optimal control problems such as those reported in [7] and [8]. Thus, a new approach is needed for this unconventional optimal control problem.

In this paper, we propose a two-level optimization approach. By choosing an ordering of system termination times, the inner problem becomes a standard time optimal control problem which can be handled by the control parametrization technique in conjunction with the time scaling transform. The choice of the optimal ordering of system termination times is the outer problem which is to be solved using a filled function method introduced in [9]. An efficient algorithm is then developed implementing this two-level optimization approach.

The rest of the paper is organized as follows. The problem formulation is given in Section 2. In Section 3, we introduce a discrete variable to represent the switching sequence and then reformulate the problem as a two-level optimization problem, where the inner problem is a standard time optimal control problem, while the outer problem is a discrete optimization problem. Section 4 discusses the solution of the inner problem by using the control parametrization technique, which is enhanced by a time scaling transform. In Section 5, a discrete filled function method is introduced for solving the outer problem which is a discrete optimization problem defined in Section 4. In Section 6, we illustrate the solution procedure developed in the previous sections through solving a numerical example.

2. Problem Statement

Consider a master process consisting of M coupled subsystems, each described by a system of ordinary differential equations:

$$\dot{\mathbf{x}}^i(t) = \mathbf{f}^i(t, \mathbf{x}^1(t), \mathbf{x}^2(t), ..., \mathbf{x}^M(t), \mathbf{u}(t))\ \chi_{[0,t_f^i]}(t), \qquad i = 1, 2, ..., M, \tag{1}$$

with the initial conditions

$$\mathbf{x}^i(0) = \mathbf{x}_0^i, \qquad i = 1, 2, ..., M, \tag{2}$$

where χ_I is the characteristic function on I defined by

$$\chi_I(t) = \begin{cases} 1, & \text{if } \mathrm{t} \in \mathrm{I}, \\ 0, & \text{otherwise}, \end{cases}$$

and for each $i = 1, 2, ..., M$, $\mathbf{x}^i(t) = [x_1^i(t), x_2^i(t), ..., x_n^i(t)]^T \in \mathbb{R}^n$ is the state of the i-th subsystem, while $\mathbf{u}(t) = [u_1(t), u_2(t), ..., u_m(t)]^T \in \mathbb{R}^m$ is a Borel measurable function such that $\mathbf{u}(t) \in U$ for $t \in [0, \infty)$, with U being defined by

$$U = \{\mathbf{u} \in \mathbb{R}^m \ ;\ u_i^{min} \leq u_i \leq u_i^{max},\ i = 1, 2, ..., m\}. \tag{3}$$

Clearly, U is a compact and convex subset of $\mathbb{R}^m$. Such a $\mathbf{u}$ is referred to as an admissible control. Let $\mathcal{U}$ be the class of all such admissible controls. For each $i = 1, 2, ..., M$, let t_f^i denote the terminal time for the i-th subsystem. The functions $\mathbf{f}^i : \mathbb{R} \times \mathbb{R}^{nM} \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2, ..., M$, are assumed to be continuously differentiable with respect to each of their respective arguments. Each subsystem terminates when it reaches a target set. This unconventional time optimal control problem may now be formally formulated as follows.

Problem ($\mathbf{P_1}$). Given the system (1) with the initial condition (2), find a control $\mathbf{u} \in \mathcal{U}$ such that the cost functional

$$J_0(\mathbf{u}) = \max\{t_f^1, t_f^2, ..., t_f^M\} \tag{4}$$

is minimized subject to the constraints:

$$\|\mathbf{x}^i(t_f^i) - \mathbf{y}^i\|^2 - \varepsilon_i^2 \leq 0, \qquad i = 1, 2, ..., M, \tag{5}$$

where $\varepsilon_i > 0$, $i = 1, 2, ..., M$, are given constants and $\mathbf{y}^i \in \mathbb{R}^n$, $i = 1, 2, ..., M$, are given vectors.

Problem (P_1) is an unconventional time optimal control problem. It is known that the time scaling transform introduced in [2] is an efficient solution method for time optimal control problems. However, Problem (P_1) cannot be solved directly by this time scaling transform, because the chronological ordering of t_f^i, $i = 1, 2, ..., M$, is not known *a priori*.

3. Reformulation as a Two-level Optimization Problem

Consider Problem (P_1) and let the ordering of the termination times be denoted as:

$$\mathbf{v} = [v_1, v_2, ..., v_M]^T.$$

The vector $\mathbf{v}$ describes the chronological order of the subsystems which are to reach their respective targets. More specifically, for this vector $\mathbf{v}$, we see that the subsystem v_1 finishes first, the subsystem v_2 finishes second, and the subsystem v_M finishes last. The vector $\mathbf{v}$ is referred to as the switching sequence. For a switching sequence $\mathbf{v}$, the dynamical system (1) with its initial conditions (2) can be restated as:

$$\dot{\mathbf{x}}^{v_i}(t) = \mathbf{f}^{v_i}(t, \mathbf{x}^1(t), \mathbf{x}^2(t), ..., \mathbf{x}^M(t), \mathbf{u}(t))\chi_{[0,t_f^{v_i}]}(t), \tag{6}$$

$$\mathbf{x}^{v_i}(0) = \mathbf{x}_0^{v_i}, \tag{7}$$

where $i = 1, 2, ..., M$.

A switching sequence $\mathbf{v} = [v_1, v_2, ..., v_M]^T$ is a permutation of the set $\{1, 2, ..., M\}$. Let $\mathbf{\Upsilon_M}$ be the set of all such switching sequences.

Now, Problem (P_1) can be reformulated as:

Problem $(\mathbf{P_2})$. Given the switched system (6) with the initial condition (7), find a switching sequence $\mathbf{v} \in \mathbf{\Upsilon_M}$ and a control $\mathbf{u} \in \mathcal{U}$ such that

$$J_0(\mathbf{u}, \mathbf{v}) = t_f^{v_M} \tag{8}$$

is minimized over $\mathbf{\Upsilon_M} \times \mathcal{U}$, subject to the constraints:

$$\|\mathbf{x}^i(t_f^{v_i}) - \mathbf{y}^i\|^2 - \varepsilon_i^2 \leq 0, \qquad i = 1, 2, ..., M. \tag{9}$$

Problem (P_2) can be viewed as a two-level optimization problem as follows:

$$\min_{\mathbf{v}\in\boldsymbol{\Upsilon}_{\mathbf{M}}} \{\min_{\mathbf{u}\in\mathcal{U}} J_0(\mathbf{u},\mathbf{v})\}.$$

To be more specific, define

$$\widehat{J}_0(\mathbf{v}) = \min_{\mathbf{u}\in\mathcal{U}} J_0(\mathbf{u},\mathbf{v})$$

for each $\mathbf{v} \in \boldsymbol{\Upsilon}_{\mathbf{M}}$. Then, it is clear that to solve Problem (P_2), it requires to solve the following inner problem for each $\mathbf{v} \in \boldsymbol{\Upsilon}_{\mathbf{M}}$.

Problem $(\mathbf{P_{3,v}})$. For a given switching sequence $\mathbf{v} \in \boldsymbol{\Upsilon}_{\mathbf{M}}$, find a control $\mathbf{u} \in \mathcal{U}$ such that the cost functional

$$J_0(\mathbf{u},\mathbf{v}) = t_f^{v_M} \tag{12}$$

is minimized subject to the switched system (6) with the initial condition (7) and the constraints:

$$\|\mathbf{x}^i(t_f^{v_i}) - \mathbf{y}^i\|^2 - \varepsilon_i^2 \leq 0, \qquad i = 1, 2, ..., M. \tag{13}$$

Now, for each $\mathbf{v} \in \boldsymbol{\Upsilon}_{\mathbf{M}}$, Problem $(P_{3,\mathbf{v}})$ is a classical time optimal control problem that can be solved by a variety of well-known techniques.

Assume that, for a given $\mathbf{v} \in \boldsymbol{\Upsilon}_{\mathbf{M}}$, there exists an optimal control of Problem $(P_{3,\mathbf{v}})$, which is denoted by $\mathbf{u}^*(.|\mathbf{v})$. Let the solution of the dynamical system (6) with the initial condition (7) under the control $\mathbf{u}^*(.|\mathbf{v})$ be denoted by $\mathbf{x}^*(.|\mathbf{v})$. Clearly, for each $\mathbf{v} \in \boldsymbol{\Upsilon}_{\mathbf{M}}$, the calculation of the cost functional $\widehat{J}_0(\mathbf{v})$ requires the solution of the Problem $(P_{3,\mathbf{v}})$. Thus, Problem (P_2) is equivalent to:

Problem $(\mathbf{P_4})$. Find a switching sequence $\mathbf{v} \in \mathbf{\Upsilon_M}$ such that the cost functional

$$\widehat{J}_0(\mathbf{v}) = J_0(\mathbf{u}^*(.|\mathbf{v})) \tag{14}$$

is minimized.

4. Solution Method for Inner Problem

We shall use the control parametrization technique, which is enhanced by the time scaling transform introduced in [2], to develop a numerical method to solve Problem $(P_{3,v})$. To this end, we approximate the control function $\mathbf{u}$ by a piecewise constant function with possible switching time points at $\{\tau_j\}_{j=1}^{qM}$, where

$$0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_{qM} = t_f^{v_M},$$

and

$$\tau_{iq} = t_f^{v_i}, \qquad i = 1, 2, ..., M.$$

Here, q is an integer specifying the number of partition points on the interval $[0, t_f^{v_M}]$. The approximate control is denoted by

$$\mathbf{u}^q = [u_1^q, u_2^q, ..., u_m^q]^T,$$

where

$$u_i^q(t) = \sum_{j=1}^{qM} \sigma_j^i \chi_{[\tau_{j-1},\tau_j)}(t), \qquad i = 1, 2, ..., m. \tag{16}$$

Here, σ_j^i, $i = 1, 2, ..., m$; $j = 1, 2, ..., qM$, are the height of the i-th component of the approximate control on the interval $[\tau_{j-1}, \tau_j)$. Let $\boldsymbol{\sigma}^i = [\sigma_1^i, \sigma_2^i, ..., \sigma_{qM}^i]^T$, $i = 1, 2, ..., m$, and $\boldsymbol{\sigma} = [(\boldsymbol{\sigma}^1)^T, (\boldsymbol{\sigma}^2)^T, ..., (\boldsymbol{\sigma}^m)^T]^T$. From (3), it is clear that

$$u_i^{min} \leq \sigma_j^i \leq u_i^{max}, \qquad i = 1, 2, ..., m; \; j = 1, 2, ..., qM.$$

Let Ξ be the set of all those $\boldsymbol{\sigma} \in \mathbb{R}^{mqM}$ that defines an approximate control $\mathbf{u}^q$, where u_i^q, $i = 1, 2, ..., m$, are as defined by (16). Here, both the heights σ_j^i, $i = 1, 2, ..., m$; $j = 1, 2, ..., qM$, and the switching time points τ_i, $i = 1, 2, ..., qM-1$, are regarded as decision variables.

However, the approximate optimal control Problem $(P_{3,v})$ with the switching time points τ_i, $i = 1, 2, ..., qM - 1$, taken as the decision variables will encounter numerical difficulties as mentioned in [2]. For this reason, a time scaling transform is introduced in [2] to map these variable switching time points into fixed switching time points in a new time scale. It is achieved by introducing a transformation from $t \in [0, t_f^{v_M}]$ to $s \in [0, qM]$ as follows:

$$\frac{dt(s)}{ds} = w(s), \tag{18}$$

$$t(0) = 0, \tag{19}$$

where $w(s)$ is a non-negative piecewise constant function with fixed switching time points at $\{1, 2, , ..., qM - 1\}$. Such a function is referred to as a time scaling control defined by

$$w(s) = \sum_{l=1}^{qM} \theta_l \chi_{[l-1,l)}(s), \tag{20}$$

where $\theta_l \geq 0$, $l = 1, 2, ..., qM$

Let $\mathcal{W}$ be the set which consists of all such time scaling controls, and let Θ be the set of all those $\boldsymbol{\theta} = [\theta_1, \theta_2, ..., \theta_{qM}]^T \in \mathbb{R}^{qM}$ such that $\theta_l \geq 0$, $l = 1, 2, ..., qM$. Clearly, each $w \in \mathcal{W}$ is determined uniquely by a $\boldsymbol{\theta} \in \Theta$ and vice versa. Thus, we write $w(.)$ as $w(.|\boldsymbol{\theta})$. Let

$$\widetilde{\mathbf{u}}^q = [\tilde{u}_1^q, \tilde{u}_2^q, ..., \tilde{u}_m^q]^T,$$

where

$$\tilde{u}_i^q(s) = u_i^q(t(s)) = \sum_{j=1}^{qM} \sigma_j^i \chi_{[j-1,j)}(s), \qquad i = 1, 2, ..., m. \tag{22}$$

Since $\widetilde{\mathbf{u}}^q$ is determined uniquely by $\boldsymbol{\sigma}$ and vice versa, it is written as $\widetilde{\mathbf{u}}^q(.|\boldsymbol{\sigma})$.

The dynamical system (6) with the initial condition (7), after the time scaling transform, becomes

$$\frac{d\mathbf{x}^{v_i}(s)}{ds} = w(s)\widetilde{\mathbf{f}}^{v_i}(s, \mathbf{x}^1(s), \mathbf{x}^2(s), ..., \mathbf{x}^M(s), \boldsymbol{\sigma}, \boldsymbol{\theta})\chi_{[0,iq]}(s), \tag{23}$$

$$\mathbf{x}^{v_i}(0) = \mathbf{x}_0^{v_i}, \tag{24}$$

where $i = 1, 2, ..., M$, $w(s) = w(s|\boldsymbol{\theta})$, $\mathbf{x}^{v_i}(s) = \mathbf{x}^{v_i}(t(s))$, and

$$\widetilde{\mathbf{f}}^{v_i}(s, \mathbf{x}^1(s), \mathbf{x}^2(s), ..., \mathbf{x}^M(s), \boldsymbol{\sigma}, \boldsymbol{\theta})$$
$$= \mathbf{f}^{v_i}(s, \mathbf{x}^1(s), \mathbf{x}^2(s), ..., \mathbf{x}^M(s), \widetilde{\mathbf{u}}^q(s|\boldsymbol{\sigma})).$$

The cost function (4), after the time scaling transform, becomes

$$\bar{J}_0(\widetilde{\mathbf{u}}(.|\boldsymbol{\sigma}), \mathbf{v}, w(.|\boldsymbol{\theta})) = t_f^{v_M} = \sum_{j=1}^{qM} \theta_j. \tag{27}$$

We write

$$\bar{J}_0(\widetilde{\mathbf{u}}(.|\boldsymbol{\sigma}), \mathbf{v}, w(.|\boldsymbol{\theta})) = \widetilde{J}_0(\boldsymbol{\sigma}, \mathbf{v}, \boldsymbol{\theta}). \tag{28}$$

Remark 4.1. The control $\mathbf{u}$ in the original time scale is obtained from $\widetilde{\mathbf{u}}(.|\boldsymbol{\sigma})$ and $w(.|\boldsymbol{\theta})$ in the transformed time scale as follows.

Solving (18)-(19) with $w(s)$ given by (20), we obtain, for $s \in [l-1, l]$,

$$t(s) = \sum_{k=1}^{l-1} \theta_k + \theta_l s, \tag{29}$$

where $l = 1, 2, ..., qM$. Thus,

$$\tau_l = t(l), \qquad\qquad l = 0, 1, ..., qM, \tag{30}$$

and

$$\tau_{iq} = t_f^{v_i}, \qquad\qquad i = 1, 2, ..., M. \tag{31}$$

The corresponding control $\mathbf{u}$, given by (16), in the original time scale is:

$$\mathbf{u}(t) = [u_1(t), u_2(t), ..., u_m(t)]^T,$$

where

$$u_i(t) = \sum_{j=1}^{qM} \sigma_j^i \chi_{[\tau_{j-1}, \tau_j)}(t), \qquad\qquad i = 1, 2, ..., m, \tag{33}$$

and τ_j, $j = 1, 2, ..., qM$, are given by (29)-(31). Such a control in the original time scale is denoted by $\mathbf{u}(.|\boldsymbol{\sigma}, \boldsymbol{\theta})$.

The original cost function (4) is denoted by

$$J_0(\mathbf{u}(.|\boldsymbol{\sigma}, \boldsymbol{\theta}), \mathbf{v}) = t_f^{v_M}. \tag{34}$$

We obtain the following approximate problem of Problem $(P_{3,v})$.

Problem ($\mathbf{P^q_{3,v}}$). For a given switching sequence $\mathbf{v} \in \boldsymbol{\Upsilon}_\mathbf{M}$, find a control parameter vector $(\boldsymbol{\sigma}, \boldsymbol{\theta}) \in \Xi \times \Theta$ such that the cost function

$$\widetilde{J}_0(\boldsymbol{\sigma}, \mathbf{v}, \boldsymbol{\theta}) = \sum_{j=1}^{qM} \theta_j \tag{35}$$

is minimized subject to the switched system (23) with the initial condition (24) and the constraints:

$$\|\mathbf{x}^{v_i}(iq) - \mathbf{y}^i\|^2 - \varepsilon_i^2 \leq 0, \qquad i = 1, 2, ..., M. \tag{36}$$

Problem $(P_{3,v}^q)$ can be viewed (see [10]) as a standard constrained optimization problem which can be solved by any efficient gradient-based optimization procedure, such as the sequential quadratic programming approximation procedure (see [11]). The gradient formulae for the cost function (35) and the constraint functions (36) with respect to $\boldsymbol{\sigma}$ and $\boldsymbol{\theta}$ can be obtained readily form Theorem 5.2.1 of [10].

The convergence results presented in the following theorems established the relationship between Problem $(P_{3,v}^q)$ and Problem $(P_{3,v})$. Their proofs are similar to those given for Theorem 6.5.1 and Theorem 6.5.2 of [10], respectively.

Theorem 4.1. Let $\mathbf{v} \in \boldsymbol{\Upsilon}_{\mathbf{M}}$ be given. Suppose that $\mathbf{u}^*$ is an optimal control of Problem $(P_{3,v})$. Let $\boldsymbol{\sigma}_q^*$ and $\boldsymbol{\theta}_q^*$ be optimal parameter vectors of Problem $(P_{3,v}^q)$ and let $\mathbf{u}_q^*$ be the piecewise constant control derived from $\boldsymbol{\sigma}_q^*$ and $\boldsymbol{\theta}_q^*$ as described in Remark 1. Then,

$$\lim_{q \to \infty} J_0(\mathbf{u}_q^*, \mathbf{v}) = J_0(\mathbf{u}^*, \mathbf{v}).$$

Theorem 4.2. Let $\mathbf{u}^*$ and $\mathbf{u}_q^*$ be defined as in Theorem 1. If $\mathbf{u}_q^*$ converges to a control $\hat{\mathbf{u}}^*$ almost everywhere, then $\hat{\mathbf{u}}^*$ is an optimal control of Problem $(P_{3,v})$.

5. Solution Method for Outer Problem

We now return to Problem (P_4). It can be considered as a discrete-valued optimization problem. To solve this discrete optimization problem, we will employ a discrete filled function method. In [12] and [9], a discrete filled function defined on a simple discrete box is introduced. For Problem (P_4), $\mathbf{\Upsilon_M}$ is not a simple discrete box. Thus, we need to introduce some additional concepts before we can apply the filled function method.

First, we define a measure $\mu_{M} : \mathbf{\Upsilon_M} \times \mathbf{\Upsilon_M} \to \mathbb{R}$ as follows:

$$\mu_{M}(\mathbf{v}, \mathbf{v}') = \sum_{i=1}^{M} (1 - \delta_{v_i v_i'}), \tag{38}$$

where δ_{ij} is Kronecker delta. It can be easily shown that the space $(\mathbf{\Upsilon_M}, \mu_{M})$ is a metric space.

For a $\mathbf{v} \in \mathbf{\Upsilon_M}$, define

$$\mathcal{N}[\mathbf{v}] = \{\mathbf{v}\} \cup \{\mathbf{v}' \in \mathbf{\Upsilon_M} \ : \ \mu_{M}(\mathbf{v}, \mathbf{v}') = M - 1\},$$

and

$$\mathcal{N}(\mathbf{v}) = \mathcal{N}[\mathbf{v}] \backslash \{\mathbf{v}\},$$

where $\mathcal{N}(\mathbf{v})$ is called a discrete neighborhood of $\mathbf{v} \in \mathbf{\Upsilon_M}$. It is clear that $\mathcal{N}(\mathbf{v}) \subset \mathbf{\Upsilon_M}$ for any $\mathbf{v} \in \mathbf{\Upsilon_M}$. Thus, each point in the space

$(\mathbf{\Upsilon_M}, \mu_{_M})$ is an interior point.

Definition 5.1. A point $\mathbf{v}^*$ is called a discrete local minimizer of J over $(\mathbf{\Upsilon_M}, \mu_{_M})$ if $J(\mathbf{v}^*) \leq J(\mathbf{v})$, $\forall \mathbf{v} \in \mathcal{N}[\mathbf{v}^*]$. In addition, if $J(\mathbf{v}^*) < J(\mathbf{v})$, $\forall \mathbf{v} \in \mathcal{N}(\mathbf{v}^*)$, then $\mathbf{v}^*$ is called a strict local minimizer of J over $(\mathbf{\Upsilon_M}, \mu_{_M})$.

Definition 5.2. A sequence $\{\mathbf{v}^i\}_{i=0}^p$ is called a discrete path in $(\mathbf{\Upsilon_M}, \mu_{_M})$ between two distinct points $\mathbf{v}^*$ and $\mathbf{v}^{**}$ if

$$\mu_{_M}(\mathbf{v}^*, \mathbf{v}^0) = \mu_{_M}(\mathbf{v}^{i-1}, \mathbf{v}^i) = \mu_{_M}(\mathbf{v}^{p-1}, \mathbf{v}^{**}) = M - 1,$$

for each $i = 1, ..., p-1$. If such a discrete path exists, then $\mathbf{v}^*$ and $\mathbf{v}^{**}$ are said to be pathwise connected in $(\mathbf{\Upsilon_M}, \mu_{_M})$. Furthermore, if every two distinct points in $(\mathbf{\Upsilon_M}, \mu_{_M})$ are pathwise connected, then $\mathbf{\Upsilon_M}$ is called a pathwise connected set.

With these definitions, we note that: (a) $\mathbf{\Upsilon_M}$ is a pathwise connected set; (b) $\mathbf{\Upsilon_M}$ is a bounded set which contains more than one point; and (c) $\widehat{J}_0 : \bigcup_{\mathbf{v} \in \mathbf{\Upsilon_M}} \mathcal{N}(\mathbf{v}) \to \mathbb{R}$ satisfies the following Lipschitz condition

$$|\widehat{J}_0(\mathbf{v}) - \widehat{J}_0(\mathbf{w})| \leq L\|\mathbf{v} - \mathbf{w}\|,$$

for every $\mathbf{v},\ \mathbf{w} \in \bigcup_{\mathbf{v} \in \mathbf{\Upsilon_M}} \mathcal{N}(\mathbf{v})$.

In the following, we present a discrete descent method for finding a local minimizer of $\widehat{J}_0$ over $(\mathbf{\Upsilon_M}, \mu_{_M})$ from any given initial point $\mathbf{v} \in \mathbf{\Upsilon_M}$.

Algorithm 5.1. (Discrete Descent Method)

Step 1: Choose any initial point $\mathbf{v}^0 \in \mathbf{\Upsilon_M}$.

Step 2: If $\mathbf{v}^0$ is a local minimizer of $\widehat{J}_0$ over $\mathcal{N}[\mathbf{v}^0]$, then stop; otherwise search the neighborhood $\mathcal{N}(\mathbf{v}^0)$ and obtain a $\mathbf{v} \in \mathcal{N}(\mathbf{v}^0) \cap \mathbf{\Upsilon_M}$ such that $\widehat{J}_0(\mathbf{v}) \leq \widehat{J}_0(\mathbf{v}^0)$.

Step 3: Let $\mathbf{v}^0 = \mathbf{v}$, goto Step 2.

Remark 5.1. In Algorithm 1, the evaluation of $\widehat{J}_0(\mathbf{v})$ for each $\mathbf{v} \in \mathbf{\Upsilon_M}$ is required to solve Problem $(P_{3,v})$. For each $\mathbf{v} \in \mathbf{\Upsilon_M}$, we take the control $\mathbf{u}^*(.|\mathbf{v})$ obtained in Section as the optimal control and write $\widehat{J}_0(\mathbf{v}) = J_0(\mathbf{u}^*(.|\mathbf{v}))$.

Clearly, Algorithm 1 only gives rise to a discrete local minimizer, which may not be a global minimizer. Thus, we need to apply a filled function method so as to jump out from the basin containing the local minimizer obtained. For this, we need some further definitions.

Let $\mathbf{v}_1^*$ be a current local minimizer of $\widehat{J}_0(\mathbf{v})$. Define

$$\mathbf{S}_1 = \{\mathbf{v} \in \mathbf{\Upsilon_M} \ : \ \widehat{J}_0(\mathbf{v}) \geq \widehat{J}_0(\mathbf{v}_1^*)\}, \tag{43}$$

and

$$\mathbf{S}_2 = \{\mathbf{v} \in \mathbf{\Upsilon_M} \ : \ \widehat{J}_0(\mathbf{v}) < \widehat{J}_0(\mathbf{v}_1^*)\}. \tag{44}$$

Definition 5.3. The function $\mathbf{P}_{\mathbf{v}_1^*}(\mathbf{v})$ is called a filled function of $\widehat{J}_0(\mathbf{v})$ at a local minimizer $\mathbf{v}_1^*$ if the following properties are satisfied.

(I) $\mathbf{P}_{\mathbf{v}_1^*}(\mathbf{v})$ has no local minimizer in the set $\mathbf{S}_1 \backslash \{\mathbf{v}^0\}$, where $\mathbf{v}^0$ is any initial point in the set $\mathbf{S}_1$, and is not necessary a local minimizer of $\mathbf{P}_{\mathbf{v}_1^*}(\mathbf{v})$.

(II) If $\mathbf{v}_1^*$ is not a global minimizer of $\widehat{J}_0(\mathbf{v})$, then there exists a local minimizer $\mathbf{v}_1$ of $\mathbf{P}_{\mathbf{v}_1^*}(\mathbf{v})$ such that $\widehat{J}_0(\mathbf{v}_1) < \widehat{J}_0(\mathbf{v}_1^*)$, i.e. $\mathbf{v}_1 \in \mathbf{S}_2$.

Here, we used the discrete filled function introduced in [9], but equipped with the measure defined by (38). Consider

$$\mathbf{P}_{A,\mathbf{v}_1^*,\mathbf{v}^0}(\mathbf{v}) = \mu_{_M}(\mathbf{v}, \mathbf{v}^0) - A(1 - e^{-[\min\{J(\mathbf{v})-J(\mathbf{v}_1^*)\ ,\ 0\}]^2}), \qquad (45)$$

where $\mathbf{v}^0 \in \mathbf{S}_1$ and $\mathbf{v}_1^*$ is the local minimizer of $\widehat{J}_0(\mathbf{v})$ over $\mathcal{N}(\mathbf{v}^0)$. Here, we take $A = \frac{Me^{(0.05)^2}}{e^{(0.05)^2}-1}$, where $M = \max_{\mathbf{v}\in\mathbf{\Upsilon_M}} \mu_{_M}(\mathbf{v}, \mathbf{v}^0)$.
In order to show that the function (45) is a filled function of $\widehat{J}_0(\mathbf{v})$ at a local minimizer $\mathbf{v}_1^*$, we need to verify that $\mathbf{P}_{A,\mathbf{v}_1^*,\mathbf{v}^0}(\mathbf{v})$ satisfies the conditions *(I)* and *(II)* given in Definition 3.

Lemma 5.1. For each $\mathbf{v} \in \mathbf{\Upsilon_M}$, if $\mu_{_M}(\mathbf{v}, \mathbf{v}^0) > 2$, then there exists a $\widehat{\mathbf{v}} \in \mathbf{\Upsilon_M}$ such that

$$\mu_{_M}(\widehat{\mathbf{v}}, \mathbf{v}^0) < \mu_{_M}(\mathbf{v}, \mathbf{v}^0). \qquad (46)$$

Proof. From (38), since $\mu_{_M}(\mathbf{v}, \mathbf{v}^0) > 2$, $I = \{1, 2, ..., M\}$ can be partitioned into $I_1 = \{i_1, i_2, ..., i_p\}$ and $I_2 = \{i_{p+1}, i_{p+2}, ..., i_M\}$ such that $v_j = v_j^0$, for each $j \in I_1$, $v_j \neq v_j^0$, for each $j \in I_2$ and $|I_2| > 2$. Now, define $\widehat{v}_j = v_j$ for each $j \in I_1$, $\widehat{v}_{i_{p+1}} = v_{i_{p+1}}^0$, and the remaining elements of $\widehat{v}$ are assigned arbitrary so that $\widehat{v} \in \mathbf{\Upsilon_M}$. Then,

$$\begin{aligned}
\mu_M(\widehat{\mathbf{v}}, \mathbf{v}^0) &= \sum_{i=1}^{M}(1 - \delta_{\widehat{v}_i v_i^0}) \\
&= \sum_{i \in I_1}(1 - \delta_{\widehat{v}_i v_i^0}) + \sum_{i \in I_2}(1 - \delta_{\widehat{v}_i v_i^0}) \\
&= 0 + \sum_{i \in I_2}(1 - \delta_{\widehat{v}_i v_i^0}) \\
&= (1 - \delta_{\widehat{v}_{i_{p+1}} v^0_{i_{p+1}}}) + \sum_{i \in I_2 \setminus \{i_{p+1}\}}(1 - \delta_{\widehat{v}_i v_i^0}) \\
&= \sum_{i \in I_2 \setminus \{i_{p+1}\}}(1 - \delta_{\widehat{v}_i v_i^0}) \\
&< \sum_{i \in I_2}(1 - \delta_{v_i v_i^0}) \\
&= \mu_M(\mathbf{v}, \mathbf{v}^0).
\end{aligned}$$

Thus, $\widehat{v}$ is the required point. □

Based on Lemma 1 and the arguments given for the proofs of Theorem 3.1 and Theorem 3.2 of [9], we can show that the results presented in the following two theorems are valid.

Theorem 5.1. Let $A = \frac{Me^{(0.05)^2}}{e^{(0.05)^2}-1}$. Then, $\mathbf{P}_{A,\mathbf{v}_1^*,\mathbf{v}^0}(\mathbf{v})$ has no local minimizer in the set $\mathbf{S}_1 \setminus \{\mathbf{v}^0\}$.

Theorem 5.2. Let the set $\mathbf{S}_2$ be nonempty. Suppose that the parameter A is chosen such that

$$A > \frac{M \ e^{([\widehat{J}_0(\mathbf{v}^*) - \widehat{J}_0(\mathbf{v}_1^*)]^2)}}{e^{([\widehat{J}_0(\mathbf{v}^*) - \widehat{J}_0(\mathbf{v}_1^*)]^2)} - 1},$$

where $\mathbf{v}^*$ is a global minimizer of $\widehat{J}_0(\mathbf{v})$. Then, $\mathbf{P}_{A,\mathbf{v}_1^*,\mathbf{v}^0}(\mathbf{v})$ has a local minimizer in the set $\mathbf{S}_2$.

We are now in a position to present an algorithm to solve Problem (P_4).

Algorithm 5.2. (Discrete Filled Function Method)
Step 1: Input any initial point $\mathbf{v}^0 \in \boldsymbol{\Upsilon}_{\mathbf{M}}$.
Step 2: Employ Algorithm 1 starting from $\mathbf{v}^0$ to obtain a local minimizer of $\widehat{J}_0$. Let the local minimizer obtained be denoted as $\mathbf{v}_1^*$.
Step 3: For each $\widehat{\mathbf{v}} \in \mathcal{N}(\mathbf{v}_1^*)$, use Algorithm 1 to minimize the filled function $\mathbf{P}_{A,\mathbf{v}_1^*,\mathbf{v}^0}(\mathbf{v})$, starting from $\widehat{\mathbf{v}}$. If at any stage, a point $\mathbf{v}$ such that $\widehat{J}_0(\mathbf{v}) < \widehat{J}_0(\mathbf{v}_0)$ is found, then set $\mathbf{v}^0 = \mathbf{v}$ and goto Step 2. Otherwise goto Step 4.
Step 4: Stop the algorithm and output $\mathbf{v}_1^*$, which is regarded as a global minimizer of $\widehat{J}_0$.

Remark 5.2. In *Step 2* and *Step 3* of Algorithm 2, the evaluation of $\widehat{J}_0(\mathbf{v})$ for each $\mathbf{v} \in \boldsymbol{\Upsilon}_{\mathbf{M}}$ involves solving Problem $(P_{3,v})$. As mentioned in Remark 2, the control $\mathbf{u}^*(.|\mathbf{v})$ obtained as described in Remark 1 is taken as the optimal control of Problem $(P_{3,v})$ and we write $\widehat{J}_0(\mathbf{v}) = J_0(\mathbf{u}^*(.|\mathbf{v}))$.

6. A Numerical Example

To demonstrate the effectiveness of the solution procedure proposed, we apply Algorithm 2, to a numerical example. During the computation, the optimal control software MISER 3.3 (see [13]) is used to solve the corresponding inner problem $(P_{3,v})$ for each corresponding $\mathbf{v} \in \mathbf{\Upsilon_M}$.

Consider a dynamical system, which is composed of the four subsystems described by

$$Subsys.\ (1): \begin{cases} \dot{x}_1(t) = \big(x_1(t)\sin(t) + u(t)\big)\chi_{[0,t_f^1]}(t) \\ \dot{x}_2(t) = \big(x_2(t)\big)\chi_{[0,t_f^1]}(t), \end{cases} \tag{48}$$

$$Subsys.\ (2): \begin{cases} \dot{x}_3(t) = \big(x_4(t)\cos(t) + u(t)\big)\chi_{[0,t_f^2]}(t) \\ \dot{x}_4(t) = \big(x_3(t)\big)\chi_{[0,t_f^2]}(t), \end{cases} \tag{49}$$

$$Subsys.\ (3): \begin{cases} \dot{x}_5(t) = \big(x_6(t)\big)\chi_{[0,t_f^3]}(t) \\ \dot{x}_6(t) = \big(x_5(t)\sin(t) + u(t)\big)\chi_{[0,t_f^3]}(t), \end{cases} \tag{50}$$

$$Subsys.\ (4): \begin{cases} \dot{x}_7(t) = \big(x_7(t)\big)\chi_{[0,t_f^4]}(t) \\ \dot{x}_8(t) = \big(x_8(t)\cos(t) + u(t)\big)\chi_{[0,t_f^4]}(t). \end{cases} \tag{51}$$

The initial conditions for the above system are

$$[x_1(0), x_2(0)]^T = [0, 1]^T, \tag{52}$$

$$[x_3(0), x_4(0)]^T = [0, 0]^T, \tag{53}$$

$$[x_5(0), x_6(0)]^T = [0, 0]^T, \tag{54}$$

$$[x_7(0), x_8(0)]^T = [1, 0]^T. \tag{55}$$

Assume that the control u is such that $u(t) \in [-10, 10]$ for all $t \geq 0$. Our time-optimal control problem is stated as follows.

Given the dynamical system (48)-(51)with the initial conditions (52)-(55), find a control u such that the cost functional

$$J_0(u) = \max\{t_f^1, t_f^2, t_f^3, t_f^4\}$$

is minimized subject to the following inequality constraints:

$$|x_1(t_f^1) - 2|^2 + |x_2(t_f^1) - 2|^2 \le 0.15,$$

$$|x_3(t_f^2) - 1|^2 + |x_4(t_f^2) - 3|^2 \le 0.15,$$

$$|x_5(t_f^3) - 3|^2 + |x_6(t_f^3) - 1|^2 \le 0.15,$$

$$|x_7(t_f^4) - 3|^2 + |x_8(t_f^4) - 3|^2 \le 0.15.$$

We apply Algorithm 2 for which we choose an initial switching sequence as:

$$\mathbf{v}_1^0 = [2 \quad 3 \quad 4 \quad 1].$$

This means that

$$t_f^2 \le t_f^3 \le t_f^4 \le t_f^1.$$

The corresponding inner problem $(P_{3,v})$ is solved by the numerical procedure described in Section , for which the optimal control software package MISER 3.3 is used.

We choose $q = 3$ and hence $qM = 12$. Thus,

$$0 = t_0 = \tau_0, \quad \tau_3 = t_f^2, \quad \tau_6 = t_f^3, \quad \tau_9 = t_f^4, \quad \tau_{12} = t_f^1 = t_f.$$

Algorithm 1 equipped with MISER 3.3 is used to minimize the cost function $\widehat{J}_0(\mathbf{v})$.

The first local minimizer $\widehat{J}_0(\mathbf{v})$ is found to be $\mathbf{v}_1^* = [4 \ 3 \ 2 \ 1]$ with the corresponding cost value $\widehat{J}_0(\mathbf{v}_1^*) = 0.79675344$. To continue, we construct the discrete filled function $\mathbf{P}_{A,\mathbf{v}_1^*,\mathbf{v}_1^0}(\mathbf{v})$. Minimizing the filled function yields a point $\mathbf{v}_{1,p}^* = [4 \ 2 \ 1 \ 3]$ with the corresponding cost value $\widehat{J}_0(\mathbf{v}_{1,p}^*) = 0.79675302$, which is less than $\widehat{J}_0(\mathbf{v}_1^*)$. That is, $\mathbf{v}_{1,p}^* \in \mathbf{S}_2 = \{\mathbf{v} \in \mathbf{\Upsilon_4} \ : \ \widehat{J}_0(\mathbf{v}) < \widehat{J}_0(\mathbf{v}_1^*)\}$. Therefore, we again minimize $\widehat{J}_0(\mathbf{v})$ using Algorithm 1 starting from $\mathbf{v}_2^0 = \mathbf{v}_{1,p}^*$.

The second local minimizer of $\widehat{J}_0(\mathbf{v})$, which is $\mathbf{v}_2^* = [4 \quad 2 \quad 1 \quad 3]$, is obtained. Again, we apply Algorithm 1 to minimize the filled function $\mathbf{P}_{A,\mathbf{v}_2^*,\mathbf{v}_2^0}(\mathbf{v})$. We obtain no local minimizer of $\mathbf{P}_{A,\mathbf{v}_2^*,\mathbf{v}_2^0}(\mathbf{v})$ in $\widetilde{\mathbf{S}}_2 = \{\mathbf{v} \in \mathbf{\Upsilon_4} \; : \; \widehat{J}_0(\mathbf{v}) < \widehat{J}_0(\mathbf{v}_2^*)\}$. Thus, we stop Algorithm 2 and take $\mathbf{v}_2^* = [4 \quad 2 \quad 1 \quad 3]$ as the global minimizer of $\widehat{J}_0(\mathbf{v})$. The corresponding cost function value is $\widehat{J}_0(\mathbf{v}_2^*) = 0.79675302$.

The obtained local minimizers and their corresponding cost function values are summarized in the following table.

Local minimizer	Cost function value
[2 3 4 1]	0.87861271
[4 3 2 1]	0.79675344
[4 2 1 3]	0.79675302

The optimal control and optimal system trajectories are depicted in Figures 1 and 2, respectively. Since the optimal switching sequence is $\mathbf{v}_2^* = [4 \quad 2 \quad 1 \quad 3]$, subsystems (4), (2) and (1) terminate before the end of the overall time horizon at $t_f^4 = 0.51$, $t_f^2 = 0.65$ and $t_f^1 = 0.73$. Due to the characteristic function appearing in (48)-(51), after termination, the states of each subsystem remain at their terminal value for the remainder of the time horizon. This can be seen clearly from Figure 1.

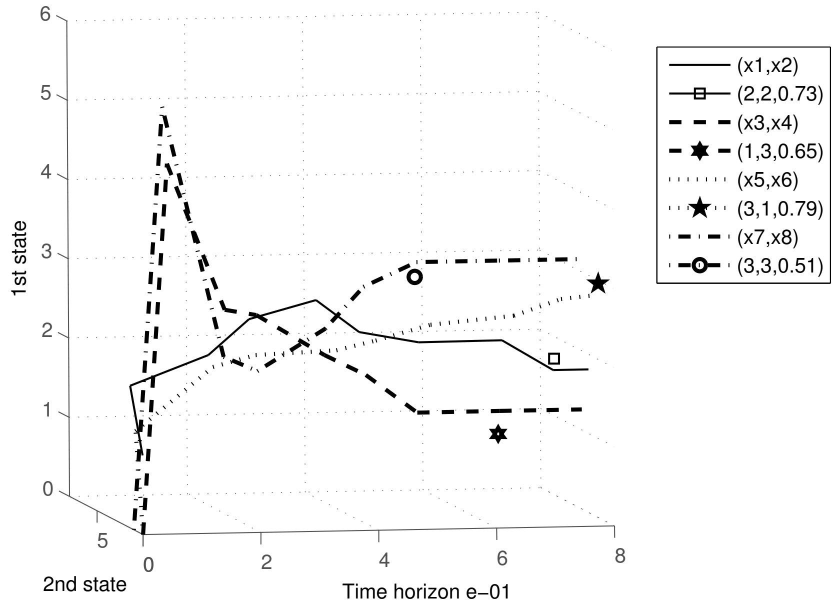

Figure 1: The optimal trajectories

7. Conclusion

In this paper, we introduced a new class of non-standard time optimal control problems in which the cost function is non-differentiable and the dynamical system are composed of M-subsystems. Each of these subsystems evolves over a different time horizon.

We developed an algorithm by combining a filled function method, discrete local search and a time scaling transform. From the numerical example solved, we see that the algorithm is highly efficient.

References

[1] Fisher, M. E., Noakes, J. L. and Teo K. L., A Minimum Trapping Time Problem in Optimal Control Theory, J. of Australian Mathematical Society, series B, Vol. 32, pp. 100-114, (1990).

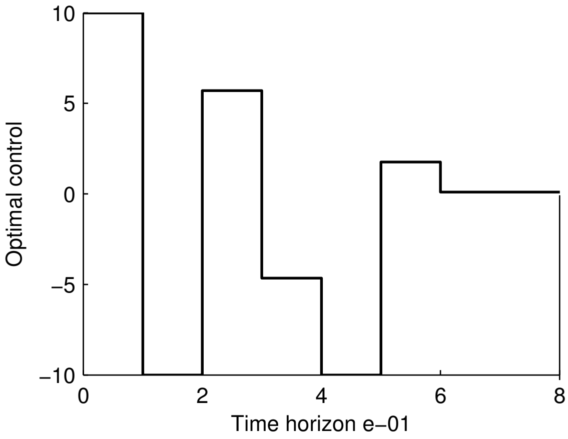

Figure 2: The optimal control

[2] Lee, H. W. J., Jennings, L. S., Teo, K. L. and Rehbock, V., Control Parametrization Enhancing Technique for Time Optimal Control Problems, Dynamic Systems and Applications, Vol. 6, pp. 243-262, (1997).

[3] Kaya, C. Y. and Noakes, J. L., Computational Method for Time-optimal Switching Control, Journal of Optimization Theory and Applications, Vol. 117, pp. 69-92, (2003).

[4] Teo, K. L. and Lim, C. C., Time Optimal Control Computation with Application to Ship Steering, J. of Optimization Theory and Applications, Vol. 56, pp. 145-156, (1988).

[5] Doyen, L. and Quincampoix M., Multi-target Control Problems, J. of Optimization Theory and Applications, Vol. 93, pp. 121-139, (1997).

[6] Mehne H. H., Farahi M. H., and Kamyad A. V., MILP Modelling for the Time Optimal Control Problem in the Case of Multiple Targets, Optimal Control Applications and Methods, Vol. 27, pp. 77-91, (2006).

[7] Rehbock, V and Livk I., Optimal Control of a Batch Crystallization Process, J. of Industrial and Managment Optimization, (to appear).

[8] Wu, C. Z. and Teo, K. L., Global Impulsive Optimal Control Computation, J. of Industrial and Managment Optimization, Vol. 2, pp. 435-450, (2006).

[9] Shang, Y. L. and Zhang L. S., A Filled Function Method for Finding a Global Minimizer on Global Integer Optimization, J. of CAM, Vol. 181, pp. 200-210, (2005).

[10] Teo, K. L., Goh, C. J., and Wong, K. H., A Unified Computational Approach to Optimal Control Problems, Longman Scientific and Technical, New York, (1991).

[11] Nocedal, J. and Wright, S. J., Numerical Optimization, Springer , New York, (1999).

[12] Levy, A. V. and Montalvo, A., The Tunneling Algorithm for the Global Mini- mization of Functions, SIAM Journal on Scientic and Statistical Computing, Vol. 6, pp. 15-29, (1985).

[13] Jennings, L. S., Fisher, M. E., Teo, K. L. and Goh, C. J. MISER 3, Optimal Control Software: Theory and User Manual, EMCOSS Pty Ltd, Perth, Australia, (1990).