manuscripta geodaetica (1995) 20:489–497

manuscripta geodaetica

The invertible GPS ambiguity transformations

P. J. G. Teunissen

Delft Geodetic Computing Centre (LGR), Faculty of Geodetic Engineering, Delft University of Technology, Thijsseweg 11, NL-2629 JA Delft, The Netherlands

Received 9 February 1994; Accepted 11 May 1994

Abstract

Certain linear combinations of the GPS-observables play a prominant role in the problem of ambiguity fixing. In particular "wide-laning" techniques have proven to be very successful. At present, however, the various integer linear combinations that are considered, are restricted to the single-channel dual-frequency case. In this contribution this class is generalized to the multi-channel case. First the two dimensional case is considered. It is shown how to infer which of the different integer ambiguities can be paired. It turns out, for instance, that one is not allowed to pair the narrow-lane ambiguity to the wide-lane ambiguity. The two dimensional case is then generalized to higher dimensions. And it is shown that admissible ambiguity transformations need to be integer and volume-preserving.

1. Introduction

The GPS observables are (P or C/A) code-derived pseudorange measurements and carrier phase measurements, which can be available on both of the two frequencies L_1 (f_1=1575.42 Mhz) and L_2 (f_2=1227.6 Mhz). In particular the very low noise behaviour of the carrier phase measurements makes high precision relative positioning possible. However, since the GPS-receivers only provide measurements of fractional phase plus the total number of cycle counts since the start of tracking, the carrier phase measurements are ambiguous by an unknown integer number of cycles, the so-called phase ambiguities. A pre-requisite for obtaining high precision relative positioning results, based on carrier phase data, is therefore that the phase ambiguities become sufficiently separable from the baseline coordinates. Different approaches are in use and have been proposed to ensure a sufficient separability between these two groups of parameters. They are either based on the use of carrier phase data only, or make use of the combination of carrier phase data and code-derived pseudorange data. One approach for static applications is to simply make use of carrier phase data that corresponds to sufficiently differing receiver-satellite geometries. But since GPS satellites are in very high altitude orbits, their relative position with respect to the receiver changes slowly, which implies that long timespans between the first and the last collected carrier phase data are necessary so as to ensure separability. A significant reduction in the timespan is possible, if one explicitly aims at resolving for the integer-values of the ambiguities. The inclusion of fast ambiguity resolution algorithms has made rapid static surveying with GPS possible, in particular for the relatively short baselines and when dual frequency carrier phase measurements are used, see e.g. [1] and [2].

Approaches that have been in use for kinematic surveying and that ensure separability between the baseline coordinates and ambiguities, are based on the use of carrier phase data of which the integer ambiguities have been estimated at an earlier, so-called initialization stage. Ways to initialize are either to start the survey from a known baseline of sufficient precision, or, if this is not possible, to use a static GPS-survey through which the baseline is determined, or to use the antenna-swap technique, see e.g. [3]. Reinitialization is needed however if continuous signal tracking is not maintained on a sufficient number of satellites.

The need for either long observational timespans or static initialization techniques is absent, if in addition to the carrier phase data, sufficiently precise ranging information can be included in the solution. Much faster integer ambiguity resolution is namely possible, both for static as well as kinematic applications, when L_1 and L_2 carrier phase data is used in combination with L_1 and L_2 P-code pseudoranges, see e.g. [4-8]. But, it is still questionable whether the P-code observables will remain available in the future for civilian users.

In all of the above mentioned approaches a prominant role is played by certain linear combinations of the original carrier phase and/or pseudorange observables. Depending on the application, derived observables can be formed with certain desirable properties, such as for instance geometry-free and ionosphere-free linear combinations. And in particular in relation to ambiguity fixing, well-known examples are the narrow-lane, the wide-lane and extra wide-lane combinations, see e.g. [9-12]. But also other wide-lane combinations have been studied [13].

For the purpose of ambiguity fixing one usually only considers those integer linear combinations of value that produce a phase observable which has a relatively long wavelength, a relatively low noise behaviour and a reasonable small ionospheric delay. And these properties are indeed very beneficial to the integer ambiguity fixing process. But what about the relative receiver-satellite geometry? For instance, the reason for aiming at a low noise behaviour for the derived phase observable is not to have a small standard deviation for this observable per se, but because the noise in the estimated ambiguities gets reduced when the noise in the observables reduces. The point is, that it is the noise in the estimated ambiguities, or better still, it is the complete variance-covariance matrix of the estimated ambiguities which becomes decisive for the ambiguity fixing process. And it is also at this level where the relative receiver-satellite geometry plays its prominant role. This observation suggests that it is of interest to generalize the current single channel integer linear combinations to multi-channel integer linear combinations. The topic of the present contribution is therefore the identification of such integer linear combinations. It will be shown, both for the single channel case as well as for the multi-channel case, which conditions an invertible ambiguity transformation needs to fulfill. And for both cases examples of admissible ambiguity transformations will be given.

2. Linear combinations of the phase observables

In the following we will restrict our attention to the carrier phase measurements only. Since it is not uncommon in the processing of phase data to difference the carrier phase measurements between satellites and between receivers to eliminate the satellite and receiver clock offsets, we will work with the double-difference phase observables. When expressed in units of range (rather than cycles) the double-difference carrier phase observables on L_1 and L_2, Φ_1 and Φ_2, can be represented as

$$(1)\qquad \begin{cases} \Phi_1 = \rho - (\lambda_1/c)^2 I + \lambda_1 N_1 + \varepsilon_1 \\ \Phi_2 = \rho - (\lambda_2/c)^2 I + \lambda_2 N_2 + \varepsilon_2 \end{cases}$$

where:

ρ	:	double-difference form of the range from receiver to satellite,
λ_i	:	wavelength of the L_i carrier ($\lambda_1 \simeq 19$cm, $\lambda_2 \simeq 24$cm),
c	:	speed of light in vacuum,
$(\lambda_i/c)^2 I$	:	double-difference form of the ionospheric phase advance on L_i,
N_i	:	integer double-difference L_i phase ambiguity,
ε	:	L_i measurement noise plus remaining unmodelled errors.

Depending on the application, various linear combinations of the phase observables Φ_1 and Φ_2 can be taken to obtain derived observables with certain desirable properties. By taking the linear combination $\alpha\Phi_1/\lambda_1+\beta\Phi_2/\lambda_2$, we obtain from (1)

$$\alpha\Phi_1/\lambda_1+\beta\Phi_2/\lambda_2 = \frac{\alpha\lambda_2+\beta\lambda_1}{\lambda_1\lambda_2}\rho - \frac{\alpha\lambda_1+\beta\lambda_2}{c^2} I + (\alpha N_1+\beta N_2) + \varepsilon_{\alpha\beta}. \qquad (2)$$

The ionospheric term, I, can be eliminated by choosing $\beta=-(\lambda_1/\lambda_2)\alpha$. This choice leads to the so-called ionosphere-free phase observable (L_3-phase)

$$(3)\qquad \Phi_3 = \rho + \frac{\lambda_1\lambda_2^2}{\lambda_2^2-\lambda_1^2}(N_1 - \frac{\lambda_1}{\lambda_2}N_2) + \varepsilon_3.$$

Instead of eliminating the ionospheric delay, one can also choose to eliminate the geometric term ρ in (2). The geometric term gets eliminated by choosing $\beta=-(\lambda_2/\lambda_1)\alpha$. This choice leads then to the so-called geometry-free phase observable (L_4-phase)

$$(4)\qquad \Phi_4 = -\frac{\lambda_1^2-\lambda_2^2}{c^2} I + \lambda_1(N_1 - \frac{\lambda_2}{\lambda_1}N_2) + \varepsilon_4.$$

It will be clear that other combinations can be taken as well, also by including the code-observables into the linear combinations. But, since our prime interest is to study linear combinations of the phase ambiguities, we will restrict our attention to the phase observables only.

It follows from (3) and (4) that the ambiguities of Φ_3 and Φ_4, N_1-$(\lambda_1/\lambda_2)N_2$ and N_1-$(\lambda_2/\lambda_1)N_2$, are generally non-integer since the coefficients λ_1/λ_2 and λ_2/λ_1 are non-integer. There does exist, however, a whole suit of linear combinations of Φ_1 and Φ_2, for which the integer-nature of the ambiguities is retained [13]. By defining

(5) $$\Phi_{\alpha\beta}=\frac{\alpha\lambda_2}{\alpha\lambda_2+\beta\lambda_1}\Phi_1+\frac{\beta\lambda_1}{\alpha\lambda_2+\beta\lambda_1}\Phi_2,$$

we obtain from (1) the derived phase observation equation

(6) $$\Phi_{\alpha\beta}=\rho-\frac{\lambda_1\lambda_2}{c^2}\frac{\alpha\lambda_1+\beta\lambda_2}{\alpha\lambda_2+\beta\lambda_1}I+\frac{\lambda_1\lambda_2}{\alpha\lambda_2+\beta\lambda_1}(\alpha N_1+\beta N_2)+\varepsilon_{\alpha\beta}.$$

And the ambiguity of $\Phi_{\alpha\beta}$ is clearly integer when α and β are chosen to be integer. Two well-known examples are the so-called narrow-lane and wide-lane phase observables, see e.g. [9-12]. The narrow-lane phase observable is obtained by setting $\alpha=\beta=1$. Its wavelength is approximately 11 cm, its ionospheric delay equals $-(\lambda_1\lambda_2/c^2)I$, and its variance is approximately half of that of Φ_1 (assuming that Φ_1 and Φ_2 are uncorrelated and have equal variance). The wide-lane phase observable, also known as L_5-phase, is obtained by setting $\alpha=-\beta=1$. Its wavelength is approximately 86 cm, its ionospheric delay equals, apart from a change of sign,that of the narrow-lane phase, and its variance is about 33 times that of Φ_1.

Apart from the narrow-lane and wide-lane phases, there are of course infinitely many other linear combinations that one might consider. For instance, a longer wavelength then that of the wide-lane is obtained by setting $\alpha=4$ and $\beta=-5$. The corresponding phase observable has a wavelength of approximately 178 cm, but an ionospheric delay that is 18 times larger then that of the wide-lane, and a variance that is 2744 times larger then that of Φ_1. Alternatively, if one wants a smaller ionospheric delay then that of the wide-lane, one might choose $\alpha=5$ and $\beta=-4$, see e.g. [9]. The corresponding phase observable has then an ionospheric delay that is 1/18 times that of the wide-lane, but a wavelength of about 10 cm, and a variance that is about 10 times larger then that of Φ_1.
The above given examples show that the wavelength, the noise behaviour and the ionospheric delay of the phase observable $\Phi_{\alpha\beta}$ are to a great extent dependent on the integer choice made for α and β. It is generally believed that, for the purpose of ambiguity fixing, only those integer linear combinations are of value that produce a phase observable which has a relatively long wavelength, a relatively low noise behaviour and a reasonable small ionospheric delay. And indeed, these properties are beneficial to the integer ambiguity fixing process. There are, however, two important additional aspects which have so far not been taken into consideration. These are the invertibility of the linear combinations and the single-channel dual-frequency restriction.

In the study of integer linear combinations one always seems to work from two phase observables, namely Φ_1 and Φ_2, towards one single derived phase observable. But why not start from two and end with two? It will be clear that the original two phase observables contain more information then the single phase observable derived from them, despite the fact that the derived phase observable may posses certain desirable properties. This observation suggests to study integer linear combinations that preserve the information content of the original two phase observables Φ_1 and Φ_2. This idea will be taken up in the next section and will automatically imply that we have to consider integer linear combinations in pairs.

The second aspect which has so far not been taken into consideration when studying the integer linear combinations, has to do with the receiver-satellite geometry. It is true that a low noise behaviour of the phase observables benefits the ambiguity fixing process. One should recognize however that this is only due to the effect the phase observation noise has on the noise of the least-squares estimated ambiguities. The noise in the estimated ambiguities gets reduced when the noise in the phase observables reduces. Hence, the point is, that not the noise of phase, but the noise of the ambiguities, or better still, the complete variance-covariance matrix of the estimated ambiguities becomes the decisive quantity for the ambiguity fixing process. And it is also at this level where the receiver-satellite geometry plays its prominent role. This being recognized, the logical next step is to study integer linear combinations that may influence the effect the receiver-satellite geometry has on the noise behaviour of the estimated ambiguities. This implies, however, the need to study integer linear combinations at the level of the complete system of observation equations.

This idea will be taken up in section four and as we will see, generalizes the results of section three.

3. Invertible integer linear combinations

In this section we will study integer linear combinations of the ambiguities in pairs. That is, instead of considering the single integer linear combination $\alpha N_1+\beta N_2$, we will start from the following two dimensional linear transformation

$$(7) \qquad \begin{bmatrix} N_{\alpha\beta} \\ N_{\gamma\delta} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

These transformed ambiguities are the ambiguities of the derived phase observables

$$(8) \qquad \begin{cases} \Phi_{\alpha\beta} = \dfrac{\alpha\lambda_2}{\alpha\lambda_2+\beta\lambda_1}\Phi_1 + \dfrac{\beta\lambda_1}{\alpha\lambda_2+\beta\lambda_1}\Phi_2 \\ \Phi_{\gamma\delta} = \dfrac{\gamma\lambda_2}{\gamma\lambda_2+\delta\lambda_1}\Phi_1 + \dfrac{\delta\lambda_1}{\gamma\lambda_2+\delta\lambda_1}\Phi_2 \end{cases}.$$

With these derived phase observables, the phase observation equations take, instead of (1), the form

$$(9) \qquad \begin{cases} \Phi_{\alpha\beta} = \rho - (\lambda_1/c)^2 m_{\alpha\beta} I + \lambda_{\alpha\beta} N_{\alpha\beta} + \varepsilon_{\alpha\beta} \\ \Phi_{\gamma\delta} = \rho - (\lambda_2/c)^2 m_{\gamma\delta} I + \lambda_{\gamma\delta} N_{\gamma\delta} + \varepsilon_{\gamma\delta} \end{cases}.$$

where:

$m_{\alpha\beta} = \dfrac{\lambda_2}{\lambda_1}\dfrac{\alpha\lambda_1+\beta\lambda_2}{\alpha\lambda_2+\beta\lambda_1}$: the ionospheric amplification factor of $\Phi_{\alpha\beta}$

$m_{\gamma\delta} = \dfrac{\lambda_1}{\lambda_2}\dfrac{\alpha\lambda_1+\beta\lambda_2}{\alpha\lambda_2+\beta\lambda_1}$: the ionospheric amplification factor of $\Phi_{\gamma\delta}$

$\lambda_{\alpha\beta} = \dfrac{\lambda_1\lambda_2}{\alpha\lambda_2+\beta\lambda_1}$: the wavelength of $\Phi_{\alpha\beta}$

$\lambda_{\gamma\delta} = \dfrac{\lambda_1\lambda_2}{\gamma\lambda_2+\delta\lambda_1}$: the wavelength of $\Phi_{\gamma\delta}$.

Note that the structure of the above transformed phase observation equations resembles that of the original observation equations (1). However, if the objective is to use these transformed phase observation equations for *ambiguity fixing*, there are three additional conditions that the ambiguity transformation (7) needs to fulfill. Firstly, in order for the transformed ambiguities $N_{\alpha\beta}$ and $N_{\gamma\delta}$ to be integers, the four scalars α, β, γ and δ of (7) need to be integers as well, since the original ambiguities N_1 and N_2 are already integers. Secondly, the transformation matrix of (7) will have to be invertible in order to guarantee a one-to-one correspondence between the original and transformed ambiguities. And finally, but most importantly, the entries of the inverse of the transformation matrix of (7) need to be integers as well. The reason for including this last condition can be made clear as follows. If the scalars α, β, γ and δ are integers, then so are the transformed ambiguities $N_{\alpha\beta}$ and $N_{\gamma\delta}$, when the original ambiguities N_1 and N_2 are integers. However, the converse of this statement is not necessarily true. That is, when the ambiguities $N_{\alpha\beta}$ and $N_{\gamma\delta}$ are integers, then the ambiguities N_1 and N_2 need not be integers, even when the scalars α, β, γ and δ are integers. But this situation is not acceptable, since it could imply that an integer fixing of the ambiguities of (9) corresponds to a fixing of the original ambiguities N_1 and N_2 on non-integer values. We therefore need to ensure that integer values of $N_{\alpha\beta}$ and $N_{\gamma\delta}$ correspond to integer values of N_1 and N_2. And this is only possible by enforcing the condition that the entries of the inverse of the transformation matrix of (7) are integers as well. The important conclusion that is reached, reads therefore that both the transformation matrix of (7) and its inverse must have entries that are integer.

With the above stated conditions, it is now possible to infer which of the different integer ambiguities can be taken as pairs. This is illustrated in the following three examples.

example 1

The transformation from the L_1-and L_2-ambiguities, N_1 and N_2, to the narrow-lane and wide-lane ambiguities reads

$$\begin{bmatrix} N_{11} \\ N_{1,-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

It will be clear that the integer values in the above matrix ensure that the ambiguities N_{11} and $N_{1,-1}$ are integer, whenever the ambiguities N_1 and N_2 are integer. Note, however, that with N_1 and N_2 being integer, the range of the above transformation is not sufficient to cover all integer-pairs N_{11} and $N_{1,-1}$. For instance, the above two linearly independent equations are inconsistent when $N_{11}=1$ and $N_{1,-1}=0$. That is, when $N_{11}=1$ and $N_{1,-1}=0$, no integer values for N_1 and N_2 can be found as a solution to the above equations. And this also happens, for instance, when $N_{11}=0$ and $N_{1,-1}=1$ or when $N_{11}=2$ and $N_{1,-1}=1$. The reason for this situation becomes clear when we consider the inverse of the above transformation. The inverse is given as

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} ½ & ½ \\ ½ & ½ \end{bmatrix} \begin{bmatrix} N_{11} \\ N_{1,-1} \end{bmatrix}.$$

And this result clearly shows that the non-integer entries of the inverse are causing the original two equations to be inconsistent for certain integer-values of N_{11} and $N_{1,-1}$. The interesting conclusion is therefore reached, that one cannot pair the narrow-lane ambiguity to the wide-lane ambiguity. Because, if one would use the narrow-lane phase together with the wide-lane phase, instead of Φ_1 and Φ_2, for ambiguity fixing, the outcome could be that by integer-fixing N_{11} and $N_{1,-1}$, one in fact is fixing N_1 and N_2 to non-integer values.

example 2

The transformation from N_1 and N_2 to N_{11} and $N_{4,-5}$ reads

$$\begin{bmatrix} N_{11} \\ N_{4,-5} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

The inverse of this transformation is given as

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} \end{bmatrix} \begin{bmatrix} N_{11} \\ N_{4,-5} \end{bmatrix}.$$

This shows that it is also not possible to pair the narrow-lane ambiguity to $N_{4,-5}$.

example 3

The previous example showed that $N_{4,-5}$ could not be paired to the narrow-lane ambiguity. But, this example will show that $N_{4,-5}$ can be paired to the wide-lane ambiguity. The transformation from N_1 and N_2 to $N_{1,-1}$ and $N_{4,-5}$ reads

$$\begin{bmatrix} N_{1,-1} \\ N_{4,-5} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

The inverse of this transformation is given as

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} N_{1,-1} \\ N_{4,-5} \end{bmatrix}.$$

Hence, whenever $N_{1,-1}$ and $N_{4,-5}$ are integer, so are N_1 and N_2, and vice versa.

In the above examples the inverse of the ambiguity transformation was explicitly given in order to check whether the entries of the inverse are integer or not. It would be more convenient, however, if we could do without the inverse and base our verification on the elements of the original transformation matrix only. This would then in particular be helpful for the higher dimensional cases.

Consider the inverse of the ambiguity transformation matrix of (7),

$$\begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{bmatrix} = \frac{1}{\alpha\delta-\beta\gamma} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}.$$

This shows that the entries of the inverse are integer, whenever the entries of the ambiguity transformation matrix are integer and in addition the condition $\alpha\delta-\beta\gamma=\pm1$ holds. The condition $\alpha\delta-\beta\gamma=\pm1$ is therefore a sufficient condition. But is it also a necessary condition? The answer to this question is in the affirmative, as the following shows. If the ambiguity transformation matrix and its inverse have integer entries, then both their determinants, $\alpha\delta-\gamma\beta$ and $\bar{\alpha}\bar{\delta}-\bar{\gamma}\bar{\beta}$, are integer as well and $(\alpha\delta-\gamma\beta)(\bar{\alpha}\bar{\delta}-\bar{\gamma}\bar{\beta})=1$. From this follows then that $\alpha\delta-\beta\gamma=\pm1$ must hold. Hence with this result, we are now in the position to rephrase our earlier conclusion. That is, the condition that the entries of both the ambiguity transformation matrix and its inverse must be integers, can now be replaced by the condition that the entries of the transformation matrix need to be integers and that its determinant needs to equal ±1. This shows that instead of considering the inverse explicitly, it suffices to check the value of the determinant of the ambiguity transformation matrix.

4. The class of ambiguity transformations

In the previous section we have looked at the transformed phase observation equations (9), having as ambiguities the integers $N_{\alpha\beta}$ and $N_{\gamma\delta}$. It is, however, not really necessary to work explicitly with the derived phase observables $\Phi_{\alpha\beta}$ and $\Phi_{\gamma\delta}$. One might work as well with the original phase observation equations (1) and then use the ambiguity transformation (7) simply to reparametrize the ambiguities from N_1, N_2 to $N_{\alpha\beta}$, $N_{\gamma\delta}$. This would then give the observation equations

$$(10)\quad \begin{cases} \Phi_1 = \rho - (\lambda_1/c)^2 I + \pm\lambda_1 \delta N_{\alpha\beta} - \pm\lambda_2 \beta N_{\gamma\delta} + \varepsilon_1 \\ \Phi_2 = \rho - (\lambda_2/c)^2 I - \pm\lambda_2 \gamma N_{\alpha\beta} + \pm\lambda_2 \alpha N_{\gamma\delta} + \varepsilon_2 . \end{cases}$$

The signs of the coefficients of the ambiguities depend on the sign of $\alpha\delta - \beta\gamma = \pm 1$.

Up till now we have only considered the two-dimensional ambiguity transformation (7). This transformation operates on a single channel basis and transforms each time a pair of L_1-and L_2-ambiguities into two new ambiguities. But, when we consider (10) for each channel and look at the problem from the point of view of an ambiguity reparametrization, there seems to be no reason why each of the two observables Φ_1 and Φ_2 should depend on only two transformed ambiguities and not on more then two. That is, there is no reason a priori to restrict the transformation to single channels only and there is also no need to assume that we need the phase observables per se on both of the two frequencies L_1 and L_2. The objective of this section is therefore to consider the multi-channel case and to show how the results of the previous section can be generalized to higher dimensions.

Let a be an m-vector, which has as its entries integer double-difference ambiguities. Then $a \in Z^m$, with Z^m being the m-space of integers. The entries of a may be ambiguities of the L_1-type only, of the L_2-type only or of both types. Let Z be an m-by-m matrix of full rank. Then, if we transform a with Z, we would like the result $b=Za$ to be integer whenever a is integer. The following result shows when this is the case:

$$\forall\ a \in Z^m,\ b = Z\,a \in Z^m \text{ iff } Z \text{ has integer entries.}$$

The "if" part of the proof is trivial, because if the entries of Z are integer and $a \in Z^m$, then clearly $b = Za \in Z^m$. The "only if" part of the proof will be given by contradiction. Let the (i,j)-th element of matrix Z be non-integer. One can then always find a vector $a \in Z^m$, $a=(0,..,1,0,..)^*$ for instance, such that $b = Za \notin Z^m$. This shows that all the entries of matrix Z need to be integers.

Using the above result we are now in the position to characterize the whole class of admissible ambiguity transformations. Since we do not only want $b=Za$ to be integer whenever a is integer, but also that $a=Z^{-1}b$ is integer whenever b is integer, it follows that both matrix Z and its inverse Z^{-1} must have entries which are integers. Note that this is in agreement with the results of the previous section. And as it was the case in the previous section for two dimensions, one can also replace the condition that the entries of Z^{-1} must be integers, by the condition that the determinant of Z must equal ± 1. This is shown as follows. First we will proof that, if Z and Z^{-1} have integer entries, then det.$Z=\pm 1$. Since the determinant of a matrix is a sum of products of the matrix entries, it follows that, if Z and Z^{-1} have integer entries, then also det.Z and det.Z^{-1} are integers. From this follows then with (det.Z) (det.Z^{-1})=1, that det.$Z=\pm 1$. Now, we will proof the converse, namely that, if Z has integer entries and its determinant equals ± 1, then also the inverse Z^{-1} has integer entries. For the inverse of Z we may write, see e.g. [14],

$$Z^{-1} = \text{adjoint}(Z)/\text{det}.Z\ .$$

Since the adjoint of matrix Z has elements which are, apart from a possible change of sign, determinants of submatrices of Z, it follows from the above equation that Z^{-1} has integer entries whenever det.$Z=\pm 1$ and Z itself has integer entries. The conclusion we reach reads therefore that matrix Z is an admissible ambiguity transformation if and only if matrix Z has integer entries and det.$Z=\pm 1$. This last property of Z implies, since the volume of the ambiguity confidence-ellipsoid is uniquely determined by the determinant of the ambiguity variance-covariance matrix, that an ambiguity reparametrization with Z leaves the volume of the confidence-ellipsoid invariant. Hence, Z is a volume-preserving transformation. And in fact, it is this property which enables one to make the confidence-ellipsoid more sphere-like, when aiming at smaller variances for the transformed ambiguities.

Now that the class of admissible ambiguity transformations has been given, it is of interest to consider groups of matrices that belong to this class. In the following examples some members from the class of ambiguity transformations are given.

example 4

The identity-matrix is of course a trivial example. But, since a permutation of rows and/or columns leaves the absolute value of the determinant unchanged, it follows that also all permutation matrices belong to the class of ambiguity transformations. This result is of course not too surprising, since in fact it is implicitly already often used in the existing ambiguity fixing algorithms. Because if one re-orders the ambiguities so as to bring the most precise ambiguity in the first slot of the ambiguity vector, one implicitly applies a permutation transformation.

example 5

It will also not come as a surprise that all ambiguity transformations that change the choice of reference satellite in the double-difference ambiguities, are admissible. As a simple example we assume that five satellites are tracked. The single-difference ambiguity related to satellite i is denoted as a_i, and the corresponding double-difference ambiguity having satellite j as reference is denoted as $a_{ji}=a_i-a_j$. The regular transformation from a_{1i} to a_{2i} reads then

$$\begin{bmatrix} a_{21} \\ a_{23} \\ a_{24} \\ a_{25} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \end{bmatrix}.$$

The matrix of this transformation has integer elements and it is easily verified that its determinant equals -1.

example 6

As a generalization of the rank-one update matrix of the previous example, consider the m-by-m matrix

$$Z=\begin{bmatrix} 1 & & & & z_1 & & & & \\ & 1 & & & z_2 & & & & \\ & & \ddots & & \vdots & & & & \\ & & & 1 & z_{i-1} & & & & \\ & & & & 1 & & & & \\ & & & & z_{i+1} & 1 & & & \\ & & & & \vdots & & \ddots & & \\ & & & & z_{m-1} & & & 1 & \\ & & & & z_m & & & & 1 \end{bmatrix}$$

where all z_j, $j=1,..,m$, are integer. Also this matrix is an admissible ambiguity transformation. This can be seen as follows. The above matrix Z can be reduced to an identity matrix, by elementary operations of subtracting an integer multiple of one column from another. And all these operations leave the determinant unchanged. Hence $\det.Z=+1$.

example 7

Also the m-by-m matrix

$$Z=\begin{bmatrix} 1 & & & & & \\ z_{21} & 1 & & & & \\ z_{31} & z_{32} & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot & \\ z_{m1} & z_{m2} & z_{m3} & \cdot & \cdot & 1 \end{bmatrix}$$

where all the z-elements are integers, can be taken as an ambiguity transformation. The determinant of a triangular matrix equals namely the product of the entries on the main diagonal. For the single channel case this immediately shows that one is allowed to pair the L_1 (or L_2) ambiguity with either the wide-lane ambiguity or with the narrow-lane ambiguity.

example 8

Once certain ambiguity transformations are identified, other ambiguity transformations can be derived from them by performing certain matrix operations, like: inversion, transposition and multiplication. For instance, when Z_1 and Z_2 are two given ambiguity transformations, then so are Z_1^{-1}, Z_1^* and Z_1Z_2.

5. Concluding remarks

In this contribution the study of integer ambiguity combinations has been extended from the single-channel dual-frequency case to the multi-channel case. As a result the class of admissible ambiguity transformations has been identified. They need to be integer and volume-preserving. As with the single-channel dual-frequency integer combinations, the purpose of having multi-channel integer combinations available is to be in a better position with regard to the problem of resolving the integer ambiguities. That is, members from the identified class of ambiguity transformations can now be used to aid the ambiguity fixing process, and in particular to reduce the effect that a slowly changing receiver-satellite geometry has on the condition of the ambiguity variance-covariance matrix. This implies that the ambiguity transformations are used to decorrelate the ambiguities. To give a brief outline of the underlying ideas and also to understand what the ambiguity transformation should achieve, it helps if we ask ourselves the question what the structure of the ambiguity variance-covariance matrix must be in order to be able to apply the simplest of all integer estimation methods. The simplest integer estimation method is clearly "rounding to the nearest integer". This method should only be used however, when the ambiguity variance-covariance matrix is diagonal. Since this situation is the best one can hope for in any ambiguity fixing problem, the idea is to search for ambiguity transformations that decorrelate the ambiguities and therefore transform the ambiguity variance-covariance matrix to a form that is as close as possible to a diagonal matrix. As a synthetic two-dimensional example, assume that the variance-covariance matrix of the ambiguities is given as

$$Q_{\hat{a}}=\begin{bmatrix} 3 & 261 \\ 261 & 23555 \end{bmatrix}.$$

The condition number of this variance-covariance matrix is approximately 2.10^5. It shows that the confidence ellipse of the ambiguities is extremely elongated. Elongations of this size can be expected for very short observational timespans and are due to the slowly changing receiver-satellite geometry. Also note that the two ambiguities are highly correlated, since their correlation coefficient equals $\rho_{\hat{a}}$=0.98. But a decorrelation and a diagonalization of the variance-covariance matrix is possible if we use an ambiguity transformation of the type of example 7 and transform the original two ambiguities into the new ambiguities $b_1=a_1$ and $b_2=a_2-87a_1$. And for this transformed problem, the computation of the integer least-squares estimates becomes rather straightforward.

6. References

[1] Frei, E., G. Beutler (1990): Rapid Static Positioning Based on the Fast Ambiguity Resolution Approach FARA: Theory and First Results. *Manuscripta Geodaetica*, Vol. 15, No. 6, 1990.

[2] Frei, E. (1991): *Rapid Differential Positioning with the Global Positioning System*. In: Geodetic and Geophysical Studies in Switzerland, Vol. 44.

[3] Hofmann-Wellenhof, B.W. Remondi (1988): The Antenna Exchange: one Aspect of High-Precision Kinematic Survey. Presented at the International GPS Workshop, *GPS Techniques Applied to Geodesy and Surveying*, Darmstadt, FRG, 10-13 April.

[4] Blewitt, G. (1990): A New Tool for Dense Monitoring of Crustal Strain: GPS Rapid Static Surveying. *EOS Transaction American Geophysical Union*, Vol. 71, No. 17, p. 483.

[5] Hatch, R. (1986): Dynamic Differential GPS at the Centimeter Level. Proceedings *4th International Geod. Symp. on Satellite Positioning*, Austin, Texas, 28 April - 2 May, pp. 1287-1298.

[6] Hatch, R. (1989): Ambiguity Resolution in the Fast Lane. Proceedings *ION GPS-89*, Colorado Springs, CO, 27-29 September, pp. 45-50.

[7] Euler, H.-J., C. Goad (1991): On Optimal Filtering of GPS Dual Frequency Observations without using Orbit Information. *Bulletin Geodesique*, Vol. 65, pp. 130-143.

[8] Seeber, G., and G. Wubbena (1989): Kinematic Positioning with Carrier Phases and "On the Way" Ambiguity Solution. Proceedings *5th Int. Geod. Symp. on Satellite Positioning*. Las Cruces, New Mexico, March 1989.

[9] Wubbena, G. (1989): The GPS Adjustment Software Package - GEONAP - Concepts and Models. Proceedings *5th Int. Geod. Symp. on Satellite Positioning*. Las Cruces, New Mexico, 13-17 March 1989, pp. 452-461.

[10] Allison, T. (1991): Multi-Observable Processing Techniques for Precise Relative Positioning. Proceedings *ION GPS-91*. Albuquerque, New Mexico, 11-13 September 1991, pp. 715-725.

[11] Euler, H.-J., H. Landau (1992): Fast GPS Ambiguity Resolution On-The-Fly for Real-Time Applications. Proceedings *6th Int. Geod. Symp. on Satellite Positioning*. Columbus, Ohio, 17-20 March 1992, pp. 650-729.

[12] Goad, C. (1992): Robust Techniques for Determining GPS Phase Ambiguities. Proceedings *6th Int. Geod. Symp. on Satellite Positioning*. Columbus, Ohio, 17-20 March 1992, pp. 245-254.

[13] Cocard, C., A. Geiger (1992): Systematic Search for all Possible Widelanes. Proceedings *6th Int. Geod. Symp. on Satellite Positioning*. Columbus, Ohio, 17-20 March 1992.

[14] Noble, B. (1969): *Applied Linear Algebra*. Prentice-Hall, Inc.