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Backward parabolic Ito equations and the second fundamental inequality

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Abstract. Regularity of solutions is studied for backward stochastic parabolic Ito equations. An analog of the second fundamental inequality (second energy estimate) and the related existence theorem are obtained for domains with boundary. This result leads to a representation theorem for non-Markov processes in bounded domains and other applications.

Keywords. Parabolic Ito equations, backward SPDEs, regularity.

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1 Introduction

The paper studies stochastic partial differential equations (SPDEs) in a cylinder $D\times[0,T]$ with a Dirichlet boundary condition on ∂D, for a region $D\subseteq\mathbb{R}^n$. We investigate regularity properties of the backward SPDEs, i.e., equations with Cauchy condition at the final time T. The difference between backward and forward equations is not that important for the deterministic equations since a deterministic backward equation can be converted to a forward equation by a time change. It cannot be done so easily for stochastic equations, because we look for solutions adapted to the driving Brownian motion. That is why the backward SPDEs require special consideration. The most common approach is to consider backward Bismut–Peng equations where the diffusion term is not given a priori but needs to be found. These approach was introduced first by Bismut [3] for ordinary linear backward stochastic equations and extended by Pardoux and Peng [26] on more general non-linear equations. The backward SPDEs with similar features were widely studied (see, e.g., Hu and Peng [18], Dokuchaev [7, 9], Yong and Zhou [31], Pardoux and Rascanu [27], Ma and Yong [23], Hu, Ma and Yong [17], Confortola [5], and the bibliography given there). Backward parabolic SPDEs represent analogs of backward parabolic Kolmogorov equations for non-Markov Ito

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processes, including the case of bounded domains, so they can be used for characterization of distributions of the first exit times in non-Markovian setting, as was shown by the author [7, 13]. A different type of backward equations was described in Chapter 5 of Rozovskii [28]. Forward SPDEs were also widely studied (see, e.g., Alós, León and Nualart [1], Bally, Gyongy and Pardoux [2], Chojnowska-Michalik and Goldys [4], Da Prato and Tubaro [6], Gyöngy [16], Krylov [20], Maslowski [24], Pardoux [25], Rozovskii [28], Walsh [30], Zhou [33], Dokuchaev [8–10], and the bibliography given there).

For linear PDEs, existence and uniqueness at different spaces is expressed traditionally via a priori estimates, when a norm of the solution is estimated via a norm of the free term. For the second order equations, there are two most important estimates based on the L_2-norm: so-called "the first energy inequality" or "the first fundamental inequality", and "the second energy inequality", or "the second fundamental inequality" (Ladyzhenskaya [21]). For instance, consider a boundary value problem for the heat equation

$$\begin{aligned} &u'_t = u''_{xx} + \varphi, \qquad \varphi = f'_x + g, \\ &u|_{t=0} = 0, \qquad u|_{\partial D} = 0, \qquad (x,t) \in Q = D \times [0,1],\ D \subset \mathbb{R}. \end{aligned} \tag{1.1}$$

The first fundamental inequality for this problems is the estimate

$$\|u'_x\|^2_{L_2(Q)} + \|u\|^2_{L_2(Q)} \le \text{const}\,(\|f\|^2_{L_2(Q)} + \|g\|^2_{L_2(Q)}).$$

Respectively, the second fundamental inequality is the estimate

$$\|u\|^2_{L_2(Q)} + \|u'_x\|^2_{L_2(Q)} + \|u''_{xx}\|^2_{L_2(Q)} \le \text{const}\,\|\varphi\|^2_{L_2(Q)}.$$

Note that the second fundamental inequality leads to an existence theorem in the class of solutions such that $u''_{xx} \in L_2(Q)$. On the other hand, the first fundamental inequality leads to an existence theorem in the class of solutions such that $u'_x \in L_2(Q)$, i.e., with generalized distributional derivatives u''_{xx} only. For the case of one-dimensional $x \in D = \mathbb{R}$ (i.e., for the state domain without boundary), the second fundamental inequality can be derived from the first fundamental inequality; it suffices to apply the first fundamental inequality for the parabolic equation that can be derived for u'_x (given that the coefficients are smooth). For the vector case of $x \in D = \mathbb{R}^n$, it would be more difficult since u'_x is a vector satisfying a system of n parabolic equations. For the case of a state domain with boundary, this approach does not work even for one-dimensional case, since the boundary values for u'_x on ∂D are unknown a priori. That is why the second fundamental inequality has to be derived separately using special methods.

For forward parabolic SPDEs, analogs of the first and the second fundamental inequalities are known. These results are summarized in Lemma 3.5 below. The first fundamental inequality for forward SPDEs in bounded domains with Dirichlet boundary condition was known for a long time (see, e.g., Rozovskii [28]). Moreover, similar results are also known for forward SPDEs of an arbitrary high order $2m \geq 2$; in this setting, the analog of "the first fundamental inequality" is an estimate for $\mathbf{E}\|u(\,\cdot\,,t)\|^2_{W_2^m(D)}$ (Rozovskii [28]). In addition, a priori estimates without Dirichlet conditions, i.e., in the entire space, are known for a general setting that covers both first and second fundamental inequalities (Krylov [20]). On the other hand, "the second fundamental inequality" for the problem with boundary conditions was more difficult to obtain. Related complications were discussed in Krylov [20, p. 237] and in Dokuchaev [10]. Kim [19] obtained a priori estimates for forward parabolic SPDEs for special weighted norms devaluating the solution and free term near the boundary. For the case of L_2-norms, these estimates can be interpreted as analogs of "the second fundamental inequality"; they are similar to the estimate $\|r_1 u''_{xx}\|_{L_2(Q)} \leq \text{const}\,\|r_2\varphi\|_{L_2(Q)}$ for problem (1.1), where r_i are some weight functions such that $r_i(x) \to 0$ for x approaching ∂D. For the standard non-weighted Sobolev norms, the second fundamental inequality for forward SPDEs in domain was obtained in Dokuchaev [10].

For the backward parabolic equations with Dirichlet boundary conditions, an analog of the first fundamental inequality is known (Zhou [33], Dokuchaev [7]). In fact, the duality relationship between forward and backward equations makes it sufficient to prove the first fundamental inequality for any one type of these two types of equations. (By the duality we mean equations (6.1) below connecting the solutions of SPDEs (3.2) and (3.3) respectively.) However, this approach does not work for the second fundamental inequality in a domain D with a boundary, since it requires to study an adjoint equation with the free term taking values in the space $(W_2^2(D))^*$ which is too wide. It was unknown if the second fundamental inequality holds in this case.

In the present paper, we study again existence, uniqueness, and a priori estimates for solutions for backward SPDEs. As was mentioned above, the first and the second fundamental inequalities for the forward SPDEs had been proved, as well as the first fundamental inequality for the backward SPDEs, so we concentrate our efforts on the remaining problem: to investigate if an analog of the second fundamental inequality holds for the backward equations. We found sufficient conditions that ensure that the second fundamental inequality and the related existence theorem hold (Theorem 4.3). To ensure this regularity, we required additional Condition 4.1.

As an examples of applications, a robustness property is established for backward SPDEs in Section 5. In addition, the second fundamental inequality leads to the representation theorem for non-Markov processes in bounded domains (Dokuchaev [14]).

This paper represents a revised version of the working paper [11]. We are happy to note that, using a different approach, Du and Tang [15] obtained recently an analog of Theorem 4.3 without Condition 4.1.

2 Definitions

2.1 Spaces and classes of functions

Assume that we are given an open domain $D \subseteq \mathbb{R}^n$ such that either $D = \mathbb{R}^n$ or D is bounded with C^2-smooth boundary ∂D. Let $T > 0$ be given, and let

$$Q \triangleq D \times (0, T).$$

We are given a standard complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a right-continuous filtration $\mathcal{F}_t$ of complete σ-algebras of events, $t \geq 0$. We are given also an N-dimensional process $w(t) = (w_1(t), \dots, w_N(t))$ with independent components such that it is a Wiener process with respect to $\mathcal{F}_t$.

We denote by $\|\cdot\|_X$ the norm in a linear normed space X, and we denote by $(\cdot, \cdot)_X$ the scalar product in a Hilbert space X.

Below, we list some notations for spaces of real-valued functions.

Let $G \subset \mathbb{R}^k$ be an open domain. Then $W_q^m(G)$ denotes the Sobolev space of functions that belong to $L_q(G)$ with the distributional derivatives up to the mth order, $q \geq 1$.

We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^k$, and we denote by $\bar{G}$ the closure of a region $G \subset \mathbb{R}^k$.

Let $H^0 \triangleq L_2(D)$, and let $H^1 \triangleq \overset{\circ}{W}{}_2^1(D)$ be the closure in the $W_2^1(D)$-norm of the set of all smooth functions $u : D \to \mathbb{R}$ such that $u|_{\partial D} \equiv 0$. Let

$$H^2 = W_2^2(D) \cap H^1$$

be the space equipped with the norm of $W_2^2(D)$. The spaces H^k are Hilbert spaces and H^k is a closed subspace of $W_2^k(D)$, $k = 1, 2$.

Let H^{-1} be the dual space to H^1, with the norm $\|\cdot\|_{H^{-1}}$ such that if $u \in H^0$, then $\|u\|_{H^{-1}}$ is the supremum of $(u, v)_{H^0}$ over all $v \in H^1$ such that $\|v\|_{H^1} \leq 1$. The space H^{-1} is a Hilbert space.

We will write $(u, v)_{H^0}$ for $u \in H^{-1}$ and $v \in H^1$, meaning the natural extension of the bilinear form from $u \in H^0$ and $v \in H^1$.

Denote by $\bar{\ell}_k$ the Lebesgue measure in $\mathbb{R}^k$ and denote by $\overline{\mathcal{B}}_k$ the σ-algebra of Lebesgue sets in $\mathbb{R}^k$.

We denote by $\overline{\mathcal{P}}$ the completion (with respect to the measure $\bar{\ell}_1 \times \mathbf{P}$) of the σ-algebra of subsets of $[0, T] \times \Omega$, generated by functions that are progressively measurable with respect to $\mathcal{F}_t$.

Let $Q_s \triangleq D \times [s, T]$. For $k = -1, 0, 1, 2$, we introduce the spaces

$$X^k(s,T) \triangleq L^2\big([s,T] \times \Omega, \overline{\mathcal{P}}, \overline{\ell}_1 \times \mathbf{P}; H^k\big),$$
$$Z_t^k \triangleq L^2\big(\Omega, \mathcal{F}_t, \mathbf{P}; H^k\big),$$
$$\mathcal{C}^k(s,T) \triangleq C\big([s,T]; Z_T^k\big).$$

Furthermore, we introduce the spaces

$$Y^k(s,T) \triangleq X^k(s,T) \cap \mathcal{C}^{k-1}(s,T), \quad k \geq 0,$$

with the norm $\|u\|_{Y^k(s,T)} \triangleq \|u\|_{X^k(s,T)} + \|u\|_{\mathcal{C}^{k-1}(s,T)}$.

In addition, we will be using the spaces

$$\mathcal{W}_r^k \triangleq L^\infty\big([0,T] \times \Omega, \overline{\mathcal{P}}, \overline{\ell}_1 \times \mathbf{P}; W_r^k(D)\big), \quad k = 0, 1, \ldots, \ 1 \leq r \leq +\infty.$$

The spaces X^k and Z_t^k are Hilbert spaces.

2.2 Stochastic integrals

Proposition 2.1. *Let $\xi \in X^0$, and let $\{\xi_k\}_{k=1}^{+\infty} \subset L^\infty([0,T] \times \Omega, \ell_1 \times \mathbf{P}; C(D))$ be a sequence such that all $\xi_k(\cdot, t, \omega)$ are progressively measurable with respect to $\mathcal{F}_t$, and let $\|\xi - \xi_k\|_{X^0} \to 0$. Let $t \in [0,T]$ and $j \in \{1, \ldots, N\}$ be given. Then the sequence of the integrals $\int_0^t \xi_k(x,s,\omega)\, dw_j(s)$ converges in Z_t^0 as $k \to \infty$, and its limit depends on ξ, but does not depend on $\{\xi_k\}$.*

Proof. The proposition follows from the completeness of X^0 and from the equality

$$\mathbf{E} \int_0^t \|\xi_k(\cdot, s, \omega) - \xi_m(\cdot, s, \omega)\|_{H^0}^2\, ds$$
$$= \int_D dx\, \mathbf{E} \left(\int_0^t \big(\xi_k(x,s,\omega) - \xi_m(x,s,\omega)\big)\, dw_j(s) \right)^2. \qquad \square$$

Definition 2.2. Let $\xi \in X^0$, $t \in [0,T]$, $j \in \{1, \ldots, N\}$. Then we define

$$\int_0^t \xi(x,s,\omega)\, dw_j(s)$$

as the limit in Z_t^0 as $k \to \infty$ of a sequence $\int_0^t \xi_k(x,s,\omega)\, dw_j(s)$, where the sequence $\{\xi_k\}$ is as in Proposition 2.1.

Sometimes we will omit ω in equations.

3 Review of existence theorems for forward and backward SPDEs

Let $(x,t) \in Q$, $\omega \in \Omega$. Consider the functions

$$\begin{aligned}
b(x,t,\omega) &: \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}^{n\times n},\\
f(x,t,\omega) &: \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}^n,\\
\lambda(x,t,\omega) &: \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R},\\
\beta_j(x,t,\omega) &: \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}^n,\\
\overline{\beta}_i(x,t,\omega) &: \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}
\end{aligned}$$

that are progressively measurable for any $x \in \mathbb{R}^n$ with respect to $\mathcal{F}_t$.

Consider the differential operators defined on functions $v : D \to \mathbb{R}$

$$\begin{aligned}
\mathcal{A}v = \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}\big(b_{ij}(x,t,\omega)\, v(x)\big)&\\
-\sum_{i=1}^n \frac{\partial}{\partial x_i}\big(f_i(x,t,\omega)\, v(x)\big) + \lambda(x,t,\omega)\, v(x),&\\
B_k v = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\big(\beta_k(x,t,\omega)\, v(x)\big) + \overline{\beta}_k(x,t,\omega)\, v(x), \quad k = 1,\dots,N.&
\end{aligned}$$

Here b_{ij}, f_i, x_i are the components of b, f, x.

In addition, consider the operators being formally adjoint to the operators $\mathcal{A}$ and B_i:

$$\begin{aligned}
\mathcal{A}^* v \triangleq \sum_{i,j=1}^n b_{ij}(x,t,\omega) \frac{\partial^2 v}{\partial x_i \partial x_j}(x)&\\
+\sum_{i=1}^n f_i(x,t,\omega) \frac{\partial v}{\partial x_i}(x) + \lambda(x,t,\omega) v(x),&\qquad (3.1)\\
B_k^* v \triangleq \frac{dv}{dx}(x)\, \beta_k(x,t,\omega) + \overline{\beta}_k(x,t,\omega)\, v(x), \quad k = 1,\dots,N.&
\end{aligned}$$

To proceed further, we assume that Conditions 3.1–3.2 remain in force throughout this paper.

Condition 3.1 (Coercivity). The matrix $b = b^\top$ is symmetric, bounded, and progressively measurable with respect to $\mathcal{F}_t$ for all x, and there exists a constant $\delta > 0$

such that

$$y^\top b(x,t,\omega)\,y - \frac{1}{2}\sum_{i=1}^{N} |y^\top \beta_i(x,t,\omega)|^2 \geq \delta |y|^2$$

for all $y \in \mathbb{R}^n$, $(x,t) \in D \times [0,T]$, $\omega \in \Omega$.

Condition 3.2. The functions $\lambda(x,t,\omega)$ and $\overline{\beta}_i(x,t,\omega)$ are bounded. The functions

$$\begin{aligned} b(x,t,\omega) &: \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}^{n\times n},\\ f(x,t,\omega) &: \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}^n,\\ \lambda(x,t,\omega) &: \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}, \end{aligned}$$

$\beta_i(x,t,\omega)$ and $\overline{\beta}_i(x,t,\omega)$ are differentiable in x and bounded in (x,t,ω), and

$$\operatorname*{ess\,sup}_{x,t,\omega}\left(\left|\frac{\partial b}{\partial x}(x,t,\omega)\right| + \left|\frac{\partial f}{\partial x}(x,t,\omega)\right| + \left|\frac{\partial \beta_i}{\partial x}(x,t,\omega)\right|\right) < +\infty, \quad i = 1,\dots,N.$$

We introduce the set of parameters

$$\begin{aligned} \mathcal{P}_1 \triangleq \Bigg(n, D, T\delta,\\ &\operatorname*{ess\,sup}_{x,t,\omega}\left[|b(x,t,\omega)| + |f(x,t,\omega)| + \left|\frac{\partial b}{\partial x}(x,t,\omega)\right| + \left|\frac{\partial f}{\partial x}(x,t,\omega)\right|\right],\\ &\operatorname*{ess\,sup}_{x,t,\omega,i}\left[|\beta_i(x,t,\omega)| + |\overline{\beta}_i(x,t,\omega)| + \left|\frac{\partial \beta_i}{\partial x}(x,t,\omega)\right|\right]\Bigg). \end{aligned}$$

Boundary value problems for forward and backward equations

Let $s \in [0,T)$, $\varphi \in X^{-1}$, $h_i \in X^0$, and $\Phi, \Psi \in Z_s^0$. Consider the boundary value problem in $D \times [s,T]$

$$\begin{aligned} d_t u &= (\mathcal{A}u + \varphi)\,dt + \sum_{i=1}^{N}(B_i u + h_i)dw_i(t), \quad t > s,\\ u|_{t=s} &= \Phi, \quad u(x,t,\omega)|_{x\in\partial D} = 0. \end{aligned} \tag{3.2}$$

Here $u = u(x,t,\omega)$, $(x,t) \in Q$, $\omega \in \Omega$. The corresponding SPDE is a forward equation.

Inequality (3.1) means that equation (3.2) is coercive or *superparabolic*, in the terminology of Rozovskii [28].

For $\xi \in X^{-1}$ and $\Psi \in Z_s^0$, consider the boundary value problem in Q

$$\begin{aligned} &d_t p + \left(\mathcal{A}^* p + \sum_{i=1}^N B_i^* \chi_i + \xi\right) dt = \sum_{i=1}^N \chi_i \, dw_i(t), \quad t < T, \\ &p|_{t=T} = \Psi, \quad p(x,t,\omega)|_{x\in\partial D} = 0. \end{aligned} \tag{3.3}$$

Here $p = p(x,t,\omega)$, $\chi_i = \chi_i(x,t,\omega)$, $(x,t) \in Q$ and $\omega \in \Omega$. The corresponding SPDE is a backward equation.

The definition of solution

Definition 3.3. Let $h_i \in X^0$ and $\varphi \in X^{-1}$. We say that equations (3.2) are satisfied for $u \in Y^1$ if

$$\begin{aligned} u(\cdot,t) = \Phi + \int_s^t \Big(\mathcal{A}u(\cdot,r) + \varphi(\cdot,r)\Big)\, dr \\ + \sum_{i=1}^N \int_s^t (B_i u(\cdot,r) + h_i(\cdot,r))\, dw_i(r) \end{aligned} \tag{3.4}$$

for all t such that $s < t \le T$, and this equality is satisfied as an equality in Z_T^{-1}.

Definition 3.4. We say that equation (3.3) is satisfied for $p \in Y^1$, $\Psi \in Z_T^0$ and $\chi_i \in X^0$ if

$$\begin{aligned} p(\cdot,t) = \Psi + \int_t^T \left(\mathcal{A}^* p(\cdot,s) + \sum_{i=1}^N B_i^* \chi_i(\cdot,s) + \xi(\cdot,s)\right) ds \\ - \sum_{i=1}^N \int_t^T \chi_i(\cdot,s)\, dw_i(s) \end{aligned}$$

for any $t \in [0,T]$. The equality here is assumed to be an equality in the space Z_T^{-1}.

Note that the condition on ∂D is satisfied in the following sense:

$$u(\cdot,t,\omega) \in H^1 \quad \text{and} \quad p(\cdot,t,\omega) \in H^1 \quad \text{for a.e. } t,\omega.$$

Further, $u, p \in Y^1$, and the value of $u(\cdot,t)$ or $p(\cdot,t)$ is uniquely defined in Z_T^0 given t, by the definitions of the corresponding spaces. The integrals with dw_i in (3.4)–(3.5) are defined as elements of Z_T^0. The integrals with ds are defined as elements of Z_T^{-1}. (Definitions 3.3–3.4 require for (3.2)–(3.3) that these integrals are equal to elements of Z_T^0 in the sense of equality in Z_T^{-1}.)

Existence theorems and known fundamental inequalities

The following lemma combines the first and the second fundamental inequalities and related existence result for forward SPDEs. It gives analogs of the so-called “energy inequalities”, or “the fundamental inequalities” known for deterministic parabolic equations (Ladyzhenskaya et al. [22]).

Lemma 3.5. *Let either $k = -1$ or $k = 0$. In addition, assume that if $k = 0$, then*

$$\beta_i(x,t,\omega) = 0 \quad \textit{for } x \in \partial D,\ i = 1,\dots,N,$$

and

$$\operatorname*{ess\,sup}_{\omega} \sup_{(x,t)\in Q} \left| \frac{\partial^2 b}{\partial x_k \partial x_m}(x,t,\omega) \right| < +\infty.$$

Let $\varphi \in X^k(s,T)$, $h_i \in X^{k+1}(s,T)$, and $\Phi \in Z_s^{k+1}$. Then problem (3.2) *has an unique solution u in the class $Y^1(s,T)$, and the following analog of the first fundamental inequality is satisfied:*

$$\|u\|_{Y^{k+2}(s,T)} \le c \left(\|\varphi\|_{X^k(s,T)} + \|\Phi\|_{Z_s^{k+1}} + \sum_{i=1}^N \|h_i\|_{X^{k+1}(s,T)} \right), \tag{3.5}$$

where $c = c(\mathcal{P}_1)$ is a constant that depends on $\mathcal{P}_1$ only.

The statement of Lemma 3.5 for $k = -1$ corresponds to the first fundamental inequality; it is a special case of Theorem 3.4.1 from Rozovskii [28]. The statement for $k = 0$ corresponds to the second energy inequality; for domains with boundary, it was obtained in Dokuchaev [10].

The following lemma gives the first fundamental inequalities and related existence results for backward SPDEs.

Lemma 3.6 (Dokuchaev [7, 13]). *For any $\xi \in X^{-1}$ and $\Psi \in Z_T^0$, there exists a pair (p,χ) such that $p \in Y^1$, $\chi = (\chi_1,\dots,\chi_N)$, $\chi_i \in X^0$, and* (3.3) *is satisfied. This pair is uniquely defined, and the following analog of the first fundamental inequality is satisfied:*

$$\|p\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} \le c(\|\xi\|_{X^{-1}} + \|\Psi\|_{Z_T^0}), \tag{3.6}$$

where $c = c(\mathcal{P}_1) > 0$ as a constant that does not depend on ξ and Ψ.

Thus, only the second fundamental inequality for backward SPDEs is missed.

4 The main result: The second fundamental inequality for backward equations

Starting from now, we assume that the following addition conditions are satisfied.

Condition 4.1. There exists a constant $\delta_1 > 0$ such that

$$\sum_{i=1}^{N} y_i^\top b(x,t,\omega)\, y_i - \frac{1}{2}\left(\sum_{i=1}^{N} y_i^\top \beta_i(x,t,\omega)\right)^2 \geq \delta_1 \sum_{i=1}^{N} |y_i|^2 \tag{4.1}$$

for all $\{y_i\}_{i=1}^N \subset \mathbb{R}^n$, $(x,t) \in D \times [0,T]$, $\omega \in \Omega$.

For an integer $M > 0$, we denote by $\Theta_b(M)$ the class of all matrix functions b such that all conditions imposed in Section 3 are satisfied, and there exists a set

$$\{t_k\}_{i=0}^M = \{t_k(M)\}_{i=0}^M$$

such that $0 = t_0 < t_1 < \cdots < t_M = T$ and

$$\max_k |t_k - t_{k-1}| \to 0 \quad \text{as } M \to +\infty,$$

and that the function $b(x,t,\omega) = b(x,\omega)$ does not depend on t for $t \in [t_k, t_{k+1})$. In particular, this means that the function $b(x,t,\cdot)$ is $\mathcal{F}_{t_k}$-measurable for all $x \in D$ and $t \in [t_k, t_{k+1})$.

Set

$$\Theta_b \triangleq \bigcup_{M>0} \Theta_b(M).$$

The following Condition 4.2 is rather technical.

Condition 4.2. The matrix b is such that all conditions imposed in Section 3 are satisfied, and that there exists a sequence $\{b^{(M)}\}_{M=1}^{+\infty} \subset \Theta_b$ such that at least one of the following conditions is satisfied:

(i) $\|b^{(M)} - b\|_{\mathcal{W}^1_\infty} \to 0$ as $M \to +\infty$.

(ii) Condition 4.1 is satisfied for b replaced by $b^{(M)}$, with the same $\delta_1 > 0$ for all M, and $\|b^{(M)}(\cdot,t,\omega) - b(\cdot,t,\omega)\|_{W^1_\infty(D)} \to 0$ for a.e. (almost every) (t,ω) as $M \to +\infty$.

We denote by $\overline{\Theta}_b$ the class of all functions b such that Condition 4.2 is satisfied.

To proceed further, we assume that Conditions 4.1–4.2 remain in force starting from here and up to the end of this paper, as well as the previously formulated conditions.

Let $\mathcal{P} = \{\mathcal{P}_1, \delta_1\}$.

Theorem 4.3. *For any* $\xi \in X^0$ *and* $\Psi \in Z_T^1$*, there exists a pair* (p, χ) *such that* $p \in Y^2$*,* $\chi = (\chi_1, \dots, \chi_N)$*,* $\chi_i \in X^1$ *and* (3.3) *is satisfied. This pair is uniquely defined, and the following analog of the second fundamental inequality holds:*

$$\|p\|_{Y^2} + \sum_{i=1}^{N} \|\chi_i\|_{X^1} \le c(\|\xi\|_{X^0} + \|\Psi\|_{Z_T^1}), \tag{4.2}$$

where $c > 0$ *is a constant that depends only on* $\mathcal{P}$.

Repeat that estimate (4.2) represents an analog of the second fundamental inequality.

On Condition 4.1

Let us discuss the properties of Condition 4.1 and compare it with Condition 3.1. First, let us note that it can happen that Condition 3.1 holds but Condition 4.1 does not hold. It can be seen from the following example.

Example 4.4. Assume that $n = 2$, $N = 2$,

$$\beta_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad b \equiv \frac{1}{2}(\beta_1\beta_1^\top + \beta_2\beta_2^\top) + 0.01 I_2 = 0.51 I_2,$$

where I_2 is the unit matrix in $\mathbb{R}^{2\times 2}$. Obviously, Condition 3.1 holds. However, Condition 4.1 does not hold for this b; to see this, it suffices to take $y_1 = \beta_1$ and $y_2 = \beta_2$.

The following theorems clarify the relations between Conditions 4.1 and 3.1.

Theorem 4.5. *If Condition* 4.1 *holds, then Condition* 3.1 *holds.*

Let us give some useful criterions of validity of Condition 4.1.

Theorem 4.6. *If* $n = 1$ *and Condition* 3.1 *holds, then Condition* 4.1 *holds.*

Theorem 4.7. *Condition* 4.1 *holds if there exist* $N_0 \in \{1, \dots, N\}$ *and* $\delta_2 > 0$ *such that* $\beta_i \equiv 0$ *for* $i > N_0$ *and*

$$y^\top b(x,t,\omega)\, y - \frac{N_0}{2} |y^\top \beta_i(x,t,\omega)|^2 \ge \delta_2 |y|^2 \tag{4.3}$$

for all $y \in \mathbb{R}^n$*,* $(x,t) \in D \times [0,T]$*,* $\omega \in \Omega$*,* $i = 1, \dots, N_0$.

Corollary 4.8. *If* $N = 1$ *and Condition* 3.1 *holds, then Condition* 4.1 *holds.*

As was mentioned before, Du and Tang [15] obtained an analog of Theorem 4.3 without Condition 4.1.

5 Applications: Robustness of the solutions

Theorem 4.3 helps to establish robustness of the solution of (3.3) with respect to disturbances of the coefficients that are small in L_∞-norm.

Consider problem (3.3), with coefficients

$$(b, f, \lambda, \xi, \beta_i, \overline{\beta}_i, \Psi) = (b^{(k)}, f^{(k)}, \lambda^{(k)}, \xi^{(k)}, \beta_i^{(k)}, \overline{\beta}_i^{(k)}, \Psi^{(k)}), \quad k = 1, 2,$$

such that Conditions 3.1–3.2 and 4.1–4.2 are satisfied for both sets of functions. Let $\mathcal{P}^{(k)}$ be the corresponding sets of parameters. Let $(p^{(k)}, \chi_1^{(k)}, \dots, \chi_N^{(k)})$ be the corresponding solutions of problem (3.3), $k = 1, 2$.

Theorem 5.1. *There exists a constant* $c = c(\mathcal{P}^{(1)}, \|u^{(2)}\|_{Y_2})$ *such that*

$$\|p^{(1)} - p^{(2)}\|_{Y^2} + \sum_{i=1}^N \|\chi_i^{(1)} - \chi_i^{(2)}\|_{X^1} \le cM,$$

where

$$\begin{aligned} M \triangleq \operatorname*{ess\,sup}_{x,t\omega} \Bigg(& |b^{(1)}(x,t,\omega) - b^{(2)}(x,t,\omega)| + |f^{(1)}(x,t,\omega) - f^{(2)}(x,t,\omega)| \\ & + |\lambda^{(1)}(x,t,\omega) - \lambda^{(2)}(x,t,\omega)| \\ & + \sum_{i=1}^N |\beta_i^{(1)}(x,t,\omega) - \beta_i^{(2)}(x,t,\omega)| \\ & + \sum_{i=1}^N |\overline{\beta}_i^{(1)}(x,t,\omega) - \overline{\beta}_i^{(2)}(x,t,\omega)| \Bigg) \\ & + \|\xi^{(1)} - \xi^{(2)}\|_{X^0} + \|\Psi^{(1)} - \Psi^{(2)}\|_{Z_0^1}. \end{aligned}$$

Note that the first fundamental inequality can help to establish robustness only with respect to disturbances of b that are small together with their derivatives in x, and this restriction is necessary even for robustness in X^0. Theorem 5.1 establishes robustness in Y^2 for the disturbances of the coefficients that are small in L_∞-norm only. For instance, if b is replaced for b_ε, where

$$\operatorname*{ess\,sup}_{x,t,\omega} |b_\varepsilon(x,t,\omega) - b(x,t,\omega)| \le \varepsilon$$

for a small $\varepsilon > 0$, then Theorem 5.1 ensures that the corresponding solution of (3.3) is close in Y^2 to the original one.

It can be added that the most important application of the second fundamental inequality for backward SPDEs is the representation theorem for functionals of non-Markov processes and their first exit times from bounded domains, when these functionals are represented via solutions of backward parabolic SPDEs. The previously known results about regularity of the solution of the backward SPDE were insufficient for the case of domains with boundary and, respectively, the representation result was never before obtained for this case. In Dokuchaev [14], it was done using the additional regularity given by the second fundamental inequality from Theorem 4.3. This representation opens ways to a systematic study of first exit times of non-Markov processes.

The rest part of the paper is devoted to the proofs of the results given above.

6 Auxiliary facts used for the proofs

In this section, we list some facts that will be used for the proof of Theorem 4.3. Lemmas 6.1–6.5 given below were obtained in Dokuchaev [13], where their proof can be found.

6.1 Decomposition of operators L and $\mathcal{M}_i$

Introduce operators

$$\begin{aligned} L(s,T) &: X^{-1}(s,T) \to Y^1(s,T),\\ \mathcal{M}_i(s,T) &: X^0(s,T) \to Y^1(s,T),\\ \mathcal{L}(s,T) &: Z_s^0 \to Y^1(s,T) \end{aligned}$$

such that

$$u = L(s,T)\varphi + \mathcal{L}(s,T)\Phi + \sum_{i=1}^{N} \mathcal{M}_i(s,T)h_i,$$

where u is the solution of problem (3.2) in $Y^1(s,T)$. These operators are linear and continuous; it follows immediately from Lemma 3.5. We will denote by L, $\mathcal{M}_i$, and $\mathcal{L}$, the operators $L(0,T)$, $\mathcal{M}_i(0,T)$, and $\mathcal{L}(0,T)$, correspondingly.

For $t \in [0,T]$, define operators $\delta_t : C([0,T]; Z_T^k) \to Z_t^k$ such that

$$\delta_t u = u(\cdot,t).$$

Lemma 6.1. *In the notation of Lemma* 3.6*, the following duality equation is satisfied:*

$$\begin{aligned} p &= L^*\xi + (\delta_T L)^*\Psi,\\ \chi_i &= \mathcal{M}_i^*\xi + (\delta_T \mathcal{M}_i)^*\Psi, \qquad (6.1)\\ p(\cdot,0) &= \mathcal{L}^*\xi + (\delta_T \mathcal{L})^*\Psi, \end{aligned}$$

where

$$L^*: X^{-1} \to X^1, \qquad \mathcal{M}_i^*: X^{-1} \to X^0,$$
$$(\delta_T L)^*: Z_0^0 \to X^1, \quad (\delta_T \mathcal{M}_i)^*: Z_0^0 \to X^0, \quad (\delta_T \mathcal{L})^*: Z_T^0 \to Z_0^0,$$

are the operators that are adjoint to the operators

$$L: X^{-1} \to X^1, \qquad \mathcal{M}_i: X^0 \to X^1,$$
$$\delta_T \mathcal{M}_i: X^{-1} \to Z_T^0, \quad \delta_T \mathcal{M}_i: X^0 \to Z_T^0, \quad \delta_T \mathcal{L}: Z_0^0 \to Z_T^0,$$

respectively.

Our method of proof of the fundamental inequalities is based on decomposition of the operators to superpositions of simpler operators.

Definition 6.2. Define the operators

$$K: Z_0^0 \to Y^1, \quad \mathcal{Q}_0: X^{-1} \to Y^1, \quad \mathcal{Q}_i: X^0 \to Y^1, \quad i = 1, \dots, N,$$

as the operators

$$\mathcal{L}: Z_0^0 \to Y^1, \quad L: X^{-1} \to Y^1, \quad \mathcal{M}_i: X^0 \to Y^1, \quad i = 1, \dots, N,$$

considered for the case when $B_i = 0$ for all i.

By Lemma 3.5, these linear operators are continuous. It follows from the definitions that

$$K\Phi + \mathcal{Q}_0\eta + \sum_{i=1}^N \mathcal{Q}_i h_i = V,$$

where $\eta \in X^{-1}$, $\Phi \in Z_0^0$, and $h_i \in X^0$, and where V is the solution of the problem

$$d_t V = (\mathcal{A}V + \eta)\, dt + \sum_{i=1}^N h_i\, dw_i(t),$$
$$V|_{t=0} = \Phi, \quad V(x,t,\omega)|_{x\in\partial D} = 0. \tag{6.2}$$

Define the operators

$$P \triangleq \sum_{i=1}^N \mathcal{Q}_i B_i, \quad P^* \triangleq \sum_{i=1}^N B_i^* \mathcal{Q}_i^*. \tag{6.3}$$

By Lemma 6.1 and the definitions, the operator $P : X^1 \to X^1$ is continuous, and $P^* : X^{-1} \to X^{-1}$ is its adjoint operator. Hence the operator $P^* : X^{-1} \to X^{-1}$ is continuous. Let

$$P_0 \triangleq \delta_T \sum_{i=1}^{N} \mathcal{Q}_i B_i, \quad P_0^* \triangleq \sum_{i=1}^{N} B_i^* (\delta_T \mathcal{Q}_i)^*.$$

By the definitions, the operator $P_0 : X^1 \to Z_T^0$ is continuous and $P_0^* : Z_T^0 \to X^{-1}$ is its adjoint operator. Hence the operator $P_0^* : Z_T^{\to} X^{-1}$ is continuous.

Lemma 6.3. *The operator $(I - P)^{-1} : X^1 \to X^1$ is continuous, and*

$$\begin{aligned} L &= (I - P)^{-1} \mathcal{Q}_0, \\ \mathcal{M}_i &= (I - P)^{-1} \mathcal{Q}_i, \\ \delta_T L &= P_0 (I - P)^{-1} \mathcal{Q}_0 + \delta_T \mathcal{Q}_0, \\ \delta_T \mathcal{M}_i &= P_0 (I - P)^{-1} \mathcal{Q}_i + \delta_T \mathcal{Q}_i, \end{aligned} \tag{6.4}$$

$i = 1, \dots, N$. The operator $(I - P^*)^{-1} : X^{-1} \to X^{-1}$ is also continuous, and

$$\begin{aligned} L^* &= \mathcal{Q}_0^* (I - P^*)^{-1}, \\ \mathcal{M}_i^* &= \mathcal{Q}_i^* (I - P^*)^{-1}, \\ (\delta_T L)^* &= \mathcal{Q}_0^* (I - P^*)^{-1} P_0^* + (\delta_T \mathcal{Q}_0)^*, \\ (\delta_T \mathcal{M}_i)^* &= \mathcal{Q}_i^* (I - P^*)^{-1} P_0^* + (\delta_T \mathcal{Q}_i)^*. \end{aligned} \tag{6.5}$$

In fact, Lemma 6.3 allows us to represent solution (3.3) via solution of a simpler problem with $B_i \equiv 0$ and an inverse operator $(I - P^*)^{-1}$. It can be illustrated as the following.

Corollary 6.4. (i) *For $\Psi = 0$, the solution $(p, \chi_1, \dots, \chi_N)$ of problem* (3.3) *can be represented as $p = \mathcal{Q}_0^* g$, $\chi_i = \mathcal{Q}_i^* g$, where $g = \xi + \sum_{i=1}^N B_i^* \chi_i$, and where $\sum_{i=1}^N B_i \chi_i = P^* g$.*

(ii) *For general Ψ, the solution $(p, \chi_1, \dots, \chi_N)$ of problem* (3.3) *can be represented as*

$$p = \mathcal{Q}_0^* g + (\delta_T \mathcal{Q}_0)^* \Psi, \quad \chi_i = \mathcal{Q}_i^* g + (\delta_T \mathcal{Q}_i)^* \Psi,$$

where $g = \xi + \sum_{i=1}^N B_i^* \chi_i$, and where $\sum_{i=1}^N B_i \chi_i = P^* g + P_0^* \Psi$. In other words, $g = (I - P^*)^{-1} \xi + (I - P^*)^{-1} P_0^* \Psi$.

It appears that this representation helps to establish the second fundamental inequality.

6.2 Semi-group property for backward equations

It is known that the forward SPDE is casual (or it has the semi-group property): if $u = L\varphi + \mathcal{L}\Phi$, where $\varphi \in X^{-1}$, $\Phi \in Z_0^0$, then

$$u|_{t\in[\theta,s]} = L(\theta, s)\varphi + \mathcal{L}(\theta, s)u(\cdot, \theta). \tag{6.6}$$

To proceed further, we need a similar property for the backward equations.

Lemma 6.5 (Dokuchaev [13]). *Let* $0 \le \theta < s < T$, *and let* $p = L^*\xi$, $\chi_i = \mathcal{M}_i\xi$, *where* $\xi \in X^{-1}$ *and* $\Psi \in Z_T^0$. *Then*

$$p|_{t\in[\theta,s]} = L(\theta, s)^*\xi|_{t\in[\theta,s]} + (\delta_s L(\theta, s))^* p(\cdot, s), \tag{6.7}$$

$$p(\cdot, \theta) = (\delta_\theta \mathcal{L}(\theta, s))^* p(\cdot, s) + \mathcal{L}(\theta, s)^*\xi, \tag{6.8}$$

$$\chi_k|_{t\in[\theta,s]} = \mathcal{M}_k(\theta, s)^*\xi|_{t\in[\theta,s]} + (\delta_s \mathcal{M}_i(\theta, s))^* p(\cdot, s), \quad k = 1, \dots, N. \tag{6.9}$$

Note that this semi-group property implies causality for backward equation (which is a non-trivial fact due the presence of χ).

6.3 A special estimate for deterministic PDEs

We use the notation $\nabla u \triangleq \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right)^\top$ for functions $u : \mathbb{R}^n \to \mathbb{R}$. In addition, we use the notation

$$(u, v)_{H^0} \triangleq \sum_{i=1}^{n} (v_i, u_i)_{H^0}$$

for functions $u, v : D \to \mathbb{R}^n$, where $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$.

For $u \in H^1$, let

$$\begin{aligned} \|u\|_{\widehat{H}^1(t,\omega)} &\triangleq (\nabla u, b(\cdot, t, \omega)\nabla u)^{1/2} \\ &= \left(\sum_{i,j=1}^{n} \int_D \frac{\partial u}{\partial x_i}(x) b_{ij}(x, t, \omega) \frac{\partial u}{\partial x_j}(x) dx \right)^{1/2}. \end{aligned} \tag{6.10}$$

For $K > 0$, introduce the operator $\mathcal{A}_K^* = \mathcal{A}^* - KI$, i.e., $\mathcal{A}_K^* u = \mathcal{A}^* u - Ku$.

Lemma 6.6. *Let* $\theta, \tau \in [0, T]$ *be given,* $0 \le \theta < \tau \le T$. *Let the function*

$$b(x, t, \omega) = b(x, \omega)$$

be constant in $t \in [\theta, \tau]$ *for a.e.* x, ω. *Let* $h = h(x, t, \omega) \in L_2(D \times [\theta, \tau])$, *and let* $u = u(x, t, \omega) : D \times [\theta, \tau] \times \Omega \to \mathbb{R}$ *be the solution of the boundary value* *problem*

$$\frac{\partial u}{\partial t} + \mathcal{A}_K^* u = -h, \quad t \in (\theta, \tau)$$
$$u(x,\tau) = 0, \quad u(x,t)|_{x\in\partial D} = 0, \tag{6.11}$$

Then for any $\varepsilon > 0$, $M > 0$, *there exists* $K = K(\varepsilon, M, \mathcal{P}) > 0$ *such that*

$$\sup_{t\in[\theta,\tau]} \|u(\cdot,t,\omega)\|^2_{\tilde{H}^1(t,\omega)} + M \sup_{t\in[\theta,\tau]} \|u(\cdot,t,\omega)\|^2_{H^0}$$
$$\leq \frac{1+\varepsilon}{2} \int_\theta^\tau \|h(\cdot,t,\omega)\|^2_{H^0} dt \quad a.s.$$

This lemma follows immediately from Dokuchaev [12, Theorem 1 and Corollary 1]; it first appeared in Dokuchaev [11].

7 The proof of Theorem 4.3

By Lemma 6.1, it suffices to show that the operators

$$L^* : X^0 \to Y^2, \quad (\delta_T L)^* : Z_T^1 \to Y^2$$

and

$$\mathcal{M}_i^* : X^0 \to X^1, \quad (\delta_T \mathcal{M}_i)^* : Z_T^1 \to X^1$$

are continuous, and that their norms are less or equal than a constant $c = c(\mathcal{P})$.

We define the operators $L^*(s,T)$, $\mathcal{M}_i^*(s,T)$, $(\delta_T L(s,T))^*$, $(\delta_T \mathcal{M}_i(s,T))^*$, similarly to L^*, $\mathcal{M}_i^*$, $(\delta_T L)^*$, $(\delta_T \mathcal{M}_i)^*$, with the time interval $[0,T]$ replaced by $[s,T]$.

We denote by $\overline{\mathcal{P}_T}$ the completion (with respect to the measure $\overline{\ell}_1 \times \mathbf{P}$) of the σ-algebra of subsets of $[0,T] \times \Omega$, generated by functions that are progressively measurable with respect to $\overline{\mathcal{B}}_1 \times \mathcal{F}_T$. Let

$$\overline{X}^k \triangleq L^2\big([0,T] \times \Omega, \overline{\mathcal{P}}_T, \overline{\ell}_1 \times \mathbf{P}; H^k\big).$$

Let $\mathcal{E}$ be the operator of projection of $\overline{X}^1$ onto X^1.

For $\xi \in X^0$ and $\Psi \in Z_T^1$, let $\overline{p}$ be the solution of the boundary value problem in Q

$$\frac{\partial \overline{p}}{\partial t} + \mathcal{A}^* \overline{p} = -\xi, \quad t \leq T,$$
$$\overline{p}|_{t=T} = \Psi, \quad \overline{p}(x,t,\omega)|_{x\in\partial D} = 0. \tag{7.1}$$

By the second fundamental inequality for deterministic parabolic equations, it follows that the solution of (7.1) is such that $\overline{p} \in \overline{X}^2 \cap \mathcal{C}^1$, (7.2) holds and

$$\|\overline{p}\|_{\overline{X}^2} + \|\overline{p}\|_{\mathcal{C}^1} \leq c\big(\|\xi\|_{X^0} + \|\Psi\|_{Z_T^0}\big), \tag{7.2}$$

where $c = c(\mathcal{P}) > 0$ is a constant. If the function $b(x,t,\omega)$ is almost surely continuous in (x,t), then inequality (7.2) follows from Theorem IV.9.1 from Ladyzenskaya et al. [22]. Since the derivative $\partial b/\partial x$ is bounded, the condition that b is continuous can be lifted; this fact is well known. In particular, (7.2) follows in this case from Dokuchaev [10, Theorem 3.1].

By the Martingale Representation Theorem, there are functions $\gamma_i(\cdot,t,\cdot) \in X^0$ such that

$$\overline{p}(x,t,\omega) = \mathbf{E}\{\overline{p}(x,t,\omega)|\mathcal{F}_0\} + \sum_{i=1}^{N} \int_0^T \gamma_i(x,t,s,\omega)dw_i(s). \tag{7.3}$$

Lemma 7.1. *Assume the function $\mu = (b, f, \lambda)$ is such that $\mu(x,t,\omega)$ is $\mathcal{F}_0$-measurable for all $x \in D$. Let $\xi \in X^0$, $\Psi \in Z_T^1$, let $\overline{p}$ be the solution of* (7.1)*, and let γ_j be the processes presented in* (7.3)*. Let $p, \chi_1, \dots, \chi_2$ be defined as*

$$p \triangleq \mathcal{E}\overline{p}, \quad \chi_i(x,s,\omega) \triangleq \gamma_i(x,s,s,\omega). \tag{7.4}$$

Then $p \in Y^1$, $\chi_i \in X^1$, and

$$\|p\|_{Y^2} + \sum_{i=1}^{N} \|\chi_i\|_{X^1} \le c\big(\|\xi\|_{X^0} + \|\Psi\|_{Z_T^0}\big), \tag{7.5}$$

where $c = c(\mathcal{P}) > 0$ is a constant. In addition,

$$p = \mathcal{Q}_0^*\xi + (\delta_T \mathcal{Q}_0)^*\Psi, \quad \chi_i = \mathcal{Q}_i^*\xi + (\delta_T \mathcal{Q}_i)^*\Psi. \tag{7.6}$$

Proof. By the Martingale Representation Theorem, there exist functions

$$\gamma_i(\cdot,t,\cdot) \in X^0, \quad \gamma_{\xi i}(\cdot,t,\cdot) \in X^0 \quad \text{and} \quad \gamma_{\Psi i}(\cdot) \in X^1,$$

such that (7.3) holds as well as

$$\xi(x,t,\omega) = \mathbf{E}\{\xi(x,t,\omega)|\mathcal{F}_0\} + \sum_{i=1}^{N} \int_0^T \gamma_{\xi i}(x,t,s,\omega)dw_i(s),$$

$$\Psi(x,\omega) = \mathbf{E}\{\Psi(x,\omega)|\mathcal{F}_0\} + \sum_{i=1}^{N} \int_0^T \gamma_{\Psi i}(x,s,\omega)dw_i(s).$$

Moreover, it follows that

$$\mathcal{D}g_i(\cdot,t,\cdot) \in X^0,$$

where either $\mathcal{D}\gamma = \partial\gamma/\partial t$ or $\mathcal{D}\gamma = \mathcal{A}^*\gamma$, and

$$\mathcal{D}\overline{p}(x,t,\omega) = \mathbf{E}\{\mathcal{D}\overline{p}(x,t,\omega)|\mathcal{F}_0\} + \sum_{i=1}^{N} \int_0^T \mathcal{D}\gamma_i(x,t,s,\omega)dw_i(s).$$

By (7.1), it follows that

$$\begin{aligned} &\frac{\partial \gamma_i}{\partial t}(\cdot,t,s,\omega) + \mathcal{A}^*\gamma_i(\cdot,t,s,\omega) = -\gamma_{\xi i}(\cdot,t,s,\omega), \quad t \in (0,T),\\ &\gamma_i(x,T,s,\omega) = \gamma_{\Psi i}(x,s,\omega), \quad \gamma_i(x,t,s,\omega)|_{x\in\partial D} = 0. \end{aligned} \tag{7.7}$$

Again, it follows from the second fundamental inequality for deterministic parabolic equations that

$$\sup_{t\in[s,T]} \|\gamma_i(\cdot,t,s,\omega)\|^2_{H^1} \le c\left(\int_s^T \|\gamma_{\xi i}(\cdot,t,s,\omega)\|^2_{H^0}dt + \|\gamma_{\Psi i}(\cdot,s,\omega)\|^2_{H^1}\right),$$

where $c = c(T,n,D) > 0$ is a constant. Hence

$$\|\gamma_i(\cdot,s,s,\omega)\|^2_{H^1} \le c\left(\int_s^T \|\gamma_{\xi i}(\cdot,t,s,\omega)\|^2_{H^0}dt + \|\gamma_{\Psi i}(\cdot,s,\omega)\|^2_{H^1}\right).$$

This estimate together with (7.2) ensures that (7.5) holds for p and χ_i defined by (7.4).

Let us show that (7.6) holds. Clearly,

$$\overline{p}(x,t,\omega) = p(x,t,\omega) + \sum_{i=1}^N \int_t^T \gamma_i(x,t,s,\omega)dw_i(s),$$

and

$$\overline{p}(\cdot,t) = \int_t^T \big(\mathcal{A}^*\overline{p}(\cdot,s) + \xi(\cdot,s)\big)\,ds.$$

Hence

$$\begin{aligned} p(\cdot,t) = \Psi &+ \int_t^T \big(\mathcal{A}^* p(\cdot,s) + \xi(\cdot,s)\big)\,ds\\ &+ \sum_{i=1}^N \Big[\int_t^T ds \int_s^T \big[\mathcal{A}^*\gamma_i(\cdot,s,r) + \gamma_{\xi i}(\cdot,s,r)\big]dw_i(r)\\ &\qquad - \int_t^T \gamma_i(\cdot,t,s)dw_i(s)\Big]\\ = \Psi &+ \int_t^T \big(\mathcal{A}^* p(\cdot,s) + \xi(\cdot,s)\big)\,ds\\ &+ \sum_{i=1}^N \Big[\int_t^T dw_i(r) \int_t^r \big[\mathcal{A}^*\gamma_i(\cdot,s,r) + \gamma_{\xi i}(\cdot,s,r)\big]ds\\ &\qquad - \int_t^T \gamma_i(\cdot,t,s)dw_i(s)\Big] \end{aligned}$$

$$= \Psi + \int_t^T \big(\mathcal{A}^* p(\cdot,s) + \xi(\cdot,s)\big)\, ds$$
$$+ \sum_{i=1}^N \int_t^T dw_i(s) \bigg[\int_t^s [\mathcal{A}^* \gamma_i(\cdot,r,s) + \gamma_{\xi i}(\cdot,r,s)]dr - \gamma_i(\cdot,t,s) \bigg].$$

By (7.7),

$$\gamma_i(\cdot,t,s) - \int_t^s [\mathcal{A}^* \gamma_i(\cdot,r,s) + \gamma_{\xi i}(\cdot,r,s)]dr = \gamma_i(\cdot,s,s).$$

By (7.4), we have selected $\gamma_i(\cdot,s,s) = \chi_i(\cdot,s)$. It follows that

$$p(\cdot,t) = \Psi + \int_t^T \big(\mathcal{A}^* p(\cdot,s) + \xi(\cdot,s)\big)\, ds - \sum_{i=1}^N \int_t^T \chi_i(\cdot,s)\, dw_i(s).$$

Finally, we obtain (7.6) from Lemma 6.1 applied to the operators $\mathcal{Q}_0^*, \mathcal{Q}_i^*, (\delta_T \mathcal{Q}_0)^*$ and $(\delta_T \mathcal{Q}_i)^*$, $i = 1,\dots,N$, considered as special cases of L^*, $\mathcal{M}_i^*$, $(\delta_T L_0)^*$ and $(\delta_T \mathcal{M}_i)^*$, respectively. This completes the proof of Lemma 7.1. □

In the following proof, we will explore the following observation. Assume that λ is replaced by

$$\lambda^{(K)}(x,t,\omega) \triangleq \lambda(x,t,\omega) + K,$$

i.e., $\mathcal{A}$ is replaced by $\mathcal{A}_K = \mathcal{A}v + KI$. In this case, the solution u of problem (3.2) has to be replaced by the process

$$u(x,t,\omega)e^{-Kt}.$$

Respectively, the solution $(p,\chi_1,\dots,\chi_N)$ of problem (3.3) has to be replaced by the process

$$\big(p(x,t,\omega)e^{K(T-t)}, \chi_1(x,t,\omega)e^{K(T-t)}, \dots, \chi_N(x,t,\omega)e^{K(T-t)}\big).$$

Therefore, it suffices to prove theorem for any case when λ is replaced for

$$\lambda^{(K)}(x,t,\omega) \triangleq \lambda(x,t,\omega) + K$$

with some $K > 0$, and this K can be taken arbitrarily large.

For linear normed spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\|\mathcal{T}\|_{\mathcal{X},\mathcal{Y}}$ the norm of an operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$.

Lemma 7.2. *Let $0 \le s < T$, and assume that the function $\mu = (b,f,\lambda)$ is such that $\mu(x,t,\cdot)$ is $\mathcal{F}_s$-measurable for all $x \in D$, $t \in [s,T)$. Moreover, we assume that $b(x,t,\omega) = b(x,\omega)$ does not depend on $t \in [s,T]$. Then there exists $K > 0$*

such that if λ *is replaced by* $\lambda(x,t,\omega)+K$*, then*

$$\|L^*(s,T)\|_{X^0(s,T),Y^2(s,T)} + \|(\delta_T L(s,T))^*\|_{Z^1_T,Y^2(s,T)} + \sum_{i=1}^n \|\mathcal{M}_i^*(s,T)\|_{X^0(s,T),X^1} + \sum_{i=1}^n \|(\delta_T \mathcal{M}_i(s,T))^*\|_{Z^1_T,X^1(s,T)} \le c,$$

where $c \in (0,+\infty)$ *depends only on* K *and* $\mathcal{P}$*.*

Proof. To simplify the notations, we consider only the case when $s=0$.

By (6.1) and (6.5), it suffices to show that the operator $(I-P^*)^{-1} : X^0 \to X^0$ is continuous. For this, it suffices to show that there exists $K>0$ such that if λ is replaced for $\lambda(x,t,\omega)+K$, then $\|P^*\|_{X^0,X^0} < 1$.

Let $\xi \in X^0$, let $\overline{p}$ be the solution of (7.1), and let γ_j be the processes presented in (7.3) with $\Psi = 0$. Let $p, \chi_1, \dots, \chi_2$ be defined by (7.4) with $\Psi=0$. In this case,

$$\begin{aligned} &\frac{\partial \gamma_i}{\partial t}(\cdot,t,s,\omega) + \mathcal{A}^*\gamma_i(\cdot,t,s,\omega) = -\gamma_{\xi i}(\cdot,t,s,\omega), \quad t\in(0,T),\\ &\gamma_i(x,T,s,\omega) = 0, \quad \gamma_i(x,t,s,\omega)|_{x\in\partial D} = 0. \end{aligned} \tag{7.8}$$

By Lemma 6.6 applied to boundary value problem (7.8), for any $\varepsilon>0$, $M>0$, there exists $K = K(\varepsilon, M, \mathcal{P}) > 0$ such that

$$\sup_{t\in[s,T]} \|\gamma_i(\cdot,t,s,\omega)\|^2_{\tilde{H}^1(t,\omega)} + M \sup_{t\in[s,T]} \|\gamma_i(\cdot,t,s,\omega)\|^2_{H^0} \le \frac{1+\varepsilon}{2}\int_s^T \|\gamma_{\xi i}(\cdot,t,s,\omega)\|^2_{H^0} dt \quad \text{a.s.}$$

Here $\|\cdot\|_{\tilde{H}^1(t,\omega)}$ is defined by (6.10). Hence

$$\int_0^T \|\gamma_i(\cdot,s,s,\omega)\|^2_{\tilde{H}^1(t,\omega)} ds + M\int_0^T \|\gamma_i(\cdot,s,s,\omega)\|^2_{H^0} ds \le \frac{1+\varepsilon}{2}\int_0^T ds \int_s^T \|\gamma_{\xi i}(\cdot,t,s,\omega)\|^2_{H^0} dt.$$

Note that

$$\mathbf{E}\sum_{i=1}^N \int_0^T dt \int_0^T \|\gamma_{\xi i}(\cdot,t,s,\omega)\|^2_{H^0} ds \le \|\xi\|^2_{X^0}.$$

Hence

$$\mathbf{E}\int_0^T \|\gamma_i(\cdot,s,s,\omega)\|^2_{\tilde{H}^1(t,\omega)} ds + M\mathbf{E}\int_0^T \|\gamma_i(\cdot,s,s,\omega)\|^2_{H^0} ds \le \frac{1+\varepsilon}{2}\|\xi\|^2_{X^0}.$$

By (7.6), it can be rewritten as

$$\begin{aligned}&\mathbf{E}\int_0^T \|\chi_i(\cdot,t,\omega)\|^2_{\tilde H^1(t,\omega)}dt+M\mathbf{E}\int_0^T \|\chi_i(\cdot,t,\omega)\|^2_{H^0}dt\\&\quad\le\frac{1+\varepsilon}{2}\|\xi\|^2_{X^0}.\end{aligned}\tag{7.9}$$

Remind that

$$P^*\xi=\sum_{j=1}^N B_j^*\chi_j.$$

By Condition 4.1, there exists $M=M(\mathcal{P})>0$ such that

$$\begin{aligned}\left\|\sum_{j=1}^N B_j^*\chi_j\right\|^2_{H^0}\le 2\sum_{j=1}^N\|\chi_j\|^2_{\tilde H^1(t,\omega)}+2M\sum_{j=1}^N\|\chi_j\|^2_{H^0}\\-2\delta_1\sum_{j=1}^N\|\nabla\chi_j\|^2_{H^0}\quad\forall t,\omega.\end{aligned}\tag{7.10}$$

By (7.9) and (7.10), it follows that a small enough $\varepsilon>0$ and a large enough $K>0$ can be found such that

$$\|P^*\xi\|^2_{X^0}=\left\|\sum_{i=1}^N B_i^*\chi_i\right\|^2_{X^0}\le c\|\xi\|^2_{X^0}$$

for this K with some $c=c(\mathcal{P},K)<1$. Hence

$$\|P^*\xi\|_{X^0}\le\sqrt{c}\|\xi\|_{X^0}.$$

Therefore, we have proved that there exists $K=K(\mathcal{P})>0$ such that if λ is replaced for $\lambda(x,t,\omega)+K$, then $\|P^*\|_{X^0,X^0}<1$, and, therefore, the operator

$$(I-P^*)^{-1}:X^0\to X^0$$

is continuous. By the first equation in (7.6), the operator $\mathcal{Q}_0^*:X^0\to Y^2$ is continuous. In addition, it follows from (7.6) and (7.9) that the operators $\mathcal{Q}_i^*:X^0\to X^1$ are continuous. Then the proof of Lemma 7.2 for the special case of $\Psi=0$ follows from the first equations for adjoint operators in (6.5).

To complete the proof of Lemma 7.2 for general Ψ, we note that by (7.5) and (7.6), it follows that it suffices to show that the operators $(\delta_T\mathcal{Q}_0)^*:Z_T^1\to Y^2$ and $(\delta_T\mathcal{Q}_i)^*:Z_T^1\to X^1$ are continuous, $i=1,\dots,N$. In addition, the upper bound of the norms of these operators depends on $\mathcal{P}$ only. Then the proof follows from the last two equations for the adjoint operators in (6.5). This completes the proof of Lemma 7.2. □

For an integer $M > 0$, denote by $\Theta(M)$ the class of all functions $\mu = (b, f, \lambda)$ such that all conditions imposed in Section 3 are satisfied and that there exists a finite set $\{t_i\}_{i=0}^{M}$ such that

- $0 = t_0 < t_1 < \cdots < t_M = T$,
- the function $\mu(x,t,\cdot) = (b(x,t,\cdot), f(x,t,\cdot), \lambda(x,t,\cdot))$ is $\mathcal{F}_{t_i}$-measurable for all $x \in D$, $t \in [t_i, t_{i+1})$,
- the function $b(x,t,\omega) = b(x,\omega)$ does not depend on t for $t \in [t_i, t_{i+1})$.

Let $\Theta \triangleq \bigcup_{M>0} \Theta(M)$.

Lemma 7.3. *Let $(b, f, \lambda) \in \Theta(M)$ for some $M > 0$. Then there exists $K > 0$ such that if λ is replaced by $\lambda(x,t,\omega) + K$, then*

$$\|L^*\|_{X^0,Y^1} + \sum_{i=1}^{n} \|\mathcal{M}_i^*\|_{X^0,X^1} + \|(\delta_T L)^*\|_{Z_T^1,Y^1} + \sum_{i=1}^{n} \|(\delta_T \mathcal{M}_i)^*\|_{Z_T^1,X^1} \le c,$$

where $c \in (0, +\infty)$ does not depend on M and depends only on K and $\mathcal{P}$.

Proof. The proof of this lemma follows immediately from Lemma 6.5 and from Lemma 7.2 applied consequently for all time intervals from the definition of $\Theta(M)$ backward from terminal time. □

Corollary 7.4. *Under the assumption of Lemma 7.3, Theorem 4.3 holds and there exists $K > 0$ such that the operators*

$$L^* : X^0 \to Y^2, \qquad \mathcal{M}_j^* : X^0 \to X^1,$$
$$(\delta_T L)^* : Z_T^1 \to Y^2, \quad (\delta_T \mathcal{M}_i)^* : Z_T^1 \to X^1, \quad j = 1, \dots, N,$$

are continuous, and their norms do not depend on M.

Up to the end of this section, we assume that λ is replaced for $\lambda(x,t,\omega) + K$ such that the conclusion of Lemma 7.3 holds.

Now we are in a position to prove Theorem 4.3 for the case of (b, f, λ) of the general kind.

Let $M = 1, 2, \dots, M \to +\infty$. Let $\varepsilon \triangleq M^{-1}$. By Condition 4.2, there exists a subsequence of M such that there exists $b_\varepsilon \in \Theta_b(M)$ for any M with the corresponding sets $\{t_k\} = \{t_k(M)\}$ with $0 = t_0 < \cdots < t_k < t_M = T$ such that $\max_k |t_k - t_{k-1}| \to 0$ as $M \to +\infty$, $b_\varepsilon(x,t,\omega) = b(t_k,t,\omega)$, $t \in [t_k, t_{k+1})$, and that there exist $q, r \in [1, +\infty]$ such that

$$b_\varepsilon \to b \quad \text{in } \mathcal{W}_{q,r}^1 \quad \text{as } \varepsilon \to 0.$$

Further, we introduce functions f_ε, λ_ε such that

$$f_\varepsilon(x,t,\omega) = \mathbf{E}\{f(x,t,\omega)|\mathcal{F}_{t_k}\},$$
$$\lambda_\varepsilon(x,t,\omega) = \mathbf{E}\{\lambda(x,t,\omega)|\mathcal{F}_{t_k}\}, \quad t \in [t_k, t_{k+1}).$$

Proposition 7.5. *Let us show that Condition* 4.1 *implies the following:*

(a) *Condition* 4.1 *is satisfied for* b *replaced by* b_ε, *with the same* $\delta_1 > 0$ *for all* ε,

(b) *without a loss of generality, we can assume that* $\sup_{\varepsilon>0} \|b_\varepsilon\|_{\mathcal{W}^1_\infty} < +\infty$.

Proof. It suffices to show that Condition 4.1 (i) implies (a) and Condition 4.1 (ii) implies (b).

Let us show that Condition 4.1 (i) implies (a). Let $\mathrm{A} = \mathrm{A}(x,t,\omega) \in \mathbb{R}^{nN\times nN}$ be the symmetric matrix that defines the quadratic form on the vectors

$$Y = (y_1,\dots,y_N) \in \mathbb{R}^{nN}$$

in (4.1), and let A_ε be the similar matrix defined for $b = b_\varepsilon$. By Condition 4.1, the minimal eigenvalue of A is positive and is separated from zero uniformly over $\varepsilon, x, t, \omega$. By the definitions, it follows that

$$\|\mathrm{A}_\varepsilon - \mathrm{A}\|_{\mathcal{W}^0_\infty} \to 0.$$

Since the minimal eigenvalue of a matrix depends continuously on its coefficients, it follows that the minimal eigenvalue of A_ε is positive and is separated from zero uniformly over $\varepsilon, x, t, \omega$. Hence Condition 4.1 (i) implies (a).

Let us show that Condition 4.1 (ii) implies (b). Let $R \stackrel{\Delta}{=} \|b\|_{\mathcal{W}^1_\infty}$, and let γ be the supremum over x, t, ω of the maximal eigenvalue of $b(x,t,\omega)$. It suffices to show that, without a loss of generality, we can assume that

$$\sup_\varepsilon \|b_\varepsilon\|_{\mathcal{W}^1_\infty} \le n\gamma + 2R + 1. \tag{7.11}$$

Suppose that (7.11) does not hold, i.e., that there exists some M such that for $\varepsilon = M^{-1}$ and some $t_k = t_k(M)$ there exists $\Gamma \subset \Omega$ such that $\Gamma \in \mathcal{F}_{t_k}$, $\mathbf{P}(\Gamma) > 0$,

$$b_\varepsilon(\,\cdot\,,t,\omega) \equiv b_\varepsilon(t_k,x,\omega), \quad t \in [t_k, t_{k+1}),$$

and

$$\|b_\varepsilon(\,\cdot\,,t_k,\omega)\|_{W^1_\infty(D)} > n\gamma + 2R + 1 \quad \text{iff} \quad \omega \in \Gamma.$$

In this case, one can replace $b_\varepsilon(\,\cdot\,,t)|_{t\in[t_k,t_{k+1})}$ by

$$\tilde{b}_\varepsilon(x,t,\omega) = b_\varepsilon(x,t,\omega)\mathbb{I}_{\Omega\setminus\Gamma}(\omega) + \gamma I_n \mathbb{I}_\Gamma(\omega), \quad t \in [t_k, t_{k+1}),$$

where $\mathbb{I}$ is the indicator function, and where I_n is the unit matrix in $\mathbb{R}^n$. Obviously, Condition 4.1 is satisfied for $\tilde{b}_\varepsilon$ replacing b_ε, with the same $\delta_1 > 0$ for all ε. In addition, we have that

$$\|\tilde{b}^{(M)} - b\|_{W^1_\infty(D)} \le \|\tilde{b}^{(M)}|_{W^1_\infty(D)} + \|b\|_{W^1_\infty(D)} \le n\gamma + R, \quad \omega \in \Gamma,$$

and

$$\begin{aligned}\|b^{(M)} - b\|_{W^1_\infty(D)} &\ge \|\tilde{b}^{(M)}\|_{W^1_\infty(D)} - \|b\|_{W^1_\infty(D)} \\ &\ge n\gamma + 2R - R = n\gamma + R, \quad \omega \in \Gamma.\end{aligned}$$

It follows that Condition 4.2 holds for the new selection $\tilde{b}_\varepsilon$. This completes the proof of Proposition 7.5. □

Further, it follows from Proposition 7.5 and from the definitions that

$$\begin{aligned}\sup_{x,t,\omega,\varepsilon}\bigg(&|b_\varepsilon(x,t,\omega)| + \left|\frac{\partial b_\varepsilon}{\partial x}(x,t,\omega)\right| + |f_\varepsilon(x,t,\omega)| + \left|\frac{\partial f_\varepsilon}{\partial x}(x,t,\omega)\right| \\ &+ |\lambda_\varepsilon(x,t,\omega)|\bigg) < +\infty.\end{aligned} \tag{7.12}$$

Let us consider a subsequence $\varepsilon = \varepsilon_i \to 0$ such that

$$\begin{aligned}&b_\varepsilon \to b, \quad f_\varepsilon \to f, \quad \lambda_\varepsilon \to \lambda, \\ &\frac{\partial b_\varepsilon}{\partial x} \to \frac{\partial b_\varepsilon}{\partial x}, \quad \frac{\partial f_\varepsilon}{\partial x} \to \frac{\partial f}{\partial x} \quad \text{in } X^0 \text{ and a.e. as } \varepsilon \to 0.\end{aligned} \tag{7.13}$$

Let

$$p_\varepsilon \triangleq L^*_\varepsilon \xi + (\delta_T \mathcal{L}_\varepsilon)^* \Psi, \quad \chi_{i\varepsilon} \triangleq \mathcal{M}^*_\varepsilon \xi + (\delta_T \mathcal{M}_{i\varepsilon})^* \Psi,$$

and let

$$p \triangleq L^* \xi + (\delta_T \mathcal{L})^* \Psi, \quad \chi_i \triangleq \mathcal{M}^*_i \xi + (\delta_T \mathcal{M}_i)^* \Psi.$$

The operators $L^*_\varepsilon : X^{-1} \to Y^1$ etc. are defined similarly to $L^* : X^{-1} \to Y^1$ etc. with substituting $(b, f, \lambda) = (b_\varepsilon, f_\varepsilon, \lambda_\varepsilon)$.

By Lemma 7.3, the sequences $\{p_\varepsilon\}$ and $\{\chi_{i\varepsilon}\}$ belong to the closed balls in the spaces X^2 and X^1 respectively with the centers at the zero and with the radius $c(\|\xi\|_{X^0} + \|\Psi\|_{Z^1_T})$, where $c = c(\mathcal{P}) > 0$ does not depend on ε. The balls mentioned are closed, concave, and bounded. It follows that these balls are weakly closed and weakly compact in the reflexible Banach spaces X^2 and X^1 respectively. It follows that the sequences $\{p_\varepsilon\}$ and $\{\chi_{i\varepsilon}\}$ have subsequences with weak limits $\tilde{p}$ and $\tilde{\chi}_i$ in the corresponding balls, i.e.,

$$\|\tilde{p}\|_{X^2} + \sum_{i=1}^N \|\tilde{\chi}_i\|_{X^1} \le c(\|\xi\|_{X^0} + \|\Psi\|_{Z^1_T}).$$

Assume that we can show that $\tilde{p}_\varepsilon \to p$ weakly in X^2 and $\chi_{i\varepsilon} \to \chi_i$ weakly in X^1 for all i. It follows that $\tilde{p} = p$ and $\tilde{\chi}_i = \chi_i$ and

$$\|p\|_{X^2} + \sum_{i=1}^N \|\chi_i\|_{X^1} \le c(\|\xi\|_{X^0} + \|\Psi\|_{Z_T^1}). \tag{7.14}$$

It follows that

$$\left\| \sum_{i=1}^N B_i^* \chi_i \right\|_{X^0} \le c_1(\|\xi\|_{X^0} + \|\Psi\|_{Z_T^1}),$$

where $c_1 = c_1(\mathscr{P})$ is a constant. Hence $g \triangleq \xi + \sum_{i=1}^N B_i^* \chi_i$ is such that

$$\|g\|_{X^0} \le c_2(\|\xi\|_{X^0} + \|\Psi\|_{Z_T^1}),$$

where $c_2 = c_2(\mathscr{P})$ is a constant. Remind that, by Lemma 6.3 and Corollary 6.4,

$$g = (I - P^*)^{-1}\xi + (I - P^*)^{-1} P_0^* \Psi, \quad p = \mathcal{Q}_0^* g + (\delta_T \mathcal{Q}_0)^* \Psi.$$

By Lemma 7.1, it follows that $p \in Y^2$ and

$$\|p\|_{Y^2} + \left\| \sum_{i=1}^N B_i^* \chi_i \right\|_{X^0} \le c_3(\|g\|_{X^0} + \|\Psi\|_{Z_T^1}),$$

where $c_3 = c_3(\mathscr{P})$. By (7.15), it follows that $p \in Y^2$ and

$$\|p\|_{Y^2} + \left\| \sum_{i=1}^N B_i^* \chi_i \right\|_{X^0} \le c_4(\|\xi\|_{X^0} + \|\Psi\|_{Z_T^1}),$$

where $c_4 = c_4(\mathscr{P})$ is a constant. Then the proof of Theorem 4.3 follows provided that the weak convergence of the sequence $\{\chi_{i\varepsilon}\}$ to χ_i is established.

Therefore, it suffices prove this weak convergence, i.e., it suffices to show that

$$I_\varepsilon \triangleq (p_\varepsilon - p, h)_{X^0} \to 0 \qquad \text{as } \varepsilon \to 0 \quad \forall h \in X^0, \tag{7.15}$$

$$J_\varepsilon \triangleq (\chi_{i\varepsilon} - \chi_i, h)_{X^0} \to 0 \quad \text{as } \varepsilon \to 0 \quad \forall h \in X^0,\ i \in \{1, \dots, N\}. \tag{7.16}$$

Let us show that (7.15) holds. Set $u_\varepsilon \triangleq L_{i\varepsilon} h$ and $u \triangleq L_i h$, where the operators $L_{i\varepsilon} : X^0 \to Y^1$ are defined similarly to the operators $L_i : X^0 \to Y^1$ with substituting $(b, f, \lambda) = (b_\varepsilon, f_\varepsilon, \lambda_\varepsilon)$. By the definitions of the corresponding adjoint operators,

$$\begin{aligned} I_\varepsilon &= (L_{i\varepsilon}^* \xi - L_i^* \xi, h)_{X^0} + ((\delta_T L_{i\varepsilon})^* \Psi - (\delta_T L_i)^* \Psi, h)_{X^0} \\ &= (\xi, u_\varepsilon - u)_{X^0} + (\Psi, u_\varepsilon(\cdot, T) - u(\cdot, T))_{Z_T^0}. \end{aligned}$$

Let the operators $\mathcal{A}_\varepsilon$ be defined similarly to $\mathcal{A}$ with substituting

$$(b, f, \lambda) = (b_\varepsilon, f_\varepsilon, \lambda_\varepsilon).$$

By the definitions, it follows that there exist functions

$$\widehat{f}_\varepsilon(x,t,\omega) : \mathbb{R}^n \times \mathbb{R}_+ \times \Omega \to \mathbb{R}^n \quad \text{and} \quad \widehat{\lambda}_\varepsilon(x,t,\omega) : \mathbb{R}^n \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$$

such that

$$\sup_{\varepsilon>0} \operatorname*{ess\,sup}_{x,t,\omega} \Big(|\widehat{f}_\varepsilon(x,t,\omega)| + |\widehat{\lambda}_\varepsilon(x,t,\omega)| \Big) < +\infty,$$

and that $\mathcal{A}_\varepsilon u - \mathcal{A} u$ is represented as

$$\mathcal{A}_\varepsilon u - \mathcal{A} u = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left([b_{ij\varepsilon} - b_{ij}] \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n \widehat{f}_{i\varepsilon} \frac{\partial u}{\partial x_i} + \widehat{\lambda}_\varepsilon u.$$

By (7.12)–(7.13), it follows that

$$\widehat{f}_\varepsilon \to 0 \quad \text{and} \quad \widehat{\lambda}_\varepsilon \to 0 \quad \text{a.e. and in } X^0.$$

The function $U_\varepsilon \triangleq u_\varepsilon - u$ is the solution in Q of the boundary value problem

$$\begin{aligned} &d_t U_\varepsilon = (\mathcal{A}_\varepsilon U_\varepsilon + F_\varepsilon(u))\, dt + \sum_{i=1}^N B_i U_\varepsilon\, dw_i(t), \\ &U_\varepsilon(x,0) = 0, \quad U_\varepsilon(x,t)|_{x\in\partial D} = 0, \end{aligned}$$

and where the linear operator $F_\varepsilon(\cdot)$ is defined as

$$\begin{aligned} F_\varepsilon(u) &\triangleq r_\varepsilon(u) + q_\varepsilon(u), \\ r_\varepsilon(u) &\triangleq \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left([\widehat{b}_{\varepsilon ij} - b_{ij}] \frac{\partial u}{\partial x_j} \right), \\ q_\varepsilon(u) &\triangleq \frac{\partial u}{\partial x} \widehat{f}_\varepsilon + \widehat{\lambda}_\varepsilon u. \end{aligned}$$

Here $\widehat{b}_{\varepsilon ij}$ are the components of the matrix $\widehat{b}_\varepsilon$. By Lemma 3.5, it follows that

$$\|U_\varepsilon\|_{Y^1} \le C \|F_\varepsilon(u)\|_{X^{-1}},$$

for a constant $C_1 = C_1(\mathcal{P})$. It follows that there exists a constant $C = C(\mathcal{P}) > 0$ such that

$$|I_\varepsilon| \le C \|U_\varepsilon\|_{Y^1} (\|\xi\|_{X^{-1}} + \|\Psi\|_{Z_0^T}) \le C \|F_\varepsilon(u)\|_{X^{-1}} (\|\xi\|_{X^0} + \|\Psi\|_{Z_1^T}).$$

We have that

$$\|r_\varepsilon(u)\|^2_{X^{-1}} = \mathbf{E}\int_0^T \left\| \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left([\widehat{b}_{\varepsilon ij} - b_{ij}] \frac{\partial u}{\partial x_j} \right) \right\|^2_{H^{-1}} dt$$
$$\le C_1 \sum_{i,j=1}^n \mathbf{E}\int_0^T \left\| [\widehat{b}_{\varepsilon ij} - b_{ij}] \frac{\partial u}{\partial x_j} \right\|^2_{H^0} dt,$$

for a constant $C = C(n)$. The functions b_ε and b are bounded, hence

$$\left| [\widehat{b}_{\varepsilon ij} - b_{ij}] \frac{\partial u}{\partial x_j} \right| \le C_1 \left| \frac{\partial u}{\partial x}(x,t,\omega) \right|$$

for a constant $C_1 = C_1(\mathcal{P})$. We have that $u \in X^1$. By Lebesgue's Dominated Convergence Theorem, it follows that

$$\left\| [\widehat{b}_{\varepsilon ij} - b_{ij}] \frac{\partial u}{\partial x_j} \right\|_{X^0} \to 0.$$

Hence $\|r_\varepsilon(u)\|_{X^{-1}} \to 0$.

Further, the functions $\widehat{f}_\varepsilon$ and $\widehat{\lambda}_\varepsilon$ are bounded, hence

$$|q_\varepsilon(u)(x,t,\omega)| \le C_1 \left(\left| \frac{\partial u}{\partial x}(x,t,\omega) \right| + |u(x,t,\omega)| \right)$$

for a constant $C_2 = C_2(\mathcal{P}) > 0$. By Lebesgue's Dominated Convergence Theorem again, it follows that $\|q_\varepsilon(u)\|_{X^0} \to 0$. Therefore, we obtain $\|U_\varepsilon(u)\|_{X^0} \to 0$. By (7), it follows that (7.15) holds.

Let us show that (7.16) holds. Set $v_\varepsilon \triangleq \mathcal{M}_{i\varepsilon} h$ and $v \triangleq \mathcal{M}_i h$, where the operators $\mathcal{M}_{i\varepsilon} : X^0 \to Y^1$ are defined similarly to the operators $\mathcal{M}_i : X^0 \to Y^1$ with substituting $(b, f, \lambda) = (b_\varepsilon, f_\varepsilon, \lambda_\varepsilon)$. By the definitions of the corresponding adjoint operators,

$$J_\varepsilon = (\mathcal{M}^*_{i\varepsilon}\xi - \mathcal{M}^*_i \xi, h)_{X^0} + ((\delta_T \mathcal{M}_{i\varepsilon})^* \Psi - (\delta_T \mathcal{M}_i)^* \Psi, h)_{X^0}$$
$$= (\xi, v_\varepsilon - v)_{X^0} + (\Psi, v_\varepsilon(\cdot, T) - v(\cdot, T))_{Z^0_T}.$$

The function $V_\varepsilon \triangleq v_\varepsilon - v$ is the solution in Q of the boundary value problem

$$d_t V_\varepsilon = (\mathcal{A}_\varepsilon V_\varepsilon + F_\varepsilon(v))\, dt + \sum_{i=1}^N B_i V_\varepsilon \, dw_i(t),$$
$$V_\varepsilon(x,0) = 0, \quad V_\varepsilon(x,t)|_{x\in\partial D} = 0,$$

where the operator $F_\varepsilon(\cdot)$ is defined as above. The remaining part of the proof of (7.16) repeats the proof of (7.15). This completes the proof of Theorem 4.3. □

8 The proof of Theorems 4.5–4.7 and 5.1

Proof of Theorem 4.5. Assume that Condition 4.1 holds. Let

$$S_N \triangleq \left\{\alpha = (\alpha_1,\dots,\alpha_N)^\top \in \mathbb{R}^N : |\alpha| = \left(\sum_{i=1}^N |\alpha_i|^2\right)^{1/2} \le 1\right\}.$$

Let $y \in \mathbb{R}^n$ be fixed and let $y_i = y_i(\alpha) \triangleq \alpha_i y$, where $\alpha \in S_N$. Let $y_i \triangleq \alpha_i y$ and $z_i = z_i(y) = \beta_i^\top y$, $z = z(y) = (z_1,\dots,z_N)^\top$. By Condition 4.1,

$$\sum_{i=1}^N y_i^\top b\, y_i \ge \frac{1}{2}\left(\sum_{i=1}^N y_i^\top \beta_i\right)^2 + \delta_1 \sum_{i=1}^N |y_i|^2$$

for all $\alpha \in S_N$, $(x,t) \in D \times [0,T]$ and $\omega \in \Omega$. Hence

$$\begin{aligned} y^\top b\, y &= \sum_{i=1}^N \alpha_i^2 y^\top b\, y \\ &\ge \frac{1}{2}\left(\sum_{i=1}^N \alpha_i y^\top \beta_i\right)^2 + \delta_1 \sum_{i=1}^N \alpha_i^2 |y|^2 \\ &= \frac{1}{2}\left(\sum_{i=1}^N \alpha_i z_i(y)\right)^2 + \delta_1 |y|^2 \sum_{i=1}^N \alpha_i^2 \\ &= \frac{1}{2}(\alpha^\top z(y))^2 + \delta_1 |y|^2 \end{aligned}$$

for any $\alpha \in S_N$. Hence

$$y^\top b\, y \ge \sup_{\alpha\in S_N} \frac{1}{2}\big(\alpha^\top z(y)\big)^2 + \delta_1 |y|^2 = \frac{1}{2}|z(y)|^2 + \delta_1 |y|^2.$$

On the other hand,

$$|z(y)|^2 = \sum_{i=1}^N |z_i(y)|^2 = \sum_{i=1}^N |y^\top \beta_i|^2.$$

Hence

$$y^\top b\, y \ge \frac{1}{2}\sum_{i=1}^N |y^\top \beta_i|^2 + \delta_1 |y|^2.$$

Hence Condition 3.1 holds with $\delta = \delta_1$. This completes the proof. □

Proof of Theorem 4.6. We have that

$$2b = \gamma + R,$$

where $\gamma = \sum_{i=1}^{n} \beta_i^2$ and $R = R(x,t,\omega) \geq 2\delta$. Let

$$D \triangleq BB^\top = \{\beta_i \beta_j\}_{i,j=1}^N, \quad \text{where } B \triangleq (\beta_1, \dots, \beta_N)^\top.$$

It suffices to show that there exists $\delta_1 > 0$ such that

$$\gamma(x,t,\omega) I_N - D(x,t,\omega) \geq 0$$

for all x, t, ω, where I_N is the unit matrix in $\mathbb{R}^{N\times N}$. Let $\lambda = \lambda(x,t,\omega)$ be the minimal eigenvalue of the matrix $\gamma(x,t,\omega) I_N - D(x,t,\omega)$. It suffices to show that $\lambda \geq 0$. Let $z = z(x,t,\omega)$ be a corresponding eigenvector such that $|z| = |B| \neq 0$ (for the trivial case $|B| = 0$, we have immediately that $\lambda = 0$). We have that

$$z = cB + B',$$

where $c \in [-1, 1]$ and $B' = B'(x,t,\omega)$ is a vector such that $B^\top B' = 0$. By the definitions, we have that $\gamma = |B|^2$ and

$$\begin{aligned} \lambda z = (\gamma I_N - D)z &= (\gamma I_N - BB^\top)(cB + B') \\ &= \gamma(cB + B') - c|B|^2 B = \gamma c B + \gamma B' - c\gamma B = \gamma B'. \end{aligned}$$

Hence $\lambda(cB + B') = \gamma B'$. It follows that either $B' \neq 0$, $c = 0$, and $\lambda = \gamma \geq 0$, or $\lambda = 0$ and $B' = 0$. This completes the proof. □

Proof of Theorem 4.7. By Hölder's inequality, we have that

$$\left(\sum_{i=1}^{N_0} y_i^\top \beta_i \right)^2 \leq N_0 \sum_{i=1}^{N_0} (y_i^\top \beta_i)^2.$$

Hence

$$\begin{aligned} \sum_{i=1}^{N} y_i^\top b\, y_i - \frac{1}{2}\left(\sum_{i=1}^{N} y_i^\top \beta_i \right)^2 &= \sum_{i=1}^{N_0} y_i^\top b\, y_i - \frac{1}{2}\left(\sum_{i=1}^{N_0} y_i^\top \beta_i \right)^2 \\ &\geq \sum_{i=1}^{N_0} y_i^\top b\, y_i - \frac{N_0}{2} \sum_{i=1}^{N_0} (y_i^\top \beta_i)^2 \\ &\geq \delta_2 \sum_{i=1}^{N} |y_i|^2. \end{aligned}$$

This completes the proof. □

Proof of Theorem 5.1. Let

$$p \triangleq p^{(1)} - p^{(2)}, \quad \chi_i \triangleq \chi_i^{(1)} - \chi_i^{(2)},$$

and let $\mathcal{A}^{(k)*}$, $B_i^{(k)*}$ be the corresponding operators (3.1), $k = 1, 2$. We have that

$$d_t p + (\mathcal{A}^{(1)*} p + \psi)\, dt + \sum_{i=1}^{N} B_i^{(1)} \chi_i\, dt + \psi = \chi_i\, dw_i(t), \quad t \le T,$$

$$p(x, 0, \omega) = \Psi^{(1)}(x, \omega) - \Psi^{(2)}(x, \omega), \quad p(x, t, \omega)\,|_{x \in \partial D} = 0.$$

Here

$$\psi \triangleq \xi^{(1)} - \xi^{(2)} + \mathcal{A}^{(1)*} p^{(2)} - \mathcal{A}^{(2)*} p^{(2)} + \sum_{i=1}^{N} (B_i^{(1)*} \chi_i^{(2)} - B_i^{(2)*} \chi_i^{(2)}).$$

By Theorem 4.3, it follows that there exists a constant $C_0 = C_0(\mathcal{P}^{(1)})$ such that

$$\|p\|_{Y^2} + \sum_{i=1}^{N} \|\chi_i\|_{X^1} \le C_0 \Big(\|\psi\|_{X^0} + \|\Psi^{(1)} - \Psi^{(2)}\|_{Z_0^1} \Big). \tag{8.1}$$

Further, we have $\psi = \sum_{m=0,1,2} \psi_m$, where

$$\psi_0 = \xi^{(1)} - \xi^{(2)},$$

$$\psi_1 \triangleq \sum_{i,j=1}^{n} [b_{ij}^{(1)} - b_{ij}^{(2)}] \frac{\partial^2 p^{(2)}}{\partial x_i \partial x_j} + \sum_{i=1}^{n} [f_i^{(1)} - f_i^{(2)}] \frac{\partial p^{(2)}}{\partial x_i} + [\lambda^{(1)} - \lambda^{(2)}] p^{(2)},$$

$$\psi_2 = \sum_{i=1}^{N} \left(\sum_{i=1}^{n} [\beta_i^{(1)} - \beta_i^{(2)}] \frac{\partial \chi_i^{(2)}}{\partial x_i} + [\widehat{\beta}^{(1)} - \widehat{\beta}^{(2)}] \chi_i^{(2)} \right).$$

Clearly,

$$\|\psi_0\|_{X^0} \le M, \quad \|\psi_1\|_{X^0} \le CM \|p^{(2)}\|_{Y_2}, \quad \|\psi_2\|_{X^0} \le CM + \sum_{i=1}^{N} \|\chi^{(2)}\|_{X^1},$$

where $C = C(n)$ is a constant. Finally, we obtain

$$\|\psi\|_{X^0} \le C_1 M \left(\|p^{(2)}\|_{Y_2} + \sum_{i=1}^{N} \|\chi_i^{(2)}\|_{X^1} + 1 \right),$$

where $C_1 = C_1(n)$ is a constant. By (8.1), the desired estimate follows. This completes the proof. □

Bibliography

[1] E. Alós, J. A. León and D. Nualart, Stochastic heat equation with random coefficients, *Probab. Theory Related Fields* **115** (1999), no. 1, 41–94.

[2] V. Bally, I. Gyongy and E. Pardoux, White noise driven parabolic SPDEs with measurable drift, *J. Funct. Anal.* **120** (1994), 484–510.

[3] J. M. Bismut, Conjugate convex functions in optimal stochastic control, *J. Math. Anal. Appl* **44** (1973), 384–404.

[4] A. Chojnowska-Michalik and B. Goldys, Existence, uniqueness and invariant measures for stochastic semilinear equations in Hilbert spaces, *Probab. Theory Related Fields* **102** (1995), no. 3, 331–356.

[5] F. Confortola, Dissipative backward stochastic differential equations with locally Lipschitz nonlinearity, *Stochastic Process. Appl.* **117** (2007), no. 5, 613–628.

[6] G. Da Prato and L. Tubaro, Fully nonlinear stochastic partial differential equations, *SIAM J. Math. Anal.* **27** (1996), no. 1, 40–55.

[7] N. G. Dokuchaev, Boundary value problems for functionals of Ito processes (in Russian), *Theory Probab. Appl.* **36** (1991), 464–481; translation in *Theory Probab. Appl.* **36** (1992), 459–476.

[8] N. G. Dokuchaev, Probability distributions of Ito's processes: Estimations for density functions and for conditional expectations of integral functionals (in Russian), *Theory Probab. Appl.* **39** (1994), no. 4, 824–833; translation in *Theory Probab. Appl.* **39** (1995), no. 4, 662–670.

[9] N. G. Dokuchaev, Nonlinear parabolic Ito's equations and duality approach (in Russian), *Theory Probab. Appl.* **48** (2002), no. 1, 22–40; translation in *Theory Probab. Appl.* **48** (2003), no. 1, 45–62.

[10] N. G. Dokuchaev, Parabolic Ito equations and second fundamental inequality, *Stochastics* **77** (2005), no. 4, 349–370.

[11] N. Dokuchaev, Backward parabolic Ito equations and second fundamental inequality, preprint (2006), `http://lanl.arxiv.org/abs/math/0606595`.

[12] N.G. Dokuchaev, Universal estimate of the gradient for parabolic equations, *J. Phys. A* **41** (2008), no. 21, 215202.

[13] N. Dokuchaev, Duality and semi-group property for backward parabolic Ito equations, *Random Oper. Stoch. Equ.* **18** (2010), 51–72.

[14] N. Dokuchaev, Representation of functionals of Ito processes in bounded domains, *Stochastics* **83** (2010), no. 1, 45–66.

[15] K. Du and S. Tang, Strong solution of backward stochastic partial differential equations in C^2 domains, *Probab. Theory Related Fields*, DOI 10.1007/s00440-011-0369-0.

[16] I. Gyöngy, Existence and uniqueness results for semilinear stochastic partial differential equations, *Stochastic Process. Appl.* **73** (1998), no. 2, 271–299.

[17] Y. Hu, J. Ma and J. Yong, On semi-linear degenerate backward stochastic partial differential equations, *Probab. Theory Related Fields* **123** (2002), no. 3, 381–411.

[18] Y. Hu and S. Peng, Adapted solution of a backward semilinear stochastic evolution equation, *Stoch. Anal. Appl.* **9** (1991), 445–459.

[19] K.-H. Kim, On stochastic partial diferential equations with variable coefficients in C^1-domains, *Stochastic Process. Appl.* **112** (2004), 261–283.

[20] N. V. Krylov, An analytic approach to SPDEs, in: *Stochastic Partial Differential Equations: Six Perspectives*, Mathematical Surveys and Monographs 64, American Mathematical Society, Providence, RI, (1999), 185–242.

[21] O. A. Ladyzhenskaia, *The Boundary Value Problems of Mathematical Physics*, Applied Mathematical Sciences 49, Springer-Verlag, New York, 1985.

[22] O. A. Ladyzenskaya, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, 1968.

[23] J. Ma and J. Yong, On linear degenerate backward stochastic PDE's, *Probab. Theory Related Fields* **113** (1999), 135–170.

[24] B. Maslowski, Stability of semilinear equations with boundary and pointwise noise, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **22** (1995), no. 1, 55–93.

[25] E. Pardoux, Stochastic partial differential equations. A review, *Bull. Sci. Math.* **117** (1993), 29–47.

[26] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation, *Systems Control Lett.* **14** (1990), 55–61.

[27] E. Pardoux and A. Rascanu, Backward stochastic differential equations with subdifferential operators and related variational inequalities, *Stochastic Process. Appl.* **76** (1998), no. 2, 191–215.

[28] B. L. Rozovskii, *Stochastic Evolution Systems. Linear Theory and Applications to Non-Linear Filtering*, Kluwer Academic Publishers, Dordrecht, 1990.

[29] G. Tessitore, Existence, uniqueness and space regularity of the adapted solutions of a backward SPDE, *Stoch. Anal. Appl.* **14** (1996), 461–486.

[30] J. B. Walsh, An introduction to stochastic partial differential equations, in: *École d'été de probabilités de Saint-Flour XIV – 1984*, Lecture Notes in Mathematics 1180, Springer-Verlag, New York (1986), 265–437.

[31] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.

[32] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, 1965.

[33] X. Y. Zhou, A duality analysis on stochastic partial differential equations, *J. Funct. Anal.* **103** (1965), no. 2, 275–293.

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