Option pricing via maximization over uncertainty and correction of volatility smile[*]

Nikolai Dokuchaev

Department of Mathematics & Statistics, Curtin University,

GPO Box U1987, Perth, 6845 Western Australia

email N.Dokuchaev@curtin.edu.au

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Abstract

The paper presents a pricing rule for market models with stochastic volatility and with an uncertainty in its evolution law. It is shown that the most common stochastic volatility models allow a possibility that the option price calculated for random volatility with an error in volatility forecasts is lower than the price for the market with zero error of volatility forecast. To eliminate this possibility, we suggest a pricing rule based on maximization of the price via a class of possible equivalent risk-neutral measures. It shown that, in a Markovian setting, this pricing rule requires to solve a parabolic Bellman equation. Some existence results and a priory estimates are obtained for this equation.

Key words: diffusion market model, volatility smile, stochastic volatility, uncertain volatility, Hamilton–Jacobi–Bellman equation.

JEL classification: C61,G13

1 Introduction

Most practitioners have adapted the famous Black-Scholes model as the premier model for pricing and hedging of options. This model consists of two assets: the risk free bond or bank

account and the risky stock. It is assumed that the dynamics of the stock is given by a random process with some standard deviation of the stock returns (the volatility coefficient, or volatility). The dynamics of the bond is deterministic and exponentially increasing with a given risk-free rate. In the classic Black-Scholes model, the volatility is assumed to be given and fixed. However, empirical research shows that the real volatility is time-varying and random. Moreover, it is commonly recognized that the Black-Scholes formula gives unbiased estimation for at-the-money options only, and it gives a systematic error for in-the-money and out-of-the-money options; in fact, that means that there is a gap between historical and implied volatility that generates so-called *volatility smile* for the implied volatility; see e.g. Black and Scholes (1972), Day and Levis (1992), Derman *et al.* (1996), Taylor and Xu (1994). A detailed review can be found in Mayhew (1995)) and Garcia *et al* (2004). Many authors emphasize that the main difficulty in modifying the Black–Scholes and Merton models is taking into account this fact.

To fill this gap, a number of deterministic and stochastic equations for volatility were proposed; see, e.g., Christie (1982), Johnson and Shanno (1987), Hull and White (1987), Masi *et al.* (1994), and the papers in Jarrow (ed.) (1998). In some other approach, a special temporal scale is used to find the time when historical volatility coincides with implied volatility; see, e.g., Geman and Ane (1996). Usually, these advanced models for volatility lead to incomplete market models.

For incomplete markets, the basic pricing method is risk neutral valuation, when the option price is given as the expected value of its future payoff with respect to a risk-neutral measure discounted back to the present time t; see, e.g., Ross (1976) and Cox and Ross (1976). This method has been developed to pricing rules based on optimal choice of the risk-neutral measures such as local risk minimization, mean variance hedging, q-optimal measures, and minimal entropy measures (see, e.g., Föllmer and Sondermann (1986), Schweizer (1992), Masi *et al.* (1994), Rheinländer and Schweizer (1997), Pham *et al.* (1998), Laurent and Pham (1999), Frittelli (2000), and others). These methods ensure optimal choice of a risk neutral measure given certain optimality criterions. Formally, all these methods define the price as

$$P_{RN}(t) = e^{-r(T-t)}\mathbf{E}_Q\{F(S(T))|\mathcal{F}_t\}, \tag{1.1}$$

2

where $S(t)$ denotes the price of the stock at time t, $\mathcal{F}_t$ is the filtration generated by observable parameters (i.e., by available information), $\mathbf{E}_Q$ is the expectation generated by a risk-neutral measure Q defined for the given volatility $\sigma(t)$ (or a evolution law of volatility), T is the terminal time, $r \in R$ is the risk free interest rate, $F(x)$ is the payoff function (for instance, $F(x) = (x - K)^+$ for the call option, where K is the strike price). The measure Q depends on the method of valuation chosen (for example, local risk minimization or mean variance hedging). Pricing rule (1.1) generates volatility smiles, or U-shape dependence of the implied volatility on K given $S(0)$.

Avellaneda *et al.* (1995), Avellaneda and Parás (1995), Dokuchaev and Savkin (1998), and Frey and Sin (1999), considered a alternative pricing model for random volatilities, where only the bounds of the volatility are given. The price for this setting ensures superreplication with probability 1; the price is obtained via maximization over the set of admissible volatility values. This pricing model does not take into account the volatility term structure and possibilities of volatility forecasting using current observations.

In this paper, we found that the classical pricing rule (1.1) implies that

$$\text{if} \quad S(t) = e^{r(t-T)}K, \text{ then } P_{RN}(t) < \bar{P}_{RN}(t), \tag{1.2}$$

for European put, call, or share-or-nothing options. Here K is the strike price, $\bar{P}_{RN}(t)$ is price (1.1) calculated for an auxiliary market model given time t such that $\sigma(s)$ is replaced by an $\mathcal{F}_t$-measurable variable for all $s \geq t$ that represent a forecast of σ given $\mathcal{F}_t$, i.e., for a model where the forecast error is zero (Theorems 3.1 and Corollary 3.1). This feature can be undesirable, because the presence of the additional risk from an error of volatility forecast should not lead to decreasing of the option price. Therefore, the volatility smile generated by pricing rule (1.1) needs a correction if one wants to eliminate (1.2).

The present paper suggests a version of pricing rule (1.1) that includes maximization over uncertainties: the option price at time t is defined as the maximum of $e^{-r(T-t)}\mathbf{E}_Q\{F(S(T))|\mathcal{F}_t\}$ over a class $\mathcal{A}$ of possible distributions of random volatilities and over a class of risk neutral measures Q. This approach is different from the superreplication method from the cited papers Avellaneda *et al.* (1995), Avellaneda and Parás (1995), Dokuchaev and Savkin (1998), and Frey and Sin (1999). For the suggested approach, superreplication is ensured in average

only and the error of hedging may contain an unhedgeable part, in the spirit of the mean variance hedging; the average error is zero for the worst case scenario. The corresponding price is no less than the Black-Scholes price (or the price with zero error for volatility forecast), and it also generates a volatility smile, for reasonable classes $\mathcal{A}$ (for instance, when $\mathcal{A}$ includes non-random volatilities). For some classes $\mathcal{A}$, this pricing rule gives the Black-Scholes price at time t when $S(t) = e^{r(t-T)}K$, where K is the strike price. Therefore, the rule suggested generates volatility smile close to one coming from risk neutral valuation, but with a correction that prevents (1.2).

For a special case of Markovian setting, our pricing rule requires to solve a non-linear parabolic Bellman equation to ensure maximization over $\mathcal{A}$. Solvability of this equation is discussed. For the worst case scenario, the resulting discounted option price process is a martingale with respect to the risk-neutral measure.

2   Definitions

We consider the diffusion model of a securities market consisting of a risk free bond or bank account with the price $B(t)$, $t \geq 0$, and a risky stock with price $S(t)$, $t \geq 0$. The prices of the stocks evolve as

$$dS(t) = S(t)\left(a(t)dt + \sigma(t)dw(t)\right), \quad t > 0, \tag{2.1}$$

where $w(t)$ is a Wiener process, $a(t)$ is a random appreciation rate, $\sigma(t)$ is a random volatility coefficient. The initial price $S(0) > 0$ is a given deterministic constant. The price of the bond evolves as

$$B(t) = e^{rt}B(0), \tag{2.2}$$

where $r \geq 0$ and $B(0)$ are given constants.

We assume that $w(\cdot)$ is a standard Wiener process on a given standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is a set of elementary events, $\mathcal{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is a probability measure.

Let $\mathcal{F}_t$ be a filtration generated by the currently observable data. We assume that $\mathcal{F}_t$ is independent from $\{w(t_2) - w(t_1)\}_{t_2 \geq t_1 \geq t}$, and $\mathcal{F}_0$ is trivial, i.e., it is the $\mathbf{P}$-augmentation of the set $\{\emptyset, \Omega\}$.

4

We assume that the process $(S(t), \sigma(t))$ is $\mathcal{F}_t$-adapted. In particular, this means that the process $(S(t), \sigma(t))$ is currently observable.

We assume that $a(t)$ is independent from $\{w(t_2) - w(t_1)\}_{t_2 \geq t_1 \geq t}$. For simplicity, we assume that $a(t)$ is a bounded process.

Let $\widetilde{S}(t) \triangleq e^{-rt} S(t)$ be the discounted price process.

Let Q be a risk-neutral measure equivalent to $\mathbf{P}$ on $\mathcal{F}$ such that the process $\widetilde{S}(t)$ is a martingale under Q, i.e., $\mathbf{E}_Q\{S(T) | \mathcal{F}_t\} = e^{r(T-t)} S(t)$, where $\mathbf{E}_Q$ is the corresponding expectation. We assume that this Q exists for any (a, σ, r) under consideration.

Black-Scholes prices

Let terminal time $T > 0$ and strike price $K > 0$ be fixed. Let $H_{BS,c}(t, x, \sigma)$ and $H_{BS,p}(t, x, \sigma)$ denote prices (3.1) for the vanilla put and call options, with the payoff functions $F(S(T)) = (S(T) - K)^+$ and $F(S(T)) = (K - S(T))^+$ respectively, under the assumption that $S(t) = x$, $(\sigma(s), r(s)) = (\sigma, r)$ ($\forall s > t$), where $\sigma \in (0, +\infty)$ is non-random. In other words, it is the Black-Scholes prices, and the celebrated Black-Scholes formula for their explicit values can be rewritten as

$$H_{BS,c}(t, x, \sigma) = x\Phi(d_+(t, x, \sigma)) - Ke^{-r(T-t)}\Phi(d_-(t, x, \sigma)), \tag{2.3}$$

$$H_{BS,p}(t, x, \sigma) = H_{BS,c}(t, x, \sigma) - x + Ke^{-r(T-t)},$$

where $\Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$, and where

$$
\begin{aligned}
d_+(t, x, \sigma) &\triangleq \frac{\ln(x/\widetilde{K}(t))}{\sigma\sqrt{(T-t)}} + \frac{\sigma\sqrt{(T-t)}}{2}, \\
d_-(t, x, \sigma) &\triangleq d_+(t, x, \sigma) - \sigma\sqrt{(T-t)}, \\
\widetilde{K}(t) &\triangleq Ke^{-r(T-t)}.
\end{aligned}
\tag{2.4}
$$

Let $H_{BS,s}(t, x, \sigma, r, K)$ denotes the price for the share-or-nothing call options with the payoff function $F(S(T)) = S(T)\mathbb{I}_{\{S(T) > K\}}$ under the assumption that $S(t) = x$, $(\sigma(s), r(s)) = (\sigma, r)$ ($\forall s > t$), where $v \in (0, +\infty)$ is non-random. The analog of Black-Scholes formula for this case is known:

$$H_{BS,s}(t, x, \sigma, r, K) = x\Phi(d_+(t, x, \sigma, r, K)).$$

For brevity, we shall denote by H_{BS} the corresponding Black-Scholes prices of different options, i.e., $H_{BS} = H_{BS,c}$ $H_{BS} = H_{BS,p}$, or $H_{BS} = H_{BS,s}$, for vanilla call, vanilla put, and share-or-nothing call respectively.

Let

$$v(t) \triangleq \frac{1}{T-t} \int_t^T \sigma(s)^2 ds.$$

Lemma 2.1 *[Hull and White (1987), p.245] Let $t \in [0,T)$ be fixed. Let $v(t)$ be $\mathcal{F}_t$-measurable. Then*

$$e^{-r(T-t)} \mathbf{E}_{Q_{\sigma(\cdot)}} \{F(S(T)) | \mathcal{F}_t\} = H_{BS}(t, S(t), \sqrt{v(t)}).$$

Note that this lemma does not exclude the case when σ and w are correlated. However, the conditions of this lemma are restrictive, since the value $\frac{1}{T-t} \int_t^T \sigma(s)^2 ds$ is not $\mathcal{F}_t$-measurable in the general case of random σ.

Corollary 2.1 *Assume that $H_{BS} = H_{BS,c}$, $H_{BS} = H_{BS,p}$, or $H_{BS} = H_{BS,s}$. Consider a market model with pricing rule (3.1). Let σ be independent from w under Q. Then*

$$P_{RN}(t, \sigma(\cdot)) = \mathbf{E}_Q \{H_{BS}(t, S(t), \sqrt{v(t)}) \,|\, \mathcal{F}_t\},$$

where $v(t)$ is defined in Lemma 2.1.

By Corollary 2.1, it is natural to accept

$$\widehat{\sigma}_1(t) \triangleq \mathbf{E}_Q \left\{ \sqrt{v(t)} \,\Big|\, \mathcal{F}_t \right\} = \mathbf{E}_Q \left\{ \left[\frac{1}{T-t} \int_t^T \sigma(s)^2 ds \right]^{1/2} \Big| \mathcal{F}_t \right\} \tag{2.5}$$

as the forecast (estimate) of $\sqrt{v(t)}$.

Another possible version of estimate for $\sqrt{v(t)}$ is

$$\widehat{\sigma}_2(t) \triangleq \left(\mathbf{E}_Q \left\{ v(t) \,\Big|\, \mathcal{F}_t \right\} \right)^{1/2} = \left(\frac{1}{T-t} \mathbf{E}_Q \left\{ \int_t^T \sigma(s)^2 ds \,\Big|\, \mathcal{F}_t \right\} \right)^{1/2}. \tag{2.6}$$

This estimate is convenient because the corresponding conditional expectation can be calculated using well developed ARCH and GARCH models for heteroscedastic time series describing stock prices with random volatility. If $v(t)$ is a martingale under Q, then $\widehat{\sigma}_2(t) \equiv \sigma(t)$.

The estimates (2.5) and (2.6) can be generalized as

$$\widehat{\sigma}_\nu(t) \triangleq \left(\mathbf{E}_Q \left\{ v(t)^{\nu/2} \,\Big|\, \mathcal{F}_t \right\} \right)^{1/\nu}, \quad \nu \geq 1. \tag{2.7}$$

By Jensen's inequality, it follows that $\widehat{\sigma}_1(t) \leq \widehat{\sigma}_\nu(t)$ with probability 1 for any $\nu \geq 1$.

6

Definition 2.1 *We say that $\sigma_{imp}(t)$ is the implied volatility at time t, if the current market option price is $H_{BS}(t, S(t), \sigma_{imp}(t))$.*

3 Pricing rules

Let $P(t)$ be the price of the option at time $t \in [0, T]$ calculated under a given rule.

Let $P_\nu(t)$ be the price calculated under the same rule applied for an auxiliary market model defined at time t such that the process $\{\sigma(s)\}_{s \geq t}$ is replaced by the process $\bar{\sigma}(t)$ such that

$$\frac{1}{T - t} \int_t^T \bar{\sigma}(s)^2 ds = \widehat{\sigma}_\nu(t)^2,$$

where $\widehat{\sigma}_\nu(t)$ is defined by (2.7), i.e., for a market where $v(t)$ is $\mathcal{F}_t$-measurable, or, in other words, it can be forecasted with zero error (for instance, one may take $\bar{\sigma}(s) \equiv \widehat{\sigma}_\nu(t)$, $s \in [t, T]$). By Lemma 2.1, $P_\nu(t) = H_{BS}(t, S(t), \widehat{\sigma}_\nu(t))$.

The following properties are desirable for a pricing rule:

(A1) The process $P(t)$ is $\mathcal{F}_t$-adapted;

(A2) $P(T) = F(S(T))$;

(A3) $P(t) \geq P_\nu(t) = H_{BS}(t, S(t), \widehat{\sigma}_\nu(t))$ for some $\nu \geq 1$.

Condition (A3) is justified from practical point of view, because the additional risk of an error of volatility forecast should lead to increasing of the option price rather than to decreasing.

The local risk minimization method, the mean variance hedging, and some other methods based on the risk-neutral valuation lead to the following pricing rule: the price is

$$P_{RN}(t, \sigma(\cdot)) \triangleq e^{-r(T-t)} \mathbf{E}_Q\{F(S(T)) \,|\, \mathcal{F}_t\}, \tag{3.1}$$

where a risk neutral measure Q is uniquely defined by (σ, a, r) and by the pricing method used .

We assume that we have chosen one of these pricing methods (for instance, local risk minimization method or mean variance hedging) Therefore, the risk neutral measure Q is uniquely defined by (σ, a, r).

When Condition (A3) does hold hold

Let $\mathcal{A}^{\perp} = \{\sigma(\cdot)\}$ be a set of processes $\sigma(\cdot)$ such that any $\sigma(\cdot) \in \mathcal{A}^{\perp}$ is independent from $w(\cdot)$ under Q.

Theorem 3.1 *Let $\sigma(\cdot) \in \mathcal{A}^{\perp}$. If $S(t) = \widetilde{K}(t) = Ke^{-r(T-t)}$, and $\frac{1}{T-t}\int_t^T \sigma(s)^2 ds$ is not $\mathcal{F}_t$-measurable, then*

$$P_{RN}(t) < H_{BS}(t, S(t), \widehat{\sigma}_{\nu}(t)) \tag{3.2}$$

for any $\nu \geq 1$, where the volatility estimate $\widehat{\sigma}_{\nu}(t)$ is defined by (2.7).

The proof is given in Appendix.

Corollary 3.1 *If $\sigma(\cdot) \in \mathcal{A}^{\perp}$, $S(t) = \widetilde{K}(t)$, and $v(t)$ is not $\mathcal{F}_t$-measurable, then the inequality in Condition (A3) does not hold for any $\nu \geq 1$, and this condition is not satisfied for rule (3.1). In this case, $\sigma_{imp}(t) < \widehat{\sigma}_{\nu}(t)$ for any $\nu \geq 1$, where $\sigma_{imp}(t)$ is the implied volatility defined by Definition 2.1, i.e., the implied volatility is less than the forecasted volatility for at-the-money options.*

A pricing rule that ensures Condition (A3)

We have assumed that the risk neutral measure Q is uniquely defined by (σ, a, r, T). Let us assume that (a, r, T) is fixed, then $Q = Q_{\sigma(\cdot)}$ is uniquely defined by $\sigma(\cdot)$. We assume that it is known that the process $\sigma(\cdot)$ is an element of a given set $\mathcal{A}$ of possible volatility processes.

We suggest the following pricing model.

Definition 3.1 *The price $P_{\max}(t, \mathcal{A})$ of the option given a class $\mathcal{A} = \{\sigma(\cdot)\}$ of possible volatility processes $\sigma(\cdot)$ is*

$$P_{\max}(t, \mathcal{A}) \triangleq e^{-r(T-t)} \sup_{\sigma(\cdot) \in \mathcal{A}} \mathbf{E}_{Q_{\sigma(\cdot)}} \{F(S(T)) | \mathcal{F}_t\}.$$

Let us describe the properties of this pricing rule for some special classes $\mathcal{A}$.

4 A class of volatilities such that $\sigma_{imp}(t) = \widehat{\sigma}_\nu(t)$ for at-the-money options

Theorem 4.1 *Let $\nu \geq 1$ and $t \in [0, T)$ be given. Let $\mathcal{A}$ be a class of volatilities such that the following holds:*

(i) the estimate $\widehat{\sigma}_\nu(t)$ defined by (2.7) is the same for all $\sigma(\cdot) \in \mathcal{A}$; and

(ii) there exists a process $\bar{\sigma}(\cdot) \in \mathcal{A}$ such that

$$\widehat{\sigma}_\nu(t)^2 = \frac{1}{T-t} \int_t^T \bar{\sigma}(s)^2 ds.$$

Then the price $P_{\max}(t, \mathcal{A})$ is such that Conditions (A1)-(A3) are satisfied with this ν; in particular,

$$P_{\max}(t, \mathcal{A}) \geq H_{BS}(t, S(t), \widehat{\sigma}_\nu(t)).$$

Moreover, if $\mathcal{A} \subset \mathcal{A}^\perp$ and $S(t) = \widetilde{K}(t)$, then

$$P_{\max}(t, \mathcal{A}) = H_{BS}(t, S(t), \widehat{\sigma}_\nu(t)),$$

where $\widehat{\sigma}_\nu(t)$ is the volatility forecast defined by (2.7).

In particular, it follows that if $S(t) \sim \widetilde{K}(t)$, then $\sigma_{imp}(t)^2 \sim \widehat{\sigma}_\nu(t)$, and the price $P_{\max}(t, \mathcal{A}) \sim H_{BS}(t, S(t), \widehat{\sigma}_\nu(t))$, i.e., it is close to the Black-Scholes price.

5 A class of volatilities in Markovian setting

We present below an example of Markovian setting, when maximization over a class of volatilities can be reduced to solution of some non-linear parabolic equations.

We consider below payoff functions $F(x)$ of a general form different from the payoffs for put, call, or share-or-nothing options.

Let us consider the case when $\sigma(t)^2 = f(\widetilde{S}(t), Y(t), t)$, where $f(\cdot) : (0, +\infty) \times \mathbf{R} \times [0, T] \to [0, +\infty)$ is a known function, and where the process $Y(t)$ evolves as

$$dY(t) = g(\widetilde{S}(t), Y(t), \eta(t), t)dt + b(\widetilde{S}(t), Y(t), \eta(t), t)d\widehat{w}(t), \quad t > 0, \quad Y(0) = Y_0. \tag{5.1}$$

Here $\widehat{w}(\cdot)$ is a standard Wiener process independent from $w(\cdot)$, and $\eta(t)$ is a n-dimensional random process such that $\eta(t)$ is independent from $\{w(t_2)-w(t_1),\widehat{w}(t_2)-\widehat{w}(t_1)\}_{t_2\geq t_1\geq t}$. The initial value Y_0 is given and deterministic. The functions $g(\cdot):(0,+\infty)\times\mathbf{R}\times\mathbf{R}\times[0,T]\to\mathbf{R}$ and $b(\cdot):(0,+\infty)\times\mathbf{R}\times\mathbf{R}\times[0,T]\to\mathbf{R}$ are given.

We assume that $\mathcal{F}_t$ is the filtration generated by $(S(t),\sigma(t)^2,Y(t),\eta(t))$.

In particular, if $g\equiv 0$ and $f(x,y,t)\equiv y$, then estimate (2.6) is such that $\widehat{\sigma}_2(t)^2\equiv\sigma(t)^2$. If, in this case, the process $Y(t)$ is independent from $S(\cdot)$ under Q, then pricing rule (3.1) is such that if $S(t)=\widetilde{K}(t)$, then the implied volatility is less than the historical volatility, i.e., $\sigma_{imp}(t)^2<\sigma(t)^2$.

Let $\mathcal{U}$ be defined as the set of all measurable functions $U:D\to\Delta$, where $\Delta\subset\mathbf{R}$ is a given compact set, $D\triangleq(0,+\infty)\times\mathbf{R}\times[0,T]$.

Let $\mathcal{A}_{\mathcal{U}}$ be defined as the set of all processes $\sigma(t)$ such that $\sigma(t)^2=f(\widetilde{S}(t),Y(t),t)$ and $\eta(t)=u(\widetilde{S}(t),Y(t),t)$ for some $u(\cdot)\in\mathcal{U}$. Clearly, $\mathcal{A}_{\mathcal{U}}$ is defined by f, g, b, and Δ.

For simplicity, we assume that there exists a constants $\delta>0$ such that $f(x,y,t)\geq\delta$ and $\sup_{u\in\Delta}b(x,y,u,t)^2>0$ for all x,y,t.

Note that $\mathcal{A}_{\mathcal{U}}$ covers models when $\sigma(t)^2$ is generated by a mean-reverting process, log-normal process, etc. For instance, the mean-reverting models can be included using $f(x,y,t)=y+\delta$, where $\delta>0$, and where $Y(t)$ is such that $dY(t)=Y(t)[\eta_1(t)dt+\eta_2(t)d\widehat{w}(t)]$, where $\eta_k(\cdot)$ are some processes. A modification of this example can include a case of the volatility that depends on the stock prices: for instance, one can take $f(x,y,t)=yx^q+\delta$, where $\delta>0$ and $q\in\mathbf{R}$, with $Y(t)$ that evolves as $dY(t)=Y(t)[\eta_1(t)dt+\eta_2(t)d\widehat{w}(t)]$ again.

We restrict our consideration by the framework of the local risk minimization method. In this framework, the risk neutral measure $Q_{\sigma(\cdot)}$ is such that

$$\frac{dQ_{\sigma(\cdot)}}{d\mathbf{P}}=\exp\left(-\int_0^T a(t)f(S(t),Y(t),t)^{-1/2}dw(t)-\frac{1}{2}\int_0^T a(t)^2f(S(t),Y(t),t)^{-1}dt\right).$$

Let us consider option price defined as

$$P_{\max}(t,\mathcal{A}_{\mathcal{U}})\quad=\quad e^{r(T-t)}\sup_{\sigma(\cdot)\in\mathcal{A}_{\mathcal{U}}}\mathbf{E}_{Q_{\sigma(\cdot)}}\{F(S(T))|\mathcal{F}_t\}.\qquad(5.2)$$

For $x,y\in\mathbf{R}$, $u\in\Delta$, $t\in[0,T]$, let

$$\phi(x,y,u,t)\triangleq(f(e^x,y,t),g(e^x,y,u,t),b(e^x,y,u,t)),$$

10

$$\widetilde{F}(x) \triangleq e^{-rT} F(e^{rT} x), \qquad \Phi(x) \triangleq \widetilde{F}(e^x).$$

Let $H(x, y, t)$ be a solution of the boundary value problem for the following nonlinear parabolic Bellman equation in D

$$
\begin{aligned}
\frac{\partial H}{\partial t}(x, y, t) \;+\;& \frac{1}{2} f(x, y, t) x^2 \frac{\partial^2 H}{\partial x^2}(x, y, t) \\
+\;& \sup_{u \in \Delta} \left\{ g(x, y, u, t) \frac{\partial H}{\partial y}(x, y, t) + \frac{1}{2} b(x, y, u, t)^2 \frac{\partial^2 H}{\partial y^2}(x, y, t) \right\} = 0, \\
H(x, y, T) \;=\;& \widetilde{F}(x).
\end{aligned}
\tag{5.3}
$$

(The equations of these type are also called Hamilton–Jacobi equations, or Hamilton–Jacobi–Bellman equations). It is shown below that (5.3) can be rewritten in more convenient form (A.6). If ϕ is bounded and continuous, then problem (5.3) has a unique viscosity solution (see Fleming and Soner (1993)).

Theorem 5.1 *Suppose that there exist $m \geq 0$, $C > 0$ such that*

$$|\phi(x, y, u, t) - \phi(x_1, y_1, u, t)| \leq C(|x - x_1| + |y - y_1|),$$

$$|\phi(x, y, u, t)| \leq C(|x| + |y| + 1),$$

$$|\Phi(x)| \leq C(|x| + 1)^m, \qquad \forall x, y, x_1, y_1 \in \mathbf{R}, u \in \Delta, \ t \in [0, T]. \tag{5.4}$$

Then the solution H of (5.3) is continuous, and

$$P_{\max}(t, \mathcal{A}_{\mathcal{U}}) = H(\widetilde{S}(t), Y(t), t). \tag{5.5}$$

In addition, suppose that there exist $m \geq 0$, $C_0 > 0$ such that

$$|\Phi(x) - \Phi(x_1)| \leq C_0(1 + R^m)|x - x_1| \quad \forall R > 0, \ \forall x, x_1 : \ |x|^2 + |x_1|^2 \leq R^2. \tag{5.6}$$

Then there exists $\widehat{C}_0 > 0$ such that

$$|x H_x'(x, y, t) + |H_y'(x, y, t)| \leq \widehat{C}_0 (1 + |\ln x| + |y|)^{2m}$$

for all t for a.e. $x > 0$, $y \in \mathbf{R}$.

In particular, it follows that H is bounded if $\widetilde{F}$ is bounded.

Let

$$G(x, y, u, t) \triangleq (f(x, y, t), g(x, y, u, t), b(x, y, u, t), \widetilde{F}(x)).$$

Theorem 5.2 *Let there exist $m \geq 0$, $C > 0$ such that*

$$|G(x,y,u,t)| + |G'_t(x,y,u,t)| + |xG'_x(x,y,u,t)| + |G'_y(x,y,u,t)| + |xG''_{xy}(x,y,u,t)|$$

$$+|x^2 G''_{xx}(x,y,u,t)| + |G''_{yy}(x,y,u,t)| \leq C(1 + |\ln x| + |y|)^m$$

$$\forall x > 0, \ y \in \mathbf{R}, \ u \in \Delta, \ t \in [0,T]$$

and the corresponding functions and derivatives are continuous. Then problem (5.3) has a unique continuous solution H, and there exists $C_1 > 0$ such that

$$|H(x,y,t)| + |H'_t(x,y,t)| + |xH'_x(x,y,t)| + |H'_y(x,y,t)| + |xH''_{xy}(x,y,t)|$$

$$+|x^2 H''_{xx}(x,y,t)| + |H''_{yy}(x,y,t)| \leq C_1(1 + |\ln x| + |y|)^{3m}$$

$$\forall x > 0, \ y \in \mathbf{R}, \ t \in [0,T].$$

In particular, the corresponding generalized derivatives are locally square integrable and locally bounded in D.

Theorem 5.3 *Let $\widetilde{F}(x) = \widetilde{F}_1(x) + M_1 x - M_2$, where $\widetilde{F}_1(x)$ is such that (5.4) is satisfied with some $m > 0$, $C > 0$ for the corresponding function $\Phi_1(x) = \widetilde{F}(e^x)$, and where M_1 and M_2 are constants. Let H_1 be the solution of (5.3) with $\widetilde{F}(\cdot) = \widetilde{F}_1(\cdot)$. Then $H(x,y,t) \triangleq H_1(x,y,t) + M_1 x + M_2$ is the solution of (5.3), and (5.5) holds for H.*

Strategies for bond-stock-options market

Let $X(0) > 0$ be the initial wealth at time $t = 0$ and let $X(t)$ be the wealth at time $t > 0$.

We assume that the wealth $X(t)$ at time $t \geq 0$ is

$$X(t) = \beta(t)B(t) + \gamma(t)S(t). \tag{5.7}$$

Here $\beta(t)$ is the quantity of the bond portfolio, $\gamma(t)$ is the quantity of the stock portfolio, $t \geq 0$. The pair $(\beta(\cdot), \gamma(\cdot))$ describes the state of the bond-stocks securities portfolio at time t. Each of these pairs is called a strategy.

The process $\widetilde{X}(t) \triangleq e^{-rt}X(t)$ is said to be the discounted wealth.

Definition 5.1 *A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible strategy if the processes $\beta(t)$ and $\gamma(t)$ are progressively measurable with respect to the filtration $\mathcal{F}_t$ and such that there*

exists a sequence of Markov times $\{T_k\}_{k=1}^{+\infty}$ with respect to the filtration $\mathcal{F}_t$ such that $T_k \to T$ a.s. and

$$\mathbf{E} \int_0^{T_k} \left(\beta(t)^2 dt + S(t)^2 \gamma(t)^2 \right) dt < +\infty \qquad \forall k = 1, 2, ...$$

Definition 5.2 *A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible self-financing strategy, if*

$$d\widetilde{X}(t) = \gamma(t) d\widetilde{S}(t). \tag{5.8}$$

In fact, (5.8) is equivalent to

$$dX(t) = \beta(t) dB(t) + \gamma(t) dS(t).$$

Theorem 5.4 *Let problem (5.3) has a unique solution such that its generalized derivatives H_t', H_x', H_y', H_{xx}'', H_{yy}'' are locally bounded in D. Then*

(i) *For $X(0) = P_{\max}(0, \mathcal{A}_{\mathcal{U}}) = H(S(0), Y(0), 0)$, there exists a self-financing strategy such that the corresponding discounted wealth is*

$$\widetilde{X}(t) = H(\widetilde{S}(t), Y(t), t) + \int_0^t \alpha(s) ds - \int_0^t \frac{\partial H}{\partial y}(\widetilde{S}(s), Y(s), s) b(\widetilde{S}(t), Y(t), \eta(t), t) d\widehat{w}(s), \tag{5.9}$$

where

$$\begin{aligned}
\alpha(t) = \alpha(t, \sigma(\cdot)) &\triangleq \sup_{u \in \Delta} \Big\{ [g(\widetilde{S}(t), Y(t), u, t) - g(\widetilde{S}(t), Y(t), \eta(t), t)] \frac{\partial H}{\partial y}(\widetilde{S}(t), Y(t), t) \\
&+ \frac{1}{2} [b(\widetilde{S}(t), Y(t), u, t)^2 - b(\widetilde{S}(t), Y(t), \eta(t), t)^2] \frac{\partial^2 H}{\partial y^2}(\widetilde{S}(t), Y(t), t) \Big\},
\end{aligned}$$

and $\alpha(\cdot) = \alpha(t, \sigma(\cdot))$ is such that $\alpha(t) \geq 0$ a.s. for a.e. t. for all $\sigma(\cdot) \in \mathcal{A}_{\mathcal{U}}$.

(ii) *The value $X(0) = H(S(0), Y(0), 0)$ is the minimal initial wealth such that, for any $\sigma(\cdot) \in \mathcal{A}_{\mathcal{U}}$, there exists a self-financing strategy and an $\mathcal{F}_t$-adapted square integrable process $\xi(\cdot) = \xi(\cdot, \sigma(\cdot))$ such that the corresponding discounted wealth $\widetilde{X}(t) = \widetilde{X}(t, \sigma(\cdot))$ satisfies*

$$\widetilde{X}(T) \geq \widetilde{F}(\widetilde{S}(T)) + \int_0^T \xi(s) d\widehat{w}(s) \quad a.s..$$

By Theorem 5.4, the initial wealth $X(0) = P_{\max}(0, \mathcal{A}_{\mathcal{U}}) = H(S(0), v(0), 0)$ gives the terminal discounted wealth

$$\begin{aligned}
\widetilde{X}(T) &= \widetilde{F}(\widetilde{S}(T)) + \int_0^T \alpha(t) dt - \int_0^T \frac{\partial H}{\partial v}(\widetilde{S}(t), v(t), t) b(\widetilde{S}(t), Y(t), \eta(t), t) d\widehat{w}(t) \\
&\geq \widetilde{F}(\widetilde{S}(T)) - \int_0^T \frac{\partial H}{\partial v}(\widetilde{S}(t), v(t), t) b(\widetilde{S}(t), Y(t), \eta(t), t) d\widehat{w}(t).
\end{aligned}$$

Appendix: Proofs

Proof of Theorem 3.1. Let us show that the function $H_{BS}(t, \widetilde{K}(t), \sigma)$ is strictly concave in $\sigma > 0$.

We have that $d_-(t, \widetilde{K}(t), \sigma) \equiv -d_+(t, \widetilde{K}(t), \sigma)$. Let

$$D(t, \sigma) \triangleq \frac{1}{4}(T - t)\sigma^2 = d_+(t, \widetilde{K}(t), \sigma)^2 = d_-(t, \widetilde{K}(t), \sigma)^2.$$

Let $H_{BS} = H_{BS,c}$ or $H_{BS} = H_{BS,p}$. By (2.3), it follows that

$$
\begin{aligned}
&\frac{\partial H_{BS}}{\partial \sigma}(t, \widetilde{K}(t), \sigma) \\
&= \frac{\widetilde{K}(t)}{\sqrt{2\pi}} e^{-\frac{D(t,\sigma)}{2}} \frac{\partial d_+}{\partial \sigma}(t, \widetilde{K}(t), \sigma) - \frac{e^{-r(T-t)}K}{\sqrt{2\pi}} e^{-\frac{D(t,\sigma)}{2}} \frac{\partial d_-}{\partial \sigma}(t, \widetilde{K}(t), \sigma) \\
&= \frac{\widetilde{K}(t)}{\sqrt{2\pi}} e^{-\frac{D(t,\sigma)}{2}} \sqrt{T - t}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial^2 H_{BS}}{\partial \sigma^2}(t, \widetilde{K}(t), \sigma) &= -\frac{\widetilde{K}(t)}{\sqrt{2\pi}} e^{-\frac{D(t,\sigma)}{2}} \frac{1}{2} \frac{\partial D(t,\sigma)}{\partial \sigma} \sqrt{T - t} \\
&= -\frac{\widetilde{K}(t)}{\sqrt{2\pi}} e^{-\frac{D(t,\sigma)}{2}} \frac{\sigma}{4}(T - t)^{3/2} < 0.
\end{aligned}
$$

Similarly, we obtain for $H_{BS} = H_{BS,s}$ that

$$\frac{\partial H_{BS,s}}{\partial \sigma}(t, \widetilde{K}(t), \sigma) = \frac{\widetilde{K}(t)}{\sqrt{2\pi}} e^{-\frac{D(t,\sigma)}{2}} \frac{\partial d_+}{\partial \sigma}(t, \widetilde{K}(t), \sigma) = \frac{\widetilde{K}(t)}{\sqrt{2\pi}} e^{-\frac{D(t,\sigma)}{2}} \frac{\sqrt{T - t}}{2}.$$

Then

$$\frac{\partial^2 H_{BS,s}}{\partial \sigma^2}(t, \widetilde{K}(t), \sigma) = -\frac{\widetilde{K}(t)}{\sqrt{2\pi}} e^{-\frac{D(t,\sigma)}{2}} \frac{1}{2} \frac{\partial D(\sigma,t)}{\partial \sigma} \frac{\sqrt{T - t}}{2} = -\frac{\widetilde{K}(t)}{\sqrt{2\pi}} e^{-\frac{D(t,\sigma)}{2}} \frac{\sigma(T - t)^{3/2}}{8} < 0.$$

Hence, the function $H_{BS}(t, \widetilde{K}(t), \sigma)$ is strictly concave in $\sigma > 0$.

To complete the proof of Theorem 3.1, it suffices to consider the case of $t = 0$.

Let us consider $\nu = 1$. By Lemma 2.1, it follows that

$$\mathbf{E}_{Q_{\sigma(\cdot)}} \widetilde{F}(\widetilde{S}(T)) = \mathbf{E}_{Q_{\sigma(\cdot)}} \mathbf{E}_{Q_{\sigma(\cdot)}} \{\widetilde{F}(\widetilde{S}(T))|v(0)\} = \mathbf{E}_{Q_{\sigma(\cdot)}} H_{BS}\left(0, S(0), \sqrt{v(0)}\right). \tag{A.1}$$

We have that $\widehat{\sigma}_1(0) = \mathbf{E}\sqrt{v(0)}$ and $\text{Var}\sqrt{v(0)} \neq 0$. By Jensen's inequality, it follows that, if $\widetilde{S}(0) = \widetilde{K}(0)$, then

$$\mathbf{E}_{Q_{\sigma(\cdot)}} \widetilde{F}(\widetilde{S}(T)) < H_{BS}(0, S(0), \widehat{\sigma}_1(0)). \tag{A.2}$$

This completes the proof for $\nu = 1$.

Let us consider $\nu > 1$. By Hölder's inequality, we obtain that

$$\widehat{\sigma}_1(0) \leq \widehat{\sigma}_\nu(0), \quad \nu > 1. \tag{A.3}$$

Further, $H_{BS}(t, x, \sigma)$ is strictly increasing in σ for vanilla put and call, and it is strictly increasing in σ for share-or-nothing call when $x = e^{-r(T-t)}K$. If $S(0) = \widetilde{K}(0)$, then

$$H_{BS}(0, S(0), \widehat{\sigma}_1(0)) < H_{BS}(0, S(0), \widehat{\sigma}_\nu(0)). \tag{A.4}$$

Therefore, the proof of Theorem 3.1 follows. $\square$

Let $\widetilde{P}_{\max}(t, \mathcal{A}) = e^{-rt}P_{\max}(t, \mathcal{A})$ be the corresponding discounted price given a class $\mathcal{A}$.

Proof of Theorem 4.1. It suffices to consider $t = 0$ only. Let $\bar{\mathbf{E}}_Q$ be the risk neutral measure defined by $\bar{\sigma}(\cdot)$, and let $\bar{v} \triangleq \frac{1}{T} \int_0^T \bar{\sigma}(s)^2 ds$. By the definition,

$$\bar{P}_{\max}(0, \mathcal{A}) \geq \bar{\mathbf{E}}_Q F(\widetilde{S}(T)) = \widetilde{H}_{BS}(0, S(0), \sqrt{\bar{v}}).$$

The second equality here follows from Lemma 2.1.

Further, let $S(0) = \widetilde{K}(0)$ and $\mathcal{A} \subset \mathcal{A}^\perp$. By Theorem 3.1, we have that

$$\mathbf{E}_{Q_{\sigma(\cdot)}} \widetilde{F}(\widetilde{S}(T)) \leq \widetilde{H}_{BS}(0, S(0), \sqrt{\bar{v}}).$$

Similarly to the proof of Theorem 3.1, it follows from (A.3) that

$$\widetilde{H}_{BS}(0, S(0), \sqrt{\bar{v}}) \leq \widetilde{H}_{BS}(0, S(0), \widehat{\sigma}_\nu(0)).$$

It follows that $\widetilde{P}_{\max}(0, \mathcal{A}) \leq \widetilde{H}_{BS}(0, \widetilde{K}, \widehat{\sigma}_\nu(0))$ in that case. Then the proof of Theorem 4.1 follows. $\square$

Proof of Theorem 5.1. Let

$$V(x, y, t) \triangleq H(e^x, y, t), \quad x \in \mathbf{R}. \tag{A.5}$$

Let $x = \ln p$. Formally,

$$\frac{\partial H}{\partial p}(p, y, t) = \frac{1}{p}\frac{\partial V}{\partial x}(x, y, t), \quad \frac{\partial^2 H}{\partial p^2}(p, y, t) = \frac{1}{p^2}\frac{\partial^2 V}{\partial x^2}(x, y, t) - \frac{1}{p^2}\frac{\partial V}{\partial x}(x, y, t).$$

Set $\widehat{D} \triangleq \mathbf{R} \times \mathbf{R} \times [0,T)$. Problem (5.3) can be rewritten for $V : \widehat{D} \to \mathbf{R}$ as

$$
\begin{aligned}
\frac{\partial V}{\partial t}(x,y,t) \;+\;& \frac{1}{2}f(e^x,y,t)\Big[\frac{\partial^2 V}{\partial x^2}(x,y,t) - \frac{\partial V}{\partial x}(x,y,t)\Big] \\
+\;& \sup_{u \in \Delta}\Big\{ g(e^x,y,u,t)\frac{\partial V}{\partial y}(x,y,t) + \frac{1}{2}b(e^x,y,u,t)^2\frac{\partial^2 V}{\partial y^2}(x,y,t)\Big\} = 0, \\
V(x,y,T) \;=\;& \Phi(x). \tag{A.6}
\end{aligned}
$$

Let $(\bar{w}(t), \widehat{w}(t))$ be a standard Wiener process in $\mathbf{R}^2$. Let $\mathcal{F}_t^w$ be the filtration generated by $(\bar{w}(t), \widehat{w}(t))$.

Let $\mathcal{V}$ be the set of all processes $\eta(t)$ that are progressively measurable with respect to $\mathcal{F}_t^w$ and such that $\eta(t) \in \Delta$ for all t a.s.. Let $\mathcal{V}_M$ be a subset of $\mathcal{V}$ such that there exists $u(\cdot) \in \mathcal{U}$ such that $\eta(t) = u(\widetilde{S}(t), Y(t), t)$ and (5.1) holds.

For $\eta(\cdot) \in \mathcal{V}$, we consider the following controlled diffusion process:

$$
\begin{aligned}
d\xi_1(t) &= -\frac{1}{2}f\Big(e^{\xi_1(t)}, \xi_2(t), t\Big)\,dt + f\Big(e^{\xi_1(t)}, \xi_2(t), t\Big)^{1/2}\,d\bar{w}(t), \\
d\xi_2(t) &= g\Big(e^{\xi_1(t)}, \xi_2(t), \eta(t), t\Big)\,dt + b\Big(e^{\xi_1(t)}, \xi_2(t), \eta(t), t\Big)\,d\widehat{w}(t).
\end{aligned}
$$

Let $\xi^{x,y,s}(t) = [\xi_1^{x,y,s}(t), \xi_2^{x,y,s}(t)]$ be the solution of this equation given $\xi^{x,y,s}(s) = (x,y)$, $x, y \in \mathbf{R}$, $s \in [0,T]$.

We have that (A.6) represents the Bellman equation for the optimal control problem

$$
\text{Maximize} \quad \mathbf{E}\Phi(\xi_1(T)) \quad \text{over} \quad \eta(\cdot) \in \mathcal{V}
$$

(see Krylov (1980), (1987)). By the assumptions on f and b, we have that

$$
\sup_{u \in \Delta}\Big[f(e^x,y,t)z_1^2 + b(e^x,y,u,t)^2 z_2^2\Big] > 0 \quad \forall x,y,t, z = (z_1,z_2) \neq 0. \tag{A.7}
$$

It follows that conditions of Theorem 5.2.5 from Krylov (1980), p. 225, are satisfied. By this theorem,

$$
V(x,y,t) = \sup_{\eta(\cdot) \in \mathcal{V}} \mathbf{E}\Phi(\xi_1^{x,y,t}(T)) = \sup_{\eta(\cdot) \in \mathcal{V}_M} \mathbf{E}\{\Phi(\xi_1^{x,y,t}(T)).
$$

By Theorem 3.1.5 from Krylov (1980), p. 132, the function V is continuous.

By Girsanov Theorem, it follows that the process $(w_Q(t), \widehat{w}(t))$ is a Wiener process under $Q_{\sigma(\cdot)}$, where

$$
w_Q(t) \triangleq w(t) + \int_0^t a(s)f(\widetilde{S}(s), Y(s), s)^{-1/2}ds.
$$

16

If $\sigma(\cdot) \in \mathcal{A}_\mathcal{U}$ and $\eta(t) = u(\widetilde{S}(t), Y(t), t)$ for the corresponding $u(\cdot) \in \mathcal{U}$, then the vector $\xi^{x,y,t}_1(T)$ has the same distribution under $\mathbf{P}$ as $(\log \widetilde{S}(T), Y(T))$ under $Q_{\sigma(\cdot)}$ given that $Y(t) = y$ and $\log \widetilde{S}(t) = x$. In this case,

$$\mathbf{E}\Phi(\xi^{x,y,t}_1(T)) = \mathbf{E}_{Q_{\sigma(\cdot)}}\{\widetilde{F}(\widetilde{S}(T))|Y(t) = y, \widetilde{S}(t) = e^x\}.$$

Hence

$$V(x,y,t) = H(e^x, y, t) = \sup_{\sigma(\cdot) \in \mathcal{A}_\mathcal{U}} \mathbf{E}_{Q_{\sigma(\cdot)}}\{\widetilde{F}(\widetilde{S}(T))|Y(t) = y, \widetilde{S}(t) = e^x\}.$$

Further, let (5.6) be satisfied. By Theorem 4.1.1 from Krylov (1980), p. 165, estimate (5.6) implies that the function V has bounded first-order generalized derivatives V'_x and V'_y. It is easy to deduct the required estimates for the first-order derivatives of H. This completes the proof of Theorem 5.1. $\square$

Proof of Theorem 5.2. By assumptions on f, F, it follows that the function $\Gamma \triangleq (\phi, \Phi)$ is such that

$$|\Gamma(x,y,t)| + |\Gamma'_t(x,y,t)| + |\Gamma'_x(x,y,t)|$$

$$+|\Gamma''_{xy}(x,y,t)| + |\Gamma''_{xx}(x,y,t)| + |\Gamma''_{yy}(x,y,t)| \leq C(1 + |x| + |y|)^m.$$

By Theorem 4.7.4 from Krylov (1980), p. 206, it follows that there exists a constant $C_1 > 0$ such that

$$|V(x,y,t)| + |V'_t(x,y,t)| + |V'_x(x,y,t)|$$

$$+|V''_{xy}(x,y,t)| + |V''_{xx}(x,y,t)| + |V''_{yy}(x,y,t)| \leq C_1(1 + |x| + |y|)^{3m} \quad \forall(x,y,t) \in \widehat{D}^*,$$

where

$$\widehat{D}^* \triangleq \{(x,y,t) \in \widehat{D} : \inf_{z \in \mathbf{R}^2: |z|=1} \sup_{u \in \Delta}\left[f(e^x, y, t)z_1^2 + b(e^x, y, u, t)^2 z_2^2\right] > 0\}.$$

Here $\widehat{D} \triangleq \mathbf{R} \times \mathbf{R} \times [0, T]$. In particular, all generalized derivatives here are locally bounded in $\widehat{D}^*$. Similarly to (A.7), we obtain that $\widehat{D}^* = \widehat{D}$. Then the function $H(x,y,t) = V(\ln x, y, t)$ has the required properties. This completes the proof of Theorem 5.2. $\square$

Proof of Theorem 5.3. Let $V(x, y, t) \triangleq V_1(x, y, t) + M_1 e^x + M_2$. It is easy to see that V is the solution of (A.6). Further,

$$V(x, y, t) = \sup_{\sigma(\cdot) \in \mathcal{A}_{\mathcal{U}}} \mathbf{E}_{Q_{\sigma(\cdot)}} \{\widetilde{F}_1(\widetilde{S}(T)) | Y(t) = y, \widetilde{S}(t) = e^x\} + M_1 e^x + M_2$$

$$= \sup_{\sigma(\cdot) \in \mathcal{A}_{\mathcal{U}}} \mathbf{E}_{Q_{\sigma(\cdot)}} \left\{ \left[\widetilde{F}_1(\widetilde{S}(T)) + M_1 \widetilde{S}(T) + M_2 \right] | Y(t) = y, \widetilde{S}(t) = e^x \right\}$$

$$= \sup_{\sigma(\cdot) \in \mathcal{A}_{\mathcal{U}}} \mathbf{E}_{Q_{\sigma(\cdot)}} \{\widetilde{F}(\widetilde{S}(T)) | Y(t) = y, \widetilde{S}(t) = e^x\}. \tag{A.8}$$

Clearly, $H(x, y, t) = V(\ln x, y, t)$ is the solution of (5.3) that has the desired form. This completes the proof of Theorem 5.3. $\square$

Proof of Theorem 5.4. By Itô formula applied to the process $\widetilde{X}(t)$ defined by (5.9), we have that

$$d\widetilde{X}(t) = \frac{\partial H}{\partial t}(\widetilde{S}(t), Y(t), t)dt + \frac{1}{2}f(\widetilde{S}(t), Y(t), t)\widetilde{S}(t)^2 \frac{\partial^2 H}{\partial x^2}(\widetilde{S}(t), Y(t), t)dt$$

$$+ \frac{1}{2}b(\widetilde{S}(t), Y(t), \eta(t), t)^2 \frac{\partial^2 H}{\partial y^2}(\widetilde{S}(t), Y(t), t)dt + \frac{\partial H}{\partial x}(\widetilde{S}(t), Y(t), t)d\widetilde{S}(t)$$

$$+ \frac{\partial H}{\partial y}(\widetilde{S}(t), Y(t), t)dY(t) + \alpha(t)dt - \frac{\partial H}{\partial y}(\widetilde{S}(t), Y(t), t)b(\widetilde{S}(t), Y(t), \eta(t), t)d\widehat{w}(t).$$

It can be rewritten as

$$d\widetilde{X}(t) = \frac{\partial H}{\partial t}(\widetilde{S}(t), Y(t), t)dt + \frac{1}{2}f(\widetilde{S}(t), Y(t), t)\widetilde{S}(t)^2 \frac{\partial^2 H}{\partial x^2}(\widetilde{S}(t), Y(t), t)dt$$

$$+ \frac{1}{2}b(\widetilde{S}(t), Y(t), \eta(t), t)^2 \frac{\partial^2 H}{\partial y^2}(\widetilde{S}(t), Y(t), t)dt + g(\widetilde{S}(t), Y(t), \eta(t), t)\frac{\partial H}{\partial y}(\widetilde{S}(t), Y(t), t)dt$$

$$+ \frac{\partial H}{\partial x}(\widetilde{S}(t), Y(t), t)d\widetilde{S}(t) + \alpha(t)dt.$$

By the definition of the process $\alpha(t)$, this equation can be rewritten as

$$d\widetilde{X}(t) = \frac{\partial H}{\partial t}(\widetilde{S}(t), Y(t), t)dt + \frac{1}{2}f(\widetilde{S}(t), Y(t), t)\widetilde{S}(t)^2 \frac{\partial^2 H}{\partial x^2}(\widetilde{S}(t), Y(t), t)dt$$

$$+ \sup_{u \in \Delta} \left\{ g(\widetilde{S}(t), Y(t), u, t)\frac{\partial H}{\partial y}(\widetilde{S}(t), Y(t), t) + \frac{1}{2}b(\widetilde{S}(t), Y(t), u, t)^2 \frac{\partial^2 H}{\partial y^2}(\widetilde{S}(t), Y(t), t) \right\} dt$$

$$+ \frac{\partial H}{\partial x}(\widetilde{S}(t), Y(t), t)d\widetilde{S}(t).$$

By (5.3), the selection of the function H ensures that the last equation can be rewritten as

$$d\widetilde{X}(t) = \frac{\partial H}{\partial x}(\widetilde{S}(t), Y(t), t)d\widetilde{S}(t).$$

18

It follows that $\widetilde{X}(t)$ is the discounted wealth for the self-financing strategy such that the quantity of stock shares at time t is $\frac{\partial H}{\partial x}(\widetilde{S}(t), Y(t), t)$. By (5.3) again, $\alpha(t) \geq 0$. This implies statement (i).

Let us prove statement (ii). Let $\bar{X}(0)$ be some other initial wealth such that $\bar{X}(0) < X(0)$ and such that, for any $\sigma(\cdot) \in \mathcal{A}_{\mathcal{U}}$, there exists an admissible strategy and an $\mathcal{F}_t$-adapted square integrable process $\bar{\xi}$ such that, for the corresponding discounted wealth $\bar{X}(t)$,

$$\bar{X}(T) \geq \widetilde{F}(\widetilde{S}(T)) + \int_0^T \bar{\xi}(s)d\widehat{w}(s) \quad \text{a.s..} \tag{A.9}$$

By statement (i), it follows that $X(0) = H(S(0), Y(0), 0) = \sup_{\sigma(\cdot) \in \mathcal{A}_{\mathcal{U}}} \mathbf{E}_{Q_{\sigma(\cdot)}} \widetilde{F}(\widetilde{S}(T))$. Hence $\bar{X}(0) < \sup_{\sigma(\cdot) \in \mathcal{A}_{\mathcal{U}}} \mathbf{E}_{Q_{\sigma(\cdot)}} \widetilde{F}(\widetilde{S}(T))$. Therefore, there exists $\bar{\sigma}(\cdot) \in \mathcal{A}_{\mathcal{U}}$ such that $\bar{X}(0) < \mathbf{E}_{Q_{\bar{\sigma}(\cdot)}} \widetilde{F}(\widetilde{S}(T))$. On the other hand, (A.9) implies that $\mathbf{E}_{Q_{\bar{\sigma}(\cdot)}} \bar{X}(T) = \bar{X}(0) \geq \mathbf{E}_{Q_{\bar{\sigma}(\cdot)}} \widetilde{F}(\widetilde{S}(T))$. Hence (A.9) does not hold. This completes the proof. $\square$

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