Department of Mathematics and Statistics

Optimal Asset Liability Management with Constraints: Theory and Application

Yan Zhang

This thesis is presented for the Degree of
Doctor of Philosophy
of
Curtin University

May 2015

Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

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Yan Zhang
May 2015

Abstract

In this thesis, we study the asset liability management with constraints under the mean-variance criterion. In addition, we also study the portfolio selection problem as a special case. This research work consists of two parts.

The first part of this research focuses on the asset liability management and portfolio selection in a jump diffusion market under the mean-variance criterion which aims at finding the optimal admissible strategy such that the variance of the terminal wealth is minimized for a given expected terminal wealth. The financial market setting we defined consists of one risk-free asset and multiple risky assets. The price of risky asset is modelled by the exponential Lévy processes; and within the cover period defined by the contract the insurer faces the risk of paying uncertain general insurance claims which follow a compound Poisson process. In the case with no liability in place, the original model can be reduced to a portfolio selection model. By applying the duality theory, the Hamilton-Jacobi-Bellman equation approach and the stochastic dynamic programming techniques, we derive the analytical closed-form expressions for the efficient investment strategy and the efficient frontier. Then we study two degenerated cases. Finally, we present some numerical examples to illustrate the theoretical results obtained in this work. Our investigation shows that the jump diffusion has significant influence on the efficient frontier and the efficient investment strategy. The effect of liabilities and key parameters are also studied.

The second part of this research focuses on asset liability management and portfolio selection with state-dependent risk aversion under the mean-variance criterion. The investor allocates the wealth among multiple assets that include one risk-free asset and multiple risky assets which are governed by an ordinary differential equation and a system of stochastic differential equations in geometric Brownian motion form, and the in-

vestor faces the risk of paying uncontrollable random liabilities within a predetermined investment horizon. Furthermore, we consider the state-dependent risk aversion in our model, which means that the risk aversion depends on the current wealth held by the investor. This innovation is inspired by the research work of Björk et al. (2014). By solving the extended Hamilton-Jacobi-Bellman system, the analytical expressions for the time-consistent optimal investment strategies, the variance versus expectation of terminal wealth and the optimal value function are derived for the case in which the risk aversion is inversely proportional to the wealth. Moreover, we also obtain the results for the portfolio selection as a special case of the asset liability management model. Finally, we numerically investigate the effects of liability and multiple risky assets on the equilibrium control policy (optimal investment strategy) and the equilibrium value function (optimal value function).

List of publications

The following papers were completed during the PhD candidature and are currently under review:

- Y. Zhang, Y. H. Wu, B. Wiwatanapataphee, "Mean-Variance Asset Liability Management with General Insurance Liability and Jumps"
- Y. Zhang, Y. H. Wu, B. Wiwatanapataphee, "Mean-Variance Asset Liability Management with Stochastic Appreciation in Liability and Jumps"
- Y. Zhang, Y. H. Wu, B. Wiwatanapataphee, "Mean-Variance Asset Liability Management with State-Dependent Risk Aversion"

Acknowledgements

The research was carried out from May 2012 to May 2015, while I was a PhD student in the Department of Mathematics and Statistics, Curtin University. I really appreciate all kinds of helps I have received from my supervisors, families, friends and colleagues during this period of time.

I would like to express my heartfelt thanks to my supervisor, Prof. Yonghong Wu. Professor Wu has guided my research during the past three years with remarkable patience and enthusiasm.

I would like to thank A/Prof. Benchawan Wiwatanapataphee in the Department of Mathematics and Statistics, Curtin University, for her advice and help in my research.

I also would like to acknowledge Dr. Qun Lin, my co-supervisor in the Department of Mathematics and Statistics, Curtin University, for her helpful suggestions on my research whenever needed.

I wish to give thanks to Prof. Kok Lay Teo, the Chair of my Thesis Committee.

In addition, I thank my fellow PhD students and friends whom I have worked with in Australia, especially, A/Prof. Adrian Wong, A/Prof. Nikolai Dokuchaev, A/Prof. Guanglu Zhou, Dr. Honglei Xu and his wife Shaoli Wang, Dr. Hai Huyen Dam, Dr. Ryan Loxton, Dr. Greg Gamble, Dr. Jingyang Zhou and his wife Yujing Wang, Dr. Changjun Yu, Dr. Kevin Fergusson, Dr. Qinqin Chai, Dr. Qian Sun, Dr. Yanli Zhou, Dr. Yanyan Yin, Dr. Nathnarong Khajohnsaksumeth, Yufei Sun, Shuang Li, Mingliang Xue, Ke Fan, Rinrada Thamchai, Pawaton Kaemawichanurat, and the department's academic visitors that I had the opportunity to meet, including Prof. Shaoyong Lai, Prof. Xiangyu Ge, Prof. Lishan Liu, Dr. Canghua Jiang, A/Prof. Xiangyu Gao, A/Prof. Tieqiao Tang, A/Prof. Fuyi Xu, Dr. Xinguang Zhang, A/Prof. Xinzhen Zhang, A/Prof. Lihui Cen, A/Prof. Guoqiang Wang, A/Prof. Ying Zhang, Prof. Feng Yang, Prof. Yonggang Li and Prof. Aiguo Wu. I

really enjoyed the time that I spent with them.

I also thank all of the staff in the Department of Mathematics and Statistics for contributing to a friendly working environment. The past and present administrative staff, in particular, Joyce Yang, Shuie Liu, Jeannie Darmago, Lisa Holling, Cheryl Cheng, Carey Ryken Rapp and Marcus Tey, deserve special thanks for providing kind and professional help on numerous occasions.

Finally, on the personal note, I sincerely thank everyone in my family, especially my parents, for their love, understanding and support during the entire period of my PhD candidature in Australia.

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CHAPTER 1

Introduction

1.1 Background

The mean-variance portfolio selection framework proposed by Markowitz (1952, 1956, 1959) paved the foundation for modern portfolio management theory. During the past half century, many researchers devoted themselves to explore the application of this mean-variance portfolio selection framework, both in academy and industry. However, due to the non-separability of the mean-variance objective function in the sense of dynamic programming, early research investigations were limited to the cases under single-period setting. No research work has provided analytical or efficient numerical method for finding the efficient investment strategy and efficient frontier under multi-period setting until 2000. A breakthrough was achieved by Li and Ng (2000) and Zhou and Li (2000); they obtained the analytical expression of efficient frontier of mean-variance portfolio selection under multi-period and continuous-time settings, respectively, by using the embedding technique. From then on, many researchers further generalized the results with consideration of various constraints. Since 1990s, many researchers proposed simplified methods for solving multi-period and continuous-time portfolio selection problems under the mean-variance framework with consideration of liabilities, which is regarded as the asset liability management optimization problem.

There are many different kinds of mean-variance criteria. In this work, we focus on the study of two types of criteria. The first one aims to determine the optimal portfolio such that the variance of the terminal wealth is minimized for a given expected terminal

wealth. The second one combines the mean and variance into one expression, which is called the mean-variance utility, by involving the risk aversion rate.

We apply different methodologies to solve the two kinds of problems. For the first problem, we use the Lagrange multiplier method, the stochastic dynamic programming and the Hamilton-Jacobi-Bellman equation approach to derive the results. The second problem is more complex. Commonly, two approaches are used to deal with this kind of time-inconsistent optimal control problems. One of the approaches is called the precommitment strategy, namely the initial point (t, x_0) is fixed and the control law $\mathbf{u}^*$ is chosen to maximize the value function $V(0, x_0, \mathbf{u})$. An alternative approach is to formulate the time-inconsistent problem in the framework of game theory. In this thesis, we take the latter approach to solve the time-inconsistent problem. In brief, our control problem in multiple periods is regarded as a game among multiple players which represent our present favor and the incarnations of our further tastes. Our objective is to find the sub-game perfect Nash equilibrium point for this game. Hence, an extended Hamilton-Jacobi-Bellman equation system is developed and applied.

The asset liability management model can be degenerated to the corresponding portfolio selection model by ignoring the dynamic process of liability. The portfolio selection problem is also a very popular topic in financial mathematics, so we investigate it in this research work as well by applying similar techniques with simplified calculation, and our analysis leads to many interesting results.

1.2 Objectives of the Thesis

The main purpose of the work is to establish an asset liability management model in an incomplete financial market to examine the effect of various constraints on the optimal investment strategy and the wealth process. As a side product, the portfolio selection model is also studied. More specifically, this thesis aims to

- investigate the impact of jump in financial market on the efficient investment strategy and the efficient frontier of the asset liability management with general insurance liability.

- examine the effect of liability on the efficient investment strategy and the efficient frontier by comparing the results for asset liability management model and portfolio selection model.

- analyze the effect of model parameters on the efficient investment strategy and the efficient frontier for both the asset liability management model and the portfolio selection model.

- develop the extended Hamilton-Jacobi-Bellman equation system for the generalized asset liability management model and the generalized portfolio selection model with multiple risky assets.

- obtain the optimal control strategy for the asset liability management model and the portfolio selection model under state-dependent risk aversion.

- develop a robust numerical scheme to simulate the evolution of the wealth, the amount of money invested on risky assets under the optimal investment strategy and risk aversion with respect to time.

1.3 Outline of the Thesis

This thesis consists of five chapters. Chapter 1 presents a brief introduction of the research and gives the objectives of the work.

Chapter 2 gives a literature review of the development of portfolio selection and asset liability management under the mean-variance framework which are the important topics in modern finance theory. Some major breakthrough works and milestone achievements are introduced in this chapter, including innovation in methodologies and application in various financial security systems.

Chapter 3 begins with the overview of the research on the asset liability management with general insurance liability model in Section 3.1. In Section 3.2, we describe the financial market setting, build the mean-variance asset liability management model with general insurance liabilities and jumps. In Section 3.3, we covert the original constrained

model to an unconstrained stochastic control problem and derive the closed-form analytical solution to the problem by applying the duality theory, the Hamilton-Jacobi-Bellman equation approach and the stochastic dynamic programming techniques. Besides, we obtain the efficient investment strategy and the efficient frontier. In section 3.4, we reduce the original asset liability management model to a portfolio selection model and derives the analytical solutions. In addition, we also study the model with no jumps and obtain the efficient investment strategy and efficient frontier of the simplified problem by applying similar techniques. Section 3.5 gives the numerical examples for the original asset liability management model and two degenerated cases.

Chapter 4 begins with an overview of the asset liability management with state-dependent risk aversion model in Section 4.1. Section 4.2 develops the mean-variance asset liability management model with state-dependent risk aversion. The verification theorem is then established and proved. In addition, the extended Hamilton-Jacobi-Bellman system for this model is derived. Section 4.3 solves for the optimal control strategy and the optimal valuation function for two different cases of risk aversion function $\gamma(x)$ as well as the theoretical results for the financial setting with only one risky asset. In Section 4.4, we degenerate the original asset liability management model to the portfolio selection model and obtain the corresponding verification theorem and the extended Hamilton-Jacobi-Bellman equation system as well as analytical solutions under two different cases of $\gamma(x)$, followed by numerical results obtained from the analytical results to illustrate the optimal investment strategy and the optimal value function.

Chapter 5 provides summaries and a discussion of possible future research.

CHAPTER 2

Literature Review

2.1 General Overview

The research work by Markowitz (1952, 1956, 1959) proposed the well-known mean-variance criterion for portfolio selection problem, which defines the weighting risk against the expected return. In financial mathematics, the risk is measured by variance, and the expected return is measured by mean. The investor aims to make rational investment choice in various candidate assets by seeking the one with lowest variance for a given expected return or the one with highest expected return for a given variance. This milestone research work paved the foundation for modern portfolio management theory. The analytical solution of efficient frontier in a single-period setting is derived in Markowitz (1956) and Merton (1972). From then on, many researchers devoted themselves to explore the extension and application of this mean-variance framework, both in academy and in industry. One of the most important frontier of this research is to generalize the results of single-period setting into a multi-period or a continuous-time setting.

Smith (1967) built a transition model which was used as a framework for portfolio revision, and gave some analysis and results of empirical tests. Mossin (1968) studied the optimal multi-period portfolio policies by seeking the optimal control to maximize the expected quadratic utility function of the terminal wealth ignoring the possibilities for intermediate consumption. Samuelson (1969) formulated and solved a multi-period generalization of portfolio selection, corresponding to lifetime planning of consumption and investment decisions. Fama (1970) investigated the risk averse consumer behaviour

in multi-period setting. Hakansson (1971) worked on a multi-period mean-variance portfolio selection problem by using a log form utility function. Merton (1972) compared the efficient frontier of single-period portfolio selection under two different financial settings, including (a) all assets are risky and (b) one of the assets is risk-free. Elton et al. (1974) synthesized the work that had been done on the multi-period consumption-investment decision in order to examine the implications for the validity of single-period models in the multi-period setting. They considered both log form and power form utility functions in the solutions. Perold (1984) developed an efficient computational approach for large-scale mean-variance portfolio optimization in the case that the covariance matrix is nonnegative. Dumas and Liucinao (1991) employed the power form utility function to work out an exact solution to a portfolio selection problem under transaction costs. Duffie and Richardson (1991) derived a closed-form minimum variance hedging policy for a given mean using quadratic utility function. For more detailed research work on mean-variance portfolio selection problem, one can refer to Elton and Gruber (1975), Kroll et al. (1984), Markowitz (1989), Grauer and Hakansson (1993), Isabella and Roland (1998).

The dominant mean-variance criterion used in the literature is maximizing the expected utility function of the terminal wealth, where the utility function is of a power form, log form, exponential form or quadratic form. With the utility function in place, the results of portfolio selection under multi-period or continuous-time settings have obvious limitations, which lies in the implicity of tradeoff information between the risk and the expected return. Therefore, the research cannot provide direct information to the investors on investment decision. Chen et al. (1971) reported the difficulties in finding optimal solutions for portfolio selection under a multi-period setting, including the major difficulty, namely the non-separability of the mean-variance objective function in the sense of dynamic programming. Elton and Gruber (1974) evaluated two portfolio selection rules based on different objective utility functions. The breakthrough research achievements by Li and Ng (2000) and Zhou and Li (2000) obtained the analytical expression of the efficient frontier of mean-variance portfolio selection under multi-period and continuous-time settings, respectively, by using the embedding technique. From then on, many researchers have been inspired to further generalize the results with consideration of various constraints.

Lim and Zhou (2002) investigated the mean-variance portfolio selection problem with random parameters in a complete market. Lim (2004) further studied the quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market. Xiong and Zhou (2007) considered the mean-variance portfolio selection problem under partial information. Elliott et al. (2010) focused on portfolio selection under a hidden Markovian regime switching model. Chiu and Wong (2011) derived a closed-form optimal strategy for portfolio selection of cointegrated assets analytically. Yao et al. (2014b) solved a continuous-time portfolio selection problem with only risky assets.

A lot of work has been carried out to study the application of portfolio selection in defined benefit pension management and defined contribution pension management. For example, Haberman and Vigna (2002) studied optimal investment strategies and risk measures in defined contribution pension schemes. Deelstra et al. (2004) investigated optimal design of the guarantee for defined contribution funds. Cairns et al. (2006) showed the advantages of stochastic lifestyle over the deterministic lifestyle in respect of defined contribution pension plans. Hainaut and Devolder (2007) focused on the management of a pension fund under mortality risk and financial risk. Delong et al. (2008) dealt with contribution rate and asset allocation strategies in a pre-retirement accumulation phase. Korn et al. (2011) considered asset allocation for a defined contribution pension fund under regime switching environment. Yao et al. (2013b) solved the efficient investment strategy and the efficient frontier of a mean-variance defined contribution pension fund management under inflation model. Liang et al. (2014) investigated the mean-variance optimization problem for a single cohort of workers in an accumulation phase of a defined benefit pension scheme. For more detailed discussion on pension fund management topic, the reader is referred to Josa-Fombellida and Rincon-Zapatera (2004), Zhang and Ewald (2010), Hainautt and Deelstra (2011), Emms (2012), Yu et al. (2012) and Yao et al. (2014a).

Since 1990s, many researchers proposed simplified methods for solving multi-period and continuous-time portfolio selection under mean-variance framework with consideration of liabilities, which is categorized as the asset liability management optimization problem. Sharp and Tint (1990) investigated the asset liability management problem in

a static setting for the first time. Kell and Müller (1995) studied efficient portfolio selection in the asset liability context. Chiu and Li (2006) focused on asset and liability management under a continuous-time mean-variance optimization framework. Chen et al. (2008) derived the efficient frontier of asset liability management with regime switching model under mean variance criterion. Xie et al. (2008) established a continuous-time mean-variance portfolio selection model with multiple risky assets and one liability in an incomplete market. Zeng and Li (2011) considered asset liability management under benchmark and mean-variance criterion in a jump diffusion market. Yao et al. (2013a) solved analytically the continuous-time mean-variance asset liability management with endogenous liabilities.

In this work, we study two main types of problems. The first one aims to find the optimal admissible strategy such that the variance of the terminal wealth is minimized for a given expected terminal wealth d. The second one combines mean and variance into one expression, which is called mean-variance utility, by introducing the risk aversion rate. It is necessary to use different techniques to solve these two kinds of optimization problems. For the first problem, we use the Lagrange multiplier method, the stochastic dynamic programming and the Hamilton-Jacobi-Bellman equation approach to derive the results. For the second problem, the sub-game perfect Nash equilibrium strategy and the extended Hamilton-Jacobi-Bellman equations are applied. Moreover, we use the general jump-diffusion process to describe insurer's wealth process and we assume that the risky assets' prices follow an exponential Lévy processes. In the following sections, we give a brief overview of the relevant theories to be used in our work and the previous research work, as well as the application of jump-diffusion process in the existing literature.

2.2 Application of Asset Liability Management

2.2.1 Asset Liability Management and Financial Security System

As the title indicates, asset liability management is a technique to manage asset and liability synthetically so as to make sure the match between asset and liability, to hedge the financial risk, to maintain liquidity and solvency according to regulator's requirements and to maximize the surplus. Hence, asset liability management is essential to all financial security systems.

Financial security systems are set up to provide risk management service for third parties, including both individual and institutions. Examples of financial security systems include life insurance companies, property insurance companies, health management institutions, social security systems, workers compensation programs, employment insurance programs, pension plans, banks, and investment companies. Financial security systems generate risks during their day-to-day business activities. Let's take insurance companies and banks as examples. The insurers receive predetermined premium from customers (asset), and pay benefits on some situations as promised in the contracts (liability). Banks do business in a contrary way. The deposit they received from customers is their liabilities and the loans they lend to individuals or corporations are their assets. So the mismatch between assets and liabilities is always the major concern. Financial security systems also bear impact due to business/economic environment, change in regulations and policies, and social/cultural values.

A financial security system's asset liability management must be conducted with inputs from both assets and liabilities. Management of assets or liabilities separately will lead to serious problems. The common problems dealt by financial security systems are as follows:

- What assets best match the existing liabilities?
- How does a change in external force affect the asset cash flow as compared to the

liability cash flow?

- Are the same scenarios being tested on a consistent basis by the asset and liability professionals?
- Will the asset and liability cash flows converge or diverge given a future market scenario?
- How do the asset-liability risks affect the pricing risk?

Asset liability management is a great dimension of risk management, in which the exposition to various risks is minimized maintaining the appropriate combination of assets and liabilities in order to satisfy the goals of financial security systems. Its main purpose is the maximization of financial security systems' profits and the minimization of the risk.

All in one word, asset liability management is very important to all financial market participants, both individuals and institutions, regardless of their size and shape. Many researchers devote themselves to generalize original models and solve them by new methodologies with the help of latest scientific tools. Till now, research in asset liability management still presents prospects for further research.

2.2.2 Asset Liability Management with Constraints

Over the last decades, the best known pioneer research work in asset liability management is to extend the asset only model to asset-liability model. A geometric approach was proposed by Leippold et al. (2004) to solve the multi-period mean-variance asset liability management model and to show the impact of taking liabilities into account on the implied mean-variance frontier, as well as the quantification of the impact of the rebalancing frequency and the determination of the optimal initial funding ratio. Two years later, the extension under continuous-time setting was achieved by Chiu and Li by establishing a stochastic linear-quadratic control framework (see Chiu and Li, 2006). They adopted the same assumptions as in Leippold et al. (2004) and derived analytical results in a complete market. In addition, the authors also examined the impact of liabilities and discussed the optimal funding ratio.

To construct more realistic models, a number of researchers studied the asset liability management in an incomplete market with various constraints, for instance, but not limited to, uncertain investment horizon, regime switching, bankrupt control, jump-diffusion in financial market, stochastic volatility and stochastic interest rate.

Uncertain investment horizon is an interesting issue that has drawn more and more attention during the past decades. The investor may be forced to exit the market due to some uncontrollable exogenous reasons. To be more specifically, the potential reasons that affect the exit time include securities markets behavior, changes in the opportunity set, uncertainty of order execution time, changes in an investor's endowment, and time of an exogenous shock to an investor's consumption process (see Blanchet-Scalliet et al. 2008). So research work is in need to study the effect of uncertain exit time on optimal investment strategy. With the given distribution of the exit time, this problem can be translated into a problem with a deterministic investment horizon which can be solved analytically by the embedding technique proposed by Li and Ng in 2000. Guo and Hu (2005) studied the portfolio selection with uncertain exit time under multi-period setting. Martellini and Urosevic (2006) generalized the Markowitz analysis to the situations involving uncertainty over time of exit. They argued that in such a case, the investor would face both an asset price risk and an exit time risk, which should be addressed simultaneously. Yi et al. (2008) further generalized the problem investigated by Guo and Hu, and extended the financial market setting from one risk-free asset and n risky assets to $n + 1$ risky assets. Yao et al. (2013c) took another factor, uncontrolled cash flow, into account and solved the multi-period asset liability management model by adopting the dynamic programming approach. Then Yao made use of the technique developed in the model with uncertain investment horizon to deal with the mortality risk in the defined contribution fund management problem (see Yao et al. 2014a). Li and Yao (2014) used Lagrange duality method and dynamic programming technique to investigate a multi-period mean-variance portfolio selection problem under uncertain exit time and stochastic market environment. This research can be extended to the asset liability management area with necessary modification in terms of liability.

In real financial market, the state of the market switches between the so-called "bullish"

market and "bearish" market. Stock price, commodity price, investor's behaviour, interest rate and inflation rate evolve in complete a different way as the market state changes. In financial mathematics, a regime-switching model is applied to describe this phenomenon. Generally, in a regime-switching model, the values of market modes are divided into a finite number of regimes. The key parameters, including appreciation rates, volatility rates and risk-free interest rates, will change according to the value of different market value. The research on the state of market by using Markov chain theory can be traced back to Pye (1966). In recent years, many researchers focused on the application of regime-switching in various financial areas, such as option pricing, bond pricing, stock returns, portfolio selection and asset liability management. Hardy (2001) established the regime-switching lognormal model (RSLN) and compared the fit of this RSLN model to the data with other common econometric models by using the monthly data from the Standard and Poor's 500 and the Toronto Stock Exchange 300 indices. The results showed that the RSLN model could provide a significantly better fit to the data than other popular models. Hence, the RSLN model could also have great advantage over other models in risk measures. Elliott (2005) developed a method to price options when the risky underlying assets are driven by the Markov-modulated geometric Brownian motion based on the regime switching random Esscher transform. Mean-variance asset liability management with regime switching under both continuous-time setting and multi-period setting were investigated by Chen et al. (2008) and Chen and Yang (2011), respectively. Xie (2009) dealt with a continuous-time mean-variance model for individual investors with stochastic liability in a Markovian regime switching financial market. Wei et al. (2013) derived the optimal equilibrium control and the corresponding equilibrium value function of the mean-variance asset liability management with regime switching model under a different type of criterion. In such a case, the extended Hamilton-Jacobi-Bellman equation system developed by Björk and Murgoci (2010) should be used to handle the time-inconsistent problem. The latest research related to regime switching includes those by Yu (2014), Zhang (2014), Bae et al. (2014) and Wu and Chen (2015).

As mentioned before, one of the key applications of asset liability management is to maintain the solvency of a financial security system. A number of literatures studied

the bankrupt control topic, for example, Zhu et al. (2004), Bielecki et al. (2005), Wei and Ye (2007) and Li and Li (2012). The generalized multi-period mean-variance model was developed through which an optimal investment policy can be generated to achieve an optimal return in the sense of a mean-variance tradeoff and also a good risk control over bankruptcy (see Zhu et al. 2004). The authors also suggested the extension of the original risk control model from discrete-time to continuous-time by imposing probability constraints at distinct time instants in the continuous-time horizon. Later, Bielecki et al. (2005) solved the continuous-time mean-variance portfolio selection with bankruptcy prohibition problem by the decomposition approach and backward stochastic differential equation method. Besides, the feasibility, existence and uniqueness of relevant results were proved; and the efficient portfolios and efficient frontier were derived as well. Wei and Ye (2007) further extended the research work by Zhu et al. (2004) by considering the multi-period mean-variance portfolio selection with bankrupt control problem in a stochastic market. Thus, Zhu's model can be embraced as a special case by adjusting the transition matrix. Based on Zhu's work, Li and Li (2012) investigated a multi-period mean-variance asset liability management with bankrupt control problem by using the Lagrangian multiplier method, the embedding technique, the dynamic programming approach and the Lagrangian duality theory. The numerical analysis showed that the bankrupt control has a significant impact on the optimal portfolio strategy. Another remarkable research work by Costa and Araujo (2008) dealt with a generalized multi-period mean-variance portfolio selection problem with market parameters subject to Markov random regime switchings. Moreover, the authors applied the results on the risk control over bankruptcy in a dynamic portfolio selection problem with Markov jump selection problem. More research need to be carried out on mean-variance asset liability management with bankrupt control under continuous-time setting with consideration of application in particular financial areas, including pension, insurance and banking.

The jump-diffusion process is a stochastic process with a mix of a jump process and a diffusion process, which has been introduced by Merton (1976) into the research of option pricing. Nowadays, it has been widely used in financial mathematics to describe the dynamics of price of risky asset in an incomplete financial market. Moreover, the

jump component in the asset is usually governed by an exponential Lévy process; while the jump component in the liability or the claim process of an insurer is often modeled by a simple Lévy process or compound Cox process. The popular processes that are used to model the logarithm of price process of risky asset include the variance Gamma process and the normal inverse Gaussian process. The notable research works including Hipp and Plum (2000), Lim (2005), Bjäuerle (2005), Delong and Gerrard (2007), Bai and Zhang (2008), Zeng and Li (2011), Coasta and Oliveria (2012), Zhang et al. (2012) and Zeng et al. (2013).

Hipp and Plum (2000) modelled the risk process by a compound Poisson process and derived explicit solution by using the Bellman equation. In the paper by Lim (2005), the jump-diffusion process of the underlying assets are driven by the Brownian motion and the doubly stochastic Poisson processes. The optimal hedging policy was obtained by using the stochastic control and backward stochastic differential equation theory. Bjäuerle (2005) used the classical Cramér-Lundberg model to describe the surplus process of an insurer with dynamic proportional reinsurance and set the mean-variance criterion as minimizing the risk of the terminal reserve for a given expected terminal reserve. Delong and Gerrard (2007) considered a collective insurance risk model with a compound Cox claim process in a financial market with an asset driven by an infinite active Lévy process. The stochastic control theory was applied to solve the classical mean-variance optimization problem and another optimization problem with consideration of a running cost in the mean-variance terminal objective. The mean-variance optimal reinsurance/new business and investment problem was investigated by Bai and Zhang (2008), in which two risk models were studied, including a classical risk model and a diffusion model. The efficient strategy and the efficient frontier were obtained by solving the Hamilton-Jecobi-Bellman equations. The mean-variance asset liability management in a jump diffusion market was modelled and solved by Zeng and Li (2011), in which the exponential Lévy process and the Lévy process were used to describe the dynamics of risky asset's price and liability, respectively.

For more detailed information of the theory and application of jump-diffusion process in finance and insurance, readers are referred to the books by Sato (1999), Bening

and Korolev (2002), Schoutens (2003), Cont and Tankov (2004), Applebaum (2009) and Øksebdal and Sulem (2009).

In most of the papers on asset liability management and portfolio selection, both the appreciation rate and the volatility rate are assumed to be constants or deterministic functions of time t. Nevertheless, many fluctuations due to various sources of uncertainty are observed in asset price and liability dynamics. Therefore, plenty of results of the effect of stochastic volatility or stochastic interest rate have been reported (see Deelstra et al. (2000), Chacko and Viceira (2005), Kraft (2005), Gao (2009) and (2010), Taksar and Zeng (2012), Li et al. (2012), Gu et al. (2012) and Zhao et al. (2014)). However, no work has been done on mean-variance asset liability management with both stochastic volatility and stochastic interest rate in a general incomplete financial market. This shows another important research direction. The typical and important stochastic volatility model and stochastic interest rate model are as follows:

Heston's Stochastic Volatility Model

$$\begin{cases} \frac{dS(t)}{S(t)} = (\mu(t)V(t) + r(t))dt + \sqrt{V(t)}dW^S(t), \\ dV(t) = k(\theta - V(t))dt + \sigma\sqrt{V(t)}dW^V(t), \end{cases}$$

where $S(t)$ is the price of risky asset, k, θ, σ are constants. For $k > 0$ and $\theta > 0$, the Heston's model is a continuous-time first-order autoregressive process where the randomly moving volatility is elastically pulled toward a central location or long-term value, θ. The parameter k indicates the speed of adjustment. The second stochastic differential equation is the famous Cox-Ingersoll-Ross (CIR) model, which can also be used to describe the evolution of short-term interest rate.

Cox-Ingerson-Ross Stochastic Interest Rate Model

$$\begin{cases} \frac{dS(t)}{S(t)} = \mu(r(t))dt + V(t)dW^S(t), \\ dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW^r(t), \\ \mu(r(t)) = \sqrt{r(t)}, \end{cases}$$

where $S(t)$ is the price of risky asset; $\mu(t)$ and $V(t)$ are the appreciate rate and the volatility rate, respectively. The other parameters in the second stochastic differential equation

have similar meanings as in $dV(t) = k(\theta - V(t))dt + \sigma\sqrt{V(t)}dW^V(t)$. There are many types of stochastic volatility models and stochastic interest rate models, for example, the constant elasticity of variance (CEV) model.

CEV Model

$$\frac{dS(t)}{S(t)} = r(t)dt + \gamma S(t)^{\beta} dW(t),$$

where γ and β are positive constants; $r(t)$ is the expected instantaneous appreciation rate of risky asset; β is the elasticity parameter; $\gamma S(t)^{\beta}$ is the instantaneous volatility of risky asset.

2.2.3 Asset Liability Management with Special Asset/Liability

In addition to the previously mentioned remarkable research work, some other contributions on asset liability management have also been made, such as those by Drijver (2005), Hilli et al. (2007), Gerstner et al. (2008), Josa-Fombellida and Rincon-Zapatero (2008), Vrontos et al. (2009), Leippold et al. (2011), Chiu and Wong (2012 and 2013), Yao et al. (2013a), concerning some special kinds of assets and/or liabilities with application in insurance and pension fund management.

The work by Drijver (2005) established the asset liability management model for pension with more reasonable constraints on key parameters, compared with the previous literature. For example, lower and upper bounds of contribution rate, risk constraints, indexing, underfunding and remedial contributions, and horizon effects. A multistage mixed-integer stochastic program was developed; the solutions were obtained by the heuristic approach. Hilli et al. (2007) developed a stochastic programming model for asset liability management of a Finnish pension insurance company with the consideration of the statutory restrictions. The problem of simultaneous minimization of risks and maximization of the terminal value of expected funds assets in a stochastic defined benefit aggregated pension plan was studied by Josa-Fombellida and Rincon-Zapatero (2008). They measured solvency risk by the variance of the terminal fund's level; and the contribution risk by a running cost associated to deviations from the evolution of the

stochastic normal cost. Vrontos et al. (2009) addressed the issue of time-varying variance and covariances/correlations of asset returns and examined the impacts in terms of asset liability management for pension funds.

Geratner et al. (2008) proposed a discrete time asset liability management model for the simulation of simplified balance sheets of life insurance products, taking into account the most important life insurance products characteristics, for example, the surrender value and the reserve-dependent bonus. Since the closed-form analytical solution to the above-mentioned pension/life insurance models cannot be obtained, efficient simulation and sensitivity tests were performed to illustrate the effectiveness of the corresponding models.

The mathematical concept, cointegration, was proposed by the Nobel Prize winner Enger and Granger. The key idea is that a linear combination of two or more non-stationary time series can be stationary. In recent years, this concept has been applied in financial mathematics, including asset management (Alexander, 1999; Chiu and Wong, 2011), option pricing (Duan and Pliska, 2004) and asset liability management (Chiu and Wong, 2012 and 2013). Chiu and Wong (2012) studied the mean-variance asset liability management considering cointegrated assets and insurance liabilities. The insurance liabilities are described by the compound Poisson process. The optimal investment strategy and the efficient frontier were solved by the backward stochastic differential equation theory. Chiu and Wong (2013) continued with the mean-variance asset liability management with cointegrated assets problem by considering random liabilities which is governed by the geometric Brownian motion. The solution strategy is a generalized technique by Lim (2004) and Lim (2005). The research of the application of cointegrated assets in life / non-life insurance and pension fund management is another interesting future topic with practical significance.

Compared with the uncontrollable exogenous liability, the endogenous liability is controllable by various financial instruments and investor's decisions. Taking endogenous liability into consideration, in other words, managing multiple assets and multiple liabilities synthetically, will make asset liability management problem much more challenging. Leippold et al. (2011) and Yao et al. (2013a) investigated mean-variance asset liability

management with endogenous liabilities under multi-period setting and continuous-time setting, respectively. Leippold also compared the asset liability management with exogenous liabilities and endogenous liabilities. Yao studied the problem in a more general market where all the assets are risky.

2.3 Methodology of Mean-Variance Optimization Problem

The two major types of mean-variance optimization problems are portfolio selection problem and asset liability management problem, which have been studied intensively and many interesting and powerful results are now available in the literature. The objective of such optimization problems is to find the optimal control policy $\mathbf{u}^*$ so as to maximize the expectation of terminal wealth for a given variance tolerance (the preselected risk level); or to minimize the variance of terminal wealth for a given range of the expectation; with the consideration of other constraints based on the problem context. The problems can be expressed mathematically as follows.

Problem $\mathbf{P_1}(\sigma)$.

$$\begin{aligned} &\text{Maximize} && E[X_T] \\ &\text{subject to} && Var[X_T] \leq \sigma, \\ & && X_{t+1} = \mathbf{f}(X_t), \quad t = 0,\ 1,\ 2, \ldots,\ T-1. \\ &\textbf{or} \\ & && dX_t = \mu(t, X_t, u_t)dt + \nu(t, X_t, u_t)dW_t + d\xi(t, X_t, u_t) \\ & && t \in [0,\ T]. \end{aligned} \tag{2.1}$$

Problem $\mathbf{P_2}(\epsilon)$.

$$\begin{aligned}
&\text{Minimize} \quad Var[X_T] \\
&\text{subject to} \quad E[X_T] \geq \epsilon, \\
&\qquad\qquad X_{t+1} = \mathbf{f}(X_t), \quad t = 0,\ 1,\ 2, \ldots,\ T-1. \\
&\textbf{or} \\
&\qquad\qquad dX_t = \mu(t, X_t, u_t)dt + \nu(t, X_t, u_t)dW_t + d\xi(t, X_t, u_t) \\
&\qquad\qquad t \in [0,\ T].
\end{aligned} \tag{2.2}$$

The optimization problems expressed in the way above is easier and more direct for investors to capture the decision-making information than the utility functions in terms of terminal wealth (Li and Ng, 2000). The wealth process X_t^* is *mean-variance efficient* if there is no other wealth process X_t which evolves according to the same dynamics, such that $E[X_T] \geq E[X_T^*]$ and $Var[X_T] \leq Var[X_T^*]$ with at least one inequality holding strictly. The corresponding control policy is the optimal control policy or *efficient investment strategy*, $\mathbf{u}^*$. The point $(Var[x(T)], E[X_T])$ corresponding to an efficient investment strategy on the variance-mean space is called an efficient point. The set of all the efficient points forms the *efficient frontier* in the variance-mean space. Problem $\mathbf{P1}(\sigma)$ and Problem $\mathbf{P2}(\epsilon)$ are the two basic optimization problems under mean-variance criterion. In the existing literature, there are many kinds of different expressions of mean-variance criterion. Here we give another two transformations of mean-variance criterion.

Problem Q1.

$$\begin{aligned}
&\text{Minimize} \quad Var[X_T] \\
&\text{subject to} \quad E[X_T] = d, \\
&\qquad\qquad X_{t+1} = \mathbf{f}(X_t), \quad t = 0,\ 1,\ 2, \ldots,\ T-1. \\
&\textbf{or} \\
&\qquad\qquad dX_t = \mu(t, X_t, u_t)dt + \nu(t, X_t, u_t)dW_t + d\xi(t, X_t, u_t) \\
&\qquad\qquad t \in [0,\ T].
\end{aligned} \tag{2.3}$$

Problem Q2.

$$\begin{aligned} &\text{Maximize} \quad E[X_T] - \tfrac{\gamma(x)}{2} Var[X_T] \\ &\text{subject to} \quad X_{t+1} = \mathbf{f}(X_t), \quad t = 0,\ 1,\ 2, \ldots,\ T-1. \\ &\textbf{or} \\ &\qquad dX_t = \mu(t, X_t, u_t)dt + \nu(t, X_t, u_t)dW_t + d\xi(t, X_t, u_t) \\ &\qquad t \in [0,\ T]. \end{aligned} \tag{2.4}$$

Problem **Q1**(σ) considers minimizing the variance in terminal wealth for a given expectation of terminal wealth (a fixed value instead of an acceptable range). Problem **Q2**(σ) consists of both variance and expectation in the objective function taking into account the state-dependent risk aversion.

In the following, we will give a brief introduction of the popular methodologies for solving the above-mentioned mean-variance optimization problems under both multi-period setting and continuous-time setting. Before we move to the next subsection, we address here that the technique developed by Lim (2005) by using the backward stochastic differential equation theory and the stochastic control method, which is very useful for the continuous-time mean-variance optimization problem of minimizing variance in the terminal wealth for a given expected terminal wealth in the case that there exists discontinuities in the underlying price processes.

2.3.1 Embedding Technique for Problem P1(σ) and Problem P2(ϵ)

The embedding technique was proposed by Li and Ng (2000) and Zhou and Li (2000) to address the difficulty of non-separability in variance and has been widely applied in portfolio selection and asset liability management area. For example, Leippold et al. (2004), Chiu and Li (2006), Yi et al. (2008), Li and Shu (2011), Li and Li (2012). The major steps of embedding technique are as follows.

Step 1: Transform Problem P1(σ) and Problem P2(ϵ) into Problem P3(ω).

According to the standard optimization theory, the equivalent formulation to either **P1**(σ) or Problem **P2**(ϵ) is transformed as follow:

Problem $\mathbf{P_3}(\omega)$.

$$\begin{aligned} &\text{Maximize} && E[X_T] - \omega Var[X_T] \\ &\text{subject to} && X_{t+1} = \mathbf{f}(X_t), \quad t = 0,\ 1,\ 2, \ldots,\ T-1. \\ &\textbf{or} \\ & && dX_t = \mu(t, X_t, u_t)dt + \nu(t, X_t, u_t)dW_t + d\xi(t, X_t, u_t) \\ & && t \in [0,\ T], \end{aligned} \tag{2.5}$$

where $\omega \in [0,\ \infty)$, which is specified by the investor as the trade-off between the expected terminal wealth and the associated risk. The relationship of the optimal control policy between Problem **P1**(σ), Problem **P2**(ϵ) and Problem **P3**(ω) are as below. If u^* solves Problem **P3**(ω), then u^* also solves Problem **P1**(σ) and Problem **P2**(ϵ) with

$$\begin{aligned} &\omega = \tfrac{\partial E[X_T]}{\partial Var[X_T]}|u^*, \\ &\sigma = Var[X_T]|u^*, \\ &\epsilon = E[X_T]|u^*. \end{aligned} \tag{2.6}$$

Because the variance operation does not satisfy the smoothing property, all the three problems are difficult to solve directly due to their non-separability in the sense of dynamic programming. So, it is necessary to embed Problem **P3**(ω) into a tractable auxiliary problem that is separable.

Step 2: Construct an Auxiliary Problem.

Problem $\mathbf{P_4}(\lambda, \omega)$.

$$\begin{aligned} &\text{Maximize} && E[-\omega X_T^2 + \lambda X_T] \\ &\text{subject to} && X_{t+1} = \mathbf{f}(X_t), \quad t = 0,\ 1,\ 2, \ldots,\ T-1. \\ &\textbf{or} \\ & && dX_t = \mu(t, X_t, u_t)dt + \nu(t, X_t, u_t)dW_t + d\xi(t, X_t, u_t) \\ & && t \in [0,\ T]. \end{aligned} \tag{2.7}$$

Problem **P4**($\lambda,\ \omega$) is separable in the sense of dynamic programming and any optimal so-

lution of Problem **P3**(ω) (as long as it exits) can be found via solving the auxiliary problem, Problem **P4**($\lambda,\ \omega$). Problem **P4**($\lambda,\ \omega$) is of a standard stochastic linear-quadratic form parameterized by λ and ω with a quadratic cost function and linear constraints. And the solution of Problem **P4**($\lambda,\ \omega$) solves Problem **P3**(ω) with $\lambda^* = 1 + 2\omega E[X_T]|u^*$ (Li and Ng 2000; Zhou and Li 2000; Leippold et al. 2004; Chiu and Li 2006).

Step 3: Derive analytically the efficient investment strategy and the efficient frontier by using dynamic programming.

Under multi-period setting, the dynamic programming algorithm starts with time $T-1$ and the optimization problem is expressed in a recursive manner. Then, u^*_{T-1} is obtained by maximizing the quadratic objective function with respect to u^*_{T-1}. Thereafter, with the u^*_{T-1} in place, the recursive expression for the expected wealth and the expected value of the squared wealth between successive time periods can be derived. Hence, the efficient investment strategy and the efficient frontier in terms of ω can be calculated by simplifying the recursive expressions.

Under continuous-time setting, the efficient investment strategy can be obtained by solving the associated stochastic Riccati equation and the partial differential equations subject to boundary conditions. We refer readers to About-Kandil et al. (2003), Butcher (2003), Ockendon et al. (2003), Quarteroni (2000), Rami et al. (2001), Wonham (1968a) and (1968b), for the methods on solving Riccati equation and partial differential equations; together with the notable papers/books on dynamic programming (see Fleming (1975), Fleming (2006), Luenberger (1968)). In the case with constraints, the associated Riccati equation and partial differential equation can be quite complex.

Step 4: Obtain the efficient investment strategy and efficient frontier of Problem P1(σ) and Problem P2(ϵ) according to the relationship between the optimal control policies.

Finally, the transformation technique is used to yield the efficient investment strategy and the efficient frontier of Problem **P1**(σ) and Problem **P2**(ϵ).

2.3.2 Stochastic Dynamic Programming Approach for Problem Q1

The application of stochastic dynamic programming in financial mathematics has long history. A great deal of work has been carried out to use the method to solve many financial mathematical problems, such as those by Merton (1969 and 1971), Fleming (1975 and 2006), Luenberger (1968), Li et al. (2002), Pliska and Ye (2007), Yao (2011). In brief, it simplifies decision-making by breaking the optimization process into a sequence of steps and finding the optimal control policy by working backwards through a recursive relationship called the Bellman equation. Finally the initial value/state of the system is the optimal value to be found. Problem **Q1** can be regarded as a special case of Problem **P2**(ϵ). Yao (2011) introduced a method to solve Problem **Q1** by using the Lagrange duality method and dynamic programming approach in order to avoid the complex calculation in the embedding technique. This method also works well in continuous-time (see Yao 2014b). Here is the Algorithm.

Step 1: Transform the constrained Problem Q1 into an unconstrained Problem Q1(θ).

Problem $\mathbf{Q_1}(\theta)$.

$$\begin{aligned} &\text{Miniimize} && E[(X_T-(d-\theta))^2] \\ &\text{subject to} && X_{t+1}=\mathbf{f}(X_t), \quad t=0,\ 1,\ 2,\ldots,\ T-1. \\ &\textbf{or} \\ & && dX_t=\mu(t,X_t,u_t)dt+\nu(t,X_t,u_t)dW_t+d\xi(t,X_t,u_t) \\ & && t\in[0,\ T]. \end{aligned} \tag{2.8}$$

According to the definition of variance, the original problem can be transformed by introducing the Lagrange multiplier θ

$$\begin{aligned} Var[X_T] &:= E[(X_T-d)^2]+2\theta(E[X_T]-d) \\ &= E[(X_T-(d-\theta))^2]-\theta^2, \end{aligned}$$

and $(-\theta^2)$ is fixed, and the original optimization Problem **Q1** is equivalent to Problem **Q1**(θ).

Step 2: Develop the Hamilton-Jacobi-Bellman equation.

This step usually starts with a truncated optimization problem and the definition of value function $V(x,\ t)$, which should satisfy the Hamilton-Jacobi-Bellman equation based on the dynamic programming principle. Under multi-period setting, the corresponding Bellman equation is in the recursive form; while under continuous-time setting, the Hamilton-Jacobi-Bellman equation is derived by using the controlled infinitesimal operator.

$$A^u = \frac{\partial}{\partial t} + \widetilde{\mu}(t, X_t, u_t)\frac{\partial}{\partial x} + \frac{1}{2}\Big\{\widetilde{\nu}^2(t, X_t, u_t)\Big\}\frac{\partial^2}{\partial x^2},$$

where $\widetilde{\mu}(t, X_t, u_t)$ and $\widetilde{\nu}(t, X_t, u_t)$ are the appreciation and volatility of the wealth process after rearrange the dynamics of wealth process.

Step 3: Derive the value function and the optimal control policy.

We start with a guess of the expression of value function and substitute this expression into the Hamilton-Jacobi-Bellman equation. Under continuous-time setting, after some calculation, the system of associated partial differential equations can be obtained with boundary conditions. Solving the partial differential equations leads to the explicit expression of the value function. Under multi-period setting, the value function can be derived by solving the recurrence relations and boundary conditions of the coefficients (see Yao 2011). At last, the first-order condition yields the optimal control policy. The optimal value of the Problem **Q1**(θ) is $V(0,\ X_0)$.

Step 4: Calculate the efficient investment strategy and the efficient frontier of the original optimization problem.

The efficient investment strategy and the efficient frontier of the original optimization problem can be obtained by eliminating the parameter θ as follows

$$Var[X_T] = \max_{\theta}\Big\{V(0,\ X_0) - \theta^2\Big\}.$$

The existence and uniqueness of the solution should be proved.

2.3.3 Nash equilibrium strategy for Problem Q2

The mean-variance criterion applied in Problem **Q2** is named as mean-variance with state-dependent risk aversion, which aims to determine the optimal portfolio so as to acquire sufficient return (by maximizing the expectation of the terminal wealth) in compensating the company's liability with minimal risk measured by the variance in the terminal wealth (Wei et al. 2013). The positive factor $\gamma(x)$, which represents the degree of risk aversion of the company, is dependent on the current wealth. In the paper by Björk et al. (2014), a natural choice of $\gamma(x)$ was proposed as inversely proportional to the current wealth.

Many researchers have investigated this kind of optimization problems. The optimal control policy is not easy to achieve due to the lack of iterated-expectation property in this mean-variance criterion, which results in the time-inconsistency difficulty in the sense that the Bellman optimality principle does not hold. Specifically speaking, an optimal control policy that optimizes the objective utility at initial time point does not remain to be optimal at any latter time. Therefore, the traditional stochastic dynamic programming approach cannot be applied directly to deal with this optimization problem under either the multi-period setting or the continuous-time setting. Time-inconsistency in optimization problem was first studied by Strotz (1955) and two major methods was proposed to tackle the problem, including (i) the strategy of precommitment and (ii) the strategy of consistent planning.

To derive the precommitment strategies, the market participants are assumed to precommit themselves to adopt the optimal strategies derived at the initial time ($t = 0$) in the future. In other words, only the feasible control policy that optimizes the initial objective function would be considered. So the chosen control policy may not be optimal at time $t = k$ ($k = 1, 2, \ldots$). Therefore, the precommitment strategy has been criticized for laking rationality. Kydland and Prescott (1977) discussed the meaning of the precommitment strategy from the economic point of view. Rechardson(1989), followed by Bajeux-Besnainou(1998), is the first one to explore the portfolio optimization under the mean-variance criterion in continuous-time even though he just set one single stock with a constant risk-free rate. Dai et al. (2010) studied the Markowitz's model with transaction cost. Besides, there are many other literatures that use the precommitment strategy to solve time-inconsistent optimization problems with the focus on extensions and improvements, including Zhou and Yin (2003), Xia (2005), Xia and Yan (2006), Celikyurt and Ozekici (2007), Fu (2010), Wu and Li (2011) and (2012).

The alternative approach to deal with the time-inconsistent optimization problem is the Nash equilibrium strategy which is formulated in game theory. Pollak (1968), followed by Peleg and Yaar (1973), provided the primitive idea of time-consistent strategy. In brief, the investors sitting at time t would consider that, starting from time $t + \Delta t$, they will follow the strategies that are optimal sitting at time $t + \Delta t$. Namely, the optimal strategy derived at time t should agree with the optimal policy derived at time $t + \Delta t$ (Zeng et al. 2013). The game theory was firstly introduced by Markowitz (1952) to analyze the mean-variance portfolio selection. Along this road, Krusell and Smith (1977) and Goldman (1980) developed their problems within the game theory framework. The Nash equilibrium strategy was then defined and refined as the time-consistent strategy for time-inconsistent optimization problems. Basak and Chabakauri (2010) constructed a mean-variance portfolio selection model in a general incomplete market. The analytical time-consistent solution was derived by using dynamic programming. Ekeland and Lazrak (2006 and 2010) analytically formalized the definition of sub-game perfect equilibrium strategy and characterized it in continuous-time. They also proved the existence of the Nash equilibrium strategy in the case of time-inconsistency. Wang (2009) gave the time-consistent solution algorithm for a multi-period mean-variance objective function of a portfolio selection problem. Björk and Murgoci (2010) investigated the time-inconsistent problem with a general controlled Markov process and a fairly general objective function and derived the corresponding Nash sub-game perfect equilibrium strategy through the development of the extended Hamilton-Jacobi-Bellman system. A numerical scheme for determining the time-consistent strategy and the precommitment strategy of a continuous-time optimal asset allocation problem with constraints was obtained by Wang and Forsyth (2011). Björk et al.(2014) further extended Basak's research in time-inconsistent optimization problem by relaxing the risk aversion rate in mean-variance utility from a constant to a state-dependent one and solved this model by the technique proposed in Björk and Murgoci (2010). Our research work is based on Björk et al. (2014). This is a provoking improvement both from modeling and economic significance. The latest research work is done by Wu and Chen (2015) who solved the analytical closed-form Nash sub-game perfect equilibrium strategy for a multi-period mean-variance portfolio selection problem with constraints.

Now, we give a brief review of the Algorithm of Nash (sub-game perfect) equilibrium strategy for problem **Q2**. This methodology involves theory on ordinary differential equations and the Hamilton-Jacobi-Bellman equation. For more details, we refer the readers to Hairer and Wanner

(1993, 2010), Yong and Zhou (1999) for detailed information.

Step 1:

Define the **Nash equilibrium Control Law**.

Step 2:

Derive the **Verification Theorem**.

Step 3:

Develop the **extended Hamilton-Jacobi-Bellman system**.

Step 4:

Solve the expression of equilibrium control law (optimal control policy) under (i) a general $\gamma(x)$ and (ii) a natural choice of $\gamma(x)$.

Step 5:

Obtain the equilibrium control law and the corresponding equilibrium value function (optimal value function) by solving the associated ordinary differential equations.

2.4 Concluding Remarks

Asset liability management with constraints, for example, uncertain exit time, regime switching, transaction cost, bankruptcy prohibition, stochastic volatility/interest rate, discontinuity in risky asset price, borrowing constraints, no-shorting constraints and special assets/liability, has been a versatile tool to describe many phenomena in real financial market, develop optimal investment strategy and improve financial standing, with a wide range of applications in many modern financial security systems, including insurance, pension fund management and investment institutes. Hence, research in asset liability management is one of the focused topics in financial mathematics. In our work, the impact of jump, state-dependent risk aversion, and parameters on the optimal investment strategy, efficient frontier and optimal value function are studied. The currently popular methodologies for mean-variance optimization problems include embedding technique, stochastic dynamic programming approach and the Nash equilibrium strategy. The latter two methods are applied in our research with further development.

CHAPTER 3

Mean-Variance Asset Liability Management with General Insurance Liability and Jumps

3.1 General Overview

In this chapter, we study the asset liability management problem with general insurance liabilities under the mean-variance criteria in a jump diffusion market. An insurer faces uncertain general insurance liability which is governed by a compound Poisson process during the cover period defined by the contract. When the insurer invests in the financial market, the wealth has both a diffusion component and a jump component from the liability (Chiu and Wong, 2012). The typical asset liability management problem assumes that the insurer can reallocate the portfolio of assets, but nothing can be done about the interim random insurance claim payments. In this regard, our trading strategy is not of the self-financing type. We formulate a stochastic linear quadratic control framework for our problem. By applying the Lagrange multiplier method and the Hamilton-Jacobi-Bellman equation approach, we convert the mean-variance asset liability management problem to a system of partial differential equations. Finally, we derive the analytical solution of efficient investment strategy and efficient frontier and illustrate it by numerical examples.

Compared with previous research work on asset liability management problem under mean-variance criterion, the main innovation of our work is that we employ an exponential Lévy process to describe the risky assets' price process and we use a compound Poisson process to describe the interim random general insurance liability payments synthetically. There appears no research work in the literature on the continuous-time version of the asset liability management problem

with general insurance liability in a jump diffusion market setting.

The remainder of the chapter is organized as follows. Section 3.2 describes the financial market setting and constructs a mean-variance asset-liability management model with general insurance liabilities following a jump diffusion process. In section 3.3, we covert the original constrained model to an unconstrained stochastic control problem and derive the closed-form analytical solution by applying the duality theory, the Hamilton-Jacobi-Bellman equation approach and the stochastic dynamic programming technique. In addition, we obtain efficient investment strategy and the efficient frontier. In section 3.4, several special cases are discussed, followed by some numerical examples in section 3.5 and conclusions in Section 3.6.

3.2 Model Formulation

Let us consider a filtered probability space $(\Omega, \mathbf{P}, \mathcal{F}, \{\mathfrak{F}_t\}_{0\leq t\leq T})$ for some finite T which denotes the investment time horizon. We assume that $\mathfrak{F}_t = \sigma\{\mathbf{W}(s); 0 \leq s \leq t\}$ be augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$, where $\mathcal{F} = \mathfrak{F}_T$. On this space, we define $\mathbf{W}(t) = (W_1(t), W_2(t), \ldots, W_m(t))'$ as an m-dimensional standard Brownian motion, $\mathbf{H}'$ as the transpose of matrix or vector $\mathbf{H}$; m_i as the ith component of any vector $\mathbf{M}$. Suppose that an insurer can allocate its wealth among $n+1$ assets that include a risk-free asset and n risky assets in a continuous-time financial market with the standard assumptions: continuous trading is allowed; no transaction cost or tax is involved in trading; and all assets are infinitely divisible. We also assume that all the functions are measurable and uniformly bounded in $[0,\ T]$. Denote by $\mathbf{L}^2_{\mathcal{F}}(t, T; \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued and measurable stochastic processes $f(s)$ adapted to $\{\mathfrak{F}_s\}_{s\geq t}$ on $[0,\ T]$ such that $E\left[\int_t^T |f(s)|^2 ds\right] < +\infty$.

Asset Price Evolution Model

The price of risk-free asset is modelled by the following ordinary differential equation.

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \quad 0 \leq t \leq T, \\ S_0(0) = s_0 > 0, \end{cases} \tag{3.1}$$

where s_0 is the initial price of the risk-free asset; $r(t)$ is a positive time-dependent deterministic risk-free interest rate. Whereas, the prices of these risky assets satisfy the following stochastic

differential equations.

$$\begin{cases} dS_i(t) = S_i(t-)\left(\alpha_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW_j(t) + d\sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}\right), \quad 0 \le t \le T, \\ S_i(0) = s_i > 0, \qquad i = 1, 2, \ldots, n, \end{cases} \tag{3.2}$$

where $(s_i, i = 1, 2, \ldots, n)$ are the initial prices of the risky assets; $(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$ and $(\sigma_{ij}(t))_{m\times m}$ are respectively the appreciation rate vector and the volatility matrix of these assets, which are assumed to be deterministic functions of time t; $\{\mathcal{N}_{Ai}(t), 0 \le t \le T\}$ are homogeneous Poisson processes with intensity $\lambda_{Ai}(\ge 0)$; and for each i, $\{y_{ik}, k = 1, 2, \ldots\}$ are independent and identically distributed (*i.i.d.*) random variables.

Remark 3.1. We assume that for each i, $P\{y_{ik} \ge -1 \text{ for all } k \ge 1\} = 1$ so as to make sure that the risky assets' prices remain positive.

Remark 3.2. According to our common sense, higher investment return usually bears more risk. Therefore, we assume that the expected return of the risky assets are greater than the risk-free interest rate, i.e., for each i, $(\alpha_i + \lambda_{Ai}E(y_{ik}) > r, k = 1, 2, \ldots)$, where E is the expectation operator.

Remark 3.3. Moreover, we assume that the two process $\{\mathbf{W}(t), 0 \le t \le T\}$ and $\{\sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}, 0 \le t \le T\}$ are independent. $\mathbf{W}(t) = (W_1(t), W_2(t), \ldots, W_m(t))'$ is used to describe all the random factors which influence the prices of risky assets. Therefore, it should have $m \ge n + 1$.

Insurance Premium

General insurance is broadly defined as non-life insurance, which is one of the two major insurance categories. Basically, there are six main forms of general insurance, including home and contents insurance, motor vehicle insurance, business insurance, mortgage related insurance, workers compensation and travel insurance (from Insurance Council of Australia). It provides payments depending on the loss from a particular financial event. It is called property and casualty insurance in the U.S. and Canada and non-life insurance in Continental Europe.

General insurance contracts are usually fixed-term renewable with the premium paid at the beginning of the covered period or by monthly instalments. Based on the results of section 4.3 in Chiu and Wong (2012), the solution for the asset liability management model with the assumption that the insurer collects a lump sum premium at the beginning of insurance contract

is equivalent to the solution for the asset liability management model with consideration of positive continuous insurance premium. Here, we let $P(t)$ be the accumulated insurance premium and let $PV(0)$ be the present value of all future premiums. So, we have

$$PV(0) = \int_0^T e^{-\int_0^s r(\tau)d\tau} dP(t). \tag{3.3}$$

We are going to give a brief derivation to prove that the two models, (i) asset liability management with $P(t)$ and (ii) asset liability management with $PV(0)$, share the same efficient investment strategy and the same efficient frontier in the section of *Wealth Process*.

Liability Evolution Model

We assume that the general insurance liability follows a compound Poisson process.

$$L(t) = \sum_{i=1}^{\mathcal{N}_L(t)} z_i, \qquad L(0) = 0. \tag{3.4}$$

where $(\mathcal{N}_L(t), 0 \leq t \leq T)$ is the Poisson process with intensity $\lambda_L(\geq 0)$; and $z_i, i = 1, 2, \ldots$ are independent positive random variables with finite second moments and independent of $\mathcal{N}_L(t)$. Let the nonnegative deterministic function $C(t)$ be the accumulated operational expenses related to the general insurance business line. Assume that $C(t)$ evolves according to the following ordinary differential equation

$$dC(t) = c(t)dt, \qquad C(0) = C_0 \geq 0. \tag{3.5}$$

Let $l(t)$ be the accumulated liability faced by the insurer. From (3.4) and (3.5), we derive the dynamics of $l(t)$ as follows

$$\begin{cases} dl(t) = c(t)dt + d\sum\limits_{i=1}^{\mathcal{N}_L(t)} z_i, \quad 0 \leq t \leq T, \\ l(0) = C_0 \geq 0. \end{cases} \tag{3.6}$$

Wealth Process

Let $u_i(t)$ denote the amount invested in asset i; and let $X(t)$ be the insurer's wealth. Then the amount invested in the $0th$ asset is $[X(t) - \sum\limits_{i=1}^{n} u_i(t)]$. Note that the strategy is not of self-financing because the insurer has to withdraw money from its wealth to pay insurance liability.

So the strategy is of insurance liability-financing. After deducing the liability, by (3.1), (3.2) and (3.6), the wealth held by the insurer $X(t)$ follows the dynamics

$$\begin{aligned}
dX(t) &= [X(t) - \sum_{i=1}^{n} u_i(t)]r(t)dt \\
&+ \sum_{i=1}^{n} u_i(t)\left(\alpha_i(t)dt + \sum_{j=1}^{m}\sigma_{ij}(t)dW_j(t) + d\sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}\right) \\
&+ dP(t) - c(t)dt - d\sum_{i=1}^{\mathcal{N}_L(t)} z_i \\
&= \sum_{i=1}^{n} u_i(t)\alpha_i(t)dt + [X(t) - \sum_{i=1}^{n} u_i(t)] \times r(t)dt + \sum_{j=1}^{m}\sum_{i=1}^{n} u_i(t)\sigma_{ij}(t)dW_j(t) \\
&+ \sum_{i=1}^{n} u_i(t)\left(d\sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}\right) + dP(t) - c(t)dt - d\sum_{i=1}^{\mathcal{N}_L(t)} z_i.
\end{aligned}$$

To rearrange $dX(t)$ above, we gain the stochastic differential equation below

$$\begin{cases}
dX(t) = [r(t)X(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t) - c(t)]dt + \sum_{j=1}^{m} \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)dW_j(t) \\
+ \sum_{i=1}^{n} u_i(t)\left(d\sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}\right) - d\sum_{i=1}^{\mathcal{N}_L(t)} z_i + dP(t), \quad 0 \le t \le T, \\
X(0) = X_0 > 0,
\end{cases} \tag{3.7}$$

where

$$\begin{cases}
\boldsymbol{u}(t) = (u_1(t), u_2(t), \ldots, u_n(t))', \\
\boldsymbol{\alpha}(t) = (\alpha_1(t) - r(t), \alpha_2(t) - r(t), \ldots, \alpha_n(t) - r(t))', \\
\boldsymbol{\delta}_j(t) = (\sigma_{1j}(t), \sigma_{2j}(t), \ldots, \sigma_{nj}(t))', \quad j = 1, 2, \ldots, m.
\end{cases}$$

Let $x(t) = X(t) + \int_t^T e^{-\int_t^s r(\tau)d\tau} dP(s)$. Now we derive $dx(t)$ according to the relationship between $x(t)$ and $X(t)$

$$\begin{aligned}
dx(t) &= dX(t) + d\left\{\int_t^T e^{-\int_t^s r(\tau)d\tau} dP(s)\right\} \\
&= dX(t) + d\left\{e^{\int_0^t r(\tau)d\tau}\int_t^T e^{-\int_0^s r(\tau)d\tau} dP(s)\right\}.
\end{aligned}$$

Then we calculate $d\left\{e^{\int_0^t r(\tau)d\tau}\int_t^T e^{-\int_0^s r(\tau)d\tau} dP(s)\right\}$ by applying the product rule in calculus.

$$
\begin{aligned}
d\left\{e^{\int_0^t r(\tau)d\tau}\int_t^T e^{-\int_0^s r(\tau)d\tau}dP(s)\right\} &= \left(r(t)e^{\int_0^t r(\tau)d\tau}dt\right)\int_t^T e^{-\int_0^s r(\tau)d\tau}dP(s) \\
&\quad + e^{\int_0^t r(\tau)d\tau}d\left\{\int_t^T e^{-\int_0^s r(\tau)d\tau}dP(s)\right\} \\
&= \left(r(t)e^{\int_0^t r(\tau)d\tau}dt\right)\int_t^T e^{-\int_0^s r(\tau)d\tau}dP(s) \\
&\quad - e^{\int_0^t r(\tau)d\tau}d\left\{\int_T^t e^{-\int_0^s r(\tau)d\tau}dP(s)\right\} \\
&= \left(r(t)e^{\int_0^t r(\tau)d\tau}dt\right)\int_t^T e^{-\int_0^s r(\tau)d\tau}dP(s) \\
&\quad - \left(e^{\int_0^t r(\tau)}\right)\left(e^{-\int_0^t r(\tau)d\tau}\right)dP(t) \\
&= \left(r(t)e^{\int_0^t r(\tau)d\tau}dt\right)\int_t^T e^{-\int_0^s r(\tau)d\tau}dP(s) - dP(t).
\end{aligned} \tag{3.8}
$$

Thus,

$$
\begin{aligned}
dx(t) &= dX(t) + \left(r(t)e^{\int_0^t r(\tau)d\tau}dt\right)\int_t^T e^{-\int_0^s r(\tau)d\tau}dP(s) - dP(t) \\
&= [r(t)X(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t) - c(t)]dt + \sum_{j=1}^m \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)dW_j(t) \\
&\quad + \sum_{i=1}^n u_i(t)\left(d\sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}\right) - d\sum_{i=1}^{\mathcal{N}_L(t)} z_i + dP(t) \\
&\quad + \left(r(t)e^{\int_0^t r(\tau)d\tau}dt\right)\int_t^T e^{-\int_0^s r(\tau)d\tau}dP(s) - dP(t) \\
&= [\boldsymbol{u}'(t)\boldsymbol{\alpha}(t) - c(t)]dt + \sum_{j=1}^m \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)dW_j(t) \\
&\quad + \sum_{i=1}^n u_i(t)\left(d\sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}\right) - d\sum_{i=1}^{\mathcal{N}_L(t)} z_i \\
&\quad r(t)\left(X(t) + \int_t^T e^{-\int_t^s r(\tau)d\tau}dP(s)\right)dt \\
&= [\boldsymbol{u}'(t)\boldsymbol{\alpha}(t) - c(t)]dt + \sum_{j=1}^m \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)dW_j(t) \\
&\quad + \sum_{i=1}^n u_i(t)\left(d\sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}\right) - d\sum_{i=1}^{\mathcal{N}_L(t)} z_i + r(t)x(t)dt.
\end{aligned} \tag{3.9}
$$

Rearrange the stochastic differential equation (3.9) and we obtain the dynamics of $dx(t)$ as follows

$$\begin{cases} dx(t) = [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t) - c(t)]dt + \sum_{j=1}^{m} \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)dW_j(t) \\ + \sum_{i=1}^{n} u_i(t)\left(d \sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}\right) - d \sum_{i=1}^{\mathcal{N}_L(t)} z_i, \quad 0 \le t \le T, \\ x(0) = X_0 + \int_0^T e^{-\int_0^s r(\tau)d\tau} dP(t) > 0. \end{cases} \tag{3.10}$$

Comparing the stochastic differential equations (3.7) and (3.10), it is easy to find that the two models share the same results, including efficient investment frontier and efficient investment strategy, with the only difference in initial value. So, in our work, we study the model (3.10) in all the remaining sections. An alternative way to simplify the model is to assume that the insurer collect premiums at the beginning of contract at the amount $PV(0)$.

Assumption 3.1. The non-degeneracy condition holds for all $t \in [0,\ T]$, i.e., there exits some $\zeta > 0$ such that $\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) \ge \zeta I_n$, where I_n is the $n \times n$ identity matrix.

Mean-Variance Asset Liability Management Optimization Problem

Definition 3.1 (Admissible Strategy)**.** *A strategy $\boldsymbol{u}(.) = \{\boldsymbol{u}(t); t \in [0,\ T]\}$ is called admissible if $\boldsymbol{u}(.) \in \mathbf{L}^2_{\mathcal{F}}(t, T; \mathbb{R}^n)$, and the pair $(x(.), \boldsymbol{u}(.))$ satisfies the stochastic differential equation (3.10). In this case, $(x(.), \boldsymbol{u}(.))$ is called an admissible pair.*

Denote by $\mathcal{U}[0,\ T]$ the set of all such admissible pairs over $[0,\ T]$. The mean-variance asset liability management problem with general insurance liability in a jump diffusion market refers to the problem of finding the optimal admissible strategy such that the variance of the terminal wealth is minimized for a given expected terminal wealth d, i.e.

$$\begin{cases} \min\limits_{(x(.),\ \boldsymbol{u}(.)) \in \mathcal{U}[0,\ T]} & Var[x(T)] := E[x(T) - d]^2, \\ s.t. & E[x(T)] = d. \end{cases} \tag{3.11}$$

The solution $\boldsymbol{u}^* = \{\boldsymbol{u}^*(t); t \in [0,\ T]\}$ for problem (3.11) is called an *efficient investment strategy* for $d \ge\ d_{SD_{min}}$, where $d_{SD_{min}}$ is the expected terminal wealth corresponding to global minimum variance of terminal wealth over all feasible strategies. The point $(Var[x(T)],\ d)$ corresponding to an efficient investment strategy on the variance-mean space is called an efficient point. The set of all the efficient points forms the *efficient frontier* in the variance-mean space.

3.3 Analytical Solution

Transformation of Problem

The equality constraint $E[x(T)] = d$ in the optimization problem (3.11) can be dealt with by the Lagrange method. By introducing a Lagrange multiplier θ, we can transform the constrained optimization problem (3.11) to the following unconstrained stochastic control problem.

$$\min_{(x(\boldsymbol{.}),\ \boldsymbol{u}(\boldsymbol{.}))\in\mathcal{U}[0,\ T]} \quad Var[x(T)] := E[x(T) - d]^2 + 2\theta(E[x(T)] - d). \tag{3.12}$$

Then we further simplify the problem and have

$$E[x(T) - d]^2 + 2\theta(E[x(T)] - d) = E[x(T) - (d - \theta)]^2 - \theta^2.$$

Since $(-\theta^2)$ is fixed, the original optimization problem is equivalent to

$$\min_{(x(\boldsymbol{.}),\ \boldsymbol{u}(\boldsymbol{.}))\in\mathcal{U}[0,\ T]} \quad Var[x(T)] := E[x(T) - (d - \theta)]^2. \tag{3.13}$$

According to the Lagrange duality theory in Luengerger (1968), if $\boldsymbol{u}(t,\ \theta)$ optimizes the Lagrangian problem (3.13), then $\boldsymbol{u}^*(t,\ \theta^*)$ optimizes the original problem (3.11) with θ^* maximizes the optimal value function of (3.12).

Development of the Hamilton-Jacobi-Bellman equation

Under the framework of stochastic dynamic programming, we consider a truncated optimization problem (3.13) with initial time $t \in [0,\ T]$ and initial state $x(t) = x$. For any time $s \in [t,\ T]$, the dynamics of $x(s)$ is as follows

$$\begin{cases} dx(s) = [r(s)x(s) + \boldsymbol{u}'(s)\boldsymbol{\alpha}(s) - c(s)]ds + \sum\limits_{j=1}^{m} \boldsymbol{u}'(s)\boldsymbol{\delta}_j(s)dW_j(s) \\ + \sum\limits_{i=1}^{n} u_i(s)\left(d \sum\limits_{k=1}^{\mathcal{N}_{Ai}(s)} y_{ik}\right) - d \sum\limits_{i=1}^{\mathcal{N}_L(s)} z_i, \\ x(t) = x. \end{cases} \tag{3.14}$$

The set of all the admissible pairs for this truncated problem is now denoted by $\mathcal{U}[t,\ T]$, i.e.,

$$\mathcal{U}[t,\ T] = \left\{(x(\boldsymbol{.}), \boldsymbol{u}(\boldsymbol{.}))|\boldsymbol{u}(\boldsymbol{.}) \in \mathbf{L}^2_{\mathcal{F}}(t, T; \mathbb{R}^n)\right\},$$

and $(x(.), \boldsymbol{u}(.))$ satisfies (3.14). Define the corresponding value function $V(t,x)$ as

$$V(t,x) = \min_{(x(.),\ \boldsymbol{u}(.))\in\mathcal{U}[t,\ T]} E_{t,x}\left[(x(T)-(d-\theta))^2\right], \tag{3.15}$$

where $E_{t,x}[.] = E[.|x(t)=x]$, and $V(T,x) = (x-(d-\theta))^2$. Then we work backwardly until $t=0$, and $V(0,x(0))$ is the optimal value of the optimization problem (3.13). Following the method described in Fleming and Soner (2006), we obtain the Hamilton-Jacobi-Bellman equation for $V(t,x)$.

$$\begin{cases} \inf\limits_{\boldsymbol{u}(t)} \Big\{ V_t + V_x[r(t)x + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t) - c(t)] + \frac{1}{2}V_{xx}\sum\limits_{j=1}^{m} \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t)\boldsymbol{u}(t) \\ + \sum\limits_{i=1}^{n} \lambda_{Ai}E[V(t,\ x+u_i(t)y_i) - V(t,\ x)] + \lambda_L E[V(t,\ x-z) - V(t,\ x)] \Big\} = 0, \\ V(T,x) = (x-(d-\theta))^2, \end{cases} \tag{3.16}$$

where $V_t = \frac{\partial V(t,\ x)}{\partial t}, V_x = \frac{\partial V(t,\ x)}{\partial x}, V_{xx} = \frac{\partial^2 V(t,\ x)}{\partial x^2}$; and z has the same distribution as z_j for an integer j; for each i, y_i have the same distribution as $\{y_{ij}, j=1,2,\ldots,n\}$.

Derivation of the value function and the optimal control policy

We guess the expression of the value function $V(t,x)$ as below and verify it later.

$$V(t,x) = \Gamma(t)x^2 + \Lambda(t)x + \Omega(t), \tag{3.17}$$

for some time-dependent deterministic functions $\Gamma(.), \Lambda(.)$, and $\Omega(.)$. The assumption $V_{xx} > 0$ implies $\Gamma(t) > 0$ for $t \in [0,\ T]$ which is to be verified too. Based on the boundary condition in (3.16) and expression of $V(t,x)$ in (3.17), we can easily obtain the boundary conditions for $\Gamma(.), \Lambda(.)$, and $\Omega(.)$ and derivatives for $V(t,x)$.

$$\begin{cases} \Gamma(T) = 1, \\ \Lambda(T) = -2(d-\theta), \\ \Omega(T) = (d-\theta)^2, \end{cases}$$

$$\begin{cases} V_t = \Gamma_t x^2 + \Lambda_t x + \Omega_t, \\ V_x = 2\Gamma x + \Lambda, \\ V_{xx} = 2\Gamma, \\ E[V(t,\ x + u_i(t)y_i) - V(t,\ x)] = u_i(t)(2x\Gamma + \Lambda)E(y_i) + u_i^2(t)\Gamma E(y_i^2), \\ E[V(t,\ x - z) - V(t,\ x)] = \Gamma E(z^2) - (2x\Gamma + \Lambda)E(z), \end{cases} \tag{3.18}$$

where $\Gamma_t = \frac{d\Gamma(t)}{dt}$, $\Lambda_t = \frac{d\Lambda(t)}{dt}$, and $\Omega_t = \frac{d\Omega(t)}{dt}$. For simplicity, let $\Gamma = \Gamma(t)$, $\Lambda = \Lambda(t)$, and $\Omega = \Omega(t)$. Substituting (3.18) into (3.16), we get

$$\begin{cases} \inf\limits_{\boldsymbol{u}(t)} \Big\{ \Gamma_t x^2 + \Lambda_t x + \Omega_t + (2\Gamma x + \Lambda)[r(t)x + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t) - c(t)] \\ +\Gamma \sum\limits_{j=1}^{m} \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t)\boldsymbol{u}(t) + \sum\limits_{i=1}^{n} \lambda_{Ai}\Big(u_i(t)(2\Gamma x + \Lambda)E(y_i) + u_i^2(t)\Gamma E(y_i^2)\Big) \\ +\lambda_L\Big(\Gamma E(z^2) - (2x\Gamma + \Lambda)E(z)\Big) \Big\} = 0. \end{cases}$$

Now, we rewrite the above equation and obtain

$$\begin{cases} \inf\limits_{\boldsymbol{u}(t)} \Big\{ \Gamma_t x^2 + \Lambda_t x + \Omega_t + (2\Gamma x + \Lambda)[r(t)x + \boldsymbol{\alpha}'(t)\boldsymbol{u}(t) - c(t)] \\ +\Gamma \sum\limits_{j=1}^{m} \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t)\boldsymbol{u}(t) + \lambda_L\Big(\Gamma E(z^2) - (2x\Gamma + \Lambda)E(z)\Big) \\ +(2x\Gamma + \Lambda)\mathbf{A}'\boldsymbol{u}(t) + \Gamma\boldsymbol{u}'(t)\mathbf{B}\boldsymbol{u}(t) \Big\} = 0, \end{cases} \tag{3.19}$$

where $\mathbf{A}$ is a nonnegative n-dimensional vector, and $\mathbf{B}$ is a nonnegative $n \times n$ symmetric matrix.

$$\mathbf{A} = (\lambda_{A1}E(y_1), \lambda_{A2}E(y_2), \ldots, \lambda_{An}E(y_n))',$$

$$\mathbf{B} = \begin{pmatrix} \lambda_{A1}E(y_1^2) & 0 & \cdots \\ 0 & \lambda_{A2}E(y_2^2) & \cdots \\ \vdots & \vdots & \ddots \\ 0 & \cdots & \lambda_{An}E(y_n^2) \end{pmatrix}.$$

In view of $\Gamma(t) > 0$, $\sum\limits_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B} > 0$, by the first-order necessary condition, which is also sufficient, we obtain the following optimal strategy $\boldsymbol{u}^*(t)$, and (3.19) attains its minimum at

$\boldsymbol{u}^*(t)$

$$\boldsymbol{u}^*(t) = -\Big(\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + B\Big)^{-1} \frac{(2x\Gamma + \Lambda)(\boldsymbol{\alpha}(t) + \mathbf{A})}{2\Gamma}. \tag{3.20}$$

Substituting (3.20) into (3.19) and rearranging the left-hand-side (LHS) of (3.19), we have

$$\begin{aligned}
LHS =& \Gamma_t x^2 + \Lambda_t x + \Omega_t + (2\Gamma x + \Lambda)(r(t)x - c(t)) + \lambda_L\Big(\Gamma E(z^2) - (2x\Gamma + \Lambda)E(z)\Big) \\
& - 2\Gamma(x + \frac{\Lambda}{2\Gamma})^2(\boldsymbol{\alpha}'(t) + \mathbf{A}')\Big(\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A}) \\
& + \Gamma(x + \frac{\Lambda}{2\Gamma})^2(\boldsymbol{\alpha}'(t) + \mathbf{A}')\Big(\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A}) \\
& = \Gamma_t x^2 + \Lambda_t x + \Omega_t + (2\Gamma x + \Lambda)(r(t)x - c(t)) + \lambda_L\Big(\Gamma E(z^2) - (2x\Gamma + \Lambda)E(z)\Big) \\
& - \Gamma(x + \frac{\Lambda}{2\Gamma})^2(\boldsymbol{\alpha}'(t) + \mathbf{A}')\Big(\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A}) \\
& = x^2\Big\{\Gamma_t + 2\Gamma r(t) - \Gamma F\Big\} \\
& + x\Big\{\Lambda_t + (r(t) - F)\Lambda - 2\Gamma G\Big\} \\
& + \Omega_t - \Lambda G - \frac{\Lambda^2 F}{4\Gamma} + \lambda_L\Gamma E(z^2) \\
& = 0,
\end{aligned}$$

where

$$\begin{cases} F = F(t) = (\boldsymbol{\alpha}'(t) + \mathbf{A}')\Big(\sum\limits_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A}), \\ G = G(t) = c(t) + \lambda_L E(z). \end{cases}$$

Now, let us compare the left-hand-side with the right-hand-side of the above equation and we should require that the coefficients of x^2, x and x^0 equal to zero. Therefore, $\Gamma(\centerdot)$, $\Lambda(\centerdot)$, and $\Omega(\centerdot)$ satisfy the following differential equations

$$\Gamma_t + 2\Gamma r(t) - \Gamma F = 0, \qquad \Gamma(T) = 1; \tag{3.21}$$

$$\Lambda_t + (r(t) - F)\Lambda - 2\Gamma G = 0, \qquad \Lambda(T) = -2(d - \theta); \tag{3.22}$$

$$\Omega_t - \Lambda G - \frac{\Lambda^2 F}{4\Gamma} + \lambda_L\Gamma E(z^2) = 0, \qquad \Omega(T) = (d - \theta)^2. \tag{3.23}$$

By solving (3.21), we have

$$\Gamma(t) = e^{-\int_t^T (F(s) - 2r(s))ds}. \tag{3.24}$$

Note that $\Gamma(t) > 0$, which satisfies our previous assumption.

From (3.22), we deduce that

$$\begin{aligned}\frac{d(\frac{\Lambda}{\Gamma})}{dt} &= \frac{\Gamma\Lambda_t - \Gamma_t\Lambda}{\Gamma^2} \\ &= \frac{\Gamma[(F - r(t))\Lambda + 2\Gamma G] - [F - 2r(t)]\Gamma\Lambda}{\Gamma^2} \\ &= r(t)\frac{\Lambda}{\Gamma} + 2G.\end{aligned}$$

Solving the above linear differential equation subject to the terminal conditions in (3.21) and (3.22), we have

$$\frac{\Lambda}{\Gamma} = -2(d-\theta)e^{-\int_t^T r(s)ds} - \frac{2G}{r(t)}\Big[1 - e^{-\int_t^T r(s)ds}\Big]. \tag{3.25}$$

This together with (3.24) yields

$$\Lambda(t) = -\frac{2G}{r(t)}e^{-\int_t^T (F(s)-2r(s))ds} + [\frac{2G}{r(t)} - 2(d-\theta)]e^{-\int_t^T (F(s)-r(s))ds}. \tag{3.26}$$

Substituting the results of $\Gamma(t)$ and $\Lambda(t)$ into (3.23) and taking integration on both sides of the equation give rise to

$$\begin{aligned}(d-\theta)^2 - \Omega(t) &= \int_t^T (d-\theta)^2 F(l)e^{-\int_l^T F(s)ds}dl \\ &\quad + \int_t^T \eta^2(l)F(l)\Gamma(l) + 2(d-\theta)F(l)\eta(l)e^{-\int_l^T (F(s)-r(s))ds}dl \\ &\quad + \int_t^T \Lambda(l)G(l) - \lambda_L\Gamma(l)E(z^2)dl \\ &= (d-\theta)^2\int_t^T de^{-\int_z^T F(s)ds} \\ &\quad + \int_t^T \eta^2(l)F(l)\Gamma(l) + 2(d-\theta)F(l)\eta(l)e^{-\int_l^T (F(s)-r(s))ds}dl \\ &\quad + \int_t^T \Lambda(l)G(l) - \lambda_L\Gamma(l)E(z^2)dl \\ &= (d-\theta)^2 - (d-\theta)^2 e^{-\int_t^T F(s)ds} \\ &\quad + \int_t^T \eta^2(l)F(l)\Gamma(l) + 2(d-\theta)F(l)\eta(l)e^{-\int_l^T (F(s)-r(s))ds}dl \\ &\quad + \int_t^T \Lambda(l)G(l) - \lambda_L\Gamma(l)E(z^2)dl,\end{aligned}$$

where $\eta(t) = \frac{G}{r(t)}\Big[1 - e^{-\int_t^T r(s)ds}\Big]$. So, we can solve for $\Omega(t)$ as

$$\Omega(t) = (d-\theta)^2 e^{-\int_t^T F(s)ds} \\ - \int_t^T \eta^2(l)F(l)\Gamma(l) + 2(d-\theta)F(l)\eta(l)e^{-\int_l^T (F(s)-r(s))ds} + \Lambda(l)G(l) - \lambda_L\Gamma(l)E(z^2)dl. \tag{3.27}$$

Substituting (3.24), (3.26) and (3.27) into (3.17) and (3.20), we obtain the optimal strategy and value function in closed-form, as follows

$$\boldsymbol{u}^*(t) = -\Big(\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A})\Big(x - (d-\theta)e^{-\int_t^T r(s)ds} - \eta(t)\Big), \tag{3.28}$$

$$\begin{aligned} V(t,x) =& x^2 \exp\Big\{-\int_t^T (F(s) - 2r(s))ds\Big\} \\ &- \frac{2Gx}{r(t)} \exp\Big\{-\int_t^T (F(s) - 2r(s))ds\Big\} \\ &+ [\frac{2Gx}{r(t)} - 2x(d-\theta)] \exp\Big\{-\int_t^T (F(s) - r(s))ds\Big\} \\ &+ (d-\theta)^2 \exp\Big\{-\int_t^T F(s)ds\Big\} \\ &- \int_t^T \eta^2(l)F(l)\Gamma(l) + 2(d-\theta)F(l)\eta(l)e^{-\int_l^T (F(s)-r(s))ds}dl \\ &- \int_t^T \Lambda(l)G(l) - \lambda_L\Gamma(l)E(z^2)dl. \end{aligned} \tag{3.29}$$

Calculate the efficient investment strategy and the efficient frontier

Now, we proceed to calculate the efficient investment strategy and the efficient frontier for the original constrained problem (3.11). As mentioned before, we have to find θ^*. When we set $t = 0$ in (3.17), the optimal value function of the optimization problem (3.13) is

$$V(0,\ x(0)) = \Gamma(0)x^2(0) + \Lambda(0)x(0) + \Omega(0). \tag{3.30}$$

By the analysis in *Transformation of Problem*, the optimal value of the equivalent Problem (3.12) is

$$\Phi(x(0),\theta) = \Gamma(0)x^2(0) + \Lambda(0)x(0) + \Omega(0) - \theta^2. \tag{3.31}$$

From our discussion about the Lagrange dual theory at the beginning of this section, the optimal

value of the original Problem (3.11), which is equivalent to the unconstrained Problem (3.12), can be obtained by maximizing $\Phi(x(0),\theta)$ with respect to θ, i.e.

$$\begin{aligned}
Var[x(T)] &= \max_{\theta} \Phi(x(0),\theta) \\
&= \max_{\theta} \Big\{ \Gamma(0)x^2(0) + \Lambda(0)x(0) + \Omega(0) - \theta^2 \Big\} \\
&= \max_{\theta} \Bigg\{ x^2(0) \exp\Big\{ -\int_0^T (F(s) - 2r(s))ds \Big\} \\
&- \frac{2G(0)x(0)}{r(0)} \exp\Big\{ -\int_0^T (F(s) - 2r(s))ds \Big\} \\
&+ [\frac{2G(0)x(0)}{r(0)} - 2x(0)(d-\theta)] \exp\Big\{ -\int_0^T (F(s) - r(s))ds \Big\} \\
&+ (d-\theta)^2 \exp\Big\{ -\int_0^T F(s)ds \Big\} - \theta^2 \\
&- \int_0^T \eta^2(l)F(l)\Gamma(l) + 2(d-\theta)F(l)\eta(l)e^{-\int_l^T (F(s)-r(s))ds} + \Lambda(l)G(l) - \lambda_L\Gamma(l)E(z^2)dl \Bigg\}.
\end{aligned} \tag{3.32}$$

Let

$$\begin{cases}
\varpi(t) = e^{-\int_t^T (F(s)-r(s))ds}, \\
\psi(t) = e^{-\int_t^T F(s)ds}, \\
\xi(t) = \int_t^T F(z)\eta(z)\varpi(z) - G(z)\varpi(z)dz, \\
\chi(t) = \int_t^T \eta^2(l)F(l)\Gamma(l) - \frac{2G^2(l)}{r(l)}\Gamma(l) + \frac{2G^2(l)}{r(l)}\varpi(l) - \lambda_L\Gamma(l)E(z^2)dl.
\end{cases} \tag{3.33}$$

Then, the above equation (3.32) can be simplified further as follows

$$\begin{aligned}
Var[x(T)] =& \max_{\theta} \Big\{ x^2(0)\Gamma(0) + [2(\theta-d)x(0) + \frac{2x(0)G(0)}{r(0)}]\varpi(0) \\
&- \frac{2x(0)G(0)}{r(0)}\Gamma(0) + (\theta-d)^2\psi(0) - \theta^2 - \chi(0) - 2(\theta-d)\xi(0) \Big\} \\
=& \max_{\theta} \Big\{ \theta^2[\psi(0)-1] + 2\theta[x(0)\varpi(0) - \xi(0) - d\psi(0)] \\
&+ x^2(0)\Gamma(0) + \frac{2x(0)G(0)\varpi(0)}{r(0)} - \frac{2x(0)G(0)\Gamma(0)}{r(0)} \\
&- \chi(0) + d^2\psi(0) - 2dx(0)\varpi(0) + 2d\xi(0).
\end{aligned} \tag{3.34}$$

The existence of the maximum of $Var[x(T)]$ depends on the coefficient of θ^2, and so we give the following proposition.

Proposition 3.1. *$\psi(0) - 1 < 0$ holds for any $t \in [0,\ T)$.*

Proof. Since $F(t) > 0$, we have $\psi(0) > 0$ and

$$\begin{aligned}\psi(0) - 1 =& e^{-\int_0^T F(s)ds} - 1\\ <& e^0 - 1\\ =& 1 - 1 = 0.\end{aligned}$$

□

With proposition 3.1, we know that the coefficient of θ^2 is strictly negative. The expression of the objective function of (3.32) is an open-down parabola as a function of θ. Hence, the optimal solution to the optimization problem (3.34) exists and is given by the first-order condition.

$$\theta^* = \frac{x(0)\varpi(0) - \xi(0) - d\psi(0)}{1 - \psi(0)}. \tag{3.35}$$

Substituting (3.35) into (3.28), we obtain the efficient investment strategy for the original optimization model (3.11) as follows

$$\begin{aligned}\boldsymbol{u}^*(t) = -\Big(\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A})\times\\ \Big(x(t) - (d - \tfrac{x(0)\varpi(0) - \xi(0) - d\psi(0)}{1-\psi(0)})e^{-\int_t^T r(s)ds} - \eta(t)\Big).\end{aligned} \tag{3.36}$$

Substituting (3.35) into (3.34), we get the optimal value of the optimization model (3.11), namely, the minimum variance as follows

$$\begin{aligned}Var^*[x(T)] &= \max_{\theta} \Phi(x(0), \theta)\\ &= \frac{[x(0)\varpi(0) - \xi(0) - d\psi(0)]^2}{1 - \psi(0)} + x^2(0)\Gamma(0) + \frac{2x(0)G(0)\varpi(0)}{r(0)}\\ &- \frac{2x(0)G(0)\Gamma(0)}{r(0)} - \chi(0) + d^2\psi(0) - 2dx(0)\varpi(0) + 2d\xi(0).\end{aligned} \tag{3.37}$$

We rewrite the above equation into a function of expected terminal wealth d as follows

$$\begin{aligned}
Var^*[x(T)] &= \max_{\theta} \Phi(x(0), \theta) \\
&= d^2 \frac{\psi(0)}{1-\psi(0)} + 2d[\xi(0) - x(0)\varpi(0) - \frac{\psi(0)x(0)\varpi(0) - \psi(0)\xi(0)}{1-\psi(0)}] \\
&+ x^2(0)\Gamma(0) + \frac{2x(0)G(0)\varpi(0)}{r(0)} - \frac{2x(0)G(0)\Gamma(0)}{r(0)} - \chi(0) + \frac{[x(0)\varpi(0) - \xi(0)]^2}{1-\psi(0)} \\
&= \frac{\psi(0)}{1-\psi(0)} \left(d - \frac{x(0)\varpi(0) - \xi(0)}{\psi(0)} \right)^2 \\
&+ x^2(0)\Gamma(0) + \frac{2x(0)G(0)\varpi(0)}{r(0)} - \frac{2x(0)G(0)\Gamma(0)}{r(0)} - \chi(0) - \frac{[x(0)\varpi(0) - \xi(0)]^2}{\psi(0)}.
\end{aligned} \tag{3.38}$$

Based on proposition 3.1, we know that the coefficient of d^2 is strictly positive. Therefore, the global minimum variance could be obtained by setting the expected terminal wealth level as $d = d_{SD_{min}} := \frac{x(0)\varpi(0) - \xi(0)}{\psi(0)}$, i.e.

$$Var^*_{min}[x(T)] = x^2(0)\Gamma(0) - \frac{2x(0)G(0)[\Gamma(0) - \varpi(0)]}{r(0)} - \chi(0) - \frac{[x(0)\varpi(0) - \xi(0)]^2}{\psi(0)}. \tag{3.39}$$

We summarize the above results in the following theorem.

Theorem 3.1. *For mean-variance asset liability management with jumps and general insurance liability problem (3.11) which aims to minimize the risk for a given expected terminal wealth $E[x(T)] = d(\geq d_{SD_{min}})$, the efficient investment strategy and the efficient frontier are given by (3.36) and (3.38), respectively.*

3.4 Analytical Results for Two Special Cases

3.4.1 Case I: The Case with No Liability

In the case with no liability, the original asset liability management problem is reduced to a portfolio selection problem with the following dynamics of wealth process

$$\begin{cases} dx(t) = [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]dt + \sum_{j=1}^{m} \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)dW_j(t) + \sum_{i=1}^{n} u_i(t)\left(d\sum_{k=1}^{\mathcal{N}_{Ai}(t)} y_{ik}\right), \quad 0 \le t \le T, \\ x(0) = x_0 > 0, \end{cases} \tag{3.40}$$

where

$$\begin{cases} \boldsymbol{u}(t) = (u_1(t), u_2(t), \ldots, u_n(t))', \\ \boldsymbol{\alpha}(t) = (\alpha_1(t) - r(t), \alpha_2(t) - r(t), \ldots, \alpha_n(t) - r(t))', \\ \boldsymbol{\delta}_j(t) = (\sigma_{1j}(t), \sigma_{2j}(t), \ldots, \sigma_{nj}(t))', \quad j = 1, 2, \ldots, m. \end{cases}$$

The mean-variance portfolio selection in a jump diffusion market refers to the problem of finding the optimal admissible strategy such that the variance of the terminal wealth is minimized for a given expected terminal wealth d, i.e.

$$\begin{cases} \min\limits_{\boldsymbol{u}(.)\in\mathcal{U}[0,\ T]} \quad Var[x(T)] := E[x(T) - d]^2, \\ \quad s.t. \qquad E[x(T)] = d. \end{cases} \tag{3.41}$$

After we transform the constrained optimization problem into an unconstrained optimization problem by introducing a Lagrange multiplier θ, we obtain the corresponding Hamilton-Jacobi-Bellman equation

$$\begin{cases} \inf\limits_{\boldsymbol{u}(t)} \Big\{ V_t + V_x[r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)] + \frac{1}{2}V_{xx}\sum_{j=1}^{m} \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t)\boldsymbol{u}(t) \\ + \sum_{i=1}^{n} \lambda_{Ai}E[V(t,\ x + u_i(t)y_i) - V(t,\ x)] \Big\} = 0, \\ V(T, x) = (x - (d - \theta))^2. \end{cases} \tag{3.42}$$

Then we guess and verify that the value function has the following solution form

$$V(t, x) = \Gamma(t)x^2 + \bar{\Lambda}(t)x + \bar{\Omega}(t). \tag{3.43}$$

Hence, the partial differential equations (3.21) - (3.23) degenerate to the following

$$\Gamma_t + 2\Gamma r(t) - \Gamma F = 0, \qquad \Gamma(T) = 1; \tag{3.44}$$

$$\bar{\Lambda}_t + (r(t) - F)\bar{\Lambda} = 0, \qquad \bar{\Lambda}(T) = -2(d-\theta); \tag{3.45}$$

$$\bar{\Omega}_t - \frac{\bar{\Lambda}^2 F}{4\Gamma} = 0, \qquad \bar{\Omega}(T) = (d-\theta)^2; \tag{3.46}$$

where

$$F = F(t) = (\boldsymbol{\alpha}'(t) + \mathbf{A}')\Big(\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A}),$$

$$\mathbf{A} = (\lambda_{A1}E(y_1), \lambda_{A2}E(y_2), \ldots, \lambda_{An}E(y_n))',$$

$$\mathbf{B} = \begin{pmatrix} \lambda_{A1}E(y_1^2) & 0 & \ldots \\ 0 & \lambda_{A2}E(y_2^2) & \ldots \\ \vdots & \vdots & \ddots \\ 0 & \ldots & \lambda_{An}E(y_n^2) \end{pmatrix}.$$

Solving (3.44) - (3.46), we get

$$\Gamma(t) = \exp\Big\{-\int_t^T (F(s) - 2r(s))ds\Big\}, \tag{3.47}$$

$$\bar{\Lambda}(t) = -2(d-\theta)\exp\Big\{-\int_t^T (F(s) - r(s))ds\Big\}, \tag{3.48}$$

$$\bar{\Omega}(t) = (d-\theta)^2 \exp\Big\{-\int_t^T F(s)ds\Big\}. \tag{3.49}$$

Then, we obtain the optimal control policy and value function in closed-form, as follows

$$\boldsymbol{u}^*(t) = -\Big(\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A})\Big(x + (\theta - d)\exp\Big\{-\int_t^T r(s)ds\Big\}\Big), \tag{3.50}$$

$$\begin{aligned} V(t,x) = {} & x^2 \exp\Big\{-\textstyle\int_t^T (F(s) - 2r(s))ds\Big\} - 2x(d-\theta)\exp\Big\{-\textstyle\int_t^T (F(s) - r(s))ds\Big\} \\ & + (d-\theta)^2 \exp\Big\{-\textstyle\int_t^T F(s)ds\Big\}. \end{aligned} \tag{3.51}$$

Thereafter, we calculate the efficient frontier by eliminating the parameter θ from $\Phi(x(0),\theta) = V(0,\ x(0)) - \theta^2$ and get θ^* as follows

$$\theta^* = \frac{x_0\varpi(0) - d\psi(0)}{1-\psi(0)}. \tag{3.52}$$

Substituting (3.52) into $\Phi(x(0),\theta)$ and rearranging the expression in terms of d, we have

$$Var^*[x(T)] = \frac{\psi(0)}{1-\psi(0)}\Big(d - \frac{\varpi(0)x_0}{\psi(0)}\Big)^2 + \frac{\Gamma(0)\psi(0) - \varpi^2(0)}{\psi(0)}x_0^2,$$

where

$$\begin{cases} \varpi(t) = \exp\Big\{-\int_t^T (F(s) - r(s))ds\Big\}, \\ \psi(t) = \exp\Big\{-\int_t^T F(s)ds\Big\}. \end{cases}$$

Since $1-\psi(0) > 0$, by setting the expected terminal wealth level as $d = d_{SD_{min}} := \frac{\varpi(0)x_0}{\psi(0)}$, we obtain the following zero global minimum variance, which means that the efficient frontier touches the expectation axis at point $d = d_{SD_{min}}$

$$Var^*_{min}[x(T)] = \frac{\Gamma(0)\psi(0) - \varpi^2(0)}{\psi(0)}x_0^2 = 0. \tag{3.53}$$

So the efficient frontier can be simplified as follows

$$Var^*[x(T)] = \frac{\psi(0)}{1-\psi(0)}\Big(d - \frac{\varpi(0)x_0}{\psi(0)}\Big)^2. \tag{3.54}$$

At last, we substitute (3.52) into (3.50) and get the efficient investment strategy

$$\begin{aligned} \boldsymbol{u}^*(t) = -\Big(\sum_{j=1}^m \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A})\times \\ \left(x(t) - \Big(d - \tfrac{x_0\varpi(0) - d\psi(0)}{1-\psi(0)}\Big)\exp\Big\{-\int_t^T r(s)ds\Big\}\right). \end{aligned} \tag{3.55}$$

From the above analysis, we have the following Theorem.

Theorem 3.2. *For a given expected terminal wealth $E[x(T)] = d(\geq d_{SD_{min}})$, the efficient investment strategy and the efficient frontier of the mean-variance portfolio selection problem (3.41) in a jump diffusion market are given by (3.55) and (3.54), respectively.*

Remark 3.4. Comparing (3.55) with the efficient investment strategy for the asset liability management problem in (3.36), we find that the expression of (3.36) has an extra term, namely

$$\begin{cases} \Big(\sum\limits_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t) + \mathbf{B}\Big)^{-1}(\boldsymbol{\alpha}(t) + \mathbf{A})\eta(t), \\ \eta(t) = \frac{c(t)+\lambda_L E(z)}{r(t)}\Big[1 - e^{-\int_t^T r(s)ds}\Big]. \end{cases}$$

This term is introduced by the general insurance liability with consideration of accumulated operational expense.

Remark 3.5. The efficient frontier of the portfolio selection problem is a straight line in the standard deviation-mean plane, with the slope $\sqrt{\frac{\psi(0)}{1-\psi(0)}}$, and it reduces to zero at the point $d = d_{SD_{min}}$. Zhou and Li (2000) showed similar results. The expression of standard deviation of our portfolio selection model is as follows

$$SD[x(T)] = \sqrt{Var[x(T)]} = \sqrt{\frac{\psi(0)}{1-\psi(0)}}\Big(d - d_{SD_{min}}\Big), \qquad d \geq d_{SD_{min}}.$$

Remark 3.6. Comparing the efficient frontier for the asset liability management problem in (3.38) and the efficient frontier for the portfolio selection in (3.54), we find that (3.38) consists of two parts. The first part, $\frac{\psi(0)}{1-\psi(0)}\left(d - \frac{x(0)\varpi(0)-\xi(0)}{\psi(0)}\right)^2$, is very similar to (3.54) except for the difference $\xi(0)$ which is related to the jump in risky assets' price and the random liability; while the second item, $x^2(0)\Gamma(0) - \frac{2x(0)G(0)[\Gamma(0)-\varpi(0)]}{r(0)} - \chi(0) - \frac{[x(0)\varpi(0)-\xi(0)]^2}{\psi(0)}$, is always positive and makes the efficient frontier no longer a straight line in the standard deviation-mean plane, but a bullet-shaped curve. The results and other features are illustrated in the numerical examples.

3.4.2 Case II: The Case with No Jump

In the case with no jump in the price of risky asset, the wealth process evolves according to the following dynamics

$$\begin{cases} dx(t) = [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t) - c(t)]dt + \sum_{j=1}^{m} \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)dW_j(t) - d\sum_{i=1}^{\mathcal{N}_L(t)} z_i, \quad 0 \le t \le T, \\ x(0) = x_0 > 0, \end{cases} \tag{3.56}$$

where

$$\begin{cases} \boldsymbol{u}(t) = (u_1(t), u_2(t), \ldots, u_n(t))', \\ \boldsymbol{\alpha}(t) = (\alpha_1(t) - r(t), \alpha_2(t) - r(t), \ldots, \alpha_n(t) - r(t))', \\ \boldsymbol{\delta}_j(t) = (\sigma_{1j}(t), \sigma_{2j}(t), \ldots, \sigma_{nj}(t))', \quad j = 1, 2, \ldots, m. \end{cases}$$

The mean-variance asset liability management refers to the problem of finding the optimal admissible strategy such that the variance of the terminal wealth is minimized for a given expected terminal wealth d, i.e.

$$\begin{cases} \min\limits_{\boldsymbol{u}(.) \in \mathcal{U}[0,\ T]} & Var[x(T)] := E[x(T) - d]^2, \\ s.t. & E[x(T)] = d. \end{cases} \tag{3.57}$$

and the corresponding Hamilton-Jacobi-Bellman equation turns out to be

$$\begin{cases} \inf\limits_{\boldsymbol{u}(t)} \Big\{ V_t + V_x[r(t)x + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t) - c(t)] + \frac{1}{2}V_{xx} \sum_{j=1}^{m} \boldsymbol{u}'(t)\boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t)\boldsymbol{u}(t) \\ +\lambda_L E[V(t,\ x - z) - V(t,\ x)] \Big\} = 0, \\ V(T, x) = (x - (d - \theta))^2. \end{cases} \tag{3.58}$$

Then we guess and verify that the value function has the following solution form

$$V(t, x) = \widetilde{\Gamma}(t)x^2 + \widetilde{\Lambda}(t)x + \widetilde{\Omega}(t). \tag{3.59}$$

Since every component in vector $\mathbf{A}$ and matrix $\mathbf{B}$ becomes zero; the partial differential equations (3.21) - (3.23) degenerate to the following system.

$$\widetilde{\Gamma}_t + 2\widetilde{\Gamma}r(t) - \widetilde{\Gamma}Q = 0, \qquad \widetilde{\Gamma}(T) = 1; \tag{3.60}$$

$$\widetilde{\Lambda}_t + (r(t) - Q)\widetilde{\Lambda} - 2\widetilde{\Gamma}G = 0, \qquad \widetilde{\Lambda}(T) = -2(d-\theta); \tag{3.61}$$

$$\widetilde{\Omega}_t - \widetilde{\Lambda}G - \frac{\widetilde{\Lambda}^2 Q}{4\widetilde{\Gamma}} + \lambda_L \widetilde{\Gamma}E(z^2) = 0, \qquad \widetilde{\Omega}(T) = (d-\theta)^2; \tag{3.62}$$

where we have used a new symbol $Q(t)$ to replace $F(t)$ to avoid confusion and

$$\begin{cases} Q = Q(t) = \boldsymbol{\alpha}'(t)\Big(\sum\limits_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t)\Big)^{-1}\boldsymbol{\alpha}(t), \\ G = G(t) = c(t) + \lambda_L E(z). \end{cases}$$

Solving (3.60) - (3.62), we get

$$\widetilde{\Gamma}(t) = \exp\Big\{-\int_t^T (Q(s) - 2r(s))ds\Big\}, \tag{3.63}$$

$$\widetilde{\Lambda}(t) = -\frac{2G}{r(t)}\exp\Big\{-\int_t^T (Q(s) - 2r(s))ds\Big\} + \Big[\frac{2G}{r(t)} - 2(d-\theta)\Big]\exp\Big\{-\int_t^T (Q(s) - r(s))ds\Big\}, \tag{3.64}$$

$$\begin{aligned}\widetilde{\Omega}(t) =& (d-\theta)^2 \exp\Big\{-\int_t^T Q(s)ds\Big\} \\ &- \int_t^T \eta^2(l)Q(l)\widetilde{\Gamma}(l) + 2(d-\theta)Q(l)\eta(l)e^{-\int_l^T (Q(s)-r(s))ds} + \widetilde{\Lambda}(l)G(l) - \lambda_L\widetilde{\Gamma}(l)E(z^2)dl.\end{aligned} \tag{3.65}$$

Substituting the results of $\widetilde{\Gamma}(t)$, $\widetilde{\Lambda}(t)$ and $\widetilde{\Omega}(t)$ into the Hamilton-Jacobi-Bellman equation and the value function, it yield the optimal investment strategy and the optimal value function. At last, we calculate θ^* and obtain the efficient investment strategy and the efficient frontier as follows by replacing θ by θ^*.

$$\begin{aligned}\boldsymbol{u}^*(t) =& -\Big(\sum_{j=1}^{m} \boldsymbol{\delta}_j(t)\boldsymbol{\delta}_j'(t)\Big)^{-1}\boldsymbol{\alpha}(t)\times \\ &\Big(x(t) - (d - \frac{x(0)\widetilde{\varpi}(0) - \widetilde{\xi}(0) - d\widetilde{\psi}(0)}{1 - \widetilde{\psi}(0)})\exp\Big\{-\int_t^T r(s)ds\Big\} - \eta(t)\Big),\end{aligned} \tag{3.66}$$

$$\begin{aligned} Var^*[x(T)] =& \max_{\theta} \Phi(x(0),\theta) \\ &= \frac{\widetilde{\psi}(0)}{1-\widetilde{\psi}(0)}\left(d - \frac{x(0)\widetilde{\varpi}(0)-\widetilde{\xi}(0)}{\widetilde{\psi}(0)}\right)^2 \\ &+ x^2(0)\widetilde{\Gamma}(0) + \frac{2x(0)G(0)\widetilde{\varpi}(0)}{r(0)} - \frac{2x(0)G(0)\widetilde{\Gamma}(0)}{r(0)} \\ &- \widetilde{\chi}(0) - \frac{[x(0)\widetilde{\varpi}(0)-\widetilde{\xi}(0)]^2}{\widetilde{\psi}(0)}. \end{aligned} \tag{3.67}$$

The global minimum variance, $Var^*_{min}[x(T)]$, is obtained at $d = d_{SD_{min}} := \frac{x(0)\widetilde{\varpi}(0)-\widetilde{\xi}(0)}{\widetilde{\psi}(0)}$.

$$Var^*_{min}[x(T)] = x^2(0)\widetilde{\Gamma}(0) - \frac{2x(0)G(0)[\widetilde{\Gamma}(0)-\widetilde{\varpi}(0)]}{r(0)} - \widetilde{\chi}(0) - \frac{[x(0)\widetilde{\varpi}(0)-\widetilde{\xi}(0)]^2}{\widetilde{\psi}(0)}, \tag{3.68}$$

where

$$\begin{cases} \eta(t) = \frac{G(t)}{r(t)}\Big[1 - \exp\Big\{-\int_t^T r(s)ds\Big\}\Big], \\ \widetilde{\varpi}(t) = \exp\Big\{-\int_t^T (Q(s)-r(s))ds\Big\}, \\ \widetilde{\psi}(t) = \exp\Big\{-\int_t^T Q(s)ds\Big\}, \\ \widetilde{\xi}(t) = \int_t^T Q(z)\eta(z)\widetilde{\varpi}(z) - G(z)\widetilde{\varpi}(z)dz, \\ \widetilde{\chi}(t) = \int_t^T \eta^2(l)Q(l)\widetilde{\Gamma}(l) - \frac{2G^2(l)}{r(l)}\widetilde{\Gamma}(l) + \frac{2G^2(l)}{r(l)}\widetilde{\varpi}(l) - \lambda_L\widetilde{\Gamma}(l)E(z^2)dl. \end{cases}$$

3.5 Numerical Investigation

In this section, we provide some numerical illustrations and sensitivity analysis for our theoretical results derived in the previous sections via several examples. We focus on the effect of liability and the effect of jump in the first two subsections. Then we present the impact of the key parameters in the last subsection, including the exit time T, the initial wealth $x(0)$, the risk-free interest rate $r(t)$ and the operational cost function $c(t)$.

For convenience, but without loss of generality, we consider the case with one risky-free asset and three risky assets to analyze the results of our original model. We assume that all the parameters are constants in this section and the values are given in Table 3.1 and (3.69).

Table 3.1: Parameter values for the original model.

Parameter	Symbol	Value
Exit Time	T	0.5
Initial Wealth	$x(0)$	10
Risk-free Interest Rate	$r(t)$	0.0217
Intensity of Poisson Process for All Risky Assets	λ_A	1
Appreciation Rate of Risky Asset One	$\alpha(t)_1$	0.0512
Appreciation Rate of Risky Asset Two	$\alpha(t)_2$	0.0475
Appreciation Rate of Risky Asset Three	$\alpha(t)_3$	0.0306
Operational Cost	$c(t)$	0.05
Intensity of Poisson Process for Insurance Liability	λ_L	4

The volatility matrix is

$$\begin{pmatrix} \delta_1(t) \\ \delta_2(t) \\ \delta_3(t) \end{pmatrix} = \begin{pmatrix} 0.6107 & 0.1488 & 0.0966 \\ 0.1488 & 0.4279 & 0.1082 \\ 0.0966 & 0.1082 & 0.1699 \end{pmatrix}. \tag{3.69}$$

Then, we calculate vector $\mathbf{A}$, matrix $\mathbf{B}$, $F(t)$ and $G(t)$. It should be addressed here that we get different values each time when we run the program due to the random jump process.

$$\mathbf{A} = (0.0916, 0.3032, 0.3996)',$$

$$\mathbf{B} = \begin{pmatrix} 0.0084 & 0 & 0 \\ 0 & 0.0919 & 0 \\ 0 & 0 & 0.1597 \end{pmatrix},$$

$$\begin{aligned} F(t) &= 0.9972, \\ G(t) &= 2.6179. \end{aligned}$$

Next, we substitute the results above into the expressions of $\eta(t)$, $\Gamma(t)$, $\varpi(t)$, $\psi(t)$, $\xi(t)$ and $\chi(t)$

and have the following results

$$\begin{cases} \eta(t) = \frac{G}{r(t)}\Big[1 - \exp\Big\{ - \int_t^T r(s)ds\Big\}\Big] = 120.64\Big[1 - e^{-0.0217(T-t)}\Big], \\ \Gamma(t) = \exp\Big\{ - \int_t^T (F(s) - 2r(s))ds\Big\} = e^{-0.9538(T-t)}, \\ \varpi(t) = \exp\Big\{ - \int_t^T (F(s) - r(s))ds\Big\} = e^{-0.9755(T-t)}, \\ \psi(t) = \exp\Big\{ - \int_t^T F(s)ds\Big\} = e^{-0.9972(T-t)}, \\ \xi(t) = \int_t^T F(z)\eta(z)\varpi(z) - G(z)\varpi(z)dz = \int_t^T 117.68e^{-0.9755(T-z)} - 120.30e^{-0.9972(T-z)}dz, \\ \chi(t) = \int_t^T \eta^2(l)F(l)\Gamma(l) - \frac{2G^2(l)}{r(l)}\Gamma(l) + \frac{2G^2(l)}{r(l)}\varpi(l) - \lambda_L\Gamma(l)E(z^2)dl \\ = \int_t^T \Big(\eta^2(l)F(l) - \frac{2G^2(l)}{r(l)} - \lambda_L E(z^2)\Big)e^{-0.9538(T-l)} + \frac{2G^2(l)}{r(l)}e^{-0.9755(T-l)}dl. \end{cases}$$

Substituting $\eta(t)$, $\Gamma(0)$, $\varpi(0)$, $\psi(0)$, $\xi(0)$ and $\chi(0)$ into (3.36) and (3.38), we obtain the efficient investment strategy and the efficient frontier as follows

$$u^*(t) = \begin{pmatrix} -0.4458 \\ 0.8352 \\ 1.8287 \end{pmatrix} \times$$

$$\left(x(t) + \left(\frac{10e^{-0.9755\times 0.5} - de^{-0.9972\times 0.5} + 0.7971}{1 - e^{-0.9972\times 0.5}} - d\right)e^{-0.0217(0.5-t)} - 120.64\Big(1 - e^{-0.0217(0.5-t)}\Big)\right),$$

$$Var^*[x(T)] = \max_{\theta} \Phi(x(0), \theta) = 1.5471(d - 11.4252)^2 + 218.3727, \tag{3.70}$$

$$SD^*[x(T)] = \sqrt{1.5471\Big(d - 11.4252\Big)^2 + 218.3727}. \tag{3.71}$$

The efficient frontier of the mean-variance asset liability management with general insurance liabilities in a jump diffusion market is depicted in Figure 3.1. From this Figure, it can be noted that the efficient frontier of the asset liability management problem with jump is no longer a

straight line, and the global minimum variance $Var^*_{min} = 218.3727$ at the point $d_{SD_{min}} = 11.4252$ is strictly positive. This is due to the risk embedded in the general insurance liability and the random jump in risky assets' prices. The investment amount in the risk-free asset, risky asset one, risky asset two and risky asset three are as follows

$$u_1(t) = -0.4458\Bigg(x(t) + \Big(\frac{10e^{-0.9755\times0.5} - de^{-0.9972\times0.5} + 0.7971}{1 - e^{-0.9972\times0.5}} - d\Big)e^{-0.0217(0.5-t)} - 120.64\Big(1 - e^{-0.0217(0.5-t)}\Big)\Bigg), \tag{3.72}$$

$$u_2(t) = 0.8352\Bigg(x(t) + \Big(\frac{10e^{-0.9755\times0.5} - de^{-0.9972\times0.5} + 0.7971}{1 - e^{-0.9972\times0.5}} - d\Big)e^{-0.0217(0.5-t)} - 120.64\Big(1 - e^{-0.0217(0.5-t)}\Big)\Bigg), \tag{3.73}$$

$$u_3(t) = 1.8287\Bigg(x(t) + \Big(\frac{10e^{-0.9755\times0.5} - de^{-0.9972\times0.5} + 0.7971}{1 - e^{-0.9972\times0.5}} - d\Big)e^{-0.0217(0.5-t)} - 120.64\Big(1 - e^{-0.0217(0.5-t)}\Big)\Bigg), \tag{3.74}$$

$$\begin{aligned} u_0(t) &= x(t) - \sum_{i=1}^{3} u_i(t) \\ &= x(t) - 2.2181\Bigg(x(t) + \Big(\frac{10e^{-0.9755\times0.5} - de^{-0.9972\times0.5} + 0.7971}{1 - e^{-0.9972\times0.5}} - d\Big)e^{-0.0217(0.5-t)} \\ &\quad - 120.64\Big(1 - e^{-0.0217(0.5-t)}\Big)\Bigg). \end{aligned} \tag{3.75}$$

We fix the value of $d(= 12)$ and plot these functions in Figure 3.2 and Figure 3.3 with time varying between 0 and 0.5 and wealth between 10 and 15. In particular, at the initial time

$t = 0$, the amount invested in the risk-free asset and risky assets are

$$\begin{aligned} u_0(0) &= 19.0004, \\ u_1(0) &= 1.8089, \\ u_2(0) &= -3.3890, \\ u_3(0) &= -7.4203. \end{aligned} \tag{3.76}$$

This means that the insurer should sell short the risky asset two and risky asset three (i.e. borrow money) for an amount 10.8093 and invest it together with initial wealth $x(0)(= 10)$ in the risk-free asset and risky asset one.

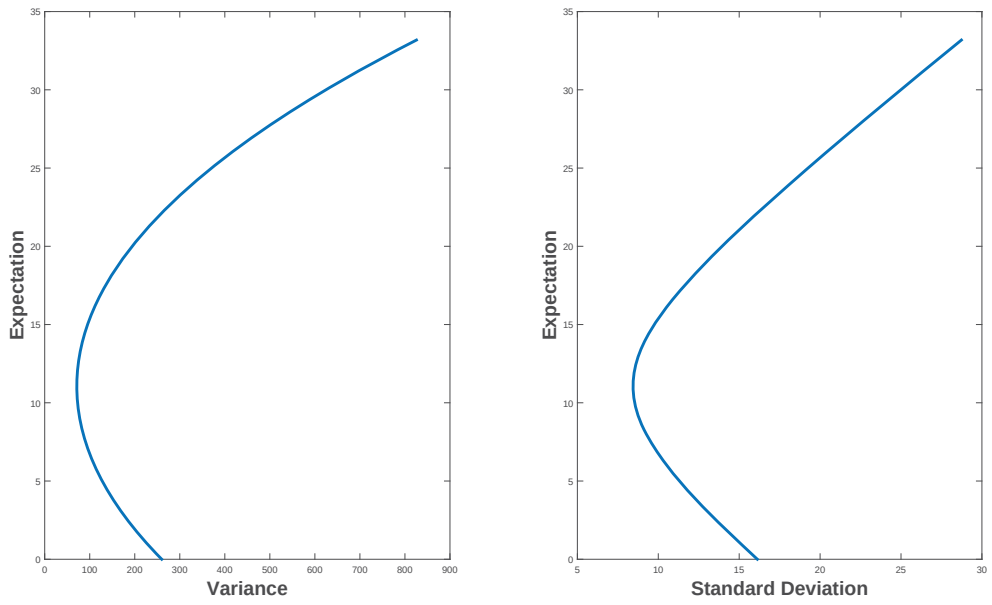

Figure 3.1: Efficient frontier of the mean-variance asset liability management with general insurance liability in a jump diffusion market

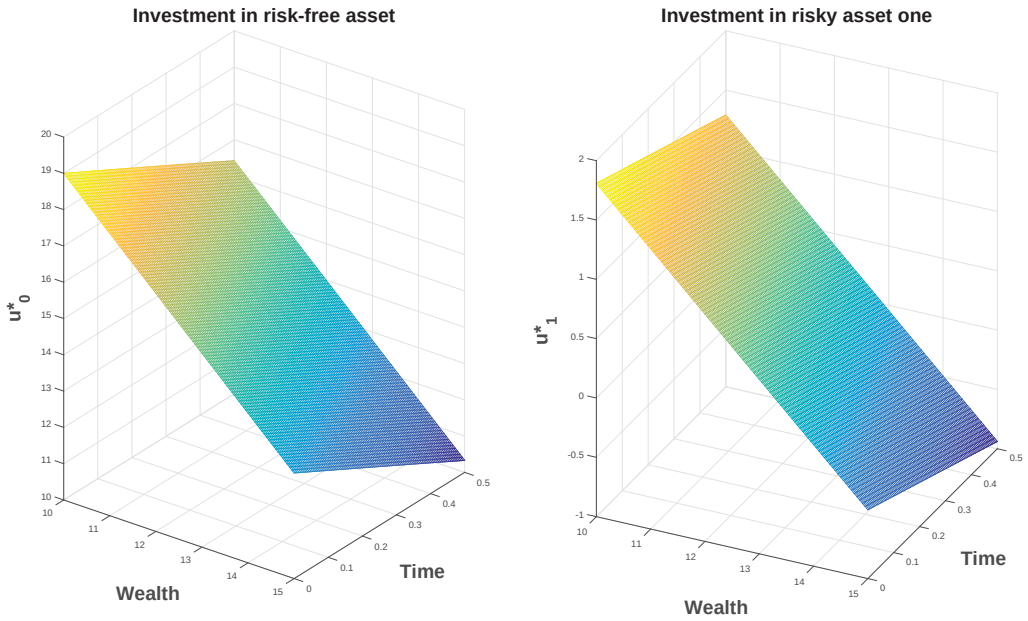

Figure 3.2: Efficient investment strategy for the risk-free asset and risky asset one

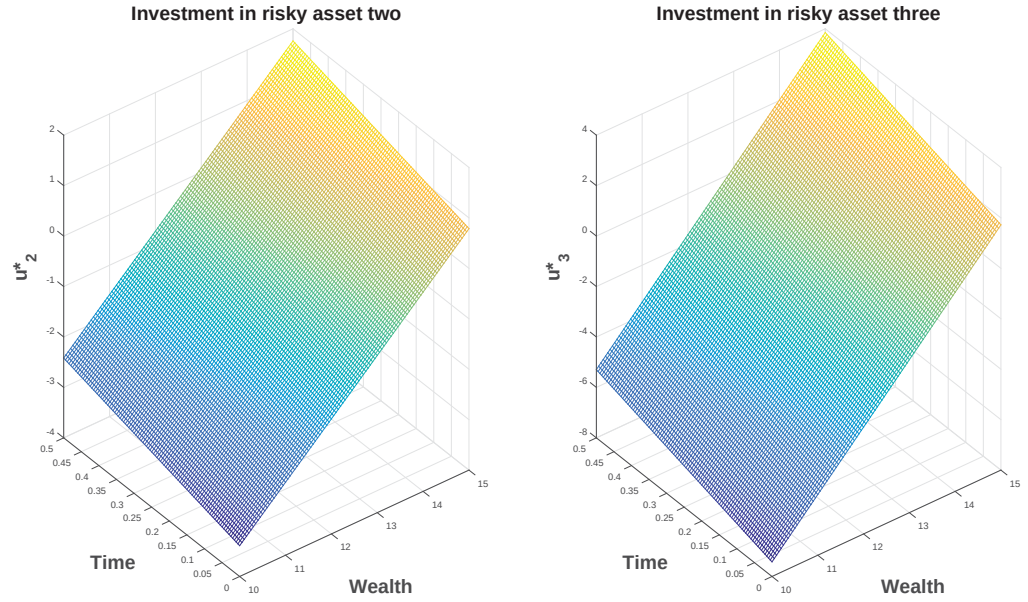

Figure 3.3: Efficient investment strategy for the risky asset two and risky asset three

3.5.1 The Effect of Liability

In the case with no liability, we set $c(t)$ and λ_L as zero. All the other parameters remain the same as given in Table 3.1. The efficient investment strategy and the efficient frontier are as

follows

$$u_1(t) = -0.4458\left(x(t) + \left(\frac{10e^{-0.9755\times0.5} - de^{-0.9972\times0.5}}{1 - e^{-0.9972\times0.5}} - d\right)e^{-0.0217(0.5-t)}\right), \tag{3.77}$$

$$u_2(t) = 0.8352\left(x(t) + \left(\frac{10e^{-0.9755\times0.5} - de^{-0.9972\times0.5}}{1 - e^{-0.9972\times0.5}} - d\right)e^{-0.0217(0.5-t)}\right), \tag{3.78}$$

$$u_3(t) = 1.8287\left(x(t) + \left(\frac{10e^{-0.9755\times0.5} - de^{-0.9972\times0.5}}{1 - e^{-0.9972\times0.5}} - d\right)e^{-0.0217(0.5-t)}\right), \tag{3.79}$$

$$\begin{aligned} u_0(t) &= x(t) - \sum_{i=1}^{3} u_i(t) \\ &= x(t) - 2.2181\left(x(t) + \left(\frac{10e^{-0.9755\times0.5} - de^{-0.9972\times0.5}}{1 - e^{-0.9972\times0.5}} - d\right)e^{-0.0217(0.5-t)}\right), \end{aligned} \tag{3.80}$$

$$d_{SD_{min}} = 10.1091, \tag{3.81}$$

$$Var^*[x(T)] = \max_{\theta} \Phi(x(0), \theta) = 1.5471(d - 10.1091)^2, \tag{3.82}$$

$$SD^*[x(T)] = \sqrt{1.5471\Big(d - 10.1091\Big)^2}. \tag{3.83}$$

At the initial time $t = 0$, the amount invested in the risk-free asset and risky assets are

$$\begin{aligned} u_0(0) &= 20.5673, \\ u_1(0) &= 2.1239, \\ u_2(0) &= -3.9790, \\ u_3(0) &= -8.7122. \end{aligned} \tag{3.84}$$

Comparing (3.84) with the efficient investment strategy with the presence of liability at $t = 0$ (3.76), we find that a larger percentage of the initial borrowed amount of money for the portfolio selection model together with the initial wealth $x(0)$ can be invested in the risk-free asset, which clearly shows the impact from the introduction of liability.

The efficient frontier is depicted in Figure 3.4 with the expectation of terminal wealth varying between 2.5 and 25. Then we fix the value of $d(= 12)$ and plot the efficient investment strategy for the portfolio selection model with time varying between 0 and 0.5 and wealth between 10 and 15 in Figure 3.5 and Figure 3.6.

As we can see in Figure 3.4, if there exits a liability, the efficient frontier never touches the

expectation axis with a positive global minimum variance and bullet-shaped curve. Nevertheless, the efficient frontier for the portfolio selection model is a straight line on the mean-standard deviation plane with zero global minimum variance. The two efficient frontiers are not tangent to each other.

Figure 3.5 shows that the optimal amount of money invested in the risk-free asset $u_0(t)$ and the safer risky asset (i.e. with lower appreciation rate) $u_1(t)$ in the case with no liability is more than the case with liability; and this investment decreases with respect to wealth $x(t)$ and time t under both conditions.

Figure 3.6 shows that the optimal amount of money invested in high-risk assets, namely $u_2(t)$ and $u_3(t)$, in the case with no liability is less than the case with liability; and this investment increases with respect to wealth $x(t)$ and time t under both conditions. The phenomena demonstrate that the insurer has to take more risk with presence of liability so as to keep a good solvency of the insurance business.

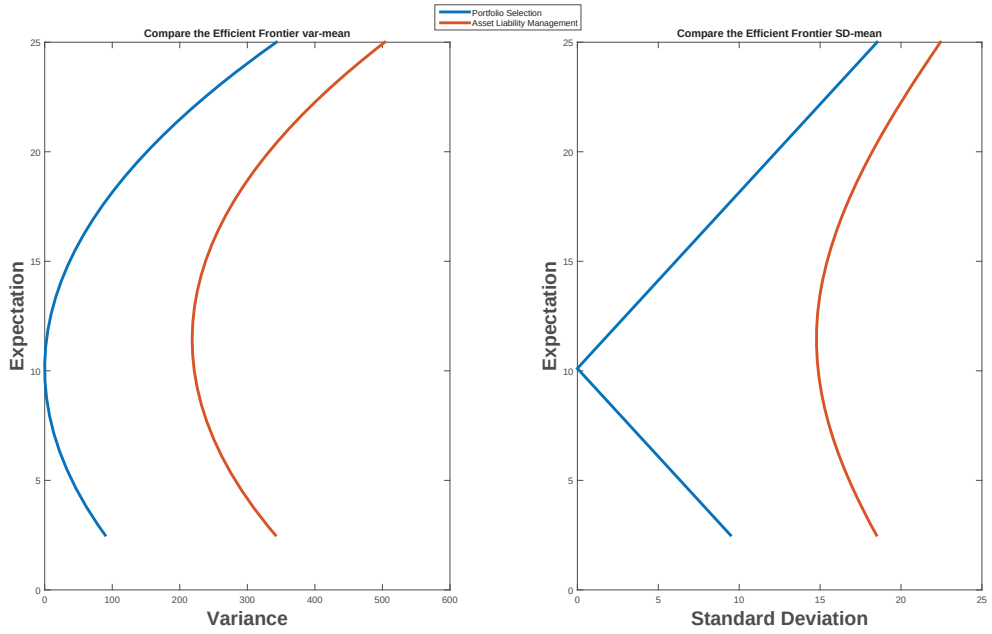

Figure 3.4: The effect of liability on the efficient frontier

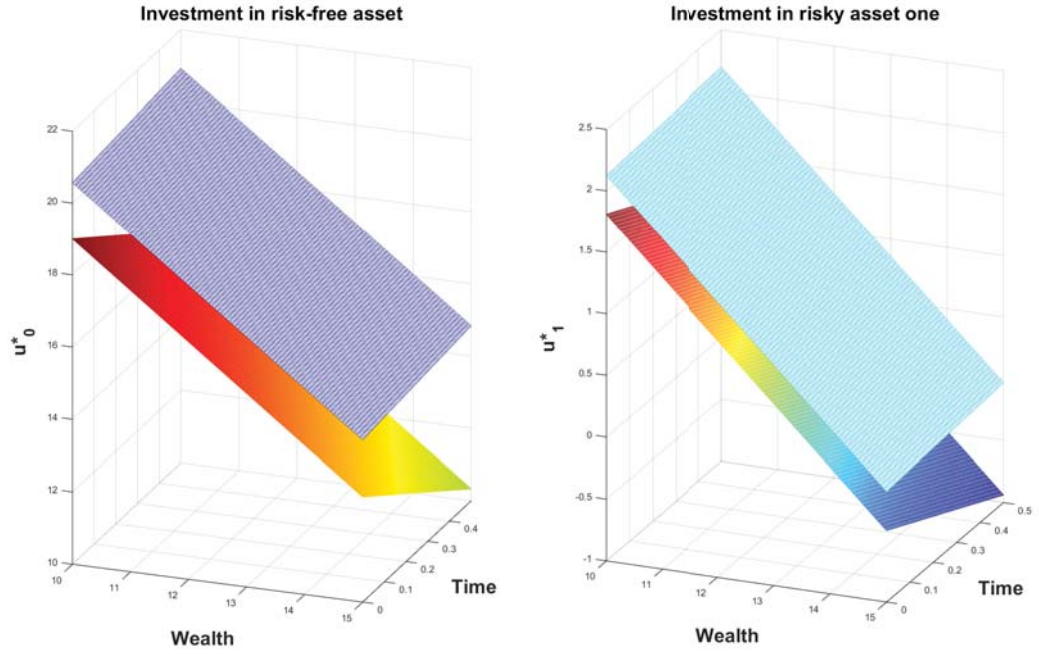

Figure 3.5: The effect of liability on the efficient investment strategy for the risk-free asset and the risky asset one. The top layer: portfolio selection. The bottom layer: asset liability management

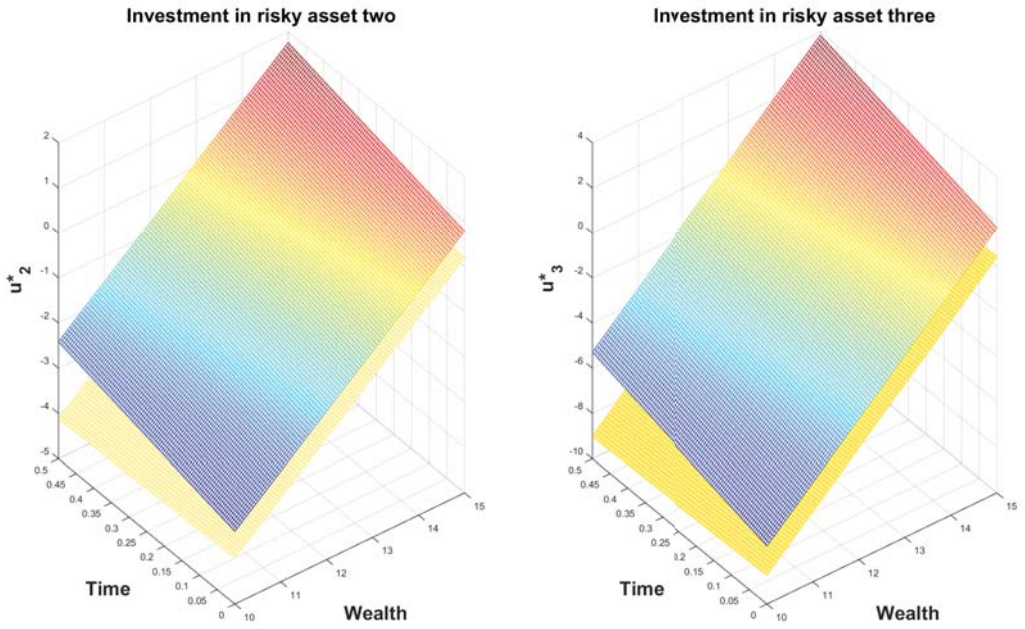

Figure 3.6: The effect of liability on the efficient investment strategy for the risky asset two and the risky asset three. The top layer: asset liability management. The bottom layer: portfolio selection

3.5.2 The Effect of Jump

In the following, we investigate the impact of the jump in the price of risky assets in financial market on the efficient frontier and the efficient investment strategy. We let the program run multiple times and plot the corresponding efficient frontiers in Figure 3.7. We find obvious discrepancy among the efficient frontiers. In the case with no jump, every component in vector

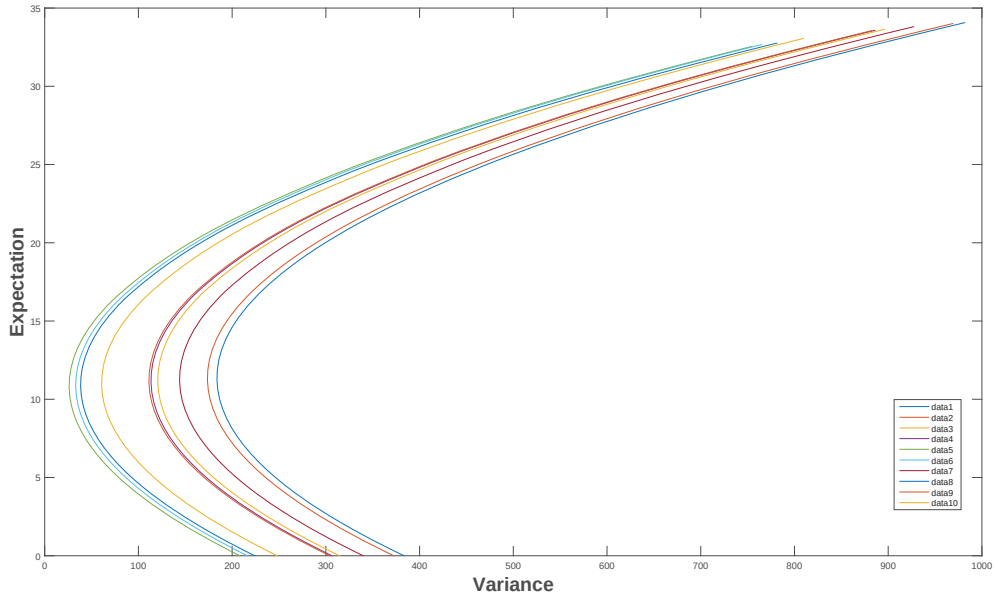

Figure 3.7: The effect of jump on the efficient frontier

A and matrix **B** becomes zero. To avoid confusion, we use another symbol $Q(t)$ to replace $F(t)$ and $Q(t) = 0.0036$. All the other parameters remain the same as given in Table 3.1. We have the following results.

$$
\begin{cases}
\widetilde{\Gamma}(t) = \exp\Big\{-\int_t^T (Q(s) - 2r(s))ds\Big\} = e^{0.0398(T-t)}, \\
\eta(t) = \frac{G(t)}{r(t)}\Big[1 - \exp\Big\{-\int_t^T r(s)ds\Big\}\Big] = 120.64\Big[1 - e^{-0.0217(T-t)}\Big], \\
\widetilde{\varpi}(t) = \exp\Big\{-\int_t^T (Q(s) - r(s))ds\Big\} = e^{0.0181(T-t)}, \\
\widetilde{\psi}(t) = \exp\Big\{-\int_t^T Q(s)ds\Big\} = e^{-0.0036(T-t)}, \\
\widetilde{\xi}(t) = \int_t^T Q(z)\eta(z)\widetilde{\varpi}(z) - G(z)\widetilde{\varpi}(z)dz = \int_t^T -2.1836e^{0.0181(T-z)} - 0.4343e^{-0.0036(T-z)}dz, \\
\widetilde{\chi}(t) = \int_t^T \eta^2(l)Q(l)\widetilde{\Gamma}(l) - \frac{2G^2(l)}{r(l)}\widetilde{\Gamma}(l) + \frac{2G^2(l)}{r(l)}\widetilde{\varpi}(l) - \lambda_L\widetilde{\Gamma}(l)E(z^2)dl \\
= \int_t^T \Big(\eta^2(l)Q(l) - \frac{2G^2(l)}{r(l)} - \lambda_L E(z^2)\Big)e^{0.0398(T-l)} + \frac{2G^2(l)}{r(l)}e^{0.0181(T-l)}dl.
\end{cases}
$$

$$u_1(t) = 0.0457\Bigg(x(t) + \Big(\frac{10e^{0.0181\times0.5} - de^{-0.0036\times0.5} + 1.3137}{1 - e^{-0.0036\times0.5}} - d\Big)e^{-0.0217(0.5-t)} - 120.64\Big(1 - e^{-0.0217(0.5-t)}\Big)\Bigg), \tag{3.85}$$

$$u_2(t) = 0.1176\Bigg(x(t) + \Big(\frac{10e^{0.0181\times0.5} - de^{-0.0036\times0.5} + 1.3137}{1 - e^{-0.0036\times0.5}} - d\Big)e^{-0.0217(0.5-t)} - 120.64\Big(1 - e^{-0.0217(0.5-t)}\Big)\Bigg), \tag{3.86}$$

$$u_3(t) = -0.0918\Bigg(x(t) + \Big(\frac{10e^{0.0181\times0.5} - de^{-0.0036\times0.5} + 1.3137}{1 - e^{-0.0036\times0.5}} - d\Big)e^{-0.0217(0.5-t)} - 120.64\Big(1 - e^{-0.0217(0.5-t)}\Big)\Bigg), \tag{3.87}$$

$$\begin{aligned} u_0(t) &= x(t) - \sum_{i=1}^{3} u_i(t) \\ &= x(t) - 0.0715\Bigg(x(t) + \Big(\frac{10e^{0.0181\times0.5} - de^{-0.0036\times0.5} + 1.3137}{1 - e^{-0.0036\times0.5}} - d\Big)e^{-0.0217(0.5-t)} \\ &\quad - 120.64\Big(1 - e^{-0.0217(0.5-t)}\Big)\Bigg), \end{aligned} \tag{3.88}$$

$$d_{SD_{min}} = 11.4252, \tag{3.89}$$

$$Var^*[x(T)] = \max_{\theta} \Phi(x(0), \theta) = 560.3334(d - 11.4252)^2 + 41.4386, \tag{3.90}$$

$$SD^*[x(T)] = \sqrt{560.3334\Big(d - 11.4252\Big)^2 + 41.4386}. \tag{3.91}$$

We notice that the value of $d_{SD_{min}}$ here is the same as the one in the original model. Now we give Corollary 3.1 followed by a proof.

Corollary 3.1. *$d_{SD_{min}}$ is independent of F or Q in the case of constant parameters.*

Proof.

$$\begin{aligned}
d_{SD_{min}} =& \frac{x(0)\varpi(0) - \xi(0)}{\psi(0)} \\
=& \frac{x(0)\exp\Big\{-\int_0^T (F(s) - r(s))ds\Big\} - \int_0^T F(z)\eta(z)\varpi(z) - G(z)\varpi(z)dz}{\exp\Big\{-\int_0^T F(s)ds\Big\}} \\
=& \frac{x(0)e^{-(F-r)T} - \int_0^T (\frac{FG}{r} - G)e^{-(F-r)(T-z)} - \frac{FG}{r}e^{F(T-z)}dz}{e^{-FT}} \\
=& x(0)e^{rT} - \frac{\frac{FG}{r} - G}{F-r}\Big(e^{FT} - e^{rT}\Big) + \frac{G}{r}\Big(e^{FT} - 1\Big) \\
=& x(0)e^{rT} - \frac{G}{r}\Big(1 - e^{rT}\Big).
\end{aligned}$$

□

The efficient frontier is depicted in Figure 3.8 with the expectation of terminal wealth varying between 0 and $3d_{SD_{min}}$. Then we fix the value of $d(= 12)$ and plot the efficient investment strategy for the portfolio selection model with time varying between 0 and 0.5 and wealth between 10 and 15 in Figure 3.9 and Figure 3.10.

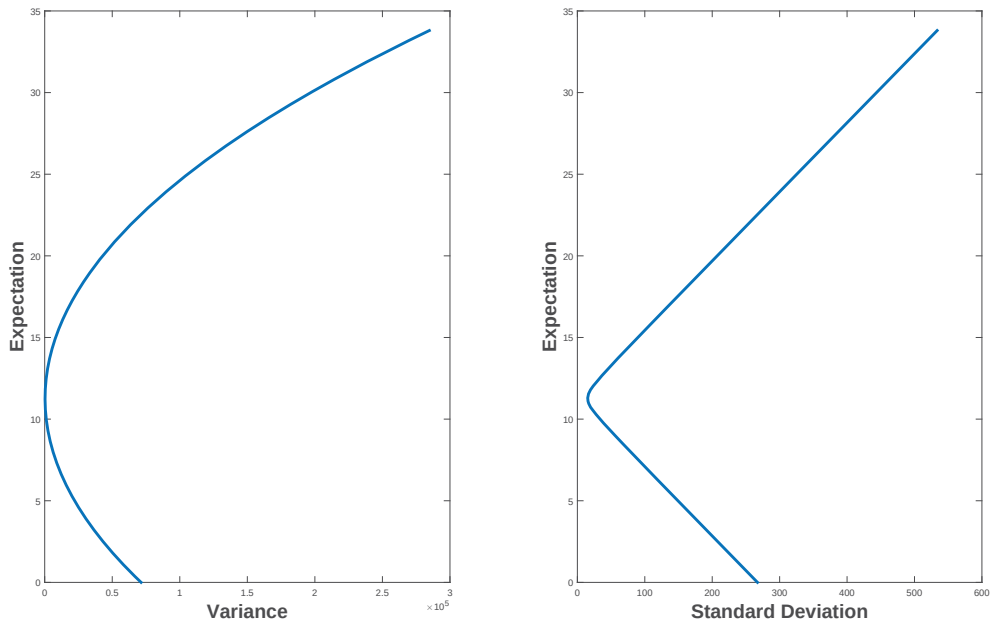

Figure 3.8: The efficient frontier in the case with no jump in risky assets' prices

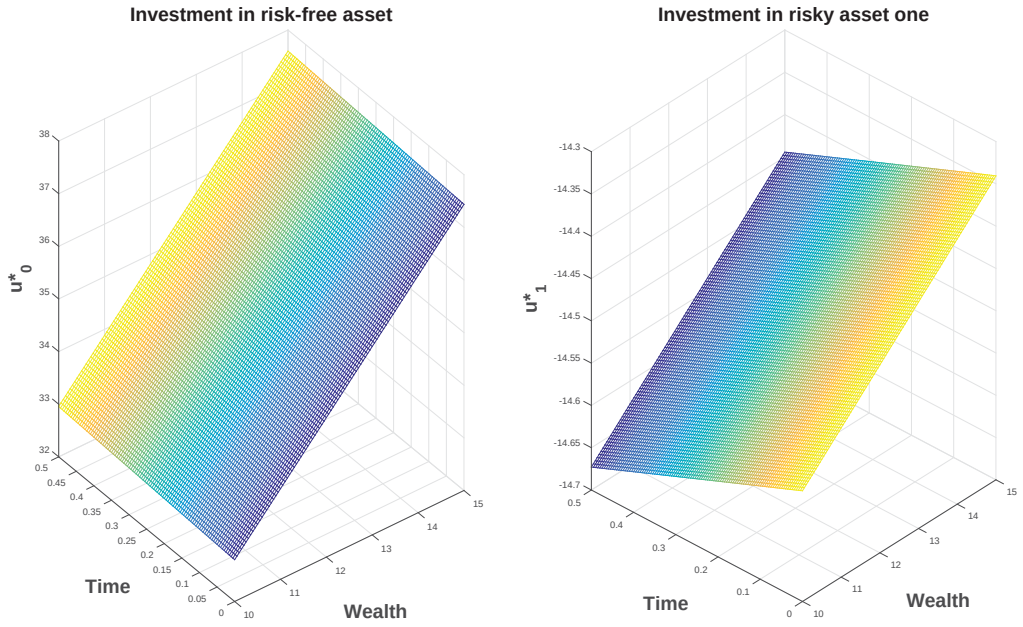

Figure 3.9: The effect of jump on the efficient investment strategy for the risk-free asset and the risky asset one. $u_0(t)$ and $u_1(t)$ increase with respect to wealth $x(t)$ and time t.

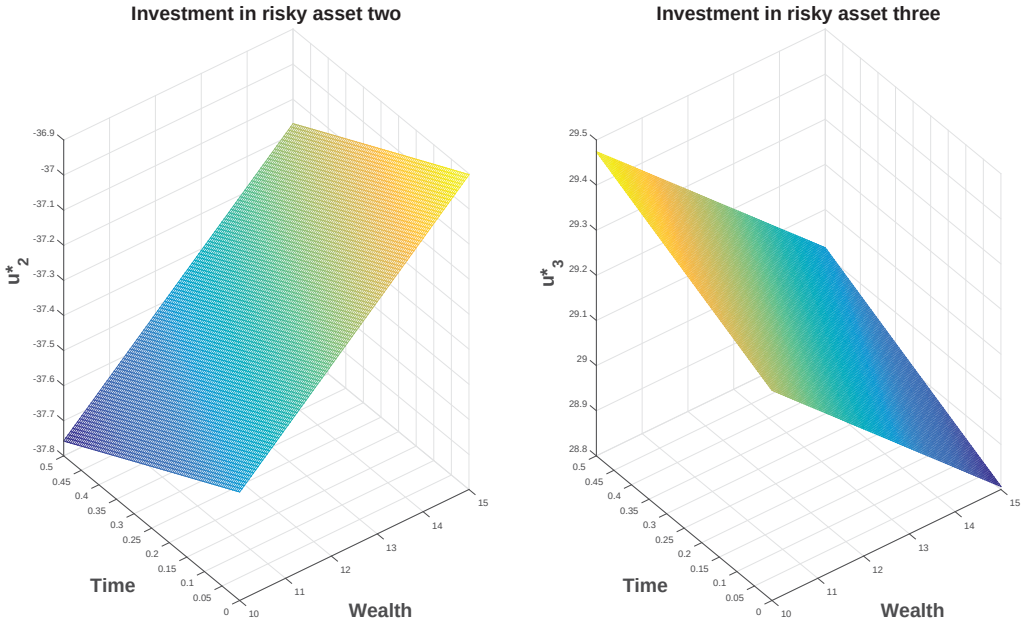

Figure 3.10: The effect of jump on the efficient investment strategy for the risky asset two and the risky asset three. $u_2(t)$ increases with respect to wealth $x(t)$ and time t. $u_3(t)$ decreases with respect to wealth $x(t)$ and time t.

3.5.3 Sensitivity Analysis

In the previous subsections, we provide some numerical illustrations of the efficient investment strategy with respect to time t and wealth $x(t)$ under different conditions. Now, we focus on the impact of parameters on the efficient frontier of the original asset liability management model. The parameters we are to analyze include the exit time T, the initial wealth $x(0)$, the risk-free interest rate $r(t)$ and the operational cost function $c(t)$. By varying one parameter and fixing the others each time, we can demonstrate the effect of T, $x(0)$, $r(t)$ and $c(t)$ in Figure 3.11, Figure 3.12, Figure 3.13 and Figure 3.14.

From Figure 3.11, we find that as the exit time T increases from 1 to 5, (i) the efficient frontier shifts toward the expectation axis; (ii) the shape of the curve changes dramatically and (iii) the longer the maturity time is, the greater the variance of the terminal wealth is for a given expectation of return. Figure 3.12 shows the impact of the initial wealth $x(0)$. We plot the efficient frontiers under five different cases $x(0) = 10,\ 15,\ 20,\ 25,\ 30$ and find that (i) the efficient frontier moves upward as $x(0)$ increases; (ii) the shape of the curve remains the same and (iii) the larger initial wealth gives higher expected return under the same risk level.

The exit time T and the initial wealth $x(0)$ give similar influence on the efficient frontier with slight difference. We hereby give the analysis as follows.

- With larger exit time, the insurer has longer time to accumulate wealth. So the insurer can expect higher expected terminal wealth for a predetermined risk tolerance decided by the board. Similarly, with greater initial wealth, the insurer can achieve the predetermined expected terminal wealth easily without taking too much risk.

- With regard to the shape of the efficient frontier, we should refer to the theoretical results in (3.38) and $\psi(0) = \exp\Big\{-\int_0^T F(s)ds\Big\}$. The value of $\frac{\psi(0)}{1-\psi(0)}$ changes as the value of T changes, which explains the difference in the shape of the curve. On the contrary, $\frac{\psi(0)}{1-\psi(0)}$ keeps the same for different values of $x(0)$.

In the following, we investigate the impact of risk-free interest rate $r(t)$ and operational cost function $c(t)$ as presented in Figure 3.13 and Figure 3.14. Because of the random jump in risky assets' price and the random insurance liability, the movement of the efficient frontier is not as stable as in the previous two tests. Basically, the curve moves upwards and away from the variance axis as the value of $r(t)$ or $c(t)$ increases, which is consistent with our common sense.

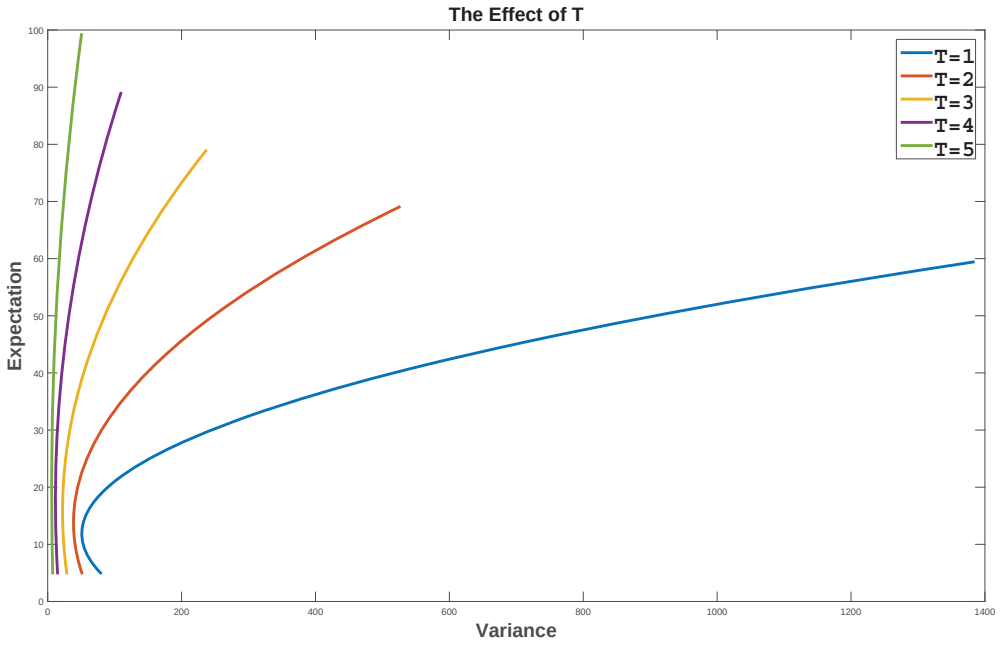

Figure 3.11: The effect of the exit time T on the efficient frontier

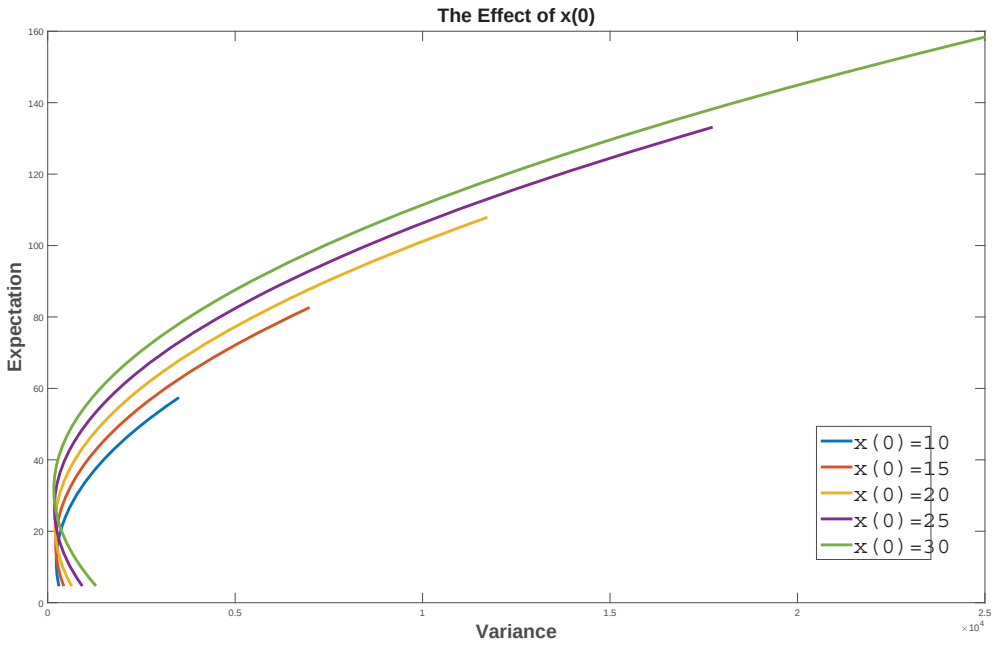

Figure 3.12: The effect of the initial wealth $x(0)$ on the efficient frontier

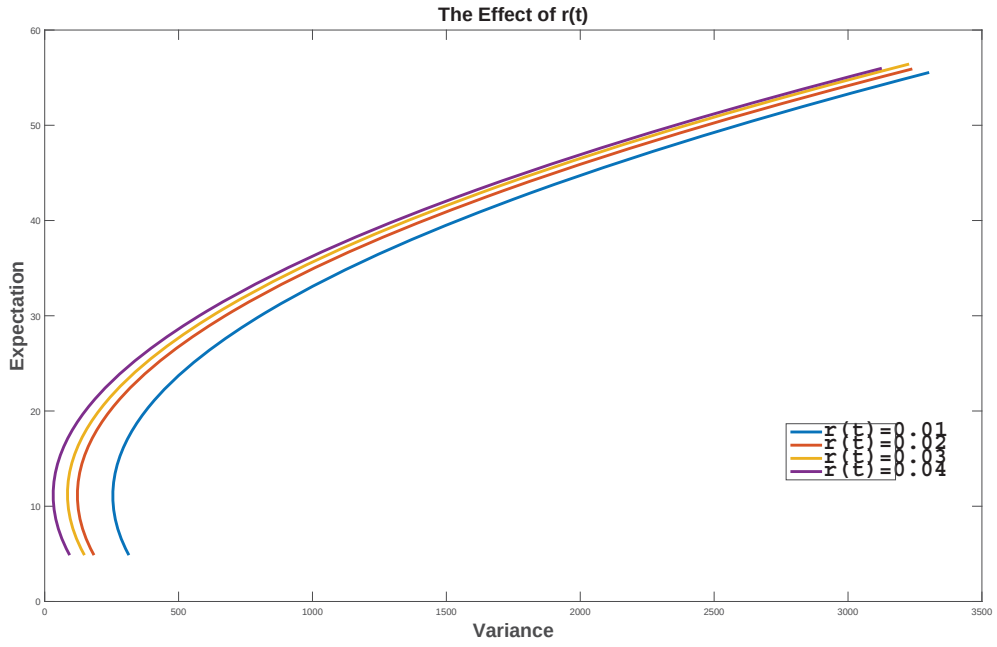

Figure 3.13: The effect of the risk-free interest rate $r(t)$ on the efficient frontier

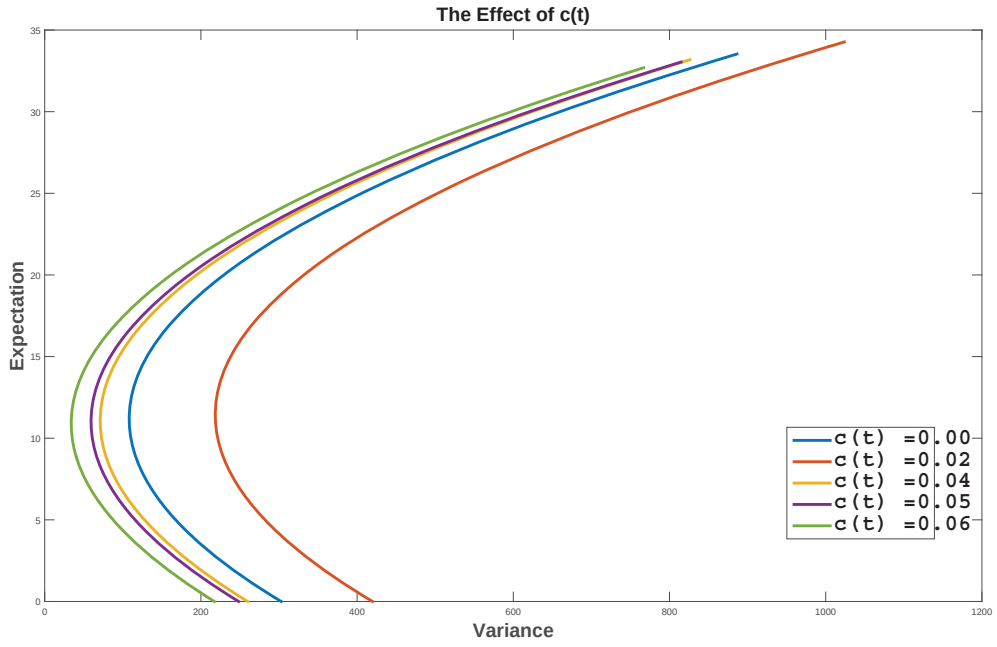

Figure 3.14: The effect of the operational cost $c(t)$ on the efficient frontier

3.6 Concluding Remarks

In Australia, the general insurers are supervised by the *Australian Prudential Regulation Authority* (APRA) under the *Insurance Act 1973*, and they are also regulated by the *International Association of Insurance Supervisors* (IAIS). It is the general insurers' responsibility to submit well-organized financial statements, including assets, liabilities, revenues, expenses, gains and losses, to the corresponding regulators and other stakeholder. All the information should be based on a *Matching Principle* and reflect a company's financial performance for the accounting period. Consequently, the asset liability management is of vital importance for all general insurers.

We here study the mean-variance asset liability management problem with general insurance liability in a jump diffusion market. We assume that there exist one risk-free asset and multiple risky assets which are governed by the exponential Lévy process. Closed-form analytical solutions for the optimal investment strategy and the efficient frontier have been obtained by applying the Lagrange duality method and the dynamic programming technique. This work can be regarded as an extension of the mean-variance portfolio selection in a jump diffusion market. Due to the existence of insurance liability, the efficient frontier is no longer a straight line and the variance of terminal wealth does not approach zero at $d_{SD_{min}}$, which is showed by the numerical examples. Some numerical analysis has been carried out to examine the impact of jump and the effect of liability on the efficient frontier. Some sensitivity analysis of the key parameters is also performed in our numerical investigation. We conclude that the initial wealth and the exit time all have significant influence on the efficient frontier. The variance of terminal wealth increases as the value of $x(0)$ or T increases. The effect of parameters, t and $x(t)$, on the optimal investment strategy is also studied. Numerical results suggest that the market fluctuations lead to remarkable challenge on the construction of the efficient and effective optimal strategy.

CHAPTER 4

Mean-Variance Asset Liability Management with State-Dependent Risk Aversion

4.1 General Overview

This chapter investigates asset liability management problem with state-dependent risk aversion under mean-variance criterion. The investor allocate the wealth among multiple assets including one risk-free asset and n risky assets which are governed by a system of geometric Brownian motion stochastic differential equations, and the investor faces the risk of paying uncontrollable random liabilities. Furthermore, we consider the state-dependent risk aversion in our model, which means that the risk aversion depends on the current wealth held by the investor. This innovation is inspired by the research work of Björk et al. (2014). By solving the extended Hamilton-Jacobi-Bellman system, the analytical closed-form expressions for the time-consistent optimal investment strategy, the variance versus expectation of terminal wealth and the optimal value function are derived. Moreover, we also obtain the results for the portfolio selection as a special case of the original model.

Compared with the other literature on asset liability management problem under mean-variance criterion, the main contributions of this work are as follows. Firstly, we introduce multiple risky assets instead of only one risky asset as in Björk et al. (2014) which is more economically relevant; and we take into account the liability, which makes the model more realistic from an economic point of view. Secondly, we derive the general extended Hamilton-Jacobi-Bellman equation taking into account multiple risky assets. Thirdly, we establish an Euler discretization scheme to demonstrate the wealth process among these assets/liability numeri-

cally under a special case of state-dependent risk aversion. We formulate the time-inconsistent problem in the framework of game theory with the objective to find the sub-game perfect Nash equilibrium point for this game.

The remainder of this chapter is organized as follows. Section 4.2 describes the financial market setting and constructs a mean-variance asset liability management model with state-dependent risk aversion. In section 4.3, we provide the verification theorem and prove it together with the extended Hamilton-Jacobi-Bellman system. By applying the game theory and solving the corresponding extended Hamilton-Jacobi-Bellman equation approach, we derive the closed-form analytical solution. In addition, we obtain the equilibrium investment strategy, the equilibrium value function and the variance versus the expectation of terminal wealth. In section 4.4, several special cases are discussed. In section 4.5, we present some numerical examples, followed by some conclusions in Section 4.6.

4.2 Model Formulation

Given a filtered probability space $(\Omega, \mathbf{P}, \mathcal{F}, \{\mathfrak{F}_t\}_{0\leq t\leq T})$, let $\mathbf{O}'$ be the transpose of a matrix or a vector $\mathbf{O}$, b_i be the ith component of any vector $\mathbf{B}$, $\mathbf{W}(t) = (W_1(t), W_2(t), \ldots, W_m(t))'$ be an m-dimensional standard Brownian motion defined on $(\Omega, \mathbf{P}, \mathcal{F})$ over $[0, T]$, and $\mathfrak{F}_t = \sigma\{\mathbf{W}(s); 0 \leq s \leq t\}$ be augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$, where $\mathcal{F} = \mathfrak{F}_T$. Some finite T denotes the investment time horizon. All random variables considered in this chapter are defined on this filtered probability space. Consider a continuous-time financial market with the standard assumptions: continuous trading is allowed; no transaction cost or tax is involved in trading; and all assets are infinitely divisible. Suppose that an investor can allocate his/her wealth among $n + 1$ assets that include one risk-free asset and n risky assets.

Asset Price Evolution Model

The price of risk-free asset is assumed to evolve according to the following ordinary differential equation.

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \quad 0 \leq t \leq T, \\ S_0(0) = s_0 > 0, \end{cases} \tag{4.1}$$

where s_0 is the initial price of the risk-free asset; $r(t)$ is a positive time-dependent deterministic risk-free interest rate. Whereas, the prices of these risky assets satisfy the following stochastic

differential equations.

$$\begin{cases} dS_i(t) = S_i(t-)\left(\alpha_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW_j(t)\right), \quad 0 \le t \le T, \\ S_i(0) = s_i > 0, \qquad i = 1, 2, \ldots, n, \end{cases} \tag{4.2}$$

where $(s_i, i = 1, 2, \ldots, n)$ are the initial prices of the risky assets; $(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$ and $(\sigma_{ij}(t))_{m\times m}$ are respectively the appreciation rate vector and the volatility matrix of these assets, which are assumed to be positive continuous bounded deterministic functions of time t.

Liability Evolution Model

The investor has to face the uncontrollable liability whose value process is modeled by

$$\begin{cases} dL(t) = \mu(t)dt + \boldsymbol{\nu}'(t)d\mathbf{W}(t), \quad 0 \le t \le T, \\ L(0) = l_0 \ge 0, \end{cases} \tag{4.3}$$

where l_0 is the initial value of the liability; $\mu(t)$ and $\boldsymbol{\nu}(t) = (\nu_1(t), \nu_2(t), \ldots, \nu_m(t))'$ are respectively the appreciation and volatility in liability, which are assumed to be deterministic functions of time t; and the liability here is in a generalized sense. We understand it as the subtraction of the real liability and the stochastic income of the investor. So we allow negative liabilities, which means that the stochastic income of the investor is greater than his/her real liability.

Remark 4.1. The price of risky asset and the liability could have correlation with each other because these dynamic processes all use the same m-dimensional standard Brownian motion $W(t)$. Besides, we consider n risky assets and one liability in the asset liability management model, so we have $m \ge n + 1$.

Wealth Process

Assume that an investor endowed with an initial wealth x_0 at time 0 plans to invest in the market dynamically in the horizon $[0,\ T]$. Let $x(t)$ denote the wealth of the investor at time t; let $u_i(t)$ denote the amount invested in asset i and $N_i(t)$ denote the number of units of asset i in the investor's portfolio, $i = 1, 2, \ldots, n$. After deducting the liability, the amount invested in the $0th$ asset is $u_0(t) = [x(t) - \sum_{i=1}^{n} u_i(t)]$. So, by (4.1) - (4.3), the wealth held by the investor at

time t, $x(t)$, follows the dynamics

$$\begin{aligned} dx(t) &= \sum_{i=0}^{n} N_i(t)dS_i(t) - dL(t) \\ &= \sum_{i=0}^{n} u_i(t)\frac{dS_i(t)}{S_i(t)} - dL(t) \\ &= \sum_{i=1}^{n} u_i(t)\alpha_i(t)dt + [x(t) - \sum_{i=1}^{n} u_i(t)]r(t)dt + \sum_{j=1}^{m}\sum_{i=1}^{n} u_i(t)\sigma_{ij}(t)dW_j(t) \\ &- \mu(t)dt - \boldsymbol{\nu}'(t)d\mathbf{W}(t). \end{aligned} \tag{4.4}$$

To further simplify $dx(t)$ in (4.4), we express the stochastic differential equation in vector form as follows

$$\begin{cases} dx(t) = [r(t)x(t) - \mu(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]dt + [\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)]d\mathbf{W}(t), \quad 0 \le t \le T, \\ x(0) = x_0 > 0, \end{cases} \tag{4.5}$$

where

$$\begin{cases} \boldsymbol{u}(t) = (u_1(t), u_2(t), \ldots, u_n(t))', \\ \boldsymbol{\alpha}(t) = (\alpha_1(t) - r(t), \alpha_2(t) - r(t), \ldots, \alpha_n(t) - r(t))', \\ \boldsymbol{\sigma}_i(t) = (\sigma_{i1}(t), \sigma_{i2}(t), \ldots, \sigma_{im}(t)), \\ \boldsymbol{\sigma}(t) = (\sigma_1'(t), \sigma_2'(t), \ldots, \sigma_n'(t))'. \end{cases}$$

We assume that all the functions are measurable and uniformly bounded in $[0,\ T]$. Denote by $\mathbf{L}^2_{\mathcal{F}}(t, T; \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued and measurable stochastic processes $f(s)$ adapted to $\{\mathfrak{F}_s\}_{s\ge t}$ on $[0,\ T]$ such that $E\left[\int_t^T |f(s)|^2 ds\right] < +\infty$.

Mean-Variance Asset Liability Management Optimization Problem

Denote by $\mathcal{U}[0,\ T]$ the set of all such admissible strategies over $[0,\ T]$. The mean-variance asset liability management problem refers to the problem of finding the optimal admissible strategy such that the mean-variance utility is maximized at terminal time T. So we have the objective function $J(t, x, u)$ described mathematically as follows

$$J(t, x, u) = E[x(T)] - \frac{\gamma(x)}{2} Var[x(T)]. \tag{4.6}$$

The target is to find the equilibrium control law (the optimal control law) $\widehat{u}$ such that the mean-variance utility is maximized at terminal time T. In other words, it is to maximize the expected

return with a penalty term for the risk, namely,

$$V(t,x(t)) = \max_{u(.)\in\mathcal{U}[0,\ T]} \Big\{ E_{t,x}[x^U(T)] - \frac{\gamma(x)}{2} Var_{t,x}[x^U(T)] \Big\}, \tag{4.7}$$

where $E_{t,x}[.] = E[. \mid x^U(t) = x]$. We extend the work of Björk et al. (2014) by considering multiple risky assets and taking into account the liability, which make the model more realistic from an economic point of view. Now we provide the definition of the equilibrium control law which is an adoption of the one given in Björk and Murgoci (2010).

Definition 4.1 (Equilibrium Control Law)**.** *Given a control law $\widehat{u}$, construct a control law $u_{\Delta t}$ by*

$$u_{\Delta t}(s,.) = \begin{cases} u, & \text{for } t \le s < t + \Delta t, \\ \widehat{u}(s,.), & \text{for } t + \Delta t \le s \le T, \end{cases}$$

where $u \in R^k$, $\Delta t > 0$, and $t \in [0,T]$ is arbitrarily chosen. If

$$\lim_{\Delta t \to 0} \frac{J(t,x,\widehat{u}) - J(t,x,u_{\Delta t})}{\Delta t} \ge 0,$$

for all $u \in R^k$, and $(t,x) \in [0,T] \times R^n$, we say that $\widehat{u}$ is the equilibrium control and we have the equilibrium value function as follows

$$V(t,x) = J(t,x,\widehat{u}).$$

Let A be the infinitesimal generator. For any fixed $u \in \mathcal{U}$, we introduce the corresponding controlled infinitesimal generator A^u defined by

$$A^u = \frac{\partial}{\partial t} + [r(t)x(t) - \mu(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial}{\partial x} + \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)]'\Big\}\frac{\partial^2}{\partial x^2}. \tag{4.8}$$

By making use of the definition of the equilibrium control as given in Definition 4.1 and the infinitesimal generator A^u in (4.8), following the application in Björk and Murgoci (2010), we can derive the extended Hamilton-Jacobi-Bellman equations and its verification theorem as follows.

Theorem 4.1 (Verification Theorem)**.** *Assume that (V,f,g) is a solution of the following extended Hamilton-Jacobi-Bellman system, and the control law $\widehat{u}$ realizes the supremum in the*

equation.

$$\begin{cases} \sup\limits_{u\in\mathcal{U}}\Big\{A^u V(t,x) - A^u f(t,x,x) + A^u f^x(t,x) \\ -A^u(G\circ g)(t,x) + H^u g(t,x)\Big\} = 0, \quad 0\le t\le T, \\ A^{\widehat{u}} f^y(t,x) = 0, \quad 0\le t\le T, \\ A^{\widehat{u}} g(t,x) = 0, \quad 0\le t\le T, \end{cases}$$

with the following conditions and definitions

$$\begin{cases} V(T,x) = F(x,x) + G(x,x), \\ f(T,x,y) = F(y,x), \\ g(T,x) = x, \\ f^y(t,x) = f(t,x,y), \\ (G\circ g)(t,x) = G(x,g(t,x)), \\ H^u g(t,x) = G_y(x,g(t,x))\times A^u g(t,x), \\ G_y(x,y) = \frac{\partial G(x,y)}{\partial y}. \end{cases}$$

Then $\widehat{u}$ is an equilibrium control law, and V is the corresponding value function. Moreover, f and g have the following probabilistic representations:

$$f(t,x,y) = E_{t,x}[F(y,x^{\widehat{u}}(T))], \tag{4.9}$$

$$g(t,x) = E_{t,x}[x^{\widehat{u}}(T)], \tag{4.10}$$

$$F(y,x(T)) = x(T) - \frac{\gamma(y)}{2}[x(T)]^2, \tag{4.11}$$

$$G(y,g) = \frac{\gamma(y)}{2}g^2. \tag{4.12}$$

Proof. As for the Hamilton-Jacobi-Bellman equations in Björk et al. (2014), by (4.7), we have

$$V(t,X(t)) = \max_{u\in U} J(t,X(t),U), \tag{4.13}$$

where

$$J(t,X(t),U) = E_{t,X(t)}[F(y,X^U(T))] + G(y,E_{t,X(t)}[X^U(T)]), \tag{4.14}$$

with F and G having the expressions in (4.11) and (4.12). For $s>t$,

$$J(s,X(s),U) = E_{s,X(s)}[F(Y(s),X^U(s))] + G(Y(s),E_{s,X(s)}[X^U(T)]). \tag{4.15}$$

From the Markovian structure and the definitions of (4.11) and (4.12), we have

$$E_{s,X(s)}[F(Y(s), X^U(T))] = f^U(s, X(s), Y(s)), \tag{4.16}$$

$$E_{s,X(s)}[X^U(T)] = g^U(s, X(s)). \tag{4.17}$$

Then, (4.15) can be written as follows

$$J(s, X(s), U) = f^U(s, X(s), Y(s)) + G(Y(s), g^U(s, X(s))). \tag{4.18}$$

Taking expectations on both sides gives

$$E_{t,X(t)}[J(s, X(s), U)] = E_{t,X(t)}[f^U(s, X(s), Y(s))] + E_{t,X(t)}[G(Y(s), g^U(s, X(s)))], \tag{4.19}$$

and going back to the definition of (4.14), we have

$$\begin{aligned} E_{t,X(t)}[J(s, X(s), U)] &= J(t, X(t), U) + E_{t,X(t)}[f^U(s, X(s), Y(s))] - E_{t,X(t)}[F(y, X^U(T))] \\ &+ E_{t,X(t)}[G(Y(s), g^U(s, X(s)))] - G(y, E_{t,X(t)}[X^U(T)]). \end{aligned} \tag{4.20}$$

From the iterated condition, we obtain

$$E_{t,X(t)}[F(y, X^U(T))] = E_{t,X(t)}[E_{s,X(s)}[F(y, X^U(T))]] = E_{t,X(t)}[f^U(s, X(s), y(s))], \tag{4.21}$$

and that

$$E_{t,X(t)}[X^U(T)] = E_{t,X(t)}[E_{s,X(s)}X^U(T)] = E_{t,X(t)}[g^U(s, X(s))]. \tag{4.22}$$

Substituting (4.21) and (4.22) back into (4.20), we obtain

$$\begin{aligned} &E_{t,X(t)}[J(s, X(s), U)] - J(t, X(t), U) - E_{t,X(t)}[f^U(s, X(s), Y(s))] + E_{t,X(t)}[f^U(s, X(s), y(s))] \\ &- E_{t,X(t)}[G(Y(s), g^U(s, X(s)))] + G(y, E_{t,X(t)}[g^U(s, X(s))]) = 0. \end{aligned} \tag{4.23}$$

Then

$$\sup_{u\in U}\{E_{t,X(t)}[J(s,X(s),U)] - J(t,X(t),U) - E_{t,X(t)}[f^U(s,X(s),Y(s))] + E_{t,X(s)}[f^U(s,X(s),y(s))]$$
$$- E_{t,X(t)}[G(Y(s),g^U(s,X(s)))] + G(y(t),E_{t,X(t)}[g^U(s,X(s))])\} = 0. \tag{4.24}$$

From (4.13) and the definition of the control law, we get that U coincides with the equilibrium law $\widehat{u}$ in $[s,T]$, then we have the following formula,

$$J(s,X(s),\widehat{u}) = V(s,X(s)), \tag{4.25}$$

$$f^U(s,X(s),y) = f(s,X(s),y), \tag{4.26}$$

$$g^U(s,X(s)) = g(s,X(s)). \tag{4.27}$$

Thus, (4.24) can be written as

$$\sup_{u\in U}\{E_{t,X(t)}[V(s,X(s))] - V(t,X(t)) - E_{t,X(t)}[f(s,X(s),Y(s))] + E_{t,X(t)}[f(s,X(s),y(s))]$$
$$- E_{t,X(t)}[G(Y(s),g(s,X(s),V(s)))] + G(y(t),E_{t,X(t)}[g^U(s,X(s))])\} = 0. \tag{4.28}$$

Denote

$$E_{t,X(t)}[V(s,X(s))] - V(t,X(t)) = A^uV, \tag{4.29}$$

$$E_{t,X(t)}[f(s,X(s),Y(s))] = A^uf, \tag{4.30}$$

$$E_{t,X(t)}[f(s,X(s),y(s))] = A^uf^y, \tag{4.31}$$

$$E_{t,X(t)}[G(Y(s),g(s,X(s)))] = A^u(G\circ g), \tag{4.32}$$

$$G(y(t),E_{t,X(t)}[g^U(s,X(s))]) = H^ug. \tag{4.33}$$

Then from the extended Hamilton-Jacobi-Bellman equation in Björk and Murgoci (2010), we extend the work to the case with stochastic liability as follows,

$$\sup_{u\in U}\{A^uV(t,x) - A^uf(t,x,x) + A^uf^x(t,x) - A^u(G\circ g)(t,x) + H^ug(t,x)\} = 0, \tag{4.34}$$

which is the Verification Theorem. □

Theorem 4.2. *The extended Hamilton-Jacobi-Bellman equations for the Nash equilibrium problem with consideration of multiple risky assets and liability are as follows*

$$
\begin{cases}
\frac{\partial V}{\partial t} + \sup\limits_{u\in\mathcal{U}} \Big\{ [r(t)x(t) - \mu(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\{\frac{\partial V}{\partial x} - \frac{\partial f}{\partial y}(t,x,x) - \frac{d\gamma(x)}{2dx}g^2\} \\
+\frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)]'\Big\}\times \\
\Big\{\frac{\partial^2 V}{\partial xx} - \frac{\partial^2 f}{\partial yy}(t,x,x) - 2\frac{\partial^2 f}{\partial xy}(t,x,x) - \frac{d^2\gamma(x)}{2dxx}g^2 - 2[\frac{d\gamma(x)}{dx}]g[\frac{\partial g}{\partial x}] - \gamma(x)[\frac{\partial g}{\partial x}]^2\Big\}\Big\} = 0, \quad 0 \le t \le T, \\
\\
\frac{\partial f}{\partial t}(t,x,y) + [r(t)x(t) - \mu(t) + \widehat{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial f}{\partial x}(t,x,y) \\
+\frac{1}{2}\Big\{[\widehat{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)][\widehat{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)]'\Big\}\frac{\partial^2 f}{\partial xx}(t,x,y) = 0, \quad 0 \le t \le T, \\
\\
\frac{\partial g}{\partial t}(t,x) + [r(t)x(t) - \mu(t) + \widehat{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial g}{\partial x}(t,x) \\
+\frac{1}{2}\Big\{[\widehat{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)][\widehat{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)]'\Big\}\frac{\partial^2 g}{\partial xx}(t,x) = 0, \quad 0 \le t \le T.
\end{cases}
$$

Proof.

$$
\begin{aligned}
A^u V(t,x) =& \frac{\partial V}{\partial t} + [r(t)x(t) - \mu(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial V}{\partial x} \\
&+ \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)]'\Big\}\frac{\partial^2 V}{\partial xx},
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
A^u f(t,x,x) =& \frac{\partial f}{\partial t}(x,x) + [r(t)x(t) - \mu(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\{\frac{\partial f}{\partial x}(t,x,x) + \frac{\partial f}{\partial y}(t,x,x)\} \\
&+ \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)]'\Big\}\times \\
&\Big\{\frac{\partial^2 f}{\partial xx}(t,x,x) + \frac{\partial^2 f}{\partial yy}(t,x,x) + 2\frac{\partial^2 f}{\partial xy}(t,x,x)\Big\},
\end{aligned} \tag{4.36}
$$

$$
\begin{aligned}
A^u f^x(t,x) =& \frac{\partial f}{\partial t}(t,x,x) + [r(t)x(t) - \mu(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial f}{\partial x}(t,x,x) \\
&+ \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)]'\Big\}\frac{\partial^2 f}{\partial xx}(t,x,x),
\end{aligned} \tag{4.37}
$$

$$
\begin{aligned}
A^u(G\circ g)(t,x) =& H^u g(t,x) + [r(t)x(t) - \mu(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial G}{\partial x}(x,g) \\
&+ \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t) - \boldsymbol{\nu}'(t)]'\Big\}\times \\
&\Big\{\frac{\partial^2 G}{\partial xx}(x,g) + \frac{\partial^2 G}{\partial yy}(x,g)[\frac{\partial g}{\partial x}]^2 + 2\frac{\partial^2 G}{\partial xy}(x,g)[\frac{\partial g}{\partial x}]\Big\}.
\end{aligned} \tag{4.38}
$$

With the results (4.35) - (4.38), the left-hand-side (LHS) of (4.34) can be rewritten as follows

$$\begin{aligned}
&\sup_{u\in\mathcal{U}}\Bigg\{\frac{\partial V}{\partial t}+[r(t)x(t)-\mu(t)+\boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial V}{\partial x}\\
&\quad+\frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)]'\Big\}\frac{\partial^2 V}{\partial xx}\\
&\quad-\frac{\partial f}{\partial t}(x,x)-[r(t)x(t)-\mu(t)+\boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\{\frac{\partial f}{\partial x}(t,x,x)+\frac{\partial f}{\partial y}(t,x,x)\}\\
&\quad-\frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)]'\Big\}\Big\{\frac{\partial^2 f}{\partial xx}(t,x,x)+\frac{\partial^2 f}{\partial yy}(t,x,x)+2\frac{\partial^2 f}{\partial xy}(t,x,x)\Big\}\\
&\quad+\frac{\partial f}{\partial t}(t,x,x)+[r(t)x(t)-\mu(t)+\boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial f}{\partial x}(t,x,x)\\
&\quad+\frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)]'\Big\}\frac{\partial^2 f}{\partial xx}(t,x,x)\\
&\quad-[r(t)x(t)-\mu(t)+\boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial G}{\partial x}(x,g)\\
&\quad-\frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)]'\Big\}\times\\
&\quad\Big\{\frac{\partial^2 G}{\partial xx}(x,g)-\frac{\partial^2 G}{\partial yy}(x,g)[\frac{\partial g}{\partial x}]^2+2\frac{\partial^2 G}{\partial xy}(x,g)[\frac{\partial g}{\partial x}]\Big\}\Bigg\}=0.
\end{aligned}\tag{4.39}$$

By further simplifying the above equation, we have

$$\begin{aligned}
&\frac{\partial V}{\partial t}+\sup_{u\in\mathcal{U}}\Bigg\{[r(t)x(t)-\mu(t)+\boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\{\frac{\partial V}{\partial x}-\frac{\partial f}{\partial y}(t,x,x)-\frac{\partial G}{\partial x}(x,g)\}\\
&\quad+\frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)-\boldsymbol{\nu}'(t)]'\Big\}\times\\
&\quad\Big\{\frac{\partial^2 V}{\partial xx}-\frac{\partial^2 f}{\partial yy}(t,x,x)-2\frac{\partial^2 f}{\partial xy}(t,x,x)-\frac{\partial^2 G}{\partial xx}(x,g)-2\frac{\partial^2 G}{\partial xy}(x,g)[\frac{\partial g}{\partial x}]-\frac{\partial^2 G}{\partial yy}(x,g)[\frac{\partial g}{\partial x}]^2\Big\}\Bigg\}=0.
\end{aligned}\tag{4.40}$$

Since,

$$\begin{aligned}
F(x,x(T)) &= x(T)-\frac{\gamma(x)}{2}[x(T)]^2,\\
G(x,g) &= \frac{\gamma(x)}{2}g^2,
\end{aligned}\tag{4.41}$$

we can derive the following results easily

$$\begin{aligned}
\frac{\partial G}{\partial x} &= \frac{d\gamma(x)}{2dx}g^2, \\
\frac{\partial^2 G}{\partial xx} &= \frac{d^2\gamma(x)}{2dxx}g^2, \\
\frac{\partial G}{\partial g} &= \gamma(x)g, \\
\frac{\partial^2 G}{\partial xg} &= \frac{d\gamma(x)}{dx}g, \\
\frac{\partial^2 G}{\partial gg} &= \gamma(x).
\end{aligned} \tag{4.42}$$

Substituting (4.42) into (4.40) gives us the first equation of the extended Hamilton-Jacobi-Bellman system. The second and the third equations can be obtained by operator A directly. □

4.3 Solution Scheme

4.3.1 The Case of a General $\gamma(x)$

Based on the extended Hamilton-Jacobi-Bellman system obtained in Theorem 4.2, we have the following system.

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \sup_{u\in\mathcal{U}}\Bigg\{[r(t)x(t) - \mu(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\{\frac{\partial V}{\partial x} - \frac{\partial f}{\partial y} - \frac{d\gamma(x)}{2dx}g^2\} \\
&\quad + \frac{1}{2}\Big\{\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t) - \boldsymbol{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t) + \boldsymbol{\nu}'(t)\boldsymbol{\nu}(t)\Big\}\times \\
&\quad \Big\{\frac{\partial^2 V}{\partial xx} - \frac{\partial^2 f}{\partial yy} - 2\frac{\partial^2 f}{\partial xy} - \frac{d^2\gamma(x)}{2dxx}g^2 - \frac{2d\gamma(x)}{dx}g[\frac{\partial g}{\partial x}] - \gamma(x)[\frac{\partial g}{\partial x}]^2\Big\}\Bigg\} = 0,
\end{aligned} \tag{4.43}$$

$$\begin{aligned}
&\frac{\partial f}{\partial t} + [r(t)x(t) - \mu(t) + \widehat{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial f}{\partial x} \\
&\quad + \frac{1}{2}\Big\{\widehat{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\widehat{\boldsymbol{u}}(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\widehat{\boldsymbol{u}}(t) - \widehat{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t) + \boldsymbol{\nu}'(t)\boldsymbol{\nu}(t)\Big\}\frac{\partial^2 f}{\partial xx} = 0,
\end{aligned} \tag{4.44}$$

$$\begin{aligned}
&\frac{\partial g}{\partial t} + [r(t)x(t) - \mu(t) + \widehat{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial g}{\partial x} \\
&\quad + \frac{1}{2}\Big\{\widehat{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\widehat{\boldsymbol{u}}(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\widehat{\boldsymbol{u}}(t) - \widehat{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t) + \boldsymbol{\nu}'(t)\boldsymbol{\nu}(t)\Big\}\frac{\partial^2 g}{\partial xx} = 0,
\end{aligned} \tag{4.45}$$

where we should address that $\widehat{u}$ is the optimal equilibrium control law and

$$
\begin{aligned}
f(T,x,x) &= x - \frac{\gamma(x)}{2}x^2, && (4.46)\\
g(T,x) &= x, && (4.47)\\
V(t,x) &= E_{t,x}[x^{\widehat{u}}(T)] - \frac{\gamma(x)}{2}Var_{t,x}[x^{\widehat{u}}(T)], && (4.48)\\
f(t,x,x) &= E_{t,x}[x^{\widehat{u}}(T)] - \frac{\gamma(x)}{2}E_{t,x}[(x^{\widehat{u}}(T))^2], && (4.49)\\
g(t,x) &= E_{t,x}[x^{\widehat{u}}(T)], && (4.50)\\
V(t,x) &= f(t,x,x) + \frac{\gamma(x)}{2}g^2(t,x). && (4.51)
\end{aligned}
$$

From (4.51), we calculate the partial derivatives of $V(t,x)$ with respect to t and x.

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{\partial f}{\partial t} + \gamma g\frac{\partial g}{\partial t}, && (4.52)\\
\frac{\partial V}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{d\gamma}{2dx}g^2 + \gamma g\frac{\partial g}{\partial x}, && (4.53)\\
\frac{\partial^2 V}{\partial xx} &= \frac{\partial^2 f}{\partial xx} + \frac{\partial^2 f}{\partial yy} + 2\frac{\partial^2 f}{\partial xy} + \frac{d^2\gamma}{2dx^2}g^2 + \frac{2d\gamma}{dx}g\frac{\partial g}{\partial x} + \gamma[\frac{\partial g}{\partial x}]^2 + \gamma g\frac{\partial^2 g}{\partial xx}. && (4.54)
\end{aligned}
$$

Substituting (4.52)-(4.54) into (4.43), we obtain

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \gamma g\frac{\partial g}{\partial t} + \sup_{u\in\mathcal{U}}\Bigg\{[r(t)x(t) - \mu(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)][\frac{\partial f}{\partial x} + \gamma g\frac{\partial g}{\partial x}]\\
&\quad + \frac{1}{2}\Big\{\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t) - \boldsymbol{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t) + \boldsymbol{\nu}'(t)\boldsymbol{\nu}(t)\Big\}[\frac{\partial^2 f}{\partial xx} + \gamma g\frac{\partial^2 g}{\partial xx}]\Bigg\} = 0.
\end{aligned} \tag{4.55}
$$

which can be rewritten as

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \gamma g\frac{\partial g}{\partial t} + \sup_{u\in\mathcal{U}}\Bigg\{[r(t)x(t) - \mu(t)][\frac{\partial f}{\partial x} + \gamma g\frac{\partial g}{\partial x}]\\
&\quad + \frac{1}{2}\boldsymbol{\nu}'(t)\boldsymbol{\nu}(t)[\frac{\partial^2 f}{\partial xx} + \gamma g\frac{\partial^2 g}{\partial xx}] + [\frac{\partial f}{\partial x} + \gamma g\frac{\partial g}{\partial x}]\boldsymbol{\alpha}'(t)\boldsymbol{u}(t)\\
&\quad + \frac{1}{2}\Big\{\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t)\Big\}[\frac{\partial^2 f}{\partial xx} + \gamma g\frac{\partial^2 g}{\partial xx}]\Bigg\} = 0.
\end{aligned} \tag{4.56}
$$

Since the LHS of (4.56) is a concave quadratic function of u, we can derive the equilibrium

control law by the first-order condition as follows

$$\begin{aligned}\widehat{\boldsymbol{u}}(t) &= -[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\times \\ &\quad \frac{[\frac{\partial f}{\partial x}(t,x,x)+\gamma(x)g(t,x)\frac{\partial g}{\partial x}(t,x)]\boldsymbol{\alpha}(t)-[\frac{\partial^2 f}{\partial xx}(t,x,x)+\gamma(x)g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)][\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t)]}{\frac{\partial^2 f}{\partial xx}(t,x,x)+\gamma(x)g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)}.\end{aligned} \tag{4.57}$$

Theorem 4.3. *Under the general state-dependent risk-aversion, the equilibrium control policy (optimal control policy) of the asset liability management is*

$$\begin{aligned}\widehat{\boldsymbol{u}}(t) &= -[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\times \\ &\quad \left\{\frac{[\frac{\partial f}{\partial x}(t,x,x)+\gamma(x)g(t,x)\frac{\partial g}{\partial x}(t,x)]\boldsymbol{\alpha}(t)}{\frac{\partial^2 f}{\partial xx}(t,x,x)+\gamma(x)g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)}-\frac{[\frac{\partial^2 f}{\partial xx}(t,x,x)+\gamma(x)g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)][\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t)]}{\frac{\partial^2 f}{\partial xx}(t,x,x)+\gamma(x)g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)}\right\},\end{aligned} \tag{4.58}$$

where $f(t,x,x)$ and $g(t,x)$ are governed by the partial differential equations (4.44)-(4.47).

Remark 4.2. Theorem 4.3 embraces the case of a constant risk aversion γ as a special case by simply replacing $\gamma(x)$ in (4.58) by the constant risk aversion γ.

Remark 4.3. The optimal control policy consists of two parts, i.e.,

$$-[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\frac{[\frac{\partial f}{\partial x}(t,x,x)+\gamma(x)g(t,x)\frac{\partial g}{\partial x}(t,x)]\boldsymbol{\alpha}(t)}{\frac{\partial^2 f}{\partial xx}(t,x,x)+\gamma(x)g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)},$$

and

$$-[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\frac{[\frac{\partial^2 f}{\partial xx}(t,x,x)+\gamma(x)g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)][\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t)]}{\frac{\partial^2 f}{\partial xx}(t,x,x)+\gamma(x)g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)}.$$

The first part in (4.58) has similar form as the results obtained by Björk et al. (2014); while the second part in (4.58) is from the liability.

4.3.2 The Case of a Natural Choice of $\gamma(x)$

Following the similar logic introduced by Björk et al. (2014), we use the same natural choice of $\gamma(x)=\frac{\gamma}{x}$ in our paper, where γ is the same as the constant used in the time-consistent

mean-variance research work, and x is the current wealth held by the investor. Hence,

$$\begin{aligned} \gamma(x) &= \frac{\gamma}{x}, \\ \frac{d\gamma(x)}{dx} &= -\frac{\gamma}{x^2}, \\ \frac{d^2\gamma(x)}{dxx} &= \frac{2\gamma}{x^3}. \end{aligned} \tag{4.59}$$

Substituting (4.59) into (4.56) and (4.57), we obtain the corresponding extended Hamilton-Jacobi-Bellman equation and the optimal policy as follows

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{\gamma}{x} g \frac{\partial g}{\partial t} + \sup_{u \in \mathcal{U}} \Bigg\{ [r(t)x(t) - \mu(t)][\frac{\partial f}{\partial x} + \frac{\gamma}{x} g \frac{\partial g}{\partial x}] \\ &\quad + \frac{1}{2} \boldsymbol{\nu}'(t)\boldsymbol{\nu}(t)[\frac{\partial^2 f}{\partial xx} + \frac{\gamma}{x} g \frac{\partial^2 g}{\partial xx}] + [\frac{\partial f}{\partial x} + \frac{\gamma}{x} g \frac{\partial g}{\partial x}]\boldsymbol{\alpha}'(t)\boldsymbol{u}(t) \\ &\quad + \frac{1}{2} \Big\{ \boldsymbol{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t) \Big\} [\frac{\partial^2 f}{\partial xx} + \frac{\gamma}{x} g \frac{\partial^2 g}{\partial xx}] \Bigg\} = 0, \end{aligned}$$

$$\begin{aligned} \widehat{\boldsymbol{u}}(t) &= -[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1} \times \\ &\quad \frac{[\frac{\partial f}{\partial x}(t,x,x) + \frac{\gamma}{x} g(t,x) \frac{\partial g}{\partial x}(t,x)]\boldsymbol{\alpha}(t) - [\frac{\partial^2 f}{\partial xx}(t,x,x) + \frac{\gamma}{x} g(t,x) \frac{\partial^2 g}{\partial xx}(t,x)][\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t)]}{\frac{\partial^2 f}{\partial xx}(t,x,x) + \frac{\gamma}{x} g(t,x) \frac{\partial^2 g}{\partial xx}(t,x)}. \end{aligned} \tag{4.60}$$

By the linear structure of the dynamics of $dx(t)$ and the boundary conditions, we guess that

$$\begin{aligned} E_{t,x}[x^{\widehat{u}}(T)] &= K(t)x + L(t), \\ E_{t,x}[(x^{\widehat{u}}(T))^2] &= M(t)x^2 + N(t)x + Q(t). \end{aligned} \tag{4.61}$$

Thus, we can obtain the expression of $f(t,x,y)$ and $g(t,x)$ accordingly.

$$\begin{aligned}
f(t,x,y) &= E_{t,x}[x^{\widehat{u}}(T)] - \frac{\gamma}{2y}E_{t,x}[(x^{\widehat{u}}(T))^2] \\
&= K(t)x + L(t) - \frac{\gamma}{2y}[M(t)x^2 + N(t)x + Q(t)] \\
&= -\frac{\gamma}{2y}M(t)x^2 + [K(t) - \frac{\gamma}{2y}N(t)]x + L(t) - \frac{\gamma}{2y}Q(t), \\
g(t,x) &= E_{t,x}[x^{\widehat{u}}(T)] = K(t)x + L(t), \\
V(t,x) &= f(t,x,x) + \frac{\gamma(x)}{2}g^2(t,x) \\
&= -\frac{\gamma}{2}M(t)x + [K(t)x - \frac{\gamma}{2}N(t)] + L(t) - \frac{\gamma}{2x}Q(t) + \frac{\gamma}{2x}[K(t)x + L(t)]^2 \\
&= \frac{\gamma}{2x}[L^2(t) - Q(t)] + \frac{\gamma x}{2}[K^2(t) - M(t)] + xK(t) + \gamma[L(t)K(t) - \frac{N(t)}{2}] + L(t).
\end{aligned} \tag{4.62}$$

With $\dot{a}$ denoting $\frac{da}{dt}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,x,y) &= -\frac{\gamma}{2y}\dot{M}(t)x^2 + [\dot{K}(t) - \frac{\gamma}{2y}\dot{N}(t)]x + \dot{L}(t) - \frac{\gamma}{2y}\dot{Q}(t), \\
\frac{\partial f}{\partial x}(t,x,y) &= -\frac{\gamma}{y}M(t)x + K(t) - \frac{\gamma}{2y}N(t), \\
\frac{\partial^2 f}{\partial xx}(t,x,y) &= -\frac{\gamma}{y}M(t), \\
\frac{\partial^2 f}{\partial xy}(t,x,y) &= \frac{\gamma}{y^2}M(t)x + \frac{\gamma}{2y^2}N(t), \\
\frac{\partial f}{\partial y}(t,x,y) &= \frac{\gamma}{2y^2}[M(t)x^2 + N(t)x + Q(t)], \\
\frac{\partial^2 f}{\partial yy}(t,x,y) &= -\frac{\gamma}{y^3}[M(t)x^2 + N(t)x + Q(t)], \\
\frac{\partial g}{\partial t}(t,x) &= \dot{K}(t)x + \dot{L}(t), \\
\frac{\partial g}{\partial x}(t,x) &= K(t), \\
\frac{\partial^2 g}{\partial xx}(t,x) &= 0.
\end{aligned} \tag{4.63}$$

Substituting (4.63) into (4.60) yields the equilibrium control policy as follows

$$\begin{aligned}
\widehat{\boldsymbol{u}}(t) &= [\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1} \times \\
&\quad \frac{[-\gamma x M(t) + xK(t) - \frac{\gamma}{2}N(t) + \gamma x K^2(t) + \gamma K(t)L(t)]\boldsymbol{\alpha}(t) + \gamma M(t)[\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t)]}{\gamma M(t)} \\
&= \mathbf{K}_1(t)x + \mathbf{K}_2(t),
\end{aligned} \tag{4.64}$$

where

$$\mathbf{K}_1(t) = [\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\frac{[\gamma[K^2(t)-M(t)]+K(t)]\boldsymbol{\alpha}(t)}{\gamma M(t)},$$

$$\mathbf{K}_2(t) = [\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\frac{[L(t)K(t)-\frac{N(t)}{2}]\boldsymbol{\alpha}(t)+M(t)\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t)}{M(t)}.$$

We should address here that $\mathbf{K}_1(t)$ and $\mathbf{K}_2(t)$ are both n-dimensional vectors. Substituting (4.63) and (4.64) into (4.44) and (4.45), we can obtain the following ordinary differential equations by separation of variables.

$$\dot{K}(t) + \mathbf{K}_1'(t)\boldsymbol{\alpha}(t)K(t) + r(t)K(t) = 0, \tag{4.65}$$

$$\dot{L}(t) + \mathbf{K}_2'(t)\boldsymbol{\alpha}(t)K(t) - \mu(t)K(t) = 0, \tag{4.66}$$

$$\dot{M}(t) + 2[\mathbf{K}_1'(t)\boldsymbol{\alpha}(t) + r(t)]M(t) + \mathbf{K}_1'(t)[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]\mathbf{K}_1(t)M(t) = 0, \tag{4.67}$$

$$\begin{aligned}\dot{N}(t) + [\mathbf{K}_1'(t)\boldsymbol{\alpha}(t) + r(t)]N(t) + 2M(t)[\mathbf{K}_2'(t)\boldsymbol{\alpha}(t) - \mu(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\mathbf{K}_1(t)\\ +\mathbf{K}_1'(t)[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]\mathbf{K}_2(t)] = 0,\end{aligned} \tag{4.68}$$

$$\begin{aligned}\dot{Q}(t) + [\mathbf{K}_2'(t)\boldsymbol{\alpha}(t) - \mu(t)]N(t) + \boldsymbol{\nu}'(t)\boldsymbol{\nu}(t)M(t) - 2M(t)\boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\mathbf{K}_2(t)\\ +\mathbf{K}_2'(t)[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]\mathbf{K}_2(t)M(t) = 0.\end{aligned} \tag{4.69}$$

By studying the variance of the terminal wealth, we obtain the following results

$$\begin{aligned}Var_{t,x}[x^U(T)] &= E_{t,x}[(x^{\widehat{u}}(T))^2] - E^2_{t,x}[x^{\widehat{u}}(T)]\\ &= [M(t) - K^2(t)]x^2 + [N(t) - 2K(t)L(t)]x + Q(t) - L^2(t).\end{aligned} \tag{4.70}$$

Theorem 4.4. *Under the natural choice of the state-dependent risk-aversion, the equilibrium control policy $\widehat{\boldsymbol{u}}(t)$, the equilibrium value function $V(t,x)$ under $\widehat{\boldsymbol{u}}(t)$, and the variance of ter-*

minal wealth under $\widehat{\boldsymbol{u}}(t)$ *are given by*

$$\begin{cases} \widehat{\boldsymbol{u}}(t) = \mathbf{K}_1(t)x + \mathbf{K}_2(t), \\ \\ V(t,x) = \frac{\gamma}{2x}[L^2(t) - Q(t)] + \frac{\gamma x}{2}[K^2(t) - M(t)] + xK(t) + \gamma[L(t)K(t) - \frac{N(t)}{2}] + L(t), \\ \\ Var_{t,x}[x^U(T)] = [M(t) - K^2(t)]x^2 + [N(t) - 2K(t)L(t)]x + Q(t) - L^2(t), \end{cases}$$

where $K(t)$*,* $L(t)$*,* $M(t)$*,* $N(t)$ *and* $Q(t)$ *are determined by the following ordinary differential equation system*

$$\begin{cases} \dot{K}(t) + \mathbf{K}_1'(t)\boldsymbol{\alpha}(t)K(t) + r(t)K(t) = 0, \\ \\ \dot{L}(t) + \mathbf{K}_2'(t)\boldsymbol{\alpha}(t)K(t) - \mu(t)K(t) = 0, \\ \\ \dot{M}(t) + 2[\mathbf{K}_1'(t)\boldsymbol{\alpha}(t) + r(t)]M(t) + \mathbf{K}_1'(t)[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]\mathbf{K}_1(t)M(t) = 0, \\ \\ \dot{N}(t) + [\mathbf{K}_1'(t)\boldsymbol{\alpha}(t) + r(t)]N(t) + 2M(t)[\mathbf{K}_2'(t)\boldsymbol{\alpha}(t) \\ -\mu(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\mathbf{K}_1(t) + \mathbf{K}_1'(t)[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]\mathbf{K}_2(t)] = 0, \\ \\ \dot{Q}(t) + [\mathbf{K}_2'(t)\boldsymbol{\alpha}(t) - \mu(t)]N(t) + \boldsymbol{\nu}'(t)\boldsymbol{\nu}(t)M(t) \\ -2M(t)\boldsymbol{\nu}'(t)\boldsymbol{\sigma}'(t)\mathbf{K}_2(t) + \mathbf{K}_2'(t)[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]\mathbf{K}_2(t)M(t) = 0, \\ \\ \mathbf{K}_1(t) = [\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\frac{[\gamma[K^2(t) - M(t)] + K(t)]\boldsymbol{\alpha}(t)}{\gamma M(t)}, \\ \\ \mathbf{K}_2(t) = [\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\frac{[L(t)K(t) - \frac{N(t)}{2}]\boldsymbol{\alpha}(t) + M(t)\boldsymbol{\sigma}(t)\boldsymbol{\nu}(t)}{M(t)}, \\ \\ K(T) = 1; \qquad L(T) = 0; \qquad M(T) = 1; \qquad N(T) = 0; \qquad Q(T) = 0. \end{cases}$$

Analytical results for the financial setting with only one risky asset

We assume that the dynamics of the risky asset is governed by

$$\begin{cases} dS_1(t) = S_1(t-)\Big(\alpha(t)dt + \boldsymbol{\sigma}_1'(t)dW(t)\Big), \quad 0 \le t \le T, \\ S_1(0) = s_1 > 0. \end{cases}$$

where s_1 is the initial price of the risky asset; $\alpha(t)$ and $\boldsymbol{\sigma}_1(t) = (\sigma_1(t), \sigma_2(t), \ldots, \sigma_m(t))'$ are respectively the appreciation rate scalar and the volatility vector, which are assumed to be positive continuous bounded deterministic functions of time t. The risk-free asset and the liability follow the same dynamics as in the original model.

Denote $\pi(t)$ as the amount of money invested in the risky asset at time t, then the investment in the risk-free asset at time t is $[x(t) - \pi(t)]$. The stochastic differential equation of $dx(t)$ is as follows

$$\begin{cases} dx(t) = [r(t)x(t) - \mu(t) + \pi(t)\alpha_1(t)]dt + [\pi(t)\boldsymbol{\sigma}_1'(t) - \boldsymbol{\nu}'(t)]d\mathbf{W}(t), \\ x(0) = x_0 > 0. \end{cases}$$

where x_0 is the initial wealth, $\alpha_1(t) = \alpha(t) - r(t)$. By applying the sub-game perfect Nash equilibrium strategy, we obtain the analytical results of the equilibrium control policy, the equilibrium value function and the variance versus expectation of terminal wealth under the case of a natural choice of $\gamma(x) = \frac{\gamma}{x}$, as given below

$$\begin{cases} \widehat{\pi}(t) = K_1(t)x + K_2(t), \\ \\ V(t,x) = \frac{\gamma}{2x}[L^2(t) - Q(t)] + \frac{\gamma x}{2}[K^2(t) - M(t)] + xK(t) + \gamma[L(t)K(t) - \frac{N(t)}{2}] + L(t), \\ \\ Var_{t,x}[x^U(T)] = [M(t) - K^2(t)]x^2 + [N(t) - 2K(t)L(t)]x + Q(t) - L^2(t), \end{cases} \tag{4.71}$$

where $K_1(t)$ and $K_2(t)$ are scalars rather than vectors.

$$K_1(t) = \frac{[\gamma[K^2(t) - M(t)] + K(t)]\alpha_1(t)}{\gamma M(t)[\boldsymbol{\sigma}_1'(t)\boldsymbol{\sigma}_1(t)]},$$

$$K_2(t) = \frac{[L(t)K(t) - \frac{N(t)}{2}]\alpha_1(t) + M(t)\boldsymbol{\sigma}_1'(t)\boldsymbol{\nu}(t)}{M(t)[\boldsymbol{\sigma}_1'(t)\boldsymbol{\sigma}_1(t)]}.$$

and $K(t)$, $L(t)$, $M(t)$, $N(t)$ and $Q(t)$ are determined by the following ordinary differential equation system.

$$\begin{cases} \dot{K}(t) + K_1(t)\alpha_1(t)K(t) + r(t)K(t) = 0, \\ \\ \dot{L}(t) + K_2(t)\alpha_1(t)K(t) - \mu(t)K(t) = 0, \\ \\ \dot{M}(t) + 2[K_1(t)\alpha_1(t) + r(t)]M(t) + K_1(t)[\boldsymbol{\sigma}_1'(t)\boldsymbol{\sigma}_1(t)]K_1(t)M(t) = 0, \\ \\ \dot{N}(t) + [K_1(t)\alpha_1(t) + r(t)]N(t) + 2M(t)[K_2(t)\alpha(t) \\ -\mu(t) - \boldsymbol{\nu}'(t)\boldsymbol{\sigma}_1(t)K_1(t) + K_1(t)[\boldsymbol{\sigma}_1'(t)\boldsymbol{\sigma}_1(t)]K_2(t)] = 0, \\ \\ \dot{Q}(t) + [K_2(t)\alpha_1(t) - \mu(t)]N(t) + \boldsymbol{\nu}'(t)\boldsymbol{\nu}(t)M(t) \\ -2M(t)\boldsymbol{\nu}'(t)\boldsymbol{\sigma}_1(t)K_2(t) + K_2(t)[\boldsymbol{\sigma}_1'(t)\boldsymbol{\sigma}_1(t)]K_2(t)M(t) = 0, \\ \\ K(T) = 1; \qquad L(T) = 0; \qquad M(T) = 1; \qquad N(T) = 0; \qquad Q(T) = 0. \end{cases} \tag{4.72}$$

4.4 Special Case: The Case with no Liability

In the case with no liability, the original asset liability management model degenerates to a portfolio selection model. The wealth held by the investor evolves according to the following stochastic differential equation.

$$\begin{cases} dx(t) = [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]dt + \boldsymbol{u}'(t)\boldsymbol{\sigma}(t)d\mathbf{W}(t), \quad 0 \le t \le T, \\ x(0) = x_0 > 0, \end{cases} \tag{4.73}$$

where

$$\begin{cases} \boldsymbol{u}(t) = (u_1(t), u_2(t), \ldots, u_n(t))', \\ \boldsymbol{\alpha}(t) = (\alpha_1(t) - r(t), \alpha_2(t) - r(t), \ldots, \alpha_n(t) - r(t))', \\ \boldsymbol{\sigma}_i(t) = (\sigma_{i1}(t), \sigma_{i2}(t), \ldots, \sigma_{im}(t)), \\ \boldsymbol{\sigma}(t) = (\sigma_1'(t), \sigma_2'(t), \ldots, \sigma_n'(t))'. \end{cases}$$

The objective function $J(t,x,u)$ and the mean-variance problem $V(t,x(t))$ are described mathematically as follows

$$J(t,x,u) = E[x(T)] - \frac{\gamma(x)}{2} Var[x(T)], \tag{4.74}$$

$$V(t,x(t)) = \max_{u(.)\in\mathcal{U}[0,\ T]} \Big\{ E_{t,x}[x^U(T)] - \frac{\gamma(x)}{2} Var_{t,x}[x^U(T)] \Big\}. \tag{4.75}$$

where $E_{t,x}[\cdot] = E[\cdot \mid x^U(t) = x]$. Now we provide the definition of an optimal equilibrium control. Let A be the infinitesimal generator. For any fixed $u \in \mathcal{U}$, we introduce the corresponding controlled infinitesimal generator A^u defined by

$$A^u = \frac{\partial}{\partial t} + [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial}{\partial x} + \frac{1}{2}\Big\{ [\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)] \times [\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]' \Big\} \frac{\partial^2}{\partial x^2}. \tag{4.76}$$

By making use of the definition of equilibrium control as given in Definition 4.1 and the infinitesimal generator A^u in (4.76), following the application in Björk and Murgoci (2010), we derive the extended Hamilton-Jacobi-Bellman equations and its verification theorem as follows.

Theorem 4.5 (Verification Theorem)**.** *Assume that (V, f, g) is a solution of the following extended Hamilton-Jacobi-Bellman system, and the control law $\widehat{u}$ realizes the supremum in the equation.*

$$\begin{cases} \sup\limits_{u\in\mathcal{U}} \Big\{ A^u V(t,x) - A^u f(t,x,x) + A^u f^x(t,x) \\ -A^u (G\circ g)(t,x) + H^u g(t,x) \Big\} = 0, \quad 0 \le t \le T, \\ A^{\widehat{u}} f^y(t,x) = 0, \quad 0 \le t \le T, \\ A^{\widehat{u}} g(t,x) = 0, \quad 0 \le t \le T, \end{cases}$$

with the following conditions and definitions

$$\begin{cases} V(T,x) = F(x,x) + G(x,x), \\ f(T,x,y) = F(y,x), \\ g(T,x) = x, \\ f^y(t,x) = f(t,x,y), \\ (G\circ g)(t,x) = G(x, g(t,x)), \\ H^u g(t,x) = G_y(x, g(t,x)) \times A^u g(t,x), \\ G_y(x,y) = \frac{\partial G(x,y)}{\partial y}. \end{cases}$$

Then $\widehat{u}$ is an equilibrium control law, and V is the corresponding value function. Moreover, f and g have the following probabilistic representations:

$$
\begin{aligned}
f(t,x,y) &= E_{t,x}[F(y, x^{\widehat{u}}(T))], && (4.77)\\
g(t,x) &= E_{t,x}[x^{\widehat{u}}(T)], && (4.78)\\
F(y,x(T)) &= x(T) - \frac{\gamma(y)}{2}[x(T)]^2, && (4.79)\\
G(y,g) &= \frac{\gamma(y)}{2}g^2. && (4.80)
\end{aligned}
$$

We omit the proof here, as the proof is similar to the proof of Theorem 4.1. Next, we derive the extended Hamilton-Jacobi-Bellman system for portfolio selection optimization problem.

Theorem 4.6. *The extended Hamilton-Jacobi-Bellman equations for the Nash equilibrium problem with consideration of multiple risk assets and liability are as follows*

$$
\begin{cases}
\frac{\partial V}{\partial t} + \sup\limits_{u\in\mathcal{U}} \Bigg\{ [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\{\frac{\partial V}{\partial x} - \frac{\partial f}{\partial y}(t,x,x) - \frac{d\gamma(x)}{2dx}g^2\} \\
+\frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)] \times [\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\times \\
\Big\{\frac{\partial^2 V}{\partial xx} - \frac{\partial^2 f}{\partial yy}(t,x,x) - 2\frac{\partial^2 f}{\partial xy}(t,x,x) - \frac{d^2\gamma(x)}{2dxx}g^2 - 2[\frac{d\gamma(x)}{dx}]g[\frac{\partial g}{\partial x}] - \gamma(x)[\frac{\partial g}{\partial x}]^2\Big\}\Bigg\} = 0, \quad 0 \le t \le T, \\
\\
\frac{\partial f}{\partial t}(t,x,y) + [r(t)x(t) + \widehat{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial f}{\partial x}(t,x,y) \\
+\frac{1}{2}\Big\{[\widehat{u}'(t)\boldsymbol{\sigma}(t)] \times [\widehat{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\frac{\partial^2 f}{\partial xx}(t,x,y) = 0, \quad 0 \le t \le T, \\
\\
\frac{\partial g}{\partial t}(t,x) + [r(t)x(t) + \widehat{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial g}{\partial x}(t,x) \\
+\frac{1}{2}\Big\{[\widehat{u}'(t)\boldsymbol{\sigma}(t)] \times [\widehat{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\frac{\partial^2 g}{\partial xx}(t,x) = 0, \quad 0 \le t \le T.
\end{cases}
$$

Proof.

$$A^u V(t,x) = \frac{\partial V}{\partial t} + [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial V}{\partial x} + \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\frac{\partial^2 V}{\partial xx}, \tag{4.81}$$

$$\begin{aligned} A^u f(t,x,x) =& \frac{\partial f}{\partial t}(x,x) + [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\{\frac{\partial f}{\partial x}(t,x,x) + \frac{\partial f}{\partial y}(t,x,x)\} \\ &+ \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\Big\{\frac{\partial^2 f}{\partial xx}(t,x,x) + \frac{\partial^2 f}{\partial yy}(t,x,x) + 2\frac{\partial^2 f}{\partial xy}(t,x,x)\Big\}, \end{aligned} \tag{4.82}$$

$$\begin{aligned} A^u f^x(t,x) =& \frac{\partial f}{\partial t}(t,x,x) + [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial f}{\partial x}(t,x,x) \\ &+ \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\frac{\partial^2 f}{\partial xx}(t,x,x), \end{aligned} \tag{4.83}$$

$$\begin{aligned} A^u(G\circ g)(t,x) =& H^u g(t,x) + [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial G}{\partial x}(x,g) \\ &+ \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\times \\ &\Big\{\frac{\partial^2 G}{\partial xx}(x,g) + \frac{\partial^2 G}{\partial yy}(x,g)[\frac{\partial g}{\partial x}]^2 + 2\frac{\partial^2 G}{\partial xy}(x,g)[\frac{\partial g}{\partial x}]\Big\}. \end{aligned} \tag{4.84}$$

With the results (4.81) - (4.84), the LHS of the first equation of the extended Hamilton-Jacobi-Bellman system could be rearranged as

$$\begin{aligned} \sup_{u\in\mathcal{U}}\Bigg\{ & \frac{\partial V}{\partial t} + [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial V}{\partial x} + \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\frac{\partial^2 V}{\partial xx} \\ & - \frac{\partial f}{\partial t}(x,x) - [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\{\frac{\partial f}{\partial x}(t,x,x) + \frac{\partial f}{\partial y}(t,x,x)\} \\ & - \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\Big\{\frac{\partial^2 f}{\partial xx}(t,x,x) + \frac{\partial^2 f}{\partial yy}(t,x,x) + 2\frac{\partial^2 f}{\partial xy}(t,x,x)\Big\} \\ & + \frac{\partial f}{\partial t}(t,x,x) + [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial f}{\partial x}(t,x,x) \\ & + \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\frac{\partial^2 f}{\partial xx}(t,x,x) \\ & - [r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\frac{\partial G}{\partial x}(x,g) \\ & - \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\}\Big\{\frac{\partial^2 G}{\partial xx}(x,g) - \frac{\partial^2 G}{\partial yy}(x,g)[\frac{\partial g}{\partial x}]^2 + 2\frac{\partial^2 G}{\partial xy}(x,g)[\frac{\partial g}{\partial x}]\Big\}\Bigg\} = 0. \end{aligned}$$

By further simplifying the above equation, we have

$$\frac{\partial V}{\partial t} + \sup_{u\in\mathcal{U}}\Bigg\{[r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)]\{\frac{\partial V}{\partial x} - \frac{\partial f}{\partial y}(t,x,x) - \frac{\partial G}{\partial x}(x,g)\}$$
$$+ \frac{1}{2}\Big\{[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)][\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)]'\Big\} \times$$
$$\Big\{\frac{\partial^2 V}{\partial xx} - \frac{\partial^2 f}{\partial yy}(t,x,x) - 2\frac{\partial^2 f}{\partial xy}(t,x,x) - \frac{\partial^2 G}{\partial xx}(x,g) - 2\frac{\partial^2 G}{\partial xy}(x,g)[\frac{\partial g}{\partial x}] - \frac{\partial^2 G}{\partial yy}(x,g)[\frac{\partial g}{\partial x}]^2\Big\}\Bigg\} = 0. \tag{4.85}$$

Since,

$$\begin{aligned} F(x,x(T)) &= x(T) - \frac{\gamma(x)}{2}[x(T)]^2, \\ G(x,g) &= \frac{\gamma(x)}{2}g^2, \end{aligned} \tag{4.86}$$

we have the following results

$$\begin{aligned} \frac{\partial G}{\partial x} &= \frac{d\gamma(x)}{2dx}g^2, \\ \frac{\partial^2 G}{\partial xx} &= \frac{d^2\gamma(x)}{2dxx}g^2, \\ \frac{\partial G}{\partial g} &= \gamma(x)g, \\ \frac{\partial^2 G}{\partial xg} &= \frac{d\gamma(x)}{dx}g, \\ \frac{\partial^2 G}{\partial gg} &= \gamma(x). \end{aligned} \tag{4.87}$$

Substituting (4.87) into (4.85) proves Theorem 4.6. The second and the third equations can be obtained by operator A directly.

□

Solution Scheme: The Case of a General $\gamma(x)$

According to the extended Hamilton-Jacobi-Bellman system in Theorem 4.6 and the relationship

between $V(t,\ x)$, f and g as given below

$$\begin{aligned} f(T,x,x) &= x - \frac{\gamma(x)}{2}x^2, && (4.88)\\ g(T,x) &= x, && (4.89)\\ V(t,x) &= E_{t,x}[x^{\widehat{u}}(T)] - \frac{\gamma(x)}{2}Var_{t,x}[x^{\widehat{u}}(T)], && (4.90)\\ f(t,x,x) &= E_{t,x}[x^{\widehat{u}}(T)] - \frac{\gamma(x)}{2}E_{t,x}[(x^{\widehat{u}}(T))^2], && (4.91)\\ g(t,x) &= E_{t,x}[x^{\widehat{u}}(T)], && (4.92)\\ V(t,x) &= f(t,x,x) + \frac{\gamma(x)}{2}g^2(t,x). && (4.93) \end{aligned}$$

We calculate the partial derivatives of $V(t,x)$ with respect to t and x as follows

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{\partial f}{\partial t} + \gamma g\frac{\partial g}{\partial t}, && (4.94)\\ \frac{\partial V}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{d\gamma}{2dx}g^2 + \gamma g\frac{\partial g}{\partial x}, && (4.95)\\ \frac{\partial^2 V}{\partial xx} &= \frac{\partial^2 f}{\partial xx} + \frac{\partial^2 f}{\partial yy} + 2\frac{\partial^2 f}{\partial xy} + \frac{d^2\gamma}{2dxx}g^2 + 2[\frac{d\gamma}{dx}]g\frac{\partial g}{\partial x} + \gamma[\frac{\partial g}{\partial x}]^2 + \gamma g\frac{\partial^2 g}{\partial xx}. && (4.96) \end{aligned}$$

Substituting (4.94)-(4.96) into the first equation of the extended Hamilton-Jacobi-Bellman system in Theorem 4.6, we obtain

$$\begin{aligned} &\frac{\partial f}{\partial t} + \gamma g\frac{\partial g}{\partial t} + \sup_{u\in\mathcal{U}}\Bigg\{[r(t)x(t) + \boldsymbol{u}'(t)\boldsymbol{\alpha}(t)][\frac{\partial f}{\partial x} + \gamma g\frac{\partial g}{\partial x}]\\ &\qquad + \frac{1}{2}[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t)][\frac{\partial^2 f}{\partial xx} + \gamma g\frac{\partial^2 g}{\partial xx}]\Bigg\} = 0. \end{aligned} \tag{4.97}$$

Now, we rewrite the above equation as follows

$$\begin{aligned} &\frac{\partial f}{\partial t} + \gamma g\frac{\partial g}{\partial t} + \sup_{u\in\mathcal{U}}\Bigg\{[r(t)x(t)][\frac{\partial f}{\partial x} + \gamma g\frac{\partial g}{\partial x}]\\ &\quad + [\frac{\partial f}{\partial x} + \gamma g\frac{\partial g}{\partial x}]\boldsymbol{\alpha}'(t)\boldsymbol{u}(t) + \frac{1}{2}[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t)][\frac{\partial^2 f}{\partial xx} + \gamma g\frac{\partial^2 g}{\partial xx}]\Bigg\} = 0. \end{aligned} \tag{4.98}$$

Hence, the first-order condition for $\boldsymbol{u}(t)$ gives the corresponding optimal control policy, as given in the theorem below.

Theorem 4.7. *Under the general state-dependent risk-aversion, the equilibrium control policy (the optimal control policy) of portfolio selection is*

$$\widehat{\boldsymbol{u}}(t) = -[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1} \times \frac{[\frac{\partial f}{\partial x}(t,x,x) + \gamma(x)g(t,x)\frac{\partial g}{\partial x}(t,x)]\boldsymbol{\alpha}(t)}{\frac{\partial^2 f}{\partial xx}(t,x,x) + \gamma(x)g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)}, \tag{4.99}$$

where $f(t,x,x)$ and $g(t,x)$ are governed by partial equations (4.88), (4.89), and the second and third equations in the extended Hamilton-Jacobi-Bellman system in Theorem 4.6.

Remark 4.4. Theorem 4.7 embraces the portfolio selection with a constant risk aversion γ as a special case by simply replacing $\gamma(x)$ in (4.99) by the constant risk aversion γ.

Solution Scheme: The Case of a Natural Choice of $\gamma(x) = \frac{\gamma}{x}$

Since,

$$\begin{aligned} \gamma(x) &= \frac{\gamma}{x}, \\ \frac{d\gamma(x)}{dx} &= -\frac{\gamma}{x^2}, \\ \frac{d^2\gamma(x)}{dxx} &= \frac{2\gamma}{x^3}. \end{aligned} \tag{4.100}$$

Substituting (4.100) into (4.99) and (4.98), we obtain the corresponding extended Hamilton-Jacobi-Bellman equation and the optimal control policy as follows

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{\gamma}{x}g\frac{\partial g}{\partial t} + \sup_{u\in\mathcal{U}}\Bigg\{ & [r(t)x(t)][\frac{\partial f}{\partial x} + \frac{\gamma}{x}g\frac{\partial g}{\partial x}] + [\frac{\partial f}{\partial x} + \frac{\gamma}{x}g\frac{\partial g}{\partial x}]\boldsymbol{\alpha}'(t)\boldsymbol{u}(t) \\ & + \frac{1}{2}[\boldsymbol{u}'(t)\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\boldsymbol{u}(t)][\frac{\partial^2 f}{\partial xx} + \frac{\gamma}{x}g\frac{\partial^2 g}{\partial xx}]\Bigg\} = 0, \end{aligned} \tag{4.101}$$

$$\widehat{\boldsymbol{u}}(t) = -[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1} \times \frac{[\frac{\partial f}{\partial x}(t,x,x) + \frac{\gamma}{x}g(t,x)\frac{\partial g}{\partial x}(t,x)]\boldsymbol{\alpha}(t)}{\frac{\partial^2 f}{\partial xx}(t,x,x) + \frac{\gamma}{x}g(t,x)\frac{\partial^2 g}{\partial xx}(t,x)}. \tag{4.102}$$

Now we make the Ansatz

$$\widehat{u}(t,x) = k(t)x, \tag{4.103}$$

where $k(t)$ is a n-dimensional vector of deterministic functions of time t. Then $x(t)$ will be a

geometric Brownian motion so we have the results as follows

$$\begin{aligned} E_{t,x}[x^{\widehat{u}}(T)] &= m(t)x, \\ E_{t,x}[(x^{\widehat{u}}(T))^2] &= n(t)x^2. \end{aligned} \tag{4.104}$$

Thus, we can obtain the expression of $f(t,x,y)$ and $g(t,x)$ accordingly as follows

$$\begin{aligned} f(t,x,y) &= E_{t,x}[x^{\widehat{u}}(T)] - \frac{\gamma}{2y}E_{t,x}[(x^{\widehat{u}}(T))^2] = m(t)x - \frac{\gamma}{2y}n(t)x^2, \\ g(t,x) &= E_{t,x}[x^{\widehat{u}}(T)] = m(t)x, \\ V(t,x) &= f(t,x,x) + \frac{\gamma(x)}{2}g^2(t,x) \\ &= \frac{\gamma}{2}m^2(t)x^2 - \frac{\gamma}{2}n(t)x + m(t)x. \end{aligned} \tag{4.105}$$

Hence,

$$\begin{aligned} \frac{\partial f}{\partial t}(t,x,y) &= -\frac{\gamma}{2y}\dot{n}(t)x^2 + \dot{m}(t)x, \\ \frac{\partial f}{\partial x}(t,x,y) &= -\frac{\gamma}{y}n(t)x + m(t), \\ \frac{\partial^2 f}{\partial xx}(t,x,y) &= -\frac{\gamma}{y}n(t), \\ \frac{\partial^2 f}{\partial xy}(t,x,y) &= \frac{\gamma}{y^2}n(t)x, \\ \frac{\partial f}{\partial y}(t,x,y) &= \frac{\gamma}{2y^2}n(t)x^2, \\ \frac{\partial^2 f}{\partial yy}(t,x,y) &= -\frac{\gamma}{y^3}n(t)x^2, \\ \frac{\partial g}{\partial t}(t,x) &= \dot{m}(t)x, \\ \frac{\partial g}{\partial x}(t,x) &= m(t), \\ \frac{\partial^2 g}{\partial xx}(t,x) &= 0. \end{aligned} \tag{4.106}$$

Substituting (4.106) into (4.102) yields the optimal control policy as follows

$$\widehat{\boldsymbol{u}}(t) = [\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1} \times \frac{[xm(t) - \gamma x n(t) + \gamma x m^2(t)]\boldsymbol{\alpha}(t)}{\gamma n(t)}. \tag{4.107}$$

Substituting (4.104) and (4.105) into the second and the third equations in the extended

Hamilton-Jacobi-Bellman system in Theorem 4.6, we obtain the following ordinary differential equations by the separation of variables

$$\dot{m}(t) + \Big\{ r(t) + \eta(t) \frac{[m(t) - \gamma n(t) + \gamma m^2(t)]}{\gamma n(t)} \Big\} m(t) = 0, \tag{4.108}$$

$$\dot{n}(t) + \Big\{ r(t) + \eta(t) \frac{[m(t) - \gamma n(t) + \gamma m^2(t)]}{\gamma n(t)} \Big\} n(t) + \eta(t) \Big\{ \frac{[m(t) - \gamma n(t) + \gamma m^2(t)]}{\gamma n(t)} \Big\}^2 = 0, \tag{4.109}$$

$$m(T) = 1; \qquad n(T) = 1. \tag{4.110}$$

where $\eta(t) = \boldsymbol{\alpha}'(t)[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\boldsymbol{\alpha}(t)$. Then, by studying the variance of the terminal wealth, we obtain the results as follows

$$Var_{t,x}[x^U(T)] = E_{t,x}[(x^{\widehat{u}}(T))^2] - E^2_{t,x}[x^{\widehat{u}}(T)] = [n(t) - m^2(t)]x^2. \tag{4.111}$$

Theorem 4.8. *Under the natural choice of state-dependent risk-aversion, the equilibrium control policy (the optimal control policy) $\widehat{\boldsymbol{u}}(t)$, the equilibrium value function $V(t,x)$ under $\widehat{\boldsymbol{u}}(t)$, and the variance of terminal wealth $Var_{t,x}[x^U(T)]$ under $\widehat{\boldsymbol{u}}(t)$ are given by*

$$\begin{cases} \widehat{\boldsymbol{u}}(t) = [\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1} \times \frac{[xm(t) - \gamma x n(t) + \gamma x m^2(t)]\boldsymbol{\alpha}(t)}{\gamma n(t)}, \\ V(t,x) = \frac{\gamma}{2} m^2(t) x^2 - \frac{\gamma}{2} n(t) x + m(t) x, \\ Var_{t,x}[x^U(T)] = [n(t) - m^2(t)]x^2. \end{cases}$$

where $m(t)$ and $n(t)$ are determined by the following ordinary differential equation system

$$\begin{cases} \dot{m}(t) + \Big\{ r(t) + \eta(t) \frac{[m(t) - \gamma n(t) + \gamma m^2(t)]}{\gamma n(t)} \Big\} m(t) = 0, \\ \dot{n}(t) + \Big\{ r(t) + \eta(t) \frac{[m(t) - \gamma n(t) + \gamma m^2(t)]}{\gamma n(t)} \Big\} n(t) + \eta(t) \Big\{ \frac{[m(t) - \gamma n(t) + \gamma m^2(t)]}{\gamma n(t)} \Big\}^2 = 0, \\ \eta(t) = \boldsymbol{\alpha}'(t)[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}\boldsymbol{\alpha}(t); \qquad m(T) = 1; \qquad n(T) = 1. \end{cases}$$

Remark 4.5. Now, we are to discuss the possibility of zero variance of terminal wealth. From (4.111), we find that the variance of terminal wealth may reduce to zero at the point $n(t) = m^2(t)$. According to the ordinary differential equations for $m(t)$ and $n(t)$ with the boundary conditions

in Theorem 4.8, we obtain the following results

$$\dot{m}(t)+\Big\{r(t)+\eta(t)\frac{[m(t)-\gamma n(t)+\gamma m^2(t)]}{\gamma n(t)}\Big\}m(t)=\dot{m}(t)+\Big\{r(t)+\frac{\eta(t)}{\gamma m(t)}\Big\}m(t)=0, \quad (4.112)$$

$$\begin{aligned}&\dot{n}(t)+\Big\{r(t)+\eta(t)\tfrac{[m(t)-\gamma n(t)+\gamma m^2(t)]}{\gamma n(t)}\Big\}n(t)+\eta(t)\Big\{\tfrac{[m(t)-\gamma n(t)+\gamma m^2(t)]}{\gamma n(t)}\Big\}^2\\&=2m(t)\dot{m}(t)+\Big\{r(t)+\tfrac{\eta(t)}{\gamma m(t)}\Big\}m^2(t)+\tfrac{\eta(t)}{\gamma^2 m^2(t)}=0.\end{aligned} \quad (4.113)$$

Substituting (4.112) into (4.113), we have

$$\begin{aligned}m(t)\dot{m}(t)+\frac{\eta(t)}{\gamma^2 m^2(t)} &= 0,\\ m^3(t)\dot{m}(t)+\frac{\eta(t)}{\gamma^2} &= 0.\end{aligned} \quad (4.114)$$

The analytical expression for $m(t)$ can be calculated by solving the ordinary differential equation (4.114) and we have $m^4(t)=1+\int_t^T \frac{4\eta(t)}{\gamma^2}dt$. So the variance reduces to zero at

$$n(t) = \sqrt{1+\int_t^T \frac{4\eta(t)}{\gamma^2}dt}, \quad (4.115)$$

$$m(t) = \sqrt[4]{1+\int_t^T \frac{4\eta(t)}{\gamma^2}dt}. \quad (4.116)$$

Remark 4.6. Compare our results, the optimal control strategy $\widehat{\boldsymbol{u}}(t)$ and the equilibrium value function $V(t,x)$, with the results obtained by Björk et al. (2014). When $\boldsymbol{\alpha}(t)$ and $[\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)]^{-1}$ are replaced by the one-dimensional time-dependent deterministic functions, our results are the same as Björk et al. (2014).

Remark 4.7. The optimal control strategy $\widehat{\boldsymbol{u}}(t)$, the equilibrium value function $V(t,x)$ and the variance of terminal wealth under the equilibrium control policy $Var_{t,x}[x^U(T)]$ are all dependent on the current wealth held by the investor because we take state-dependent risk aversion into account.

4.5 Numerical Investigation

In this section, we carry out numerical investigations to analyze (i) the effect of liability and (ii) the effect of multiple risky assets on the equilibrium control policy (optimal investment

strategy), the equilibrium value function and the variance versus expectation of terminal wealth based on the theoretical results we obtained under the case of risk aversion function $\gamma(x) = \frac{\gamma}{x}$.

For convenience, but without loss of generality, we consider the cases with one risky-free asset and three risky assets. We assume that all the parameters are constants in this section and the values are given in Table 4.1 and (4.117) and (4.118).

Table 4.1: Parameter values for the original model.

Parameter	Symbol	Value
Exit Time	T	5
Initial Wealth	$x(0)$	10
Risk-free Interest Rate	$r(t)$	0.05
Liability Appreciation Rate	$\mu(t)$	0.07
Appreciation Rate of Risky Asset One	$\alpha(t)_1$	0.08
Appreciation Rate of Risky Asset Two	$\alpha(t)_2$	0.10
Appreciation Rate of Risky Asset Three	$\alpha(t)_3$	0.12
State-Dependent Risk Aversion	$\frac{\gamma}{x(t)}$	$\frac{6}{x(t)}$

The asset volatility matrix is

$$\boldsymbol{\sigma}(t) = \begin{pmatrix} 0.19 & 0.18 & 0.23 \\ 0.18 & 0.25 & 0.21 \\ 0.23 & 0.20 & 0.16 \end{pmatrix}. \tag{4.117}$$

The liability volatility vector is

$$\boldsymbol{\nu}(t) = (0.11, 0.15, 0.09)'. \tag{4.118}$$

Figure 4.1 shows the wealth process from time $t = 0$ to exit time $t = T$. It has a clear increasing trend with fluctuations due to the random factors embedded in risky assets and liability.

The equilibrium investment strategy for the three risky assets and the risk-free asset are depicted in subgraphs (a), (b), (c) and (d) in Figure 4.2 and Figure 4.3. The subgraphs (a) and (b) illustrate that the investments in risky asset one and risky asset two decrease with respect to time t. Whereas, the investments in risky asset three and the risk-free asset show an increasing trend in the subgraphs (c) and (d). In addition, the investment strategy u_1 is always negative and the other three investment strategies, u_2, u_3 and u_0, are always positive. This means (i) the

investor should sell short risky asset one and invest it together with the net wealth that he/she holds at time t in the other three assets, and (ii) the borrowed amount of money increases with respect to time t.

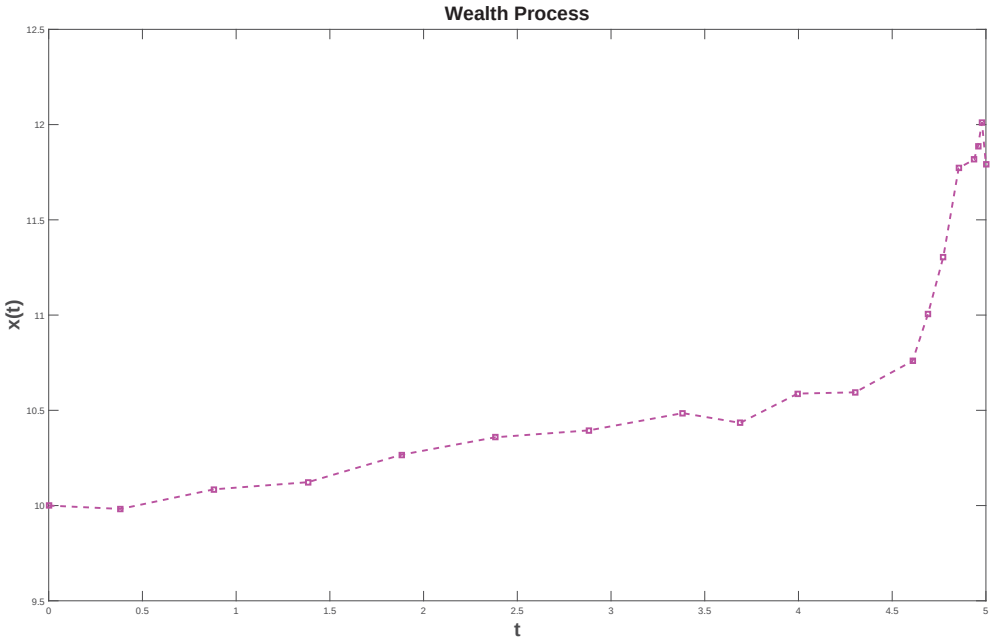

Figure 4.1: The wealth process versus time for the asset liability management model

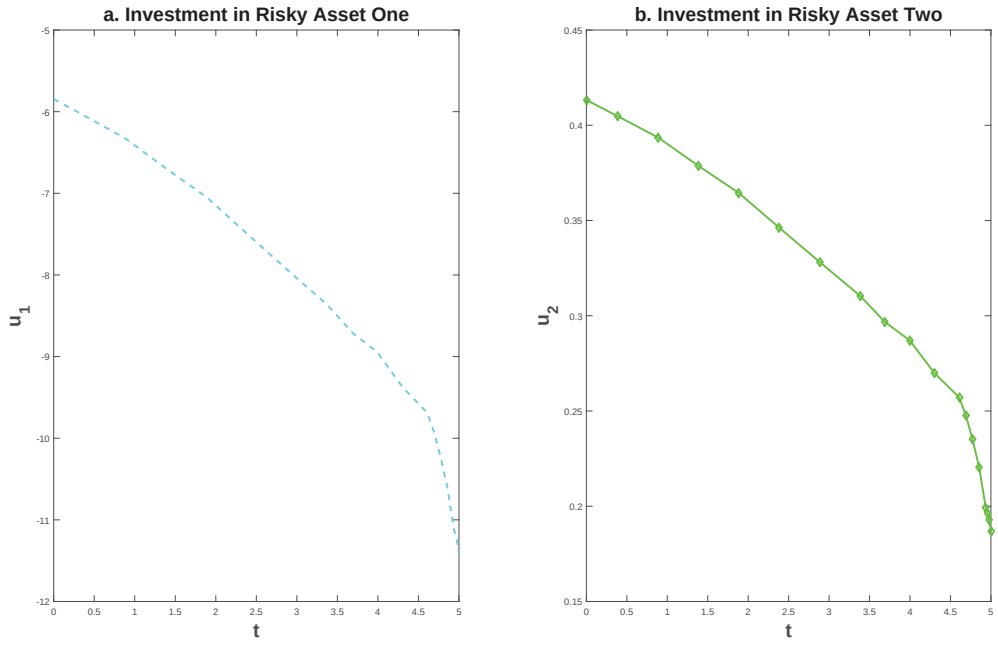

Figure 4.2: The equilibrium control policy (optimal investment strategy) of the risky asset one and the risky asset two for the asset liability management model

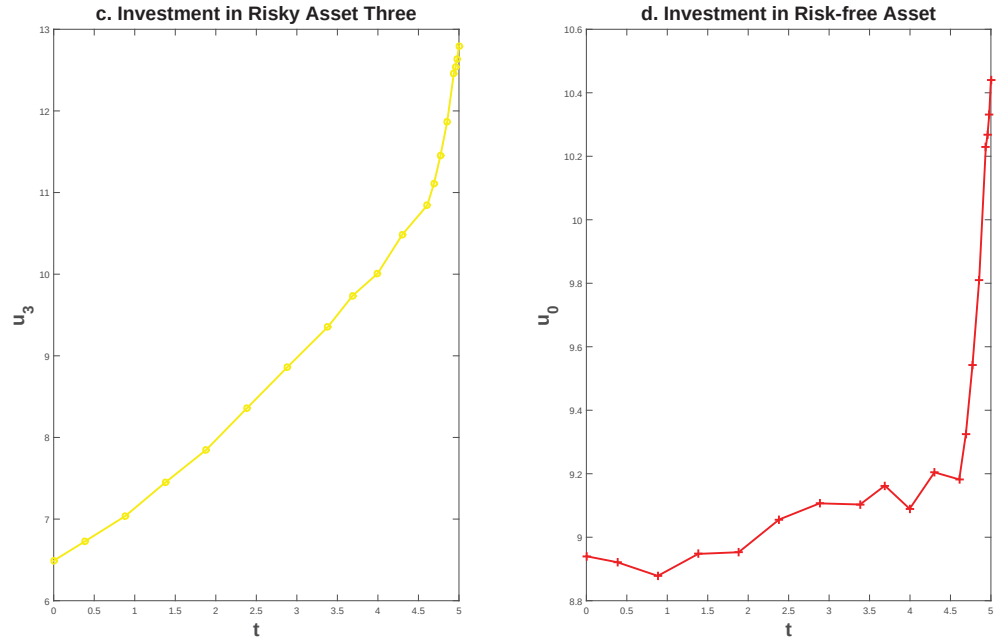

Figure 4.3: The equilibrium control policy (optimal investment strategy) of the risky asset three and the risk-free asset for the asset liability management model

Next, we proceed to study the equilibrium value function and the relationship between the expectation and the variance for the original model. Figure 4.4 demonstrates the equilibrium

value function with respect to time t. Figure 4.5 presents the variance versus the expectation of terminal wealth on the variance-expectation plane.

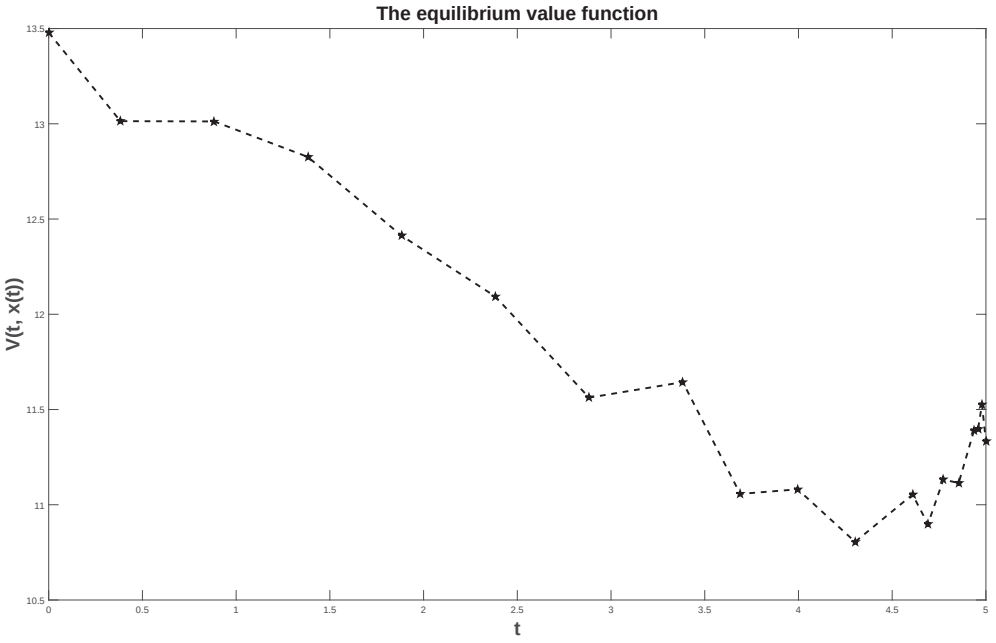

Figure 4.4: The equilibrium value function for the asset liability management model

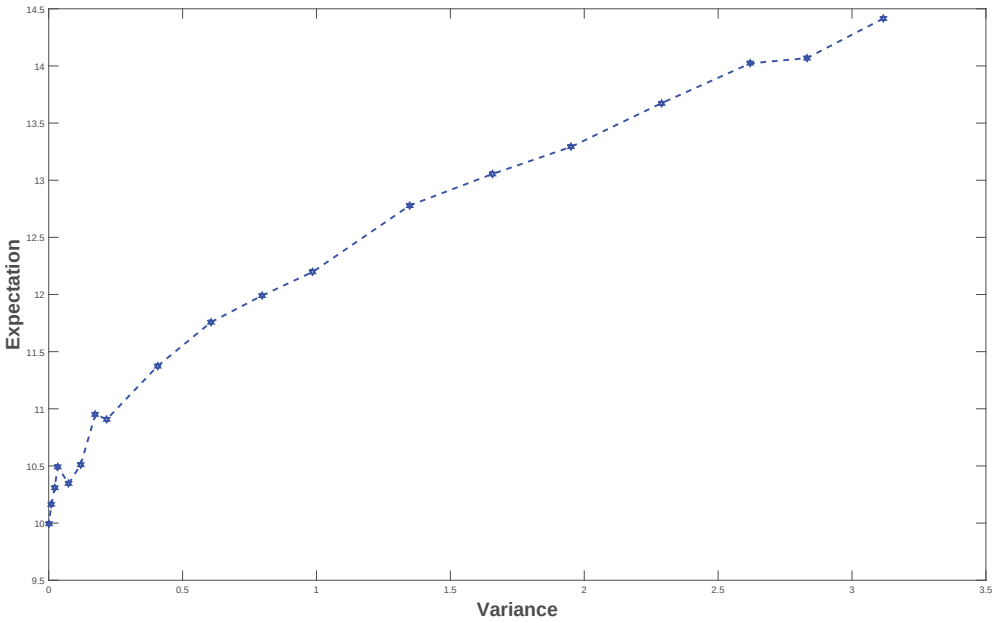

Figure 4.5: The expectation of terminal wealth versus the variance of terminal wealth

4.5.1 The Effect of Liability

In the case with no liability, the original asset liability management model reduces to a portfolio selection model. Except for the parameters introduced by the liability, we keep all the other parameters at the same values as in Table 4.1 and (4.117). According to the analytical expression we obtained in the previous section, we have the following results.

$$\begin{cases} \widehat{\boldsymbol{u}}(t) = \begin{pmatrix} -5.5266 \\ -0.2246 \\ 5.2578 \end{pmatrix} \times \frac{[x(t)m(t) - 6x(t)n(t) + 6x(t)m^2(t)]}{6n(t)}, \\ \\ V(t,x) = 3m^2(t)x^2(t) - 3n(t)x(t) + m(t)x(t), \\ \\ Var_{t,x}[x^U(T)] = [n(t) - m^2(t)]x^2(t). \end{cases}$$

We plot the wealth process, the equilibrium control policy and the equilibrium value function in Figure 4.6 to Figure 4.9. Comparing the numerical results for the portfolio selection model with the numerical results for the asset liability management model, we observe that the wealth increases faster under the case with no liability as shown in Figure 4.6. This is consistent with our intuition. The investor does not need to withdraw money to cover the liability, he/she has more net wealth to invest and earns higher return.

The investment strategy for the portfolio selection model is significantly different from the one for the asset liability management model. Figure 4.7 shows that the investment in risky asset one and risky asset two increase as time elapses and the amount is always negative, which means (i) the investor has to sell short (borrow money) risky asset one and risky asset two during the investment horizon and (ii) the borrowed amount of money reduces with respect to time t in the absence of a liability. This phenomenon demonstrates the effect of liability. Figure 4.8 shows that the investment in risky asset three decreases with respect to time t, whereas the investment in the risk-free asset increases greatly, which means that the investor does not need to take that much risk.

The portfolio selection model gives a much greater equilibrium value function, which is quite reasonable because the investor has more money to invest and he/she should be more optimistic with no liability risk, and thus a greater current utility will be resulted.

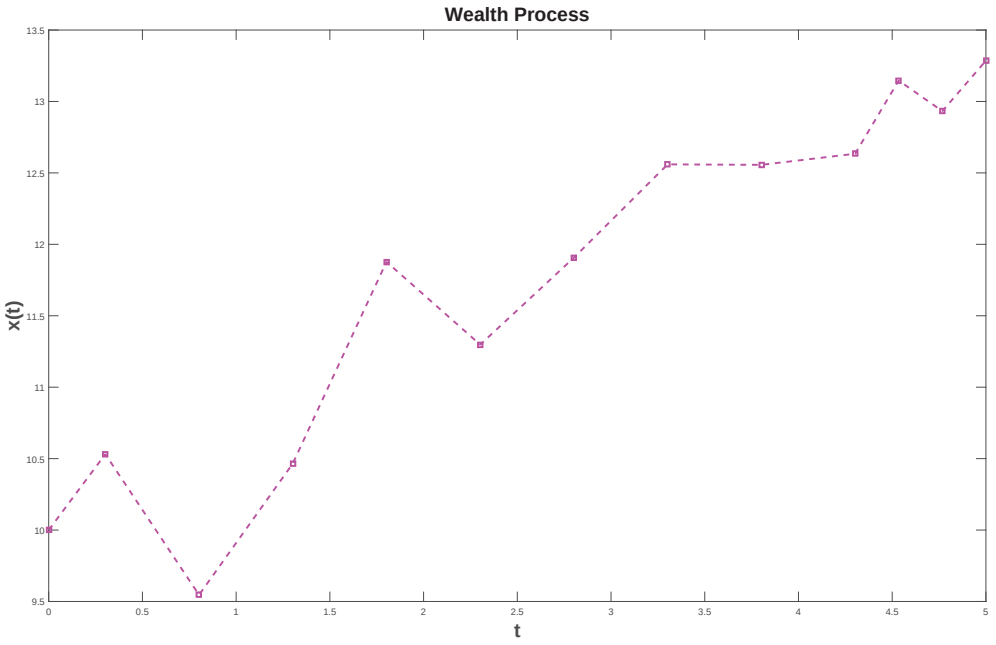

Figure 4.6: The wealth process v.s. time for the portfolio selection model

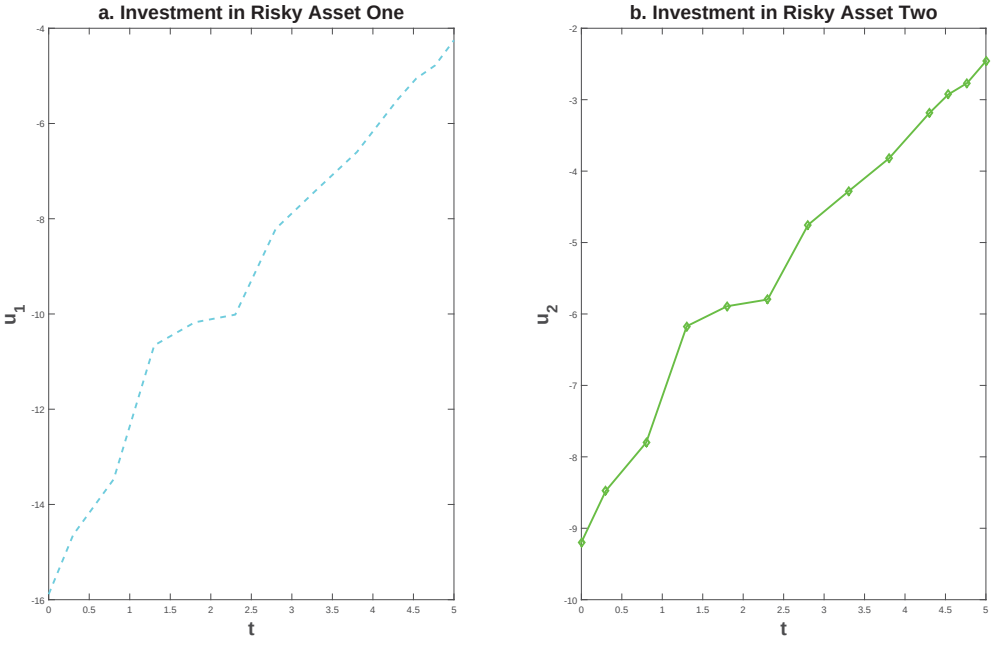

Figure 4.7: The equilibrium control policy (optimal investment strategy) of the risky asset one and the risky asset two for the portfolio selection model

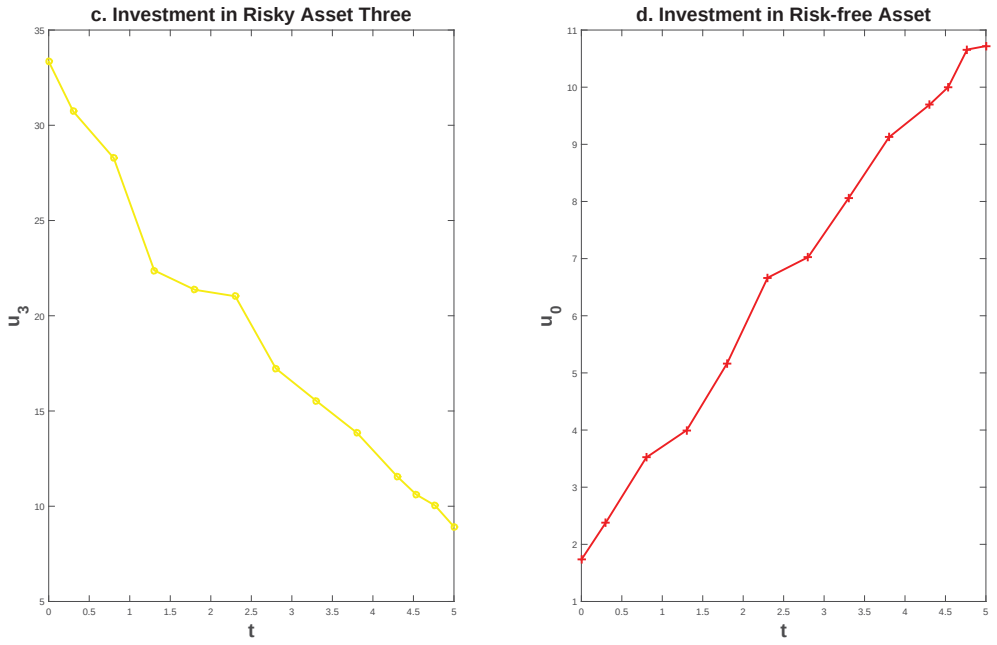

Figure 4.8: The equilibrium control policy (optimal investment strategy) of the risky asset three and the risk-free asset for the portfolio selection model

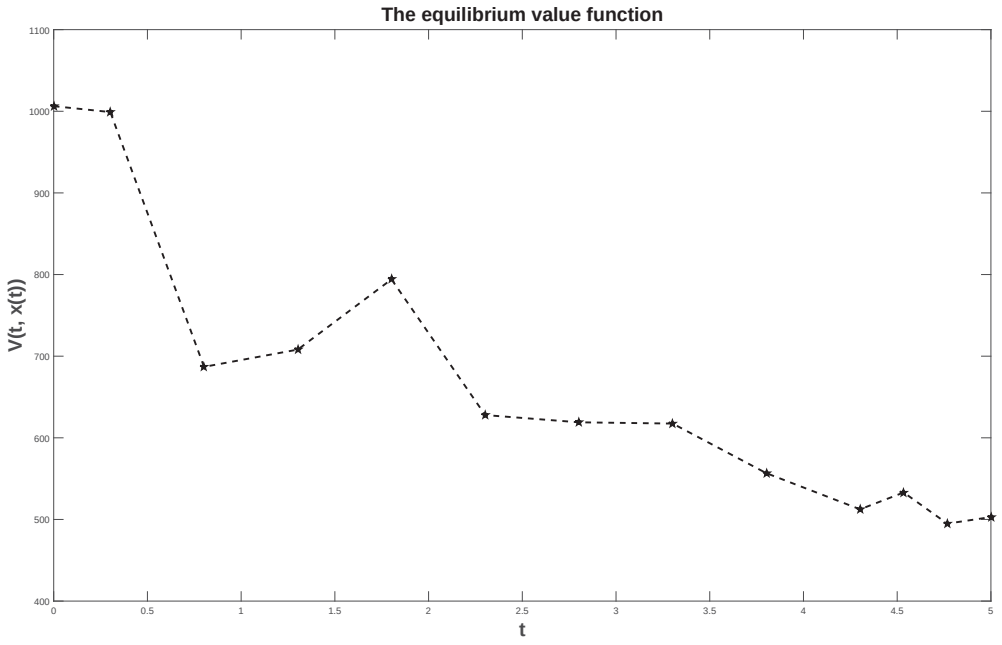

Figure 4.9: The equilibrium value function for portfolio selection model

4.5.2 Multiple Risky Assets Versus Single Risky Asset

In the case with only one risky asset, we assume the appreciation rate to be $\alpha(t) = 0.10$, the volatility vector is $\sigma_1 = (0.20, 0.21, 0.19)'$. All the other parameters remain the same as given

in Table 4.1 and (4.118). Thus, $\alpha_1(t) = \alpha(t) - r(t) = 0.05$. According to the results in (4.71) and (4.72), we plot the wealth process, the equilibrium control policy and the equilibrium value function in Figure 4.10 to Figure 4.13.

Compared with the wealth process for the asset liability management model with three risky assets in Figure 4.1, the return at exit time $t = T$ in Figure 4.10 is roughly 50% lower due to the lack of risky assets with higher appreciation rate. In reality, there are many kinds of investment available, for example, corporate bond, corporate stocks, property, commodity and financial derivatives, with different return rate, risk, expire date and so on. One thus can not use only one risky asset to simulate the financial market setting properly.

Figure 4.11 and Figure 4.12 show that the optimal amount invested in the risky asset and the risk-free asset are always positive and increase with respect to time t. This indicates that the equilibrium control policy does not involve any short sell strategy. The net amount of investment is just the current net wealth held by the investor, which restricts the investor to take full use of the financial instruments and thus leads to a lower investment return.

We also observe a slightly lower equilibrium value function under the case with only one risky asset in Figure 4.13 than the one under the case with multiple risky assets as shown in Figure 4.4. The reason behind this phenomenon is similar to what has mentioned above, including limited investment amount and limited investment choice.

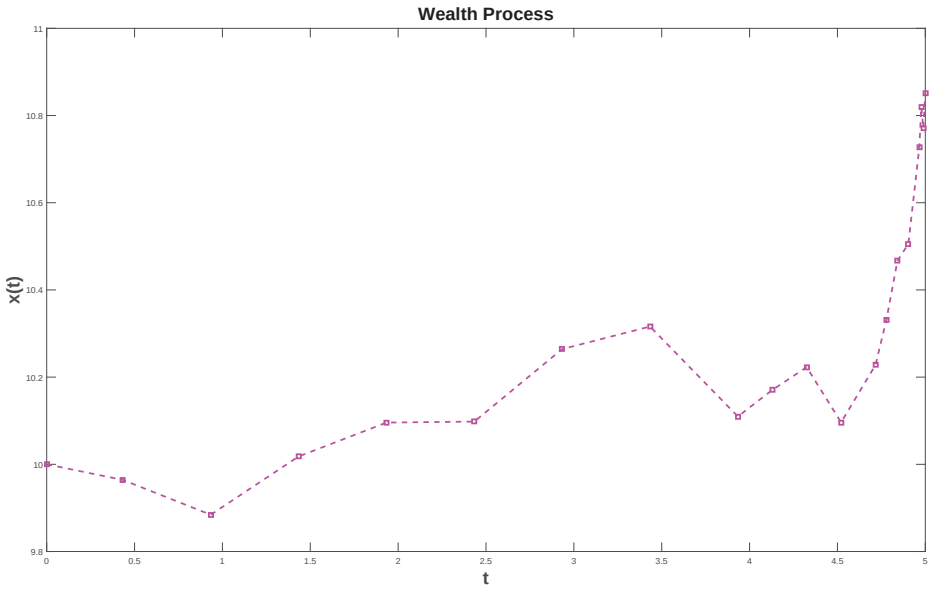

Figure 4.10: The wealth process versus time under the case with only one risky asset

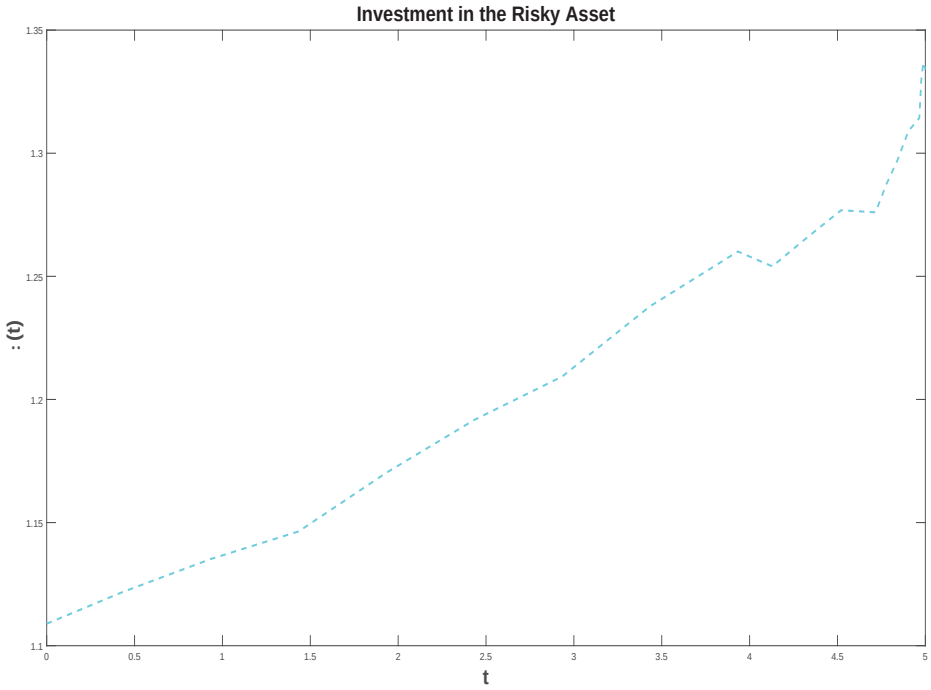

Figure 4.11: The equilibrium control policy (optimal investment strategy) in the risky asset

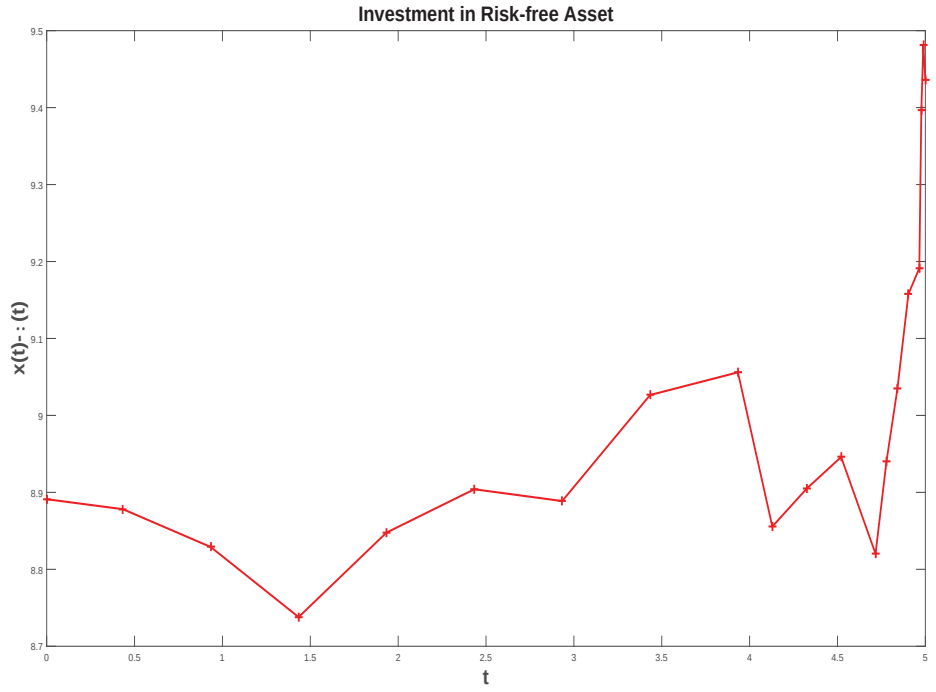

Figure 4.12: The equilibrium control policy (optimal investment strategy) in the risk-free asset

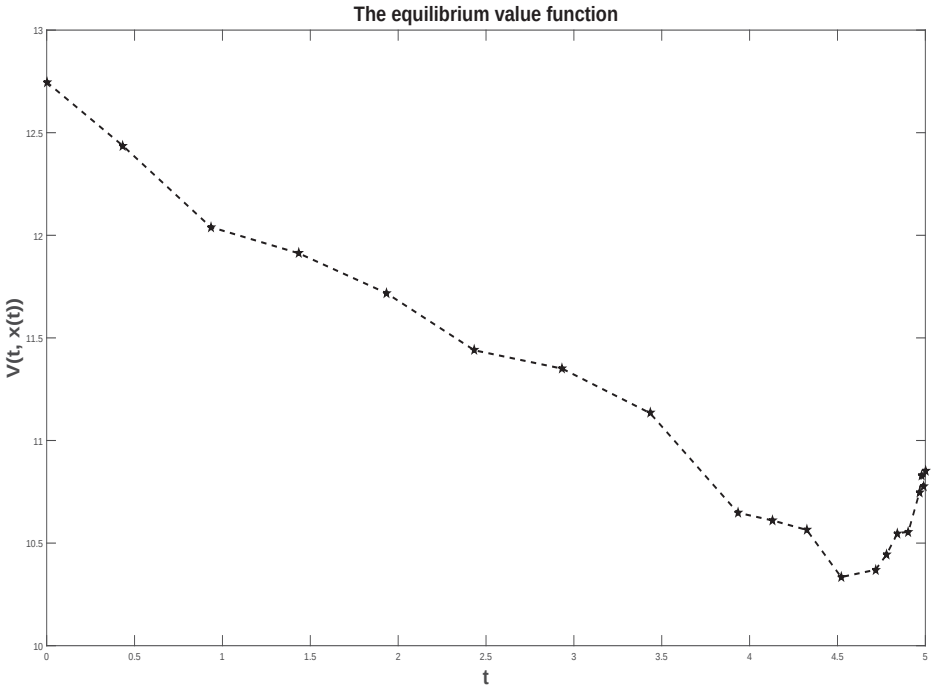

Figure 4.13: The equilibrium value function under the case with only one risky asset

4.6 Concluding Remarks

In this chapter, we derive the analytical solutions for the asset liability management model with state-dependent risk aversion. The closed-form expressions of the equilibrium control policy, the equilibrium value function and the variance versus expectation of terminal wealth under the equilibrium control policy are obtained under the sub-game perfect Nash equilibrium strategy framework. Besides, we derive the results for the corresponding portfolio selection model, and investigate the impact of liability. Moreover, we derive the results for a simplified financial setting which has only one risky asset. Then, we compare the results under two different financial settings, namely (i) the one with multiple risky assets and (ii) the one with single risky asset. Our numerical results suggest that the financial setting with multiple risky assets is more economically relevant than the one with single risky asset.

CHAPTER 5

Summary and future research directions

5.1 Summary of the Main Contributions

In this thesis, we study the asset liability management and portfolio selection under two different kinds of mean-variance criteria. For the criterion which aims at finding the optimal admissible strategy such that the variance of the terminal wealth is minimized for a given expected terminal wealth d, we use the Hamilton-Jacobi-Bellman equation approach and the stochastic dynamic programming technique to solve the model analytically. For the criterion which compiles the mean and variance of the wealth process in one expression by taking the state-dependent risk aversion into account, we use the sub-game perfect Nash equilibrium strategy and the extended Hamilton-Jacobi-Bellman system to derive the analytical expressions for the time-consistent optimal investment strategies, the efficient frontier and the optimal value function. The results obtained can be summarized in two aspects as follows.

(i) Development of solutions for the asset liability management model and the portfolio selection model under the mean-variance criterion in a jump diffusion market

- We obtain the analytical expressions of efficient investment strategy and efficient frontier by applying the above-mentioned methods. This involves the process of solving the associated partial differential equations derived from the Hamilton-Jacobi-Bellman equation.
- The numerical analysis shows that the efficient frontier of the asset liability management problem with jump is no longer a straight line, and the global minimum variance is strictly positive. This is due to the risks embedded in the general insurance liability and the random jump in risky assets' prices.

- In the case with no liability, the asset liability management model reduces to a portfolio selection model. The numerical investigations show that the efficient frontier of the portfolio selection problem with jump is a straight line in the mean-standard deviation coordinated plane, and the variance reaches zero at the point $d = d_{SD_{min}}$.

- In the case with no jump in risky assets' price, we derive the optimal investment strategy and the efficient frontier for the degenerated model. Then, we compare the results with the solution to the original model and illustrate the impact of market fluctuations.

- We analyze the trend showed in the numerical results and establish possible reasons as well as proof of some interesting coincidence under special conditions.

- In order to study the impact of exit time T on the efficient frontier of terminal wealth, we plot the efficient frontier for the case with five different exit times $T = 1, 2, 3, 4,$ and 5. The results demonstrate that (i) when the exit time T increases from 1 to 5, the corresponding efficient frontier moves to the upper right; (ii) the longer the exit time is, the larger the global minimum variance and its corresponding expected terminal wealth are. We analyze the other key parameters in a similar way.

(ii) Development of solutions for the asset liability management model and the portfolio selection model with state-dependent risk aversion

- We obtain the analytical expressions of equilibrium control strategy and equilibrium value function for (a) the case of a general risk aversion function $\gamma(x)$ and (b) the case in which the risk aversion is inversely proportional to the wealth, by applying the above-mentioned methods. This involve the process of solving the associated ordinary differential equations derived from the extended Hamilton-Jacobi-Bellman equation system.

- In the case with no liability, the asset liability management model reduces to a portfolio selection model, and the associated ordinary differential equation system reduces significantly from five equations to two equations, with boundary conditions. We then derive the equilibrium control policy, the equilibrium value function and the variance versus expectation of terminal wealth for the simplified model.

- We give the numerical results for the asset liability management model and the portfolio selection model by discretizing the wealth process under the Euler discretization scheme.

We analyze the value of wealth, the amount of money invested in multiple risky assets and risk aversion parameters versus time for both models and hence demonstrate the effect of liability.

- We consider multiple risky assets as well as only one risky asset, and obtain theoretical results for both cases under a natural choice of $\gamma(x)$ and demonstrate the effect of using multiple risky assets.

5.2 Future Research Directions

The work in this thesis has opened several interesting new areas for future research. We discuss some of them below.

- This research work focuses on optimization of asset liability management & portfolio selection with constraints, including jump diffusion in asset price and state-dependent risk aversion. This optimization problem can be extended by considering further constraints synthetically, including Markov regime switching, bankruptcy prohibition, transaction cost, trading risk and mean-CaR criterion.

- This research can also be extended to the study of real world practical problems, including defined contribution pension fund management, defined benefit pension fund management, annuity management and insurance/reinsurance products. The methodology that is applied and further developed in this thesis are also applicable for reinsurance, financial engineering and financial risk management problems. All of those issues are interesting future research topics with great practical significance.

- In this research, we use the stochastic dynamic programming and the Hamilton-Jacobi-Bellman equation approach or extended Hamilton-Jacobi-Bellman equation approach to solve the asset liability management model and portfolio selection model. Then, we obtain the analytical solution for the efficient optimal control strategy by solving the associated partial differential equations or ordinary differential equations. Another future research direction is to develop more complex stochastic differential equations by taking time delay or fractional Brownian motion or cointegrated assets into account and apply the theory

of backward stochastic differential equations and the stochastic control technique to solve this kind of problems.

- Risk management for both individuals and enterprises has drawn more and more attention in recent years. So the relevant topics are also on the research agenda, for example, (i) risk measurement, such as value-at-risk and (ii) risk hedging of operational risk, financial market risk, strategy risk, social and political risk.

- The impact of multi-scale stochastic volatility on asset liability management and portfolio selection problems should be investigated with the focus on the correlation between stochastic volatility and risk aversion as well as the optimal investment strategy, the efficient frontier and the optimal value function.

- In our work, we use the simple Lévy process to describe the market fluctuations. In future work, we plan to involve more general or complex processes to model risky assets' price, for example, the variance Gamma process and the normal inverse Gaussian process. Furthermore, the market parameters usually can hardly be expressed by explicit functions, therefore advanced parameter estimation techniques should be developed with both higher order convergence and better stability property. Asset liability management under ambiguity is another interesting topic in future work.

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