Probabilistic Design of Ground Improvement by Vertical Drains for Soil of Spatially Variable Coefficient of Consolidation

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Abstract: The design of soil consolidation via prefabricated vertical drains (PVDs) has been traditionally carried out deterministically and thus can be misleading due to the ignorance of the uncertainty associated with the inherent variability of soil properties. To treat such uncertainty in the course of design of soil improvement by PVDs, more rational probabilistic methods are necessary. In this paper, a simplified probabilistic method is proposed in which the inherent variability of the coefficient of consolidation, which is the most significant uncertain soil parameter that affects the consolidation process, is considered. An easy-to-use design procedure and charts are provided for routine use by practitioners.

Keywords: Probabilistic design; Soil consolidation; Vertical drains; Soil spatial variability.
Introduction

Over the past decade or so, the development activities in areas of soft soils have increased significantly so as to suit the demands of the increased population in many countries and to ensure the marginal use of limited land space. Construction over soft soils often requires a pre-construction treatment of the existing soft sub-soils so that soil strength and stiffness are improved, thus, eliminating the undue risks of excessive post-construction deformations and associated instability. Among a number of available ground improvement techniques, the use of prefabricated vertical drains (PVDs) with preloading has become the most viable for stabilization of soft soils (Indraratna et al., 2012). PVDs accelerate the consolidation process and prevent post-construction build-up of excess pore water pressure, leading to improved soil strength and reduced lateral and differential settlements (Rowe and Taechakumthorn, 2008).

Despite the fact that the theoretical design aspects of soil consolidation by PVDs are well established (e.g. Barron, 1948; Hansbo, 1981; Hird et al., 1992; Onoue, 1988; Yoshikuni and Nakanodo, 1974), reliable predictions of the soil consolidation rates remain difficult to obtain due to the uncertainty associated with the factors affecting the consolidation process (Hong and Shang, 1998; Rowe, 1972; Zhou et al., 1999). The uncertainty associated with the design of any geotechnical engineering system including soil consolidation can be divided into three main sources (Phoon and Kulhawy, 1999): inherent variability; measurement error and transformation uncertainty. The inherent variability (also called aleatoric uncertainty) is due to the natural geologic processes caused by the complex characteristics of transport of raw materials, layered deposition
and common weathering (Vanmarcke, 1977). The measurement error is mainly due to
the inadequate equipment and poor testing procedures. The transformation uncertainty
(also called model uncertainty) occurs during the translation of the field or laboratory
measurements into design, using empirical or correlation models. Collectively, the
measurement error and model uncertainty can be described as epistemic uncertainty. To
obtain a reliable design for a geotechnical system, all of the above sources of
uncertainty should be taken into consideration. However, the measurement error and
transformation uncertainty can be reduced or even removed by improving the
measurement methods and enhancing the calculation models (Lacasse and Nadim,
1996). Therefore, the inherent variability is the most significant source of uncertainty
that needs to be addressed in design of geotechnical engineering systems and this is the
main focus of the present paper for soil improvement by PVDs.

The degree of consolidation achieved via PVDs is greatly controlled by some soil
properties (e.g. soil permeability and volume compressibility) that are spatially variable
and potentially induce inherent variability in their characterization, which provides
significant geotechnical uncertainty. However, given the analytical and numerical
complexity of the problem of soil improvement by PVDs, available research into soil
consolidation considering geotechnical uncertainty has been very limited and also failed
to accommodate the true nature of inherent soil variability. For example, Hong and
Shang (1998) and Zhou et al. (1999) presented a probabilistic design method based on
available analytical solutions considering the geotechnical uncertainty associated with
the coefficient of horizontal (radial) consolidation. However, this method is inadequate
as soil variability was characterized by random values rather than by random fields.
Walker and Indraratna (2006) proposed an analytical model based on Hansbo (1981) theory incorporating a parabolic horizontal permeability distribution in the smear zone. Basu et al. (2006) and Abuel-Naga et al. (2012) have extended the work done by Walker and Indraratna (2006) to include a transition zone of linearly varying permeability between the smear and undisturbed zones, but the permeability in the smear and undisturbed zones are assumed to be constant. Therefore, there is a need to develop an alternative probabilistic design method that considers the true nature of soil inherent variability in the course of design of soil improvement by PVDs and this paper will fill in this gap.

In order to include the inherent soil variability in design of soil improvement by PVDs, a computational numerical stochastic scheme that combines the finite element method and Monte Carlo (FEMC) technique can be employed. However, such numerical scheme is complex and requires a large number of simulations that are computationally intensive and time consuming, and it is not uncommon that practicing engineers have neither the time nor the resources to perform such FEMC simulations. Consequently, in this work, an alternative approximate simplified probabilistic design method (PDM) that can be readily used by practitioners is developed in which the inherent soil variability of the coefficient of consolidation is explicitly incorporated in a systematic and economically viable manner. The proposed PDM is verified by comparing its results with those obtained from the FEMC solutions and the results are found to be in good agreement. In the sections that follow, detailed description of the proposed PDM is demonstrated followed by the stochastic FEMC approach. Finally, a comparison between the results obtained from the PDM and FEMC is presented and discussed.
Probabilistic Design Method of Soil Consolidation by PVDs

The probabilistic design method (PDM) described herein considers, for the first time, incorporation of the true nature of soil variability in design of soil improvement by PVDs. The soil property considered to be randomly variables is the horizontal coefficient of consolidation, $c_h$, as it is the most significant soil property that affects soil consolidation by PVDs, as explained in the next section. It should be noted that the proposed PDM method is an extension of the previous work done by Hong and Shang (1998) and Zhou et al. (1999) but in the current study the inherent soil variability of $c_h$ is explicitly incorporated and appropriately implemented. The proposed PDM is approximate and can be used to estimate the drain spacing by employing a factored design value of $c_h$ so as to satisfy a specific target probability level of the degree of consolidation that needs to be achieved in a specified timeframe. The proposed PDM involves the following steps:

1. Identification and characterization of soil properties that are spatially variable in the ground;
2. Development of analytical formulation for the design factors taking into account the associated uncertainty due to spatially variable soils;
3. Estimation of correlation structure of soils; and
4. Development of probabilistic design procedure and charts for routine use by practitioners.

Details of the above steps are described and discussed below.
Identification and characterization of spatially variable soil properties

As mentioned earlier, spatial variability of soil properties affects the behavior of soil consolidation, and among all soil properties affecting soil consolidation, the coefficient of vertical consolidation, $c_v$, and coefficient of horizontal consolidation, $c_h$, are the most significant. Both $c_v$ and $c_h$ may vary substantially in the ground, even in a uniform soil layer (Chang, 1985). For example, based on experimental data, Terzaghi et al. (1996) reported that the coefficient of variation (COV) of $c_v$ for Mexico City clay, San Francisco clay and clay deposit in Pisa (Italy) are 12%, 35% and 69%, respectively. By analyzing $c_v$ values obtained from oedometer tests carried out on Kawasaki clay, Chang (1985) estimated COV of $c_v$ to be 30%. According to data reported by Lumb (1974), the COV of $c_v$ and $c_h$ are estimated to range from 25% to 50%, and based on data reported in the literature, Lee et al. (1983) found that the extent to which $c_v$ and $c_h$ vary may range from 25% to 100%.

As mentioned above, both $c_v$ and $c_h$ exhibit inherent variability and may be considered as random variables in design of soil stabilization by PVDs. However, in accordance with the sensitivity analyses carried out by Hong and Shang (1998) and Zhou et al. (1999) considering several uncertain soil properties, it was found that $c_h$ is the most significant random soil property affecting the degree of soil consolidation by PVDs. In addition, the consolidation of soil by PVDs can take place by simultaneous vertical and horizontal (radial) drainage of water. However, as the drainage length in the vertical direction is significantly higher than that of the horizontal direction and water flow resistance in the horizontal direction is often much lower than that of the vertical
direction (Bergado et al., 1993; Hansbo, 1981), soil consolidation due to vertical drainage is much less than that of horizontal drainage. Furthermore, it has been shown by Crawford et al. (1992) from a back-analysis of an instrumented test embankment in Canada that the rates of consolidation are very sensitive to $c_h$, and that $c_h$ is the most significant design parameter. Accordingly, in the proposed probabilistic design approach, $c_h$ is considered to be the only spatially variable soil property, while the other soil properties are held constant and treated deterministically, including $c_v$, so as to reduce the superfluous complexity of the problem.

Inherent variability of soil properties can be mathematically characterized by treating the soil properties as random variables. In statistics, a random variable is described by a probability distribution (usually referred to as the ‘PDF’ or probability density function). The PDF of a random variable can be represented by several classical statistical parameters, namely, the mean value, $\mu$, variance, $\sigma^2$ (the variance can also be represented by standard deviation, $\sigma$, or COV, $\nu$ and $\nu = \sigma/\mu$). However, inherent variability of soil properties is not entirely random and spatial dependencies also exist (Fenton and Vanmarcke, 1990; Jaksa et al., 1997; Vanmarcke, 1977). That is, a soil property at two separate spatial locations could be similar or otherwise, depending on the distance they are located apart and this is known as spatial correlation. Vanmarcke (1977) pointed out that adequate characterization of spatially variable soil properties requires consideration (incorporation) of such spatial correlation. The mean and standard deviation are the point statistical measures with no consideration of the spatial correlation structure of soil properties. Therefore, a third parameter (i.e. the scale of fluctuation, SOF) is usually introduced as an additional statistic to consider the spatial
correlation of soil properties. The scale of fluctuation is also known as the correlation length and is usually denoted as $\theta$. Generally speaking, a large value of $\theta$ indicates smooth spatial variation of soil property of interest, whereas a small value of $\theta$ implies erratic variation. All soils by nature exhibits SOF due to the geological process of transport of raw materials, layer deposition and common weathering process. An extensive literature review suggested that the amount of information on SOF is relatively limited in comparison to the amount of information on the COV of soil variability. Phoon and Kulhway (1999) reported suggested guidelines of SOF for a range of soil properties (e.g. undrained shear strength, cone tip resistance, water content, effective unit weight, etc.) based on a comprehensive review of various test measurements and it was found that the vertical SOF generally ranges between 0.1 and 12.7 m, while the horizontal SOF typically ranges between 3 and 80 m. Several other researchers (Lacasse and Lamballerie, 1995; Vanmarcke, 1977) also found similar ranges of vertical and horizontal scales of fluctuation to those reported by Phoon and Kulhway (1999) for the cone penetration resistance of sand and clay. However, to the authors’ best knowledge, no records have been published in relation to SOF of $c_h$. It is believed that SOF of $c_h$ should be within a range similar to that of other soil properties reported in the literature. This is due to the fact that spatial correlation structure of a soil mass is caused by changes in the constitutive nature of soil over the ground; therefore, $c_h$ would have similar scales of fluctuation to other soil properties.

In order to model the spatial randomness of $c_h$, a probability distribution function of its variation should be used and a number of different probability distributions for $c_h$ have been suggested in the literature. For example, Chang (1985) and Hong (1992) used
both lognormal and gamma distributions, and a study carried out by Zhou et al. (1999) considered the Weibull distribution together with the lognormal and gamma distributions. However, in the current study, the variability of $c_h$ is characterized by a lognormal distribution because the observation from field tests data reported by Chang (1985) suggested that the variation of $c_h$ can be adequately modeled by a lognormal distribution. In addition, the lognormal distribution offers mathematical convenience because of having simple relationship with the normal distribution. The probability density function of the assumed lognormally distributed $c_h$ with a mean, $\mu_{c_h}$, and standard deviation, $\sigma_{c_h}$, can thus be given by:

$$f(c_h) = \frac{1}{c_h \sigma_{\ln c_h} \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{\ln c_h - \mu_{\ln c_h}}{\sigma_{\ln c_h}} \right)^2 \right]$$

(1)

where: $\mu_{\ln c_h}$ and $\sigma_{\ln c_h}$ are, respectively, the mean and standard deviation of the underlying normally distributed $c_h$, i.e. $\ln(c_h)$, obtained from the specified $\mu_{c_h}$ and $\sigma_{c_h}$ of the lognormally distributed $c_h$ using the following transformation functions (Fenton and Griffiths, 2008):

$$\mu_{\ln c_h} = \ln \mu_{c_h} - \frac{1}{2} \sigma_{\ln c_h}^2$$

(2)

and
\[
\sigma_{\ln c_h} = \sqrt{\ln\left(1 + \frac{\sigma_{c_h}^2}{\mu_{c_h}^2}\right)} = \sqrt{\ln(1 + \nu_{c_h}^2)}
\]  

(3)

where: \( \nu_{c_h} = \frac{\sigma_{c_h}}{\mu_{c_h}} \) is the coefficient of variation of horizontal coefficient of consolidation. Therefore, in order to properly acknowledge and quantify the spatial variability of \( c_h \), the following spatial variability characteristics of \( c_h \) need to be identified: the mean, \( \mu_{c_h} \), standard deviation, \( \sigma_{c_h} \), and correlation length, \( \theta_{c_h} \).

In order to determine the above statistical characteristics of \( c_h \) for a certain site, a carefully-controlled geotechnical site investigation program of closely-spaced soil boring and testing needs to be undertaken, and obtained data be analyzed. However, such a comprehensive site investigation program is often beyond the scope of most projects and in the absence of such field information, traditional site investigation of limited soil testing together with information from the geological maps and knowledge from previous site investigation of nearby locations can be used to assign reasonable level of soil variability for the site in question. In addition, the typical ranges of statistical parameters of \( c_h \) available in the literature (Beacher and Christian, 2003) can be used to assign reasonable values of soil variability of \( c_h \), provided that they are extracted from similar geologic origins and collected over limited spatial extents.

Detailed description of methods that can be used for evaluating the spatial variation of soil properties is beyond the scope of the present paper and can be found in many publications (e.g. DeGroot and Baecher, 1993; Jaksa et al., 1997; Phoon and Kulhawy, 1999; Vanmarcke, 1984).
As indicated earlier, soil consolidation by PVDs takes place by simultaneous vertical and horizontal (radial) drainage of water. The analytical solution for the degree of consolidation due to the horizontal drainage \( U_h(t) \), is given by Hansbo (1981) as follows (see Fig. 1 for demonstration of parameters):

\[
U_h(t) = 1 - \exp \left( -\frac{2c_y t}{r^2 \alpha} \right)
\]

(4)

and

\[
\alpha = F_n + F_s + F_r
\]

(5)

where: \( F_n \), \( F_s \) and \( F_r \) are the drain spacing, smear and well-resistance factors, respectively, which can be expressed as follows:

\[
F_n = \frac{n^2}{(n^2 - 1)} \left[ \ln(n) - \frac{3}{4} + \frac{1}{n^2} - \frac{1}{4n^2} \right] \approx \ln(n) - \frac{3}{4}
\]

(6)

\[
F_s = \left( \frac{k_h}{k_h - 1} \right) \ln \left( \frac{r_w}{r_w} \right) = \left( \frac{k_h}{k_h - 1} \right) \ln(s)
\]

(7)

\[
F_r = \frac{2\pi n^2}{3} \frac{k_h}{q_w}
\]

(8)
where: $c_h$ is the coefficient of consolidation in the horizontal direction; $r_e$ is the radius of equivalent soil cylinder with impermeable perimeter or the radius of zone of influence; $t$ is the consolidation time; $\alpha$ is a group parameter representing the smear effects and geometry of the PVD system; $n = r_e/r_w$ is the drain spacing ratio ($r_w$ is the equivalent radius of the drain); $s = r_s/r_w$ is the smear ratio ($r_s$ is the radius of the smear zone); $k'_h$ is the horizontal permeability in the smear zone; $L$ is the maximum vertical drainage distance; and $q_w$ is the vertical discharge capacity of the drains. If the effects of both the horizontal and vertical drainage are considered, the analytical solution for the overall degree of consolidation, $U(t)$, can be obtained as follows (Carillo, 1942):

\[
U(t) = 1 - [1 - U_v(t)] \left[1 - U_h(t)\right]
\] (9)

where: $U_v(t)$ is the degree of consolidation due to vertical drainage at any time $t$, that can be determined as follows (Lambe and Whitman, 1969):

\[
U_v(t) = 1 - \sum_{i=0}^{\infty} \frac{2}{M^2} \exp\left(-M^2 \frac{c_v t}{L^2}\right)
\] (10)

where: $M = \pi/2(2i+1)$; $c_v$ is the coefficient of consolidation in the vertical direction.

The design of PVD systems using the analytical solutions set out above has been traditionally carried out deterministically by assuming that the consolidating soil mass surrounding the PVDs is homogeneous (i.e. soil variability is ignored) with constant mean values of soil properties across the soil mass. In such case, the drain spacing, $S$, or
the radius of influence zone, \( r_e \), can be determined by iteratively solving Eq. (11) which is derived by substituting Eq. (4) into Eq. (9), as follows:

\[
\frac{r_e^2 \alpha(r_e)}{\mu_{c_e}} = \frac{2t_s \mu_{c_e}}{\ln\left(\frac{1 - U_s(t_s)}{1 - U_s(t_s)}\right)}
\]

It should be noted that the term \( \alpha(r_e) \) in Eq. (11) represents \( \alpha \) given in Eq. (4) but used herein to highlight the fact that the factor \( \alpha \) is a function of \( r_e \).

As \( c_h \) is spatially variable, using the deterministically estimated value of \( r_e \) as described above, the degree of consolidation achieved at time \( t_s \) may not satisfy the target degree of consolidation \( U_s(t_s) \), resulting in the entire ground improvement process to be unsuccessful. In order to take into account the spatial variability of \( c_h, r_e \) or \( S \) is computed from the proposed probabilistic approach. The approach involves determination of a factored \( c_h \) (i.e. \( c_{hf} \)) in such a way that the probability of \( U(t_s) \geq U_s(t_s) \), i.e. \( P[U(t_s) \geq U_s(t_s)] \), is equal to the specified target probability of achieving certain degree of consolidation \( P_s \), in which \( P_s = P[U(t_s) \geq U_s(t_s)] \). In other words, if \( c_{hf} \) is used in computing the spacing of PVD, the target degree of consolidation \( U_s(t_s) \) can be achieved at \( P_s \). The factored \( c_{hf} \) can then be expressed in the form of the design factor, \( D_h \), times the mean value of the horizontal coefficient of consolidation, \( \mu_{c_e} \), i.e.

\[
c_{hf} = D_h \mu_{c_e}
\]

Since \( U(t) \) is a monotonically increasing function of \( c_h \), the following relationship holds (Benjamin and Cornell, 1970; Zhou et al., 1999):
where: \( P[\cdot] \) = probability of its argument.

It has mentioned earlier that the probability distribution of \( c_h \) is assumed to be lognormally distributed. Therefore, Eq. (12) can be rewritten in the form of the following lognormal probability distribution transformation:

\[
P_s = P[c_h \geq c_{hf}] = 1 - \Phi \left( \frac{\ln c_{hf} - \mu_{ln c_h}}{\sigma_{ln c_h}} \right)
\]

where: \( \Phi(\cdot) \) is the standard normal cumulative distribution function; \( \mu_{ln c_h} \) and \( \sigma_{ln c_h} \) are respectively, the locally averaged mean and standard deviation of the underlying normally distributed \( c_h \). In order to obtain \( c_{hf} \) from Eq. (13) for a given \( P_s \), \( \mu_{ln c_h} \) and \( \sigma_{ln c_h} \) must be obtained by expressing them in terms of the known statistical input parameters of \( c_h \) (i.e. \( \mu_{c_h} \), \( \sigma_{c_h} \) and \( \theta_{c_h} \)). Before finding \( \mu_{ln c_h} \) and \( \sigma_{ln c_h} \), a brief discussion in regards to these two parameters is essential and presented below. It should be noted that the basis of using the locally averaged soil properties to incorporate spatial variations of soil was first introduced by Fenton and Griffiths (2008), who successfully used this concept in some applications in geotechnical engineering (e.g. design of shallow foundations) which was found to give compatible results to those obtained from the finite element Monte Carlo solutions.
The enormous significance of spatial variability to soil consolidation is that during the consolidation process water must escape and during the escape process the water must circumnavigate low permeability areas in favor of high permeability areas. In other words, in a consolidating heterogeneous soil mass, high flow rates in some regions of high $k_h$ are offset by lower flow rates in other regions of low $k_h$, meaning that the total flow from the vicinity of PVD is effectively an averaging process. This is to say that the overall behavior of PVD system is not governed by the soil properties at discrete points but by the average soil properties of the soil volume within the soil domain. It should be noted that, the random fields are characterized by their point statistics, meaning that $\mu_{c_h}$, $\sigma_{c_h}$ and $\theta_{c_h}$ of $c_h$ are defined at the point level. However, soil properties are rarely measured at a point and most engineering measurements concerned with soil properties are performed on samples of some finite volume. Therefore, the measured soil properties are actually locally averaged over the sample volume. For example, the permeability of soil is generally estimated using a laboratory sample of some volume which involves measuring the amount of water which passes through the sample in some time interval. The paths that the water takes to escape from the sample are not considered individually; rather all the flow paths are taken into account together. It should be note that the permeability of soil at the point level is either infinite (in case of a void) or zero (in case of solid). In the light of this, the flow of water through the spatially variable soil into the drain is essentially a process governed by the locally averaged soil properties (Fenton and Griffiths, 2008). The main effects of the local averaging are to reduce the point variance and damp the contribution from the high frequency components. If the point distribution of the soil property of interest is normally distributed, the local averaging process will lead to a reduction in the point
variance but the mean will not be affected. For the lognormal distribution, however, both the mean and standard deviation will be reduced by the local averaging. This is because the mean of a lognormal distribution depends on both the mean and variance of the underlying normal distribution. Based on the above discussion, the locally averaged mean of the underlying normally distributed $c_h$ (i.e. $\mu_{ln}\tau_a$) is unaltered by the local averaging and can be given by:

$$\mu_{ln}\tau_a = \mu_{ln}\tau_a$$  \hspace{1cm} (14)

Using Eqs. (2) and (3), $\mu_{ln}\tau_a$ can be expressed in terms of the known point statistics of $c_h$, as follows:

$$\mu_{ln}\tau_a = \mu_{ln}\tau_a = \ln \mu_{\tau_a} - \frac{1}{2} \ln \left( 1 + \nu_{\tau_a}^2 \right)$$  \hspace{1cm} (15)

According to the local averaging theory (Vanmarcke, 1984), the locally averaged standard deviation of the underlying normally distributed $c_h$ (i.e. $\sigma_{ln}\tau_a$) can be given by:

$$\sigma_{ln}\tau_a = \sigma_{ln}\tau_a \sqrt{\gamma(D)}$$  \hspace{1cm} (16)

where: $\gamma(D)$ is the “variance reduction function” that defines the amount by which the point variance is reduced as a result of the local averaging over the domain $D$ and is a
function of the size of the averaging domain and scale of fluctuation. Using Eq. (3),
σ_{ln \epsilon_s} in Eq. (16) can be expressed in terms of specified point statistics of \epsilon_s, as follows:

$$\sigma_{ln \epsilon_s} = \sqrt{\gamma(D) \ln(1 + \nu_c^2)}$$  \hspace{1cm} (17)

Using Eqs. (15) and (17), and replacing \epsilon_f by \epsilon_f \mu_{\epsilon_s}, Eq. (13) gives:

$$P_s = P[\epsilon_s \geq \epsilon_f] = 1 - \Phi \left( \frac{\ln D_f + \ln(1 + \nu_c^2)}{\sqrt{\gamma(D) \ln(1 + \nu_c^2)}} \right)$$  \hspace{1cm} (18)

Eq. (18) can now be rewritten as follows:

$$D_f = \frac{1}{\sqrt{1 + \nu_c^2}} \exp\left[ \Phi^{-1}(1 - P_s) \sqrt{\gamma(D) \ln(1 + \nu_c^2)} \right]$$  \hspace{1cm} (19)

where: \Phi^{-1}(\bullet) is the inverse normal cumulative distribution function.

If sufficient data at the site is available, the mean and variance of the soil property of interest can be established with reasonable confidence. However, unlike the mean and variance, reasonable estimation of the statistics regarding the spatial correlation structure is not utterly straightforward. In the following section, the procedure for obtaining the correlation structure of soil is described and discussed.
Estimation of correlation structure of soil

As mentioned earlier, soil properties measured at two spatial locations may be correlated or uncorrelated depending on the correlation structure of a soil. The concept of spatial correlation between soil properties at two disjoint points can be captured mathematically using the theoretical correlation function, which is generally fitted to the sample correlation function (obtained from measured data) to determine the scale of fluctuation. The sample correlation function, \( \rho(\tau) \), is the plots of sample correlation against separation or lag distances, \( \tau \). To obtain this, the sample covariance, \( \text{Cov}[X_i, X_{i+\tau}] \), is generally normalized by the sample variance, \( \text{Var}[X] \), as follows:

\[
\rho(\tau) = \frac{\text{Cov}[X_i, X_{i+\tau}]}{\text{Var}[X]} \tag{20}
\]

where: \( X \) represents a spatially variable soil property.

The computed sample correlation function is then fitted with an assumed theoretical correlation function to determine the scale of fluctuation, \( \theta \). A number of theoretical correlation functions are indicated in the literature (e.g. Fenton and Griffiths, 2008; Vanmarcke, 1977) to represent the correlation of a soil property in the ground. Upon deciding on the theoretical correlation function that represents the correlation structure of the soil property in question, the variance reduction factor can be estimated from the corresponding variance function of the selected correlation structure. In the following section, a simplified procedure for obtaining the variance reduction factor is described and discussed.
Approximation to the variance reduction factor

Since Eq. (4) only accounts for soil consolidation due to radial drainage from a horizontal plane passes through the vertical drain, $c_h$ can be considered as 2D random field. This means that on a horizontal ($x$-$y$) plane, $c_h$ is spatially variable and in the vertical ($z$) direction it is spatially constant. In other words, $\theta_{c_h}$ is finite in $x$ and $y$ directions, and infinite ($\infty$) in $z$ direction. The spatial correlation structure of $c_h$ is also assumed to be statistically isotropic, i.e. the scales of fluctuation in the $x$ and $y$ directions on a horizontal plane are assumed to be the same (i.e. $\theta_{c_h(x)} = \theta_{c_h(y)} = \theta_{c_h}$).

Although the correlation structures in any spatial direction are usually different, the reason for assuming $c_h$ as an isotropic random field is that the correlation structure is more related to the formation process (i.e. layer deposition). Therefore, on a horizontal plane the spatial correlation structure of $c_h$ would have similar scales of fluctuation in any direction. In addition, the scale of fluctuation is a difficult parameter to estimate in practice and assuming an isotropic condition with smaller scale of fluctuation will provide slightly conservative results (Fenton and Griffiths 2008). Based on the above discussion, the correlation structure of $c_h$ is idealized by applying an isotropic two-dimensional (2D) exponentially decaying (Markovian) spatial correlation function of the following form:

$$\rho_{\text{inc}_h}(r) = \exp \left( -\frac{2|r|}{\theta_{c_h}} \right)$$

(21)
where: $|r| = \sqrt{\tau_x^2 + \tau_y^2}$ is the absolute distance between two points in the soil domain ($\tau_x$ and $\tau_y$ are, respectively, the difference between the $x$ and $y$ coordinates of any two points in the random field). Although any correlation function can be employed, the Markov correlation function is selected because of its simplicity and ease of implementation.

Now the variance reduction factor can be estimated from the corresponding variance function of the 2D Markov correlation function shown in Eq. (21), as follows:

$$\gamma(D) = \gamma(X, Y) = \frac{1}{X^2Y^2} \int_0^X \int_0^Y \int_0^Y \int_0^Y \rho(\zeta_1 - \eta_1, \zeta_2 - \eta_2) d\zeta_1 d\eta_1 d\zeta_2 d\eta_2$$

where: $X$ and $Y$ are the dimensions of the averaging domain in the $x$ and $y$ directions, respectively (i.e. $D = X \times Y$). As mentioned earlier, the variance reduction factor is a function of the size of the averaging domain and scale of fluctuation. Numerical integration of the function shown in Eq. (22) leads to the variance reduction factor that varies between 0 and 1, depending on the values of $S$ (or $r_c$) and $\theta_{\tau_x}$. The detailed calculation procedure of the variance reduction factor is given in Appendix A.

**Development of chart for the variance reduction factor**

Using the algorithm shown in Appendix A, a chart for the variance reduction factor estimated over a wide range of $S$ and $\theta_{\tau_x}$ that are likely to be encountered in reality is presented in Fig. 2. As in practice, PVDs are installed with a drain spacing of 1–3 m (Walker, 2011), thus, the variance reduction factor is estimated over $S$ ranges between
1–3 m. Theoretically, $\theta_{cs}$ can have values ranging from zero to infinity, when $\theta_{cs}$ tends to be zero $\gamma(D)$ also tends to be zero, whereas when $\theta_{cs}$ to be infinity $\gamma(D)$ tends to be unity. Therefore, the minimum and maximum values of $\theta_{cs}$ are chosen so as to cover all the values of $\gamma(D)$ from 0 to 1. It should be noted that, for the interest of generality, $\theta_{cs}$ in Fig. 2 is expressed in the non-dimensional form $\Theta (\Theta = \theta_{cs} / S)$. It should also be noted that, in practice, PVDs are installed in a square or triangular pattern, thus, the geometry of the influence area of each drain is either square (for square pattern installation) or hexagonal (for triangular pattern installation) depending on the installation pattern. The influence area of an individual PVD can also be represented by an equivalent circular area for both installation patterns. However, a closed form solution for the variance function does not exist for most of the correlation function in two or higher dimensions. In such case, it can be obtained by numerical integration (see Eq. (22)). Therefore, to avoid superfluous complexity in the numerical integration, $\gamma(D)$ plotted in Fig. 2 is approximated based on an equivalent square influence area of side length $S$ irrespective of the installation fashion.

Development of probabilistic design procedure and charts

As can be seen in Eq. (19), for a given $P_s$ and $\nu_{cs}$, $D_f$ can be expressed as a function of $\gamma(D)$. The calculated values of $D_f$ over the range of $\gamma(D)$ for different values of $P_s$ and $\nu_{cs}$ are presented in Fig. 3. It can be seen from each individual figure that for any certain $\nu_{cs}$, $D_f$ is a decreasing function of $P_s$ and $\gamma(D)$. What this means is that a lower value of $c_h$ is required if the specified reliability of achieving a target degree of
consolidation is high. Comparison between Figs. 3a–e reveals that for a certain $P_s$ and
$\gamma(D)$, $D_f$ is also a decreasing function of $\nu_{e_b}$. In other words, if the uncertainty of $c_h$
increases, smaller value of $c_h$ is required.

Using the appropriate value of $D_f$ based on the prescribed values of $P_s$, $\nu_{e_b}$ and $\gamma(D)$ can
be obtained from the chart shown in Fig. 3 and the design of soil consolidation via
PVDs can be carried out in a similar way to that described by Zhou et al. (1999). The
design process involves estimation of the drain spacing, $S$, to satisfy a target degree of
consolidation, $U_s(t_s)$. In order to determine $S$, the required values of $r_e$ (i.e. $r_{es}$) using the
design values of $c_h$ (i.e. $c_{hf}$) need to be calculated first. Then $S$ can be obtained from the
calculated $r_{es}$ using the available conversion formulae. It should be noted that for the
rectangular pattern installation $S = r_{es}/0.565$, while $S = r_{es}/0.525$ for the triangular
pattern.

By replacing $c_h$ with its design factored value $c_{hf}$ in Eq. (11) and dividing both sides of
Eq. (11) by $r_w^2$, the following equation yields:

$$
\left(\frac{r_{es}}{r_w}\right)^2 \alpha(r_{es}) = \frac{2t_s D_f \mu_{e_b}}{\ln \left( \frac{1-U_s(t_s)}{1-U_s(t_s)} \right)} \frac{1}{r_w^2}
$$  \hspace{1cm} (23)

Eq. (23) can be expressed in a graphical form for different values of $F_t + F_r$ as shown
in Fig. 4 where the horizontal and the vertical axes are defined by:
\[
\psi = \frac{2t_s D_f \mu_{cs}}{\ln \left( \frac{1 - U_c(t_s)}{1 - U_c(t_r)} \right) r_w^2}
\]

and \( n = r_c/r_w \), respectively. Now Figs. 3 and 4 can be used for design of ground improvement by PVDs using the following steps:

1. Select an appropriate PVD on the basis of discharge capacity, \( q_w \), jacket filter characteristics, material strength and durability, and calculate its equivalent drain radius \( r_w = (a + b)/\pi \), where: \( a \) and \( b \) are the width and thickness of the PVD, respectively;

2. Determine the characteristic values of all other deterministic parameters involved in the design including \( c_v, k_h \), and \( L \) and estimate \( k_h' \) and \( r_s \) based on the installation procedure, mandrel size, and shape and soil micro fabric then calculate \( r_s/r_w, k_h/q_w \) and \( k_h/k_h' \);

3. Find the mean, standard deviation and scale of fluctuation of \( c_h \), i.e. \( \mu_{c_h}, \sigma_{c_h} \) and \( \theta_{c_h} \); and calculate \( v_{c_h} = \sigma_{c_h}/\mu_{c_h} \);

4. Specify certain consolidation time, \( t_s \), and corresponding degree of consolidation, \( U_s(t_s) \);

5. Calculate \( U_s(t_s) \) using Eq. (10);

6. Assume an initial drain spacing, \( S_o \), for the probabilistic design and using \( S_o \) and \( \theta_{c_h} \) find the variance reduction factor, \( \gamma(D) \) via chart provided in Fig. 2 if there is any lack of improved site information;
7. For a given probability of achieving the target degree of consolidation, $P_s$, determine the design factor $D_f$ from Fig. 3 and if $\nu_{c,s}$ is such that it stands between two consecutive figures in Fig. 3, use linear interpolation to determine an appropriate $D_f$;

8. Calculate $\psi$ from Eq. (24) and $F_s + F_r$ using Eqs. (7) and (8);

9. Find the value of $n$ from Fig. 4 using the values of $\psi$ and $F_s + F_r$ obtained in Step 8, and calculate $r_{es} = nr_{nw}$;

10. Obtain the design drain spacing, $S$, using $S = r_{es}/0.565$ for the rectangular pattern or $S = r_{es}/0.525$ for the triangular pattern;

11. If the calculated $S$ in Step 11 is equal to the initially assumed $S_a$ in Step 6, then the design process ends; otherwise repeat Steps 6–11 until $S \approx S_a$.

It can be noticed that the probabilistic design procedure set out above is iterative and this is due to the fact that $\gamma(D)$, which is required for the determination of $D_f$, is a function of the design parameter $S$. Therefore, to reduce the number of iteration for obtaining $S$, it is instructive to proceed with the deterministic design by setting the design factor $D_f$ equal to unity in Eq. (23). The deterministically designed drain spacing, $S_D$, or a slightly lower value of $S_D$ (e.g. 25% smaller than $S_D$) can be used as $S_a$ for the first iteration. Following this procedure, the required number of iteration may be reduced to be 3 or 4 iterations. It is noteworthy mentioning that the solution proposed by Hong and Shang (1998) and Zhou et al. (1999) is a special case of the PDM presented in this work. If $D_f$ is obtained from Fig. 3 by setting $\gamma(D) = 1$, it will give the same solution as that of the design procedure presented by Hong and Shang (1998) and Zhou et al. (1999). Therefore, the design procedure of Hong and Shang (1998) and Zhou et al. (1999) is applicable for soil variability only when $c_h$ within the influence...
zone becomes perfectly correlated (i.e. when $\theta_{sv} \to \infty$). To further facilitate the use of
the proposed PDM, an executable computer program suitable for use by practitioners is
developed and can be provided upon request.

**Stochastic Finite Element Monte Carlo Approach**

To demonstrate the validity of the proposed PDM, a series of stochastic FEMC analyses
are performed and used for comparison with the PDM. The stochastic FEMC approach
merges the local average subdivision (LAS) method (to generate random $c_h$ fields) and
finite element modeling (to calculate soil consolidation by PVDs) into a Monte Carlo
framework using the following steps:

1. Create a virtual soil profile for specified site conditions which comprises a grid of
elements allowing arbitrary distributions of coefficient of consolidation to be
modeled across the grid. This is achieved by generating spatially variable random
coefficient of consolidation field using the local average subdivision (LAS) method
developed by Fenton and Vanmarcke (1990);

2. Incorporate the generated spatial variability of the coefficient of consolidation into
the finite element modeling of soil consolidation by PVDs; and

3. Repeat Steps 1 and 2 many times using the Monte Carlo technique so that a series of
consolidation responses can be obtained from which the probability of achieving a
target degree of consolidation at a specified time can be estimated.

As the geometry and drain spacing of the problem at hand need to be known for the
finite element analysis, the consolidation problem under consideration with given $U(t_s)$,
Ps, ts, υ, and θυ is designed first by the PDM. The above steps are then applied to a consolidation problem of a known geometry. Within the scope of the paper the above steps are briefly described in the following sections, and detailed description of the steps used can be found in Shahin and Bari (2012).

**Generation of spatially variable horizontal coefficient of consolidation field**

It has already mentioned earlier that ch is considered as the only random variable in the proposed PDM and can be reasonably idealized as a 2D random field. Accordingly, by employing the specified values of $\mu_{c_h}$, $\sigma_{c_h}$ and $\theta_{c_h}$, 2D random field of $c_h$ is generated using the LAS method (Fenton and Vanmarcke, 1990). The LAS algorithm generates $c_h$ field in the form of a grid of cells that are assigned locally averaged values of $c_h$ different from one another across the grid, albeit remain constant within each element of the soil domain ($c_h$ is generally measured using some representative volume, the constant value of $c_h$ within each element is deemed to be such measure). Using the LAS code, random field of $c_h$ is generated in such a way that the number of grid cells is equal to the number of the finite elements of the soil mass, taking full account of the finite element size in the local averaging process.

In the process of simulating random field of $c_h$, correlated local averages of standard normal random field $G(x)$ are first generated with zero mean, unit variance and spatial correlation function using the LAS technique. The correlation coefficient between $c_h$ measured at a point $x_1$ and a second point $x_2$ is specified by the correlation function.
shown previously in Eq. (21). As \( c_h \) is assumed to be characterized statistically by a lognormal distribution, the correlated standard normal random field, \( G(x) \), generated by the LAS method is then transformed into a lognormal distribution using the following transformation function (Fenton and Griffiths, 2008):

\[
c_h = \exp\left\{ \mu_{\ln c_h} + \sigma_{\ln c_h} G(x) \right\}
\]

(25)

where: \( x_i \) and \( k_i \) are, respectively, the vector containing the coordinates of the center of the \( i \)th element and the soil property value assigned to that element. It should be noted that the random fields of \( c_h \) are generated using the free available 2D LAS computer code (http://www.engmath.dal.ca/rfem/) implying that \( \theta_{z_n} \) in the vertical direction (z-direction) is infinite (i.e. the soil properties in this direction remain constant).

Finite element modeling incorporating spatial variability of horizontal coefficient of consolidation

With the complete subsurface profile having been simulated in the previous step, the spatial variability of \( c_h \) is now known and can be employed as input in a finite element consolidation modeling of soil improvement by PVDs. All numerical analyses are carried out using a modified version of the finite element computer program “p86” from the book by Smith and Griffiths (2004) in which soil consolidation is treated as a 2D uncoupled problem under axisymmetric condition. Originally program “p86” was for general two (plane) or three dimensional analysis of the uncoupled consolidation equation using implicit time integration with the “theta” method. The authors modified
the source code of “p86” to allow an axisymmetric and repetitive Monte-Carlo analyses. Although the modified version of “p86” can also be used for 3D analysis, 2D finite element analysis is conducted as 3D FEMC analysis is computationally too intensive. Since only 2D axisymmetric analysis is performed, FE analysis considers only the consolidation due to drainage in the horizontal direction (i.e. consolidation due to drainage in the vertical direction is not estimated from the FE solution). In this study, it is assumed that if the vertical drainage is considered in the FE analysis, it will give the same results as $U_v(t)$, as shown in Eq. (10).

It can be noticed that the probabilistic design procedures proposed earlier in this paper take the smear effect into account through a constant ratio of permeability in the undisturbed zone to the smear zone (i.e. $k_h / k_h'$). However, no explicit permeability parameter is considered in the FE analysis. To simulate such reduced permeability condition in the smear zone during the FE analysis, two independent random fields of $c_h$ are generated separately. By employing the specified $\mu_{c_h}$, $\sigma_{c_h}$ and $\theta_{c_h}$ into the LAS method, a random field of $c_h$ is generated first for the whole soil domain and mapped onto the corresponding grid in the finite element mesh. Then another random filed of $c_h$ is generated (for the whole influence zone) with modified mean, $\mu_{c_h}'$, and $\sigma_{c_h}'$ in such a way that $k_h / k_h' = \mu_{c_h} / \mu_{c_h}'$ and $\sigma_{c_h} / \mu_{c_h} = \sigma_{c_h}' / \mu_{c_h}'$ (i.e. same coefficient of variation is employed for both fields). However, for both random fields, the same value of $\theta_{c_h}$ is used. Now from the second random field, only the corresponding elements to the smear zone are mapped onto the finite element mesh. This process of random field generation ensures original random nature of $c_h$ over the soil domain and reasonably reflects the
smear effect as well. It should be noted that, for simplicity, the well resistance factor which may affect the rate of consolidation is not considered in the FE analysis. This is due to the fact that the discharge capacities of most PVDs available in the market are relatively high, and hence the well resistance effect can be ignored in most practical cases (Abuel-Naga et al., 2012; Chu, 2004).

An initial pore water pressure of 100 kPa dissipates in a square domain of side length $S$ (spacing of PVD system) is considered in all FE analyses. The selection of square influence area (irrespective of PVD installation pattern) instead of the equivalent circular influence area is to avoid the unfavourable mesh shape as the LAS method requires square (or rectangular) elements to accurately compute locally averaged values of $c_h$ for each element across the grid. For the same reason, square shaped smear zone of side length $S' = \sqrt{\frac{\pi}{\sqrt{3}}} S$ and PVD of side length $S_w = \frac{\pi S_w}{2}$ are employed. A single generation of a random field and the subsequent finite-element analysis of that field are termed “realization”. For an individual realization, the degree of consolidation, $U_h(t)$, at any certain consolidation time, $t$, is calculated with the help of the following expression:

$$U_h(t) = 1 - \frac{\bar{u}}{u_0}$$ \hspace{1cm} (26)

where: $u_0 =$ initial pore pressure; and $\bar{u} =$ average pore pressures at any time of the consolidation process. It has to be emphasized that the average pore pressure ($\bar{u}$) at any time of the consolidation process is calculated by numerically integrating the pore pressure across the volume of each element at a particular time, summing the
contribution of each element and dividing by the total mesh area (element areas are also
calculated by numerical integration).

As the accuracy of the finite element analysis is dependent on the mesh density, a
sensitivity analysis is carried out for both the deterministic and stochastic solutions on
various mesh dimensions to ensure reasonable refinement with minimal discretization
error and to produce reliable and reproducible statistics of the output quantities. The
consolidation problem under consideration for the mesh sensitivity analysis implies an
axisymmetric unit cell of geometry $r_e = 1.467 \text{ m} \ (\text{i.e. } S = 2.6 \text{ m}), r_s = 0.5642 \text{ m} \ (\text{i.e. } S_s =
1 \text{ m})$ and $r_w = 0.1273 \text{ m} \ (\text{i.e. } S_w = 0.2 \text{ m}).$ The mean value of the horizontal coefficient
of consolidation for the undisturbed zone, $\mu_{c_u}$, is selected to be equal to $4 \text{ m}^2/\text{year}.$ It is
assumed that $k_h/k'_h = 2.$ To simulate such condition, the mean value of the horizontal
coefficient of consolidation for the smear zone, $\mu'_{c_s}$, is used. In order to
determine an optimum mesh density, deterministic analysis (using the respective
constant mean value of $c_h$ for undisturbed and smear zones) of the above problem is
conducted first for three different mesh densities with element size of: $0.2 \text{ m} \times 0.2 \text{ m};$
$0.1 \text{ m} \times 0.1 \text{ m}$; and $0.05 \text{ m} \times 0.05 \text{ m}.$ The obtained results are shown in Fig. 5a, which
also includes the results of the mesh size analysis for the stochastic analysis (will be
described below). It can be seen that the degree of consolidation, $U_h,$ obtained from the
mesh size $0.1 \text{ m} \times 0.1 \text{ m}$ is identical to that of the mesh size $0.05 \text{ m} \times 0.05 \text{ m},$ and is
marginally different from that of the mesh size $0.2 \text{ m} \times 0.2 \text{ m}.$ Accordingly, the mesh
size $0.1 \text{ m} \times 0.1 \text{ m}$ is deemed to be suitable for the deterministic analysis. On the other
hand, the two mesh sizes: $0.1 \text{ m} \times 0.1 \text{ m}$ and $0.05 \text{ m} \times 0.05 \text{ m}$ are further investigated
for the stochastic analysis. After performing a suite of 2000 Monte Carlo simulations
for a case study of $\nu_s = 100\%$ and $\theta_s = 1.0$, the mean, $\mu_{U_s}$, and standard deviation, $\sigma_{U_s}$, of the degree of consolidation over the 2000 simulations are estimated using the method of moments and the results are shown in Fig. 5. It can be seen that there is little or no change in $\mu_{U_s}$ (Fig. 5a) and $\sigma_{U_s}$ (Fig. 5b) from the mesh size $0.1 \text{ m} \times 0.1 \text{ m}$ to $0.05 \text{ m} \times 0.05 \text{ m}$ and thus the mesh size $0.1 \text{ m} \times 0.1 \text{ m}$ is also deemed to give reasonable precision for the stochastic consolidation analysis. However, to comply with the minimum correlation length reported in the literature, it was decided to discretize the soil domain in the current study into the more refined mesh of element size of $0.05 \text{ m} \times 0.05 \text{ m}$.

Repetition of process based on the Monte Carlo technique

The accuracy of the estimated statistics of the output quantities of interest is dependent on the number of Monte Carlo simulations. Therefore, to maintain accuracy and run time efficiency, the sensitivity of results to the number of Monte Carlo simulations is examined. The sensitivity analysis indicated that 2000 realizations are sufficient to give reasonably stable output statistics for each analysis of interest. Based on this observation, the process of generating a random field of $c_h$ and the subsequent finite element analysis is repeated 2000 times. Huang et al. (2010) also performed successful probabilistic analysis on soil consolidation using a $0.05 \text{ m} \times 0.05 \text{ m}$ square element mesh with 2000 simulations. At each “realization” of the Monte Carlo process, the random field of $c_h$ is generated using the same $\nu_s$ and $\theta_s$, but with spatial distribution of $c_h$ that varies from one realization to the next. Fig. 6 shows a typical example of the discretized FE mesh and the corresponding soil domain represented by a grey scale of a
typical realization of the random field of $c_h$ in which the magnitude of $c_h$ remains
constant within each element but differs from one element to another. The obtained
outputs from the suite of 2000 realizations of the Monte Carlo process are collated and
statistically analyzed to make a comparative study between the proposed PDM and the
FEMC, as will be seen later.

**Probabilistic interpretation of the FEMC results**

In the PDM, the drain spacing $S$ is calculated based on a given $U_s(t_s)$, $t_s$ and $P_s$, while in
the FE analysis $S$ is a known parameter. Therefore, to compare the PDM with the
FEMC, the probability of achieving $U_s(t_s)$ (i.e. $P[U \geq U_s]$) at $t_s$ or the probability that $t$ is
less than or equal to $t_s$ (i.e. $P[t \leq t_s]$) that achieves $U_s(t_s)$ is to be estimated. In this study,
the later process is employed, i.e. $P[t \leq t_s]$ is estimated. This is because determining
probability from a set of data requires establishment of a reasonable probability
distribution for the data set. However, the obtained fit using the raw data of $U(t_s)$ was
typically poor while the distribution of $t$ at $U_s(t_s)$ obtained from the suite of the 2000
realizations is reasonably fitted with lognormal distribution and gives sufficiently
reasonable approximation to the $P[t \leq t_s]$. By accepting the lognormal distribution for $t$
at any given $U_s(t_s)$, the statistical moments $\mu_t$ and $\sigma_t$ that are representing the mean and
standard deviation of the lognormally distributed $t$ are calculated from the suite of 2000
realizations using the following transformation functions:

$$\mu_t = \frac{1}{n_{\text{sim}}} \sum_{i=1}^{n_{\text{sim}}} t_i$$  \hspace{1cm} (27)
\[ \sigma_i = \sqrt{\frac{1}{n_{sim} - 1} \sum_{i=1}^{n_{sim}} (t_i - \mu_i)^2} \]  

(28)

where: \( t_i \) is the \( t \) from the \( i \)'th realization \( (i = 1, 2, 3, \ldots, n_{sim}) \) at given \( U_s(t_s) \) and \( n_{sim} \) = total number of realizations = 2000. The probability that \( t \) is less than or equal to \( t_s \) (i.e. \( P[t \leq t_s] \)) that achieves \( U_s(t_s) \) can then be obtained from the following lognormal probability distribution transformation:

\[ P[t \leq t_s] = \Phi \left( \frac{\ln t_s - \mu_{ln}}{\sigma_{ln}} \right) \]  

(29)

where: \( P[.\] is the probability of its argument; \( \Phi(.) \) is the standard normal cumulative distribution function; \( \mu_{ln} \) and \( \sigma_{ln} \) are, respectively, the mean and standard deviation of the underlying normally distributed \( \ln t \) and can be estimated from \( \mu_t \) and \( \sigma_t \) with reference to Eqs. (2) and (3), as follows:

\[ \mu_{ln} = \ln \mu_t - \frac{1}{2} \sigma_{ln}^2 \]  

(30)

\[ \sigma_{ln} = \sqrt{\ln \left( 1 + \frac{\sigma_t^2}{\mu_t^2} \right)} \]  

(31)
Comparison between PDM and FEMC through Illustrative numerical example

The design methodology described earlier for soil improvement by PVDs is illustrated by the following numerical example. A PVD type is selected with \( r_w = 0.06366 \) m having a discharge capacity \( q_w = 350 \) m\(^3\)/year. The required deterministic parameters are assumed to be: \( c_v = 2 \) m\(^2\)/year, \( k_h = 1.577 \times 10^{-2} \) m/year, \( k'_h = 7.88 \times 10^{-3} \) m/year, \( r_s = 0.22568 \) m, \( L = 10 \) m. Based on the above parameters, the calculated \( r_s/r_w = 3.545 \), \( k_h/k'_h = 2 \) and \( k_h/q_w = 4.5 \times 10^{-5} \). For the site of interest, it is assumed that \( \mu_{c_h} = 4 \) m\(^2\)/year and \( \sigma_{c_h} = 4 \) m\(^2\)/year. The calculated \( \nu_{c_h} \) is therefore equal to 100%. It is further assumed that the correlation structure of \( c_h \) in any spatial direction can be reasonably represented by the Markov correlation function given in Eq. (21) and the obtained \( \theta_{c_h} = 1 \) m. The design is to be carried out for achieving 90% consolidation, i.e. \( U_s(t_s) = 0.90 \), with a probability of 95% confidence (i.e. \( P_s = 0.95 \)) after 1 year (i.e. \( t_s = 1 \) year). The rectangular drain pattern is selected for the drain installation. Following Step 5 of the outlined design procedures, \( U_s(t_s) \) is calculated using Eq. (10), which is found to be equal to 0.16.

It has suggested in the previous section that prior to commencing the probabilistic design, deterministic design would be beneficial from the view point of computational requirement. For deterministic design, the design factor \( D_f \) is required to be set equal to unity. Substituting all the parameters calculated earlier in Eq. (24) gives:
As in the finite element analysis, no well resistance effect is considered (i.e. the drain is modeled as a perfect drain), thus $r_F = 0$. Note that, for the selected drain, the estimated well resistance factor using Eq. (8) is very negligible ($F_r = 0.0094$) and the assumption of no well resistance in this case is reasonably accepted. The value of $F_s + F_r$ can then be calculated using Eq. (7) as follows:

$$F_s + F_r = (2.0 - 1.0) \times \ln(3.545) + 0 = 1.2655$$

Using the values of $\psi$ and $F_s + F_r$ as calculated above, the value of $n$ reads from Fig. 4 as 16.689. Using the value of $n$ in Step 10, the obtained deterministically designed $r_e$ is equal to 1.0624 m (i.e. 16.689 × 0.06366 = 1.0624). Since the rectangular pattern is preferred, the corresponding drain spacing, $S_D$ obtained in Step 11 is equal to $1.0624/0.565 = 1.88$ m. Let us now proceed to the probabilistic design.

**Iteration-1**

Following Step 6, it is assumed that $S_a = 1.4$ m (25% smaller than $S_D$). Therefore, $\Theta = 0.714$ ($\Theta = \frac{\theta_c}{S_a}$). The value $\gamma(D)$ corresponding to $\Theta = 0.714$ in Fig. 2 is 0.291. Using $\gamma(D) = 0.291$, $\nu_{c_o} = 100\%$ and $P_s = 0.95$, the value of $D_f$ reads from Fig. 3e as $0.3378$. Substituting $D_f = 0.3378$ in Eq. (24) yields:
\[
\psi = \frac{2t_s D_i \mu_{\ell_s}}{2 \ln \left( \frac{1 - U_s(t_s)}{1 - U_i(t_i)} \right)} \frac{1}{r_w^2} = 2 \times 1.0 \times 0.3378 \times 4.0 \times \frac{1}{0.06366^2} = 313.33
\]

and \( F_s + F_r = 1.2655 \)

It should be noted that \( F_s + F_r \) is constant for a selected ground improvement problem irrespective of the design approach. By employing the calculated values of \( \psi \) and \( F_s + F_r \), the value of \( n \) reads from Fig. 4 as 10.461. Using the value of \( n \) in Step 10, the obtained \( r_{ep} \) from the first iteration is equal to 0.6659 m (i.e. 10.461 \( \times \) 0.06366 = 0.6659). For the rectangular pattern, the corresponding drain spacing, \( S \), obtained in Step 11 is equal to 0.6659/0.565 = 1.178 m. As \( S \neq S_a \), more iterations are required.

**Iteration-2**

Assuming \( S_a = 1.18 \) m gives \( \Theta = 0.8474 \) and \( \gamma(D) = 0.3436 \). Using \( \gamma(D) = 0.3436 \), the value of \( D_i \) reads from Fig. 3e as 0.3168. For \( D_i = 0.3168 \), \( \psi = 293.848 \) and \( F_s + F_r = 1.2655 \). Using these values of \( \psi \) and \( F_s + F_r \), Fig. 4 yields \( n = 10.18 \).

Therefore, the obtained \( r_{ep} \) from the second iteration is equal to 0.648 m. For the rectangular pattern, the corresponding drain spacing, \( S \), obtained in Step 11 is equal to 0.648/0.565 = 1.147 m. As \( S \neq S_a \), proceed to the next iteration.
### Iteration-3

Assuming $S_a = 1.14$ m gives $\Theta = 0.8772$ and $\gamma(D) = 0.3545$. Using $\gamma(D) = 0.3545$, the value of $D_f$ obtained from Fig. 3e is 0.313. For $D_f = 0.313$, $\psi = 290.212$ and $F_s + F_r = 1.2655$. Using these values of $\psi$ and $F_s + F_r$, Fig. 4 yields $n = 10.127$.

Therefore, the obtained $r_{cp}$ from the third iteration is equal to 0.644 m. As the drains are installed in a rectangular fashion, the corresponding drain spacing, $S$, is equal to 1.14 m.

It can be seen that the calculated $S$ is now equal to the initially assumed $S_a$. Therefore, the designed drain spacing that achieves 90% consolidation with 95% confidence probability ($\nu_{e_s} = 100\%$ and $\theta_{e_s} = 1$ m) is 1.14 m. It is interesting to note that the deterministically designed drain spacing for this problem was 1.88 m.

To test the proposed PDM, the FEMC analysis for the above design example is also performed. In the FE analysis, a mesh of square averaging area of side length equal to the drain spacing $S = 1.14$ m as estimated by the PDM is discretized. The same deterministic and random field parameters (i.e. $r_w = 0.06366$ m, $r_s = 0.22568$ m, $r_s/r_w = 3.545$, $\mu_{e_s} = 4$ m$^2$/year, $\sigma_{e_s} = 4$ m$^2$/year and $\theta_{e_s} = 1$ m) as that used for the PDM is utilized for the FE analysis. It should be noted that, to simulate $k_h/k'_h = 2$, the random field for the smear zone (2nd random field) is generated with $\mu'_{e_s} = 2$ m$^2$/year, $\sigma'_{e_s} = 2$ m$^2$/year and $\theta_{e_s} = 1$ (i.e. $\nu_{e_s}$ and $\theta_{e_s}$ are kept constant for both random fields). Since soil consolidation due to the vertical drainage is not considered in the FEMC, the time, $t$, required for achieving $U_h(t_s)$ at each Monte-Carlo simulation is taken as the problem output. It should be noted that, for $U_s(t_s) = 0.90$ and $U_v(t_s) = 0.16$, the calculated $U_h(t_s)$ using Eq. (9) is equal to 0.881. In Fig. 7, the legitimacy of the lognormal distribution...
The hypothesis for $t$ is examined by the well-known Chi-square test through frequency density plot of $t$ data at $U_s(t_s) = 0.881$ obtained from the 2000 realizations and a fitted lognormal distribution is superimposed. The visual inspection of Fig. 7 suggests that the lognormal distribution fits the $t$ histogram very well. The goodness-of-fit test using the chi-square test yielded $p$-value of 0.1 implying that the lognormal distribution hypothesis for $t$ is valid. With reference to Fig. 7, the probability that $t$ is less than or equal to $t_s$, i.e. $P[t \leq t_s]$, can be estimated as follows:

$$P[t \leq t_s] = P[t \leq 1] = \Phi \left( \frac{\ln 1 - (-0.82)}{0.46} \right) = 96.3\%$$

It should be noted that, in case of the PDM, the target probability, $P_s$, for the problem was 95% implying an excellent agreement between the proposed PDM and the FEMC approaches. Following the same procedure as illustrated above, both the PDM and FEMC are performed over a range of $\nu_{c_s}$ and $\theta_{c_s}$ as shown in Table 1. The material and geometric properties used for the analyses are shown in Table 2. By specifying $U_s(t_s) = 90\%$, $P_s = 95\%$ and $t_s = 1$ year, the drain spacing $S$ is estimated for each combination of $\nu_{c_s}$ and $\theta_{c_s}$ using the PDM and listed in Table 3. It can be seen that, for certain $\nu_{c_s}$, the calculated drain spacing $S$ decreases with the increase of $\theta_{c_s}$. The explanation behind this behavior lies in the fact that for a vanishingly small $\theta_{c_s}$, soil becomes infinitely rough, i.e. any point at which soil has low $c_h$ will be surrounded by points where the soil has high $c_h$. What this means is that the flow path initially becomes increasingly tortuous with longer drainage length, hence, the flow is forced to find a shorter passage cutting through the low $c_h$ regions. In contrast, for larger $\theta_{c_s}$, ...
regions of low $c_h$ are bunched together and as a result, the draining pore water detour
the bunched up low $c_h$ regions instead of cutting through them which leads to a longer
drainage length and consequently slower increasing rate consolidation, subsequently
requires a smaller drain spacing.

The FEMC analysis for each combination of $\nu_{c_h}$ and $\theta_{c_g}$ is also conducted by
employing the estimated $S$ obtained through the PDM. The estimated $P[t \leq t_s]$ at $U_s(t_s) =$
90% from the FEMC analysis is displayed in the last row of Table 3 (referred as $P_{FEMC}$).
It can be seen that the target probability $P_s$ (5th row) for the proposed PDM agrees very
well with $P_{FEMC}$ (last row). However, the agreement between the proposed PDM and
FEMC in Table 3 is examined at a single target probability level, i.e. $P_s = 95%$.
Therefore, to make a general comment regarding the validity of the proposed PDM, it
was tested over a wider range of $P_s$. For this purpose, the same example problem (see
Table 2 for the material and geometric properties of the problem) considered earlier to
illustrate the design steps of the proposed PDM is used for a design spacing, $S = 1.14$ m
(i.e. $r_e = 0.644$ m) and spatial variability parameters of $\nu_{c_h} = 100\%$ and $\theta_{c_g} = 1.0$. Two
sets of FEMC analyses are conducted, in the first set, $\nu_{c_h}$ is kept constant at 50% for
various $\theta_{c_g}$ of 0.25 m, 0.5 m and 1.0 m, while in the second set $\theta_{c_g}$ is kept constant at
1.0 for various $\nu_{c_h}$ of 25%, 50% and 100%. It is assumed that $U_s = 90\%$, therefore, the
required $t$ of achieving $U_h = 88.1\%$ is targeted at each Monte-Carlo simulation and $P[t \leq$
$t_s]$ is calculated as a function of $t$. The time $t$ to satisfy a certain $P_s$ of achieving $U_s =$
90% is determined from the proposed PDM and its results are compared with those
obtained from the FEMC. It should be noted that as all geometric properties of the
problem is known, $\gamma(D)$ for a certain soil variability is also known. Therefore, using the
known $\gamma(D)$, $D_f$ for a certain $P_s$ can be evaluated from Eq. (19). The calculated $D_f$ can
then be used to obtain $t$ for the PDM using Eq. (23). The results obtained from both the
proposed PDM and FEMC for each set of soil spatial variability parameters mentioned
above are shown in Fig. 8 in which $P[t \leq t_s]$ is expressed as a function of $t$. It can be
seen that all curves start from the deterministic time of achieving $U_h = 88.1\%$ (i.e.
t_{D88.1}). This is due to the fact that $P[t \leq t_s]$ at any time less than the time for the
deterministic target degree of consolidation has little meaning as it implies $D_f > 1.0$
which is physically irrational. For the same reason, a minimum value of $P_s = 70\%$ is
used in developing the chart of $D_f$ shown in Fig. 3.

The effect of increasing $\nu_{c_h}$ on $P[t \leq t_s]$ at a fixed value of $\theta_{c_h} = 1.0$ is illustrated in Fig.
8a, which indicates that the predicted $P[t \leq t_s]$ obtained from the proposed PDM agrees
very well with those obtained from the FEMC for all cases of $\nu_{c_h}$. On the other hand,
Fig. 8b demonstrates that there is also good agreement between $P[t \leq t_s]$ estimated via
the FEMC and proposed PDM, for various values of $\theta_{c_h}$ at a fixed value of $\nu_{c_h} = 50\%$, 
although a slight discrepancy in $P[t \leq t_s]$ exists when $\theta$ is as small as 0.25. As also can
be seen in Table 3, at any certain $\nu_{c_h}$, $P_{FEMC}$ is slightly lower than $P_s$ when $\theta_{c_h}$ is too
small (i.e. for erratic soil) implying that the estimated drain spacing using the PDM will
be reasonably un-conservative for very small $\theta_{c_h}$. Close view to the results presented in
Table 3 reveals that $P_{FEMC}$ is higher than $P_s$ when $\theta_{c_h} \geq r_s/2$. What this means is that the
estimated drain spacing using the PDM will be slightly conservative when $\theta_{c_h} \geq r_s/2$.
Since $\theta_{c_h}$ is a difficult parameter to estimate in practice, it is suggested to use $\theta_{c_h} \approx r_s$ in
the absence of improved site information which will allow slightly conservative estimate of the drain spacing using the PDM.

It should be noted that although the abovementioned PDM method is mainly developed to consider the inherent soil variability (i.e. aleatoric variability) of \( c_h \), it can also be used to account for the epistemic uncertainty (i.e. measurement error and model uncertainty) of \( c_h \). This can be made by characterizing the probability distribution and COV of \( c_h \), and assuming SOF of \( c_h \) to be equal to infinity. The probabilistic design can then be carried out by determining \( D_f \) from Fig. 3 corresponding to \( \gamma(D) = 1.0 \).

Conclusions

This paper presents a simplified probabilistic design (PDM) method for ground improvement by prefabricated vertical drains (PVDs) taking into account the spatial variability of the horizontal coefficient of consolidation which is deemed to be the most significant random variable affecting soil consolidation. The proposed design approach involves determination of the drain spacing employing a factored design value of the horizontal coefficient of consolidation that achieves a target probability level of specific degree of consolidation at certain timeframe. Simplified design procedure and charts were developed from the proposed probabilistic approach for routine use in practice. The method was verified by a 2D finite element Monte Carlo (FEMC) approach. Although a slight discrepancy between the probabilities of achieving a target degree of consolidation estimated via the PDM and FEMC was observed for very erratic soil (i.e. soil that has very small value of spatial correlation or scale of fluctuation), the overall
agreement between this two methods is very good, indicating that the simpler PDM is reliable and can be used with confidence.
Notations

\( c_h \)  
horizontal coefficient of consolidation

\( c_v \)  
vertical coefficient of consolidation

\( c_{bf} \)  
factored value of \( c_h \)

\( \text{Cov}[...] \)  
covariance operator

\( D_f \)  
design factor

\( F_n \)  
drain spacing factor

\( F_r \)  
well-resistance factor

\( F'_r \)  
average well resistance factor over the entire drain length

\( F_s \)  
smear factor

\( k_h \)  
coefficients of permeability in the horizontal (radial) direction

\( k'_h \)  
horizontal permeability of the smear zone

\( k_v \)  
coefficients of permeability in the vertical direction

\( L \)  
maximum vertical drainage distance, length of the vertical drain

\( m_v \)  
coefficient of volume compressibility

\( n \)  
drain spacing ratio

\( q_w \)  
vertical discharge capacity of drains

\( r_e \)  
radius of the equivalent soil cylinder, radius of the influence zone

\( r_{es} \)  
required radius of the equivalent soil cylinder for achieving target degree of consolidation

\( r_w \)  
equivalent radius of the drain

\( r_s \)  
radius of the smear zone

\( S \)  
spacing of the drain
\( S_D \) deterministically designed drain spacing

\( s \) smear zone ratio

\( t \) consolidation time

\( t_s \) specified time of consolidation

\( U(t) \) degree of consolidation at time \( t \)

\( U_h(t) \) degree of consolidation due to horizontal drainage at time \( t \)

\( U_v(t) \) degree of consolidation due to vertical drainage at time \( t \)

\( U_s(t_s) \) target degree of consolidation at specified time \( t_s \)

\( U_v(t_s) \) degree of consolidation due to vertical drainage at specified time \( t_s \)

\( \nu \) coefficient of variation of a random variable

\( \nu_c \) coefficient of variation of horizontal coefficient of consolidation

\( \gamma_w \) unit weight of water

\( \alpha \) a group parameter representing the smear effects and geometry of the PVD system

\( \gamma(D) \) variance function giving variance reduction due to averaging over domain \( D \)

\( \theta \) correlation length or scale of fluctuation

\( \theta_c \) scale of fluctuation of horizontal coefficient of consolidation

\( \mu \) mean value of a random variable

\( \mu_{c_h} \) mean of the lognormally distributed \( c_h \)

\( \mu_{\ln c_h} \) mean of the underlying normally distributed \( c_h \)

\( \mu_{\ln \tau_c} \) mean of the logarithm of locally averaged \( c_h \)

\( \sigma/\sigma^2 \) standard deviation/variance of a random variable

\( \sigma_{c_h} \) standard deviation of the lognormally distributed \( c_h \)
\( \sigma_{\ln c_h} \) standard deviation of the underlying normally distributed \( c_h \)

\( \sigma_{\ln \tau_s} \) standard deviation of the logarithm of locally averaged \( c_h \)

\( \rho_{\ln c_h}(\tau) \) correlation function giving correlation between two \( \ln(c_h) \) data separated by a distance \( \tau \)

\( \tau \) absolute distance between two points in the soil domain

\( P_s \) target probability of achieving degree of consolidation

\( P(\bullet) \) probability of its argument

\( \Phi(\bullet) \) standard normal cumulative distribution function

\( \Phi^{-1}(\bullet) \) inverse normal cumulative distribution function

**Appendix A. Determination of variance reduction factor**

The amount by which the variance is reduced from the point variance as a result of the local averaging can be estimated as follows. The fourfold integration in Eq. (22) can be condensed to a twofold integration by taking advantage of the quadrant symmetry \((\rho(\tau_1, \tau_2) = \rho(\tau_1, -\tau_2) = \rho(-\tau_1, \tau_2) = \rho(-\tau_1, -\tau_2))\) of the correlation function in Eq. (21) and can be expressed as:

\[
\gamma(X, Y) = \frac{4}{X^2 Y^2} \times \int_0^X \int_0^Y (X - \tau_1)(Y - \tau_2) \rho(\tau_1, \tau_2) d\tau_1 d\tau_2 \quad (A.1)
\]

Eq. (A.1) can be computed numerically with reasonable accuracy using sixteen-point Gaussian quadrature integration scheme as follows:
\[ \gamma(X,Y) = \frac{1}{4} \sum_{i=1}^{16} \omega_i (1 - \vartheta_i) \sum_{j=1}^{16} \omega_j (1 - \vartheta_j) \rho(\zeta_i, \eta_j) \]  
(A.2)

where

\[ \zeta_i = \frac{X}{2} (1 + \vartheta_i), \eta_i = \frac{Y}{2} (1 + \vartheta_i) \]  
(A.3)

and \( \omega_i \) and \( \vartheta_i \) are the weights and Gauss points respectively.
References


Figure Captions:

Fig. 1. Schematic diagram of a unit cell soil cylinder with prefabricated vertical drain

Fig. 2. Variance reduction factor as a function of scale of fluctuation

Fig. 3. Design factor $D_f$ for (a) $\nu_{cv} = 10\%$; (b) $\nu_{cv} = 25\%$; (c) $\nu_{cv} = 50\%$; (d) $\nu_{cv} = 75\%$; (e) $\nu_{cv} = 100\%$

Fig. 4. Design graph for drain spacing [adapted from Zhou et al. (1999)]

Fig. 5. Influence of mesh density on (a) $U_h$ (deterministic) and $\mu_{U_h}$ (stochastic); (b) $\sigma_{U_h}$

Fig. 6. Typical realization of the random field $c_h$ with discretized finite element mesh

Fig. 7. Frequency density histogram and fitted distribution of $t$ at $U_h = 88.1\%$ for $\nu_{cv} = 100\%$, $\theta_{cv} = 1.0$

Fig. 8. Comparison between FEMC and PDM for the effect of: (a) $\nu_{cv}$ on $P[t \leq t_s]$ for $\theta_{cv} = 1.0$ and (b) $\theta_{cv}$ on $P[t \leq t_s]$ for $\nu_{cv} = 50\%$; at $U_h = 88.1\%$
Table 1

Random field parameters for the PDM and FEMC analyses.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_{c_h} ) (m(^2)/year)</td>
<td>4</td>
</tr>
<tr>
<td>( \nu_{c_h} ) (%)</td>
<td>50, 100</td>
</tr>
<tr>
<td>( \theta_{c_h} ) (m)</td>
<td>0.25, 0.5, 1.0</td>
</tr>
</tbody>
</table>
Table 2

Material and Geometric properties for the PDM and FEMC analyses.

<table>
<thead>
<tr>
<th>Property</th>
<th>PDM</th>
<th>FEMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_v$ (m$^2$/year)</td>
<td>2</td>
<td>–</td>
</tr>
<tr>
<td>$k_h$ (m/year)</td>
<td>$1.577 \times 10^{-2}$</td>
<td>–</td>
</tr>
<tr>
<td>$k'_h$ (m/year)</td>
<td>$7.88 \times 10^{-3}$</td>
<td>–</td>
</tr>
<tr>
<td>$L$ (m)</td>
<td>10</td>
<td>–</td>
</tr>
<tr>
<td>$r_w$ (m)</td>
<td>0.06366</td>
<td>0.06366</td>
</tr>
<tr>
<td>$r_s$ (m)</td>
<td>0.22568</td>
<td>0.22568</td>
</tr>
<tr>
<td>$q_w$ (m$^3$/year)</td>
<td>350</td>
<td>–</td>
</tr>
</tbody>
</table>
Table 3
Summary of results obtained from the PDM and FEMC analyses.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_s(t_s)$ (%)</td>
<td>90</td>
</tr>
<tr>
<td>$t_s$ (year)</td>
<td>1</td>
</tr>
<tr>
<td>$\nu_{e_s}$ (%)</td>
<td>50% 100%</td>
</tr>
<tr>
<td>$\theta_s$ (m)</td>
<td>0.25 0.5 1.0 0.25 0.5 1.0</td>
</tr>
<tr>
<td>$P_s$ (%)</td>
<td>95 95 95 95 95 95</td>
</tr>
<tr>
<td>$S$ (m)</td>
<td>1.69 1.61 1.5 1.44 1.3 1.14</td>
</tr>
<tr>
<td>$P_{FEMC}$ (%)</td>
<td>90.9 94.8 96.1 91.7 95.2 96.3</td>
</tr>
</tbody>
</table>
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