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Research Article

Multiple Positive Solutions of Singular Nonlinear Sturm-Liouville Problems with Carathéodory Perturbed Term

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By employing a well-known fixed point theorem, we establish the existence of multiple positive solutions for the following fourth-order singular differential equation $Lu = p(t)f(t, u(t), u''(t)) - g(t, u(t), u''(t)), 0 < t < 1, \alpha_1 u(0) - \beta_1 u'(0) = 0, \gamma_1 u(1) + \delta_1 u'(1) = 0, \alpha_2 u''(0) - \beta_2 u'''(0) = 0, \gamma_2 u''(1) + \delta_2 u'''(1) = 0$, with $\alpha_i, \beta_i, \gamma_i, \delta_i \geq 0$ and $\beta_i\gamma_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0,\ \ i = 1, 2$, where L denotes the linear operator $Lu := (ru''')' - qu'', r \in C^1([0,1], (0,+\infty))$, and $q \in C([0,1], [0,+\infty))$. This equation is viewed as a perturbation of the fourth-order Sturm-Liouville problem, where the perturbed term $g : (0,1) \times [0,+\infty) \times (-\infty,+\infty) \to (-\infty,+\infty)$ only satisfies the global Carathéodory conditions, which implies that the perturbed effect of g on f is quite large so that the nonlinearity can tend to negative infinity at some singular points.

1. Introduction

In this paper, we consider the existence of multiple positive solutions for the following fourth-order singular Sturm-Liouville boundary value problem involving a perturbed term

$$\begin{aligned} Lu = p(t)f\left(t, u(t), u''(t)\right) - g\left(t, u(t), u''(t)\right), \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \qquad \gamma_1 u(1) + \delta_1 u'(1) = 0, \\ \alpha_2 u''(0) - \beta_2 u'''(0) = 0, \qquad \gamma_2 u''(1) + \delta_2 u'''(1) = 0, \end{aligned} \tag{1.1}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \geq 0$ and $\beta_i\gamma_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$, $i = 1, 2$, and L denotes the linear operator $Lu := (ru''')' - qu''$, $r \in C^1([0,1], (0, +\infty))$ and $q \in C([0,1], [0, +\infty))$ and $q \in C([0,1], [0, +\infty))$. The perturbed term, $g : (0,1) \times [0, +\infty) \times (-\infty, +\infty) \to [0, +\infty)$, satisfies global Carathéodory's conditions.

Equation (1.1) arises from many branches of applied mathematics and physics; for details, see [1–16]. It mainly describes the deformation of an elastic beam for $g(t, u, u'') \equiv 0$; for example, under the Lidstone boundary condition,

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

problem (1.1) is used to model such phenomena as the deflection of an elastic beam simply supported at the endpoints; see [1, 3, 5, 7–11]. Also, if the boundary condition of (1.1) is a Focal boundary condition, then it describes the deflection of an elastic beam having both end-points fixed, or having one end simply supported and the other end clamped with sliding clamps. In addition, the derivative u'' in f is the bending moment term which represents the bending effect, see [1, 3, 5, 7–11, 13, 14, 16]. A brief discussion of the physical interpretation under some boundary conditions associated with the linear beam equation can be found in Zill and Cullen [17].

Recently, for the case where the nonlinearity f does not contain the bending moment term u'', Ma and Wang [1] studied the existence of positive solutions for (1.1) subject to boundary conditions $u(0) = u(1) = u''(0) = u''(1) = 0$ and $u(0) = u'(1) = u''(0) = u'''(1) = 0$ if f is superlinear or sublinear. In the case where f contains the bending moment term u'' and under the particular boundary conditions, the authors of papers [9, 12] studied the existence of positive solutions for (1.1) when f satisfies the following growth condition:

$$\left| f(t, x, y) - (\alpha x - \beta y) \right| \leq a|x| + b\left|y\right| + c, \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}$ $a, b, c > 0, a, b$ is enough small. But most of the above works were done on base of the assumptions that the nonlinearity is nonnegative and has no any singularity. In recent years, one found that the fourth-order changing-sign nonlinear problems also occur to the classical model for the elastic beam fixed at both ends, especially in the medium span or large span bridge constructions, this implies that it is necessary and quite natural to study fourth-order changing-sign boundary value problems.

In this paper, we focus on the particularly difficult and interesting situation, when (1.1) is singularly perturbed, so that the nonlinearity is allowed to change sign, even may tend to negative infinity. This problem has essential difference from those unperturbed problems of [1–16]. We quote in the sequel some papers from the relevant bibliography devoted to this subject. In [18], Loud considered the existence of T-periodic solutions for a first-order perturbed system of ordinary differential equations by employing the so-called bifurcation function

$$f_0(\theta) = \int_0^T \langle z_0(\tau), \phi(\tau - \theta, x_0(\tau), 0) \rangle d\tau. \tag{1.4}$$

Moreover, the author of [18] also considered the case when θ_0 is not a simple zero of f_0, and the existence of T-periodic solutions of the above problem is associated with the existence of the roots of a certain quadratic equation. Recently, by using the exponential dichotomies and contraction mapping principle, Xia et al. [19] established some sufficient conditions of the existence and uniqueness of almost periodic solution for a forced perturbed system with piecewise constant argument. The other works, such as Khanmamedov [20], Wu and Gan [21], Makarenkov and Nistri [22], Liu and Yang [23], Clavero et al. [24], and Cui and Geng [25], are rich sources for application of perturbed problems.

Our main tool used for the analysis here is known as Guo-Krasnoselskii's fixed point theorem, for the convenience of the reader, we now state it as follows.

Lemma 1.1 (see, [26]). *Let E be a real Banach space, $P \subset E$ a cone. Assume Ω_1, Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(1) $\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_1$, *and* $\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_2$, *or*

(2) $\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_1$, *and* $\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_2$.

Then, T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries and Lemmas

The following definition introduces global Carathéodory's conditions imposed on a map.

Definition 2.1. A map $g : (t, x, y) \mapsto g(t, x, y)$ is said to satisfy global Crathéodory's conditions if the following conditions hold:

(i) for each $(x, y) \in \mathbb{R} \times \mathbb{R}$, the mapping $t \mapsto g(t, x, y)$ is Lebesgue measurable;

(ii) for a.e. $t \in [0, 1]$, the mapping $(x, y) \mapsto g(t, x, y)$ is continuous on $\mathbb{R} \times \mathbb{R}$;

(iii) there exists a $\rho \in L^1[0, 1]$ such that, for a.e. $t \in [0, 1]$ and $(x, y) \in \mathbb{R} \times \mathbb{R}$, we have

$$\left|g(t, x, y)\right| \leq \rho(t). \tag{2.1}$$

The following lemmas play an important role in proving our main results.

Lemma 2.2 (see, [27]). *Let ψ_2 and ϕ_2 be the solutions of the linear problems*

$$\begin{gathered}
-\left(r(t)\phi_2'(t)\right)' + q(t)\phi_2(t) = 0, \quad 0 < t < 1, \\
\phi_2(0) = \beta_2, \phi_2'(0) = \alpha_2, \\
-\left(r(t)\psi_2'(t)\right)' + q(t)\psi_2(t) = 0, \quad 0 < t < 1, \\
\psi_2(0) = \delta_2, \psi_2'(0) = -\gamma_2,
\end{gathered} \tag{2.2}$$

respectively. Then,

(i) ϕ_2 *is strictly increasing on* $[0, 1]$ *and* $\phi_2(t) > 0$ *on* $(0, 1]$;

(ii) ψ_2 *is strictly decreasing on* $[0, 1]$ *and* $\psi_2(t) > 0$ *on* $[0, 1)$.

Set

$$w_2 = -r(t)\left(\phi_2(t)\psi_2'(t) - \psi_2(t)\phi_2'(t)\right), \tag{2.3}$$

by Liouville's formula, one can easily show $w_2 = \text{constant} > 0$.

As [27], we define Green's function for the BVP:

$$\begin{gathered} -\left(r(t)u'(t)\right)' + q(t)u(t) = 0, \quad 0 < t < 1, \\ \alpha_2 u(0) - \beta_2 u'(0) = 0, \qquad \gamma_2 u(1) + \delta_2 u'(1) = 0, \end{gathered} \tag{2.4}$$

by

$$G_2(t,s) = \frac{1}{w_2}\begin{cases} \phi_2(t)\psi_2(s), & 0 \le t \le s \le 1, \\ \phi_2(s)\psi_2(t), & 0 \le s \le t \le 1, \end{cases} \tag{2.5}$$

then we have the following lemma.

Lemma 2.3. *For any* $(t,s) \in [0,1] \times [0,1]$, $i = 1,2$, *we have*

$$\theta_2 G_2(s,s) G_2(t,t) \le G_2(t,s) \le G_2(s,s), \quad (\text{or } G_2(t,t)), \tag{2.6}$$

where

$$\theta_2 = \frac{w_2}{\phi_2(1)\psi_2(0)}. \tag{2.7}$$

Proof. It follows from the monotonicity of $\phi_2(t)$ and $\psi_2(t)$ that the right-hand side of (2.6) holds. For the left hand side, by the monotonicity of $\phi_2(t)$ and $\psi_2(t)$, we have

$$\begin{aligned} G_2(t,s) &= \frac{1}{w_2}\begin{cases} \phi_2(t)\psi_2(s), & 0 \le t \le s \le 1, \\ \phi_2(s)\psi_2(t), & 0 \le s \le t \le 1, \end{cases} \\ &\ge \frac{1}{w_2}\begin{cases} \phi_2(t)\psi_2(s)\dfrac{\phi_2(s)\psi_2(t)}{\phi_2(1)\psi_2(0)}, & 0 \le t \le s \le 1, \\ \phi_2(s)\psi_2(t)\dfrac{\phi_2(t)\psi_2(s)}{\phi_2(1)\psi_2(0)}, & 0 \le s \le t \le 1, \end{cases} \\ &= \frac{w_2}{\phi_2(1)\psi_2(0)} G_2(t,t) G_2(s,s), \\ &= \theta_2 G_2(t,t) G_2(s,s). \end{aligned} \tag{2.8}$$

The proof is completed. □

Also, it is well known the Green function for the boundary value problem

$$\begin{gathered} -u'' = 0, \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \qquad \gamma_1 u(1) + \delta_1 u'(1) = 0, \end{gathered} \tag{2.9}$$

is

$$G_1(t,s) = \frac{1}{w_1} \begin{cases} (\beta_1 + \alpha_1 s)(\gamma_1 + \delta_1(1-t)), & 0 \le s \le t \le 1, \\ (\beta_1 + \alpha_1 t)(\gamma_1 + \delta_1(1-s)), & 0 \le t \le s \le 1, \end{cases} \tag{2.10}$$

where $w_1 = \beta_1\gamma_1 + \alpha_1\gamma_1 + \alpha_1\delta_1$. Let

$$e(t) = \frac{1}{w_1}(\beta_1 + \alpha_1 t)(\gamma_1 + \delta_1(1-t)), \tag{2.11}$$

clearly,

$$e(t)e(s) \le G_1(t,s) \le e(s). \tag{2.12}$$

Now, we define an integral operator $S : C[0,1] \to [0,1]$ by

$$Sv(t) = \int_0^1 G_1(t,\tau)v(\tau)d\tau, \tag{2.13}$$

and, then, by (2.9), we have

$$\begin{gathered} (Sv)''(t) = -v(t), \quad 0 < t < 1, \\ \alpha_1 (Sv)(0) - \beta_1 (Sv)'(0) = 0, \qquad \gamma_1 (Sv)(1) + \delta_1 (Sv)'(1) = 0. \end{gathered} \tag{2.14}$$

In order to obtain existence of positive solutions to problem (1.1), we will consider the existence of positive solutions to the following modified problem

$$\begin{gathered} -(rv')'(t) + qv(t) = p(t)f(t, Sv(t), -v(t)) - g(t, Sv(t), -v(t)), \quad 0 < t < 1, \\ \alpha_2 v(0) - \beta_2 v'(0) = 0, \qquad \gamma_2 v(1) + \delta_2 v'(1) = 0. \end{gathered} \tag{2.15}$$

Lemma 2.4. *Let $u(t) = Sv(t), v(t) \in C[0,1]$. Then, we can transform* (1.1) *into* (2.15)*. Moreover, if $v \in C([0,1],[0,+\infty)$ is a solution of problem* (2.15)*, then the function $u(t) = Sv(t)$ is a positive solution of problem* (1.1)*.*

Proof. It follows from (2.9) that $u''(t) = -v(t)$, put $u''(t) = -v(t)$ and $u(t) = Sv(t)$ into (1.1), we can transform (1.1) into (2.15).

Conversely, if $v \in C([0,1],[0,+\infty))$ is a solution of (2.15), let $u(t) = Sv(t)$, we have $u''(t) = -v(t)$, thus $u = Sv$ is a solution of (1.1). The proof of Lemma 2.4 is completed. □

In the rest of the paper, we always suppose that the following assumptions hold.

(B1) $p : (0,1) \to [0,+\infty)$ is continuous and satisfies

$$0 < \int_0^1 G_2(s,s)p(s)ds < +\infty. \tag{2.16}$$

(B2) $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to [0,+\infty)$ is continuous.

(B3) $g : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to [0,+\infty)$ satisfies global Crathéodory's condition and

$$\int_0^1 \rho(s)ds > 0. \tag{2.17}$$

Remark 2.5. It follows from (B1), (B3) and from the monotonicity of $\phi_2(t), \psi_2(t)$ that there exists $(a,b) \subset (0,1)$ such that

$$\begin{aligned}
0 &< \int_a^b G_2(s,s)p(s)ds \le \int_a^b G_2(s,s)\left[p(s) + \rho(s)\right]ds \\
&\le \int_0^1 G_2(s,s)\left[p(s) + \rho(s)\right]ds \\
&\le \int_0^1 G_2(s,s)p(s)ds + \frac{\phi_2(1)\psi_2(0)}{w_2}\int_0^1 \rho(s)ds \\
&< +\infty.
\end{aligned} \tag{2.18}$$

So for convenience, in the rest of this paper, we define serval constants as follows:

$$\begin{gathered}
K = \int_0^1 G_2(s,s)\left[p(s) + \rho(s)\right]ds, \qquad \mu_2 = \frac{\phi_2(a)\psi_2(b)}{\phi_2(1)\psi_2(0)}, \qquad l = \mu_2\int_a^b G_2(s,s)p(s)ds, \\
\mu_1 = \frac{(\beta_1 + \alpha_1 a)(\gamma_1 + \delta_1(1-b))\theta_2}{w_1}\int_0^1 G_1(s,s)G_2(s,s)ds.
\end{gathered} \tag{2.19}$$

Lemma 2.6. *Assume* (B3) *is satisfied. Then, the boundary value problem*

$$\begin{gathered}
-(r(t)y')'(t) + q(t)y(t) = \rho(t), \quad 0 < t < 1, \\
\alpha_2 y(0) - \beta_2 y'(0) = 0, \qquad \gamma_2 y(1) + \delta_2 y'(1) = 0,
\end{gathered} \tag{2.20}$$

has a unique solution

$$y(t) = \int_0^1 G_2(t,s)\rho(s)ds, \tag{2.21}$$

which satisfies

$$y(t) \le G_2(t,t)\int_0^1 \rho(s)ds. \tag{2.22}$$

Proof. First, $y(t) = \int_0^1 G_2(t,s)\rho(s)ds$ solves the BVP (2.20), and it is the unique solution of the BVP (2.20), since $-(r(t)y')'(t) + q(t)y(t) = 0$ with boundary conditions

$$\alpha_2 y(0) - \beta_2 y'(0) = 0, \qquad \gamma_2 y(1) + \delta_2 y'(1) = 0 \tag{2.23}$$

only has a trivial solution. Finally, it follows from (2.6) and (B3) that (2.22) holds. □

Define a modified function $[\cdot]^*$ for any $z \in C[0,1]$ by

$$[z(t)]^* = \begin{cases} z(t), & z(t) \ge 0, \\ 0, & z(t) < 0. \end{cases} \tag{2.24}$$

We consider the following approximating problem

$$\begin{aligned} -(r(t)x')'(t) + q(t)x(t) &= p(t)f\left(t, S[x(t)-y(t)]^*, -[x(t)-y(t)]^*\right) \\ &\quad - g\left(t, S[x(t)-y(t)]^*, -[x(t)-y(t)]^*\right) + \rho(t), \quad 0<t<1, \\ \alpha_2 x(0) - \beta_2 x'(0) = 0, &\qquad \gamma_2 x(1) + \delta_2 x'(1) = 0. \end{aligned} \tag{2.25}$$

Lemma 2.7. *If $x(t) \ge y(t)$ for any $t \in [0,1]$ is a positive solution of the BVP* (2.25), *then $S(x-y)$ is a positive solution of the singular perturbed differential equation* (1.1).

Proof. In fact, if x is a positive solution of the BVP (2.25) such that $x(t) \ge y(t)$ for any $t \in [0,1]$, then, from (2.25) and the definition of $[z(t)]^*$, we have

$$\begin{aligned} -(r(t)x')'(t) + q(t)x(t) &= p(t)f\left(t, S(x(t)-y(t)), -(x(t)-y(t))\right) \\ &\quad - g\left(t, S(x(t)-y(t)), -(x(t)-y(t))\right) + \rho(t), \quad 0<t<1, \\ \alpha_2 x(0) - \beta_2 x'(0) = 0, &\qquad \gamma_2 x(1) + \delta_2 x'(1) = 0. \end{aligned} \tag{2.26}$$

Let $v = x - y$, then $-(r(t)v')'(t) + q(t)v(t) = -(rx')'(t) + qx(t) + (ry')'(t) - qy(t)$, which implies that

$$\begin{aligned} -(r(t)x')'(t) + q(t)x(t) &= -(r(t)v')'(t) + q(t)v(t) - (r(t)y')'(t) + q(t)y(t) \\ &= -(r(t)v')'(t) + q(t)v(t) + \rho(t), \\ \alpha_2 x(0) - \beta_2 x'(0) = 0, &\qquad \gamma_2 x(1) + \delta_2 x'(1) = 0. \end{aligned} \tag{2.27}$$

Thus, (2.26) becomes (2.15), that is, $x - y$ is a positive solution of the differential equation (2.15). By Lemma 2.4, $u = S(x-y)$ is a positive solution of the singular perturbed differential equation (1.1). This completes the proof of Lemma 2.7. □

Thus, the BVP (2.25) is equivalent to the integral equation

$$\begin{aligned} x(t) = \int_0^1 & G_2(t,s)\big[p(s)f\big(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\big) \\ & -g\big(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\big) + \rho(s)\big]ds. \end{aligned} \tag{2.28}$$

Hence, we will look for fixed points $x(t) \geq y(t), t \in [0,1]$, for the mapping T defined on $E := C([0,1],[0,+\infty))$ by

$$\begin{aligned} (Tx)(t) = \int_0^1 & G_2(t,s)\big[p(s)f\big(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\big) \\ & -g\big(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\big) + \rho(s)\big]ds. \end{aligned} \tag{2.29}$$

The basic space used in this paper is $E = C([0,1];\mathbb{R})$, where $\mathbb{R}$ is a real number set. Obviously, the space E is a Banach space if it is endowed with the norm as follows:

$$\|u\| = \max_{t\in[0,1]}|u(t)|, \tag{2.30}$$

for any $u \in E$. Let

$$P = \{x \in E : x(t) \geq \theta_2 G_2(t,t)\|x\|\}, \tag{2.31}$$

where θ_2 is defined by (2.7), then P is a cone of E.

Lemma 2.8. *Assume that* (B1)–(B3) *hold. Then,* $T : P \to P$ *is well defined. Furthermore,* $T : P \to P$ *is a completely continuous operator.*

Proof. For any fixed $x \in P$, there exists a constant $L > 0$ such that $\|x\| \leq L$. And then,

$$\begin{gathered} [x(s) - y(s)]^* \leq x(s) \leq \|x\| \leq L, \\ \big|S[x(s) - \mathrm{y}(s)]^*\big| \leq L\int_0^1 G_1(t,s)ds \leq \frac{(\beta_1+\alpha_1)(\gamma_1+\delta_1)}{w_1}L. \end{gathered} \tag{2.32}$$

On the other hand, since g satisfies global Carathéodory's condition, we have $g(\cdot, u(\cdot), v(\cdot)) \in L^1(0,1)$. Accordingly, Tx in (2.29) is continuous on $[0,1]$, and, by (2.32),

$$\begin{aligned}(Tx)(t) &= \int_0^1 G_2(t,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)\right.\\ &\qquad \left. -g\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)+\rho(s)\right]ds\\ &\le \int_0^1 G_2(s,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)+2\rho(s)\right]ds\\ &\le N\int_0^1 G_2(s,s)\left[p(s)+\rho(s)\right]ds < +\infty,\end{aligned} \tag{2.33}$$

where

$$N = \max_{(t,u,v)\in[0,1]\times[0,((\beta_1+\alpha_1)(\gamma_1+\delta_1)/w_1)L]\times[-L,0]} f(t,u,v)+2. \tag{2.34}$$

This implies that the operator $T : P \to E$ is well defined.

Next, for any $x \in P$, by (2.6), we have

$$\begin{aligned}\|Tx\| &= \max_{0\le t\le 1}\int_0^1 G_2(t,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)\right.\\ &\qquad \left. -g\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)+\rho(s)\right]ds\\ &\le \int_0^1 G_2(s,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)\right.\\ &\qquad \left. -g\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)+\rho(s)\right]ds.\end{aligned} \tag{2.35}$$

On the other hand, from (2.6), we also have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G_2(t,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)\right.\\ &\qquad \left. -g\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)+\rho(s)\right]ds\\ &\ge \theta_2 G_2(t,t)\int_0^1 G_2(s,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)\right.\\ &\qquad \left. -g\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)+\rho(s)\right]ds.\end{aligned} \tag{2.36}$$

So

$$(Tx)(t) \ge \theta_2 G(t,t)\|Tx\|, \quad t \in [0,1], \tag{2.37}$$

which yields that $T(P) \subset P$.

At the end, according to the Ascoli-Arzela Theorem, using standard arguments, one can show $T : P \to P$ is a completely continuous operator. □

3. Main Results

Theorem 3.1. *Suppose* (B1)–(B3) *hold. In addition, assume that the following conditions are satisfied.*

(S1) *There exists a constant*

$$r > \max\left(2K, \frac{\int_0^1 \rho(s)ds}{\theta_2}\right) \tag{3.1}$$

such that for any $(t,u,v) \in [0,1] \times [0,((\beta_1+\alpha_1)(\gamma_1+\delta_1)/w_1)r] \times [-r,0]$, $f(t,u,v) \leq (r/K) - 2$, *where* θ_2 *and* K *are defined by* (2.7) *and* (2.19), *respectively.*

(S2) *There exists a constant* $R > 2r$ *such that, for any* $(t,u,v) \in [0,1] \times [(1/2)\mu_1 R,((\beta_1+\alpha_1)(\gamma_1+\delta_1)/w_1)R] \times [-R,-(1/2)\mu_2 R]$,

$$f(t,u,v) \geq \frac{R}{l}, \tag{3.2}$$

where μ_1, μ_2, l *are defined by* (2.19).

(S3)

$$\lim_{|u|+|v|\to+\infty} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|} = 0. \tag{3.3}$$

Then, the singular perturbed differential equation (1.1) *has at least two positive solutions* u_1, u_2, *and there exist two positive constants* n_1, n_2 *such that* $u_1(t) \geq n_1 e(t), u_2(t) \geq n_2 e(t)$, *for any* $t \in [0,1]$.

Proof. Let $\Omega_1 = \{x \in P : \|x\| < r\}$. Then, for any $x \in \partial\Omega_1, s \in [0,1]$, we have

$$\begin{gathered} [x(s)-y(s)]^* \leq x(s) \leq \|x\| \leq r, \\ \left|S[x(s)-y(s)]^*\right| \leq r\int_0^1 G_1(t,s)ds \leq \frac{(\beta_1+\alpha_1)(\gamma_1+\delta_1)}{w_1}r. \end{gathered} \tag{3.4}$$

It follows from (S1) that

$$\begin{aligned}
\|Tx\| &= \max_{t\in[0,1]}(Tx)(t) \\
&= \max_{0\le t\le 1}\int_0^1 G_2(t,s)\left[p(s)f\left(s,S[x(s)-y(s)]^*,-[x(s)-y(s)]^*\right)\right. \\
&\qquad\left. -g\left(s,S[x(s)-y(s)]^*,-[x(s)-y(s)]^*\right)+\rho(s)\right]ds \\
&\le \int_0^1 G_2(s,s)\left[\left(\frac{r}{K}-2\right)p(s)+2\rho(s)\right]ds \\
&\le \frac{r}{K}\int_0^1 G_2(s,s)\left[p(s)+\rho(s)\right]ds \\
&= r = \|x\|.
\end{aligned} \tag{3.5}$$

Therefore,

$$\|Tx\| \le \|x\|, \quad x\in P\cap\partial\Omega_1. \tag{3.6}$$

On the other hand, let $\Omega_2 = \{x\in P : \|x\| < R\}$ and $\partial\Omega_2 = \{x\in P : \|x\| = R\}$. Then, for any $x\in\partial\Omega_2, t\in[0,1]$, noticing $R > 2r$ and (2.22), we have

$$\begin{aligned}
x(t)-y(t) &\ge x(t)-G_2(t,t)\int_0^1\rho(s)ds \\
&\ge x(t)-\frac{x(t)}{\theta_2 R}\int_0^1\rho(s)ds \\
&\ge x(t)-\frac{r}{R}x(t)\ge\frac{1}{2}x(t) \\
&\ge \frac{1}{2}\theta_2 G_2(t,t)R.
\end{aligned} \tag{3.7}$$

So by (3.7), for any $x\in\partial\Omega_2, t\in[a,b]$, we have

$$\begin{aligned}
\frac{1}{2}\mu_2 R &= \frac{\phi_2(a)\psi_2(b)}{2w_2}\theta_2 R \le x(t)-y(t)\le R, \\
\frac{1}{2}\mu_1 R &= \frac{(\beta_1+\alpha_1 a)(\gamma_1+\delta_1(1-b))\theta_2 R}{2w_1}\int_0^1 e(s)G_2(s,s)ds \\
&\le \frac{1}{2}\theta_2 R\int_0^1 G_1(t,s)G_2(s,s)ds \le S(x(t)-y(t)) \\
&\le \frac{(\beta_1+\alpha_1)(\gamma_1+\delta_1)}{w_1}R.
\end{aligned} \tag{3.8}$$

It follows from (S2), (3.8), and (2.6) that, for any $x \in \partial\Omega_2, t \in [a,b]$,

$$\begin{aligned}
\|Tx\| &\geq \int_0^1 G_2(t,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)\right.\\
&\qquad\left. - g\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right) + \rho(s)\right]ds\\
&\geq \theta_2 G_2(t,t)\int_0^1 G_2(s,s)p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)ds\\
&\geq \theta_2 G_2(t,t)\int_a^b G_2(s,s)p(s)f\left(s, S[x(s)-y(s)], x(s)-y(s)\right)ds\\
&\geq \theta_2\frac{\phi_2(a)\psi_2(b)}{w_2}\int_a^b G_2(s,s)p(s)ds\frac{R}{l} = \mu_2\int_a^b G_2(s,s)p(s)ds\frac{R}{l}\\
&= R = \|x\|.
\end{aligned} \tag{3.9}$$

So we have

$$\|Tx\| \geq \|x\|, \quad x \in P \cap \partial\Omega_2. \tag{3.10}$$

Next, let us choose $\varepsilon > 0$ such that

$$\varepsilon\int_0^1 G_2(s,s)p(s)ds < 1. \tag{3.11}$$

Then, for the above ε, by (S3), there exists $N > R > 0$ such that, for any $t \in [0,1]$ and for any $|u| + |v| \geq N$,

$$f(t,u,v) \leq \varepsilon(|u| + |v|). \tag{3.12}$$

Let

$$\sigma = \max_{(t,u,v)\in[0,1]\times[0,((\beta_1+\alpha_1)(\gamma_1+\delta_1)/w_1)N]\times[-N,0]} f(t,u,v) + 2, \tag{3.13}$$

take

$$R^* = \frac{\sigma\int_0^1 G_2(s,s)\left[p(s)+\rho(s)\right]ds + 2\int_0^1 G_2(s,s)\rho(s)ds}{1 - \varepsilon\left(1 + \left((\beta_1+\alpha_1)(\gamma_1+\delta_1)/w_1\right)\right)\int_0^1 G_2(s,s)p(s)ds} + N, \tag{3.14}$$

then $R^* > N > R$.

Now let $\Omega_3 = \{x \in P : \|x\| < R^*\}$ and $\partial\Omega_3 = \{x \in P : \|x\| = R^*\}$. Then, for any $x \in P \cap \partial\Omega_3$, we have

$$\begin{aligned}
\|Tx\| &= \max_{0\le t\le 1}\int_0^1 G_2(t,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)\right.\\
&\qquad \left. -g\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right) + \rho(s)\right]ds\\
&\le \int_0^1 G_2(s,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, [x(s)-y(s)]^*\right) + 2\rho(s)\right]ds\\
&\le \left(\max_{(t,u,v)\in[0,1]\times[0,((\beta_1+\alpha_1)(\gamma_1+\delta_1)/w_1)N]\times[-N,0]} f(t,u,v) + 2\right)\int_0^1 G_2(s,s)\left[p(s)+\rho(s)\right]ds\\
&\quad + \int_0^1 G_2(s,s)\left[p(s)\varepsilon\left(S[x(s)-y(s)]^* + [x(s)-y(s)]^*\right) + 2\rho(s)\right]ds\\
&\le \sigma\int_0^1 G_2(s,s)\left[p(s)+\rho(s)\right]ds + \int_0^1 G_2(s,s)\left[p(s)\varepsilon\left(1 + \frac{(\beta_1+\alpha_1)(\gamma_1+\delta_1)}{w_1}\right)\|x\|\right.\\
&\qquad \left. +2\rho(s)\vphantom{\frac{1}{1}}\right]ds\\
&= \sigma\int_0^1 G_2(s,s)\left[p(s)+\rho(s)\right]ds + 2\int_0^1 G_2(s,s)\rho(s)ds + \varepsilon\left(1 + \frac{(\beta_1+\alpha_1)(\gamma_1+\delta_1)}{w_1}\right)\\
&\quad \times \int_0^1 G_2(s,s)p(s)ds\|x\|\\
&< R^* = \|x\|,
\end{aligned} \tag{3.15}$$

which implies that

$$\|Tx\| \le \|x\|, \quad x \in P \cap \partial\Omega_3. \tag{3.16}$$

By Lemma 1.1, T has two fixed points x_1, x_2 such that $r \le \|x_1\| \le R \le \|x_2\|$.

It follows from

$$r > \frac{\int_0^1 \rho(s)ds}{\theta_2} \tag{3.17}$$

that

$$\begin{aligned}
x_1(t) - y(t) &\ge x_1(t) - G_2(t,t)\int_0^1 \rho(s)ds\\
&\ge x_1(t) - \frac{x_1(t)}{\theta_2 r}\int_0^1 \rho(s)ds
\end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{\int_0^1 \rho(s)ds}{\theta_2 r}\right) x_1(t) \\
&\geq \theta_2 r \left(1 - \frac{\int_0^1 \rho(s)ds}{\theta_2 r}\right) G_2(t,t) \\
&= m_1 G_2(t,t) > 0, \quad t \in (0,1).
\end{aligned} \tag{3.18}
$$

As for (3.18), we also find a positive constant m_2 such that

$$x_2(t) - y(t) \geq m_2 G_2(t,t) > 0, \quad t \in (0,1). \tag{3.19}$$

Let $u_i(t) = S(x_i - y)(t), (i = 1,2)$, then

$$
\begin{aligned}
&u_i(t) > 0, \quad t \in (0,1)(i = 1,2), \\
&u_i(t) = S\left(x_i - y\right)(t) \geq m_i \int_0^1 G_1(t,s) G_2(s,s) ds \\
&\geq m_i \int_0^1 e(s) G_2(s,s) ds e(t) \\
&= n_i e(t).
\end{aligned} \tag{3.20}
$$

By Lemma 2.7, we know that the singular perturbed differential equation (1.1) has at least two positive solutions u_1, u_2 satisfying

$$u_1(t) \geq n_1 e(t), \quad u_1(t) \geq n_2 e(t), \quad t \in [0,1], \tag{3.21}$$

for some positive constants n_1, n_2. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Suppose* (B1)–(B3) *hold. In addition, assume that the following conditions are satisfied.*

(S4) *There exists a constant*

$$r > \frac{2\int_0^1 \rho(s)ds}{\theta_2} \tag{3.22}$$

such that, for any $(t,u,v) \in [0,1] \times [(1/2)\mu_1 r, ((\beta_1+\alpha_1)(\gamma_1+\delta_1)/w_1) r] \times [-r, -(1/2)\mu_2 r]$,

$$f(t,u,v) \geq \frac{r}{l}, \tag{3.23}$$

where θ_2 *and* μ_1, μ_2, l *are defined by* (2.7) *and* (2.19), *respectively.*

(S5) *There exists a constant* $R > \max\{r, ((r/l) + 2)K\}$ *such that, for any* $(t, u, v) \in [0, 1] \times [0, ((\beta_1 + \alpha_1)(\gamma_1 + \delta_1)/w_1)R] \times [-R, 0]$,

$$f(t, u, v) \leq \frac{R}{K} - 2, \tag{3.24}$$

where K, l *are defined by* (2.19) *and* r *is defined by* (S4).

(S6)

$$\lim_{|u|+|v| \to \infty} \min_{t \in [a,b]} \frac{f(t, u, v)}{|u| + |v|} = +\infty. \tag{3.25}$$

Then, the singular perturbed differential equation (1.1) *has at least two positive solutions* u_1, u_2, *and there exist two positive constants* n_1, n_2 *such that* $u_1(t) \geq n_1 e(t), u_2(t) \geq n_2 e(t)$, *for any* $t \in [0, 1]$.

Proof. Firstly, let $\Omega_1 = \{x \in P : \|x\| < r\}$. Then, for any $x \in \partial\Omega_1, t \in [0, 1]$, by (2.22), we have

$$\begin{aligned} x(t) - y(t) &\geq x(t) - G_2(t, t) \int_0^1 \rho(s) ds \\ &\geq x(t) - \frac{x(t)}{\theta_2 r} \int_0^1 \rho(s) ds \\ &\geq \frac{1}{2} x(t) \geq \frac{1}{2} \theta_2 G_2(t, t) r. \end{aligned} \tag{3.26}$$

So, by (3.26), for any $x \in \partial\Omega_1, t \in [a, b]$, we have

$$\begin{aligned} \frac{1}{2}\mu_2 r &= \frac{\phi_2(a)\psi_2(b)}{2w_2} \theta_2 r \leq x(t) - y(t) \leq r, \\ \frac{1}{2}\mu_1 r &= \frac{(\beta_1 + \alpha_1 a)(\gamma_1 + \delta_1(1 - b))\theta_2 r}{2w_1} \int_0^1 e(s) G_2(s, s) ds \\ &\leq \frac{1}{2}\theta_2 r \int_0^1 G_1(t, s) G_2(s, s) ds \leq S\big(x(t) - y(t)\big) \\ &\leq \frac{(\beta_1 + \alpha_1)(\gamma_1 + \delta_1)}{w_1} r. \end{aligned} \tag{3.27}$$

It follows from (S4), (3.27), and (2.6) that, for any $x \in \partial\Omega_2, t \in [a,b]$,

$$\begin{aligned}
\|Tx\| &\geq \int_0^1 G_2(t,s)\big[p(s)f\big(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\big) \\
&\qquad -g\big(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\big) + \rho(s)\big]ds \\
&\geq \theta_2 G_2(t,t)\int_0^1 G_2(s,s)p(s)f\big(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\big)ds \\
&\geq \theta_2 G_2(t,t)\int_a^b G_2(s,s)p(s)f\big(s, S[x(s)-y(s)], x(s)-y(s)\big)ds \\
&\geq \theta_2 \frac{\phi_2(a)\psi_2(b)}{w_2}\int_a^b G_2(s,s)p(s)ds\frac{r}{l} = \mu_2\int_a^b G_2(s,s)p(s)ds\frac{r}{l} \\
&= R = \|x\|.
\end{aligned} \tag{3.28}$$

Therefore, we have

$$\|Tx\| \geq \|x\|, \quad x \in P \cap \partial\Omega_1. \tag{3.29}$$

Next, by (S5), we have $R > r$ and

$$\frac{R}{K} - 2 > \frac{r}{l} > 0. \tag{3.30}$$

Let $\Omega_2 = \{x \in P : \|x\| < R\}$. Then, for any $x \in \partial\Omega_2, s \in [0,1]$, we have

$$\begin{gathered}
[x(s)-y(s)]^* \leq x(s) \leq \|x\| \leq R, \\
\big|S[x(s)-y(s)]^*\big| \leq R\int_0^1 G_1(t,s)ds \leq \frac{(\beta_1+\alpha_1)(\gamma_1+\delta_1)}{w_1}R.
\end{gathered} \tag{3.31}$$

It follows from (S5) that

$$\begin{aligned}
\|Tx\| &= \max_{t\in[0,1]}(Tx)(t) \\
&= \max_{0\leq t\leq 1}\int_0^1 G_2(t,s)\big[p(s)f\big(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\big) \\
&\qquad -g\big(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\big) + \rho(s)\big]ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 G_2(s,s)\left[\left(\frac{R}{K}-2\right)p(s)+2\rho(s)\right]ds \\
&\leq \frac{R}{K}\int_0^1 G_2(s,s)\left[p(s)+\rho(s)\right]ds \\
&= R = \|x\|.
\end{aligned} \tag{3.32}$$

Therefore,

$$\|Tx\| \leq \|x\|, \quad x \in P \cap \partial\Omega_2. \tag{3.33}$$

On the other hand, choose a large enough real number $M > 0$ such that

$$\frac{1}{2}\mu_2^2 M \int_a^b G_2(s,s)p(s)ds \geq 1. \tag{3.34}$$

From (S6), there exists $N > R$ such that, for any $t \in [a,b]$,

$$f(t,u,v) \geq M(|u|+|v|), \quad |u|+|v| \geq N. \tag{3.35}$$

Take

$$R^* > \max\left\{\frac{2N}{\mu_2}, R\right\}, \tag{3.36}$$

then $R^* > R > r$. Let $\Omega_3 = \{x \in P : \|x\| < R^*\}$, for any $x \in P \cap \partial\Omega_3$ and for any $t \in [a,b]$, we have

$$\begin{aligned}
x(t)-y(t) &\geq x(t) - G_2(t,t)\int_0^1 \rho(s)ds \\
&\geq x(t) - \frac{\int_0^1 \rho(s)ds}{\theta_2 R^*}x(t) \\
&\geq \frac{1}{2}x(t) \geq \frac{\theta_2 G_2(t,t)}{2}R^* \\
&\geq \frac{1}{2}\mu_2 R^* \geq N > 0.
\end{aligned} \tag{3.37}$$

So, for any $x \in P \cap \partial\Omega_3, t \in [a,b]$, by (3.35), (3.37), we have

$$\begin{aligned}
\|Tx\| &\geq \int_0^1 G_2(t,s)\left[p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)\right.\\
&\qquad \left. -g\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right) + \rho(s)\right]ds\\
&\geq \theta_2 G_2(t,t)\int_0^1 G_2(s,s)p(s)f\left(s, S[x(s)-y(s)]^*, -[x(s)-y(s)]^*\right)ds\\
&\geq \theta_2 G_2(t,t)\int_a^b G_2(s,s)p(s)f\left(s, S[x(s)-y(s)], x(s)-y(s)\right)ds\\
&\geq \theta_2 \frac{\phi_2(a)\varphi_2(b)}{w_2}\int_a^b G_2(s,s)p(s)M\left[\left|S(x(s)-y(s))\right| + \left|(x(s)-y(s))\right|\right]ds\\
&\geq \frac{\theta_2\phi_2(a)\varphi_2(b)}{w_2}\int_a^b G_2(s,s)p(s)M\left(\left|x(s)-y(s)\right|\right)ds\\
&= \mu_2 M\int_a^b G_2(s,s)p(s)ds \times \frac{1}{2}\mu_2 R^*\\
&= \frac{1}{2}\mu_2^2 M\int_a^b G_2(s,s)p(s)ds \times R^*\\
&\geq R^* = \|x\|.
\end{aligned} \tag{3.38}$$

Thus,

$$\|Tx\| \geq \|x\|, \quad x \in P \cap \partial\Omega_3. \tag{3.39}$$

By Lemma 1.1, T has two fixed points x_1, x_2 such that $r \leq \|x_1\| \leq R \leq \|x_2\|$. Noticing that

$$r > \frac{2\int_0^1 \rho(s)ds}{\theta_2}, \tag{3.40}$$

we have

$$\begin{aligned}
x_1(t) - y(t) &\geq x_1(t) - G_2(t,t)\int_0^1 \rho(s)ds\\
&\geq x_1(t) - \frac{x_1(t)}{\theta_2 r}\int_0^1 \rho(s)ds\\
&= \left(1 - \frac{\int_0^1 \rho(s)ds}{\theta_2 r}\right)x_1(t)
\end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{2}x_1(t) \geq \frac{1}{2}\theta_2 r G_2(t,t) \\ &= m_1 G_2(t,t) > 0, \quad t \in (0,1). \end{aligned} \tag{3.41}$$

As for (3.41), we also can find a positive constant m_2 such that

$$x_2(t) - y(t) \geq m_2 G_2(t,t) > 0, \quad t \in (0,1). \tag{3.42}$$

Let $u_i(t) = S(x_i - y)(t), (i = 1, 2)$, then

$$\begin{aligned} &u_i(t) > 0, \quad t \in (0,1)(i = 1,2), \\ &u_i(t) = S\left(x_i - y\right)(t) \geq m_i \int_0^1 G_1(t,s)G_2(s,s)ds \\ &\geq m_i \int_0^1 e(s)G_2(s,s)dse(t) \\ &= n_i e(t). \end{aligned} \tag{3.43}$$

By Lemma 2.7, we know that the singular perturbed differential equation (1.1) has at least two positive solutions u_1, u_2 satisfying

$$u_1(t) \geq \mathrm{n}_1 e(t), \quad u_1(t) \geq n_2 e(t), \quad t \in [0,1], \tag{3.44}$$

for some positive constants n_1, n_2. The proof of Theorem 3.2 is completed. □

An example Consider the following singular perturbed boundary value problem

$$\begin{gathered} -\left(e^t u'''\right)'(t) + 2e^t u''(t) = \frac{1}{t(1-t)} f\left(t, u(t), u''(t)\right) - \frac{\sin(u(t)) + \arctan(u''(t))}{(1 + (\pi/2))\sqrt{t}}, \\ u(0) = 0, \qquad u(1) = 0, \qquad u''(0) - 2u'''(0) = 0, \qquad u''(1) + u'''(1) = 0, \end{gathered} \tag{3.45}$$

where

$$f\left(t,x,y\right) = \begin{cases} tx + \dfrac{y^2}{100}, & \left(t,x,y\right) \in [0,1] \times [0,10] \times [-10,0], \\ 41.88x + 4.075y, & \left(t,x,y\right) \in [0,1] \times [10,13.93] \times [-100,-10], \\ 200(1+t^2)e^{x/13.93} + \sin y, & \left(t,x,y\right) \in [0,1] \times [13.93,100] \times [-100,-422.89], \\ 23.2469\left(x + \left|y\right|\right)^{1/2}, & \left(t,x,y\right) \in [0,1] \times [100,+\infty) \times (-\infty,-422.89]. \end{cases} \tag{3.46}$$

Then, the BVP (3.45) has at least two positive solutions $u_1(t)$ and $u_2(t)$ such that

$$u_1(t) \geq 2.3924t(1-t), \quad t \in [0,1], \quad u_2(t) \geq 24.833t(1-t), \quad t \in [0,1]. \tag{3.47}$$

Proof. In fact, let

$$r(t) \equiv e^t, \qquad q(t) \equiv 2e^t, \qquad p(t) = \frac{1}{t(1-t)}, \qquad \rho(t) = \frac{1}{\sqrt{t}}, \qquad g(\mathrm{t}, x, y) = \frac{\sin x + \arctan y}{(1+(\pi/2))\sqrt{t}}, \tag{3.48}$$

then

$$\left|g(t,x,y)\right| \leq \frac{1}{\sqrt{t}}. \tag{3.49}$$

The corresponding Green's functions can be written by

$$G_1(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$$
$$G_2(t,s) = \frac{1}{3e}\begin{cases} \left(\dfrac{5}{3}e^s + \dfrac{1}{3}e^{-2s}\right)\left(\dfrac{1}{3}e^t + \dfrac{2}{3}e^{-2t}\right), & 0 \leq s \leq t \leq 1, \\ \left(\dfrac{5}{3}e^t + \dfrac{1}{3}e^{-2t}\right)\left(\dfrac{1}{3}e^s + \dfrac{2}{3}e^{-2s}\right), & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.50}$$

and, for $(t,s) \in [0,1] \times [0,1]$, we have $t(1-t)s(1-s) \leq G_1(t,s) \leq s(1-s)$ and

$$\frac{1}{5e^2 + e^{-1}} G_2(s,s)G_2(t,t) \leq G_2(t,s) \leq G_2(s,s) \quad (\text{or } G_2(t,t)). \tag{3.51}$$

□

Now, take $[1/4, 3/4] \subset [0,1]$, then we have

$$\begin{aligned} K &= \int_0^1 G_2(s,s)\left[p(s) + \rho(s)\right] ds \\ &= \int_0^1 \frac{\left((5/3)e^s + (1/3)e^{-2s}\right)\left((1/3)e^s + (2/3)e^{-2s}\right)}{3e}\left[\frac{1}{s(1-s)} + \frac{1}{\sqrt{s}}\right] ds \approx 0.6206, \\ \mu_2 &= \frac{\phi_2(a)\psi_2(b)}{\phi_2(1)\psi_2(0)} = \frac{\phi_2(1/4)\psi_2(3/4)}{\phi_2(1)\psi_2(0)} = \frac{\left((5/3)e^{1/4} + (1/3)e^{-1/2}\right)\left((1/3)e^{3/4} + (2/3)e^{-3/2}\right)}{(5/3)e + (1/3)e^{-2}} \\ &\approx 0.4374, \\ l &= \mu_2 \int_a^{\mathrm{b}} G_2(s,s)p(s)ds = 0.4374 \int_{1/4}^{3/4} \frac{\left((5/3)e^s + (1/3)e^{-2s}\right)\left((1/3)e^s + (2/3)e^{-2s}\right)}{3es(1-s)} ds \end{aligned}$$

$$\begin{aligned}
&\approx 0.2786,\\
\mu_1 &= \frac{(\beta_1+\alpha_1 a)(\gamma_1+\delta_1(1-b))\theta_2}{w_1}\int_0^1 G_1(s,s)G_2(s,s)ds\\
&= \frac{1}{16(5e^2+e^{-1})}\int_0^1 s(1-s)\frac{((5/3)e^s+(1/3)e^{-2s})((1/3)e^s+(2/3)e^{-2s})}{3e}ds \approx 8.4578,\\
\theta_2 &= \frac{w_2}{\phi_2(1)\psi_2(0)} = \frac{p(0)[\psi_2(0)\phi_2'(0)-\psi_2'(0)\phi_2(0)]}{\phi_2(1)\psi_2(0)} \approx 4.938.
\end{aligned} \tag{3.52}$$

On the other hand, since

$$\int_0^1 \rho(s)ds = \int_0^1 \frac{1}{\sqrt{s}}ds = 2, \tag{3.53}$$

then (B1)–(B3) are satisfied.

Choose $r = 10$, then

$$10 = r > \max\left\{2K, \frac{\int_0^1 \rho(s)ds}{\theta_2}\right\} = \{1.2412, 0.4050\}, \tag{3.54}$$

and, for any $(t,x,y) \in [0,1]\times[0,10]\times[-10,0]$,

$$f(t,x,y) \le 11 \le \frac{r}{K} - 2 \approx 14.1134. \tag{3.55}$$

So the condition (S1) is satisfied.

On the other hand, we take $R = 100$, then $R > 2r = 20$, and, for any $(t,x,y) \in [0,1]\times[(1/2)\mu_1 R, ((\beta_1+\alpha_1)(\gamma_1+\delta_1)/w_1)R]\times[-R, -(1/2)\mu_2 R] = [0,1]\times[13.93,100]\times[-100,-422.89]$, we have

$$f(t,x,y) \ge 542.65 \ge \frac{R}{l} \approx 358.93, \tag{3.56}$$

this implies that the condition (S2) holds. Next, we have

$$\lim_{|x|+|y|\to+\infty}\max_{t\in[0,1]}\frac{f(t,x,y)}{|x|+|y|} = \lim_{|x|+|y|\to+\infty} 23.2469(|x|+|y|)^{-1/2} = 0. \tag{3.57}$$

Thus, (S3) also holds. By Theorem 3.1, the BVP (3.45) has at least two positive solutions $u_1(t)$ and $u_2(t)$. Since

$$\begin{aligned} n_1 &= \frac{1}{3e}\left(\theta_2 r - \int_0^1 \rho(s)ds\right)\int_0^1 s(1-s)\left(\frac{5}{3}e^s + \frac{1}{3}e^{-2s}\right)\left(\frac{1}{3}e^s + \frac{2}{3}e^{-2s}\right)ds \approx 2.3924, \\ n_2 &= \frac{1}{3e}\left(\theta_2 R - \int_0^1 \rho(s)ds\right)\int_0^1 s(1-s)\left(\frac{5}{3}e^s + \frac{1}{3}e^{-2s}\right)\left(\frac{1}{3}e^s + \frac{2}{3}e^{-2s}\right)ds \approx 24.833, \end{aligned} \tag{3.58}$$

thus

$$u_1(t) \geq 2.3924t(1-t), \quad t \in [0,1], \qquad u_2(t) \geq 24.833t(1-t), \quad t \in [0,1]. \tag{3.59}$$

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