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MINIMAX PASSBAND GROUP DELAY NONLINEAR PHASE PEAK CONSTRAINED FIR FILTER DESIGN WITHOUT IMPOSING DESIRED PHASE RESPONSE

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Abstract. *In this paper, a nonlinear phase finite impulse response (FIR) filter is designed without imposing a desired phase response. The maximum passband group delay of the filter is minimized subject to a positivity constraint on the passband group delay response of the filter as well as a specification on the maximum absolute difference between the desired magnitude square response and the designed magnitude square response over both the passband and the stopband. This filter design problem is a quadratic NP hard functional inequality constrained optimization problem. To tackle this problem, first, the one norm functional inequality constraint of the optimization problem is approximated by a smooth function so that the quadratic NP hard functional inequality constrained optimization problem is converted to a nonconvex functional inequality constrained optimization problem. Then, a modified filled function method is applied for finding the global minimum of the nonconvex optimization problem. By using a local minimum of the corresponding unconstrained optimization problem as the initial condition of our proposed global optimization algorithm, computer numerical simulation results show that our proposed approach could efficiently and effectively design a minimax passband group delay nonlinear phase peak constrained FIR filter without imposing a desired phase response.*
Keywords: Nonlinear phase peak constrained FIR filter design, Minimax passband group delay, Functional inequality constrained optimization problem, Quadratic NP hard optimization problem, Modified filled function method

1. **Introduction.** Nonlinear phase FIR filters are attractive in signal processing applications as they could achieve better frequency selectivities than linear phase filters for the same filter lengths. Bounded input bounded output stability of FIR filters is guaranteed. Consequently, nonlinear phase FIR filters are found in many science and engineering applications [15,16].

Although many nonlinear phase peak constrained FIR filter designs could be found in literature, most of these designs minimize the maximum absolute differences between the desired magnitude square responses and the designed magnitude square responses over both the passbands and the stopbands of the filters [3-8]. However, these designs have not considered the maximum passband group delays of the filters. For designs tackling the maximum passband group delays of the filters [9-11], they require the desired phase responses of the filters. Unlike linear phase filter designs, the desired phase responses of nonlinear phase filters are usually unknown. By imposing certain desired phase responses to the designs, the maximum passband group delays of the designed filters are not minimized. Also, the frequency selectivities of the designed filters could be very poor. The main advantage of the design proposed in this paper is that no desirable phase response is required.

Since a fast response filter is always preferred, the maximum passband group delay of the filter is minimized in this paper. As the desired phase response of the filter is unknown, a specification is defined on the maximum absolute difference between the desired magnitude square response and the designed magnitude square response over both the passband and the stopband of the filter. Besides, as the passband group delay response of a filter has to be positive [1,2], the positivity constraint on the passband group delay response of the filter is imposed. In fact, this filter design problem is a quadratic NP hard functional inequality constrained optimization problem. Thus, it is very challenge to find the global minimum of the optimization problem. In this paper, the one norm functional inequality constraint of the quadratic NP hard optimization problem is first approximated by a smooth function so that the quadratic NP hard functional inequality constrained optimization problem is converted to a nonconvex functional inequality constrained optimization problem [12]. Then, a modified filled function method is applied for finding the global minimum of the nonconvex optimization problem [13].

The outline of this paper is as follows. The problem formulation and the solution method are presented in Section 2. In Section 3, a condition on a local minimum of the corresponding unconstrained optimization problem is derived. The obtained local minimum is employed as the initial condition of the global optimization algorithm. Computer numerical simulation results are presented in Section 4. Finally, a conclusion is drawn in Section 5.

2. Problem Formulation and Solution Method.

2.1. Problem formulation.

Denote the transpose operator as the superscript ^{T}. Denote the magnitude response, the phase response, the group delay response, the desired magnitude response, the impulse response, the passband, the stopband, the length, the specification on the maximum absolute difference between the designed magnitude square response and the desirable magnitude square response of a nonlinear phase peak constrained FIR filter as $|H(\omega)|$, $\angle H(\omega)$, $\tau(\omega)$, $D(\omega)$, $h(n)$, B_p, B_s, N and $(\delta(\omega))^2$, respectively. Denote the vector of the filter coefficients as $\mathbf{x} \equiv [h(0), h(1), \cdots, h(N-1)]^T$. Denote the frequency response kernels as $\boldsymbol{\iota}_s(\omega) \equiv [0, \sin\omega, \cdots, \sin((N-1)\omega)]^T$, $\boldsymbol{\iota}_c(\omega) \equiv [1, \cos\omega, \cdots, \cos((N-1)\omega)]^T$, $\boldsymbol{\iota'}_s(\omega) \equiv [0, \sin\omega, \cdots, (N-1)\sin((N-1)\omega)]^T$ and $\boldsymbol{\iota'}_c(\omega) \equiv [0, \cos\omega, \cdots, (N-1)\cos((N-1)\omega)]^T$. Define $\mathbf{Q}_1(\omega) \equiv \boldsymbol{\iota}_c(\omega)\boldsymbol{\iota'}^T_c(\omega) + \boldsymbol{\iota}_s(\omega)\boldsymbol{\iota'}^T_s(\omega)$ and $\mathbf{Q}_2(\omega) \equiv \boldsymbol{\iota}_c(\omega)\boldsymbol{\iota}^T_c(\omega) + \boldsymbol{\iota}_s(\omega)\boldsymbol{\iota}^T_s(\omega)$, then we have $\angle H(\omega) = -\tan^{-1}\frac{\boldsymbol{\iota}^T_s(\omega)\mathbf{x}}{\boldsymbol{\iota}^T_c(\omega)\mathbf{x}}$ and

$$\begin{aligned}\tau(\omega) &= -\frac{d}{d\omega}\angle H(\omega) \\ &= \frac{d}{d\omega}\tan^{-1}\frac{\boldsymbol{\iota}^T_s(\omega)\mathbf{x}}{\boldsymbol{\iota}^T_c(\omega)\mathbf{x}}\end{aligned}$$

$$= \frac{1}{1+\left(\frac{\boldsymbol{\iota}_s^T(\omega)\mathbf{x}}{\boldsymbol{\iota}_c^T(\omega)\mathbf{x}}\right)^2} \frac{\left(\boldsymbol{\iota}_c^T(\omega)\mathbf{x}\right)\left(\boldsymbol{\iota}'^T_c(\omega)\mathbf{x}\right)+\left(\boldsymbol{\iota}_s^T(\omega)\mathbf{x}\right)\left(\boldsymbol{\iota}'^T_s(\omega)\mathbf{x}\right)}{\left(\boldsymbol{\iota}_c^T(\omega)\mathbf{x}\right)^2}$$

$$= \frac{\left(\boldsymbol{\iota}_c^T(\omega)\mathbf{x}\right)\left(\boldsymbol{\iota}'^T_c(\omega)\mathbf{x}\right)+\left(\boldsymbol{\iota}_s^T(\omega)\mathbf{x}\right)\left(\boldsymbol{\iota}'^T_s(\omega)\mathbf{x}\right)}{\left(\boldsymbol{\iota}_c^T(\omega)\mathbf{x}\right)^2+\left(\boldsymbol{\iota}_s^T(\omega)\mathbf{x}\right)^2}$$

$$= \frac{\mathbf{x}^T\mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}}.$$

A specification on the maximum absolute difference between the designed magnitude square response and the desirable magnitude square response of the filter is defined as follows. $\left||H(\omega)|^2-(D(\omega))^2\right| \le (\delta(\omega))^2\ \forall\omega\in B_p\cup B_s$. As $|H(\omega)|^2=\mathbf{x}^T\boldsymbol{\iota}_c(\omega)\boldsymbol{\iota}_c^T(\omega)\mathbf{x}+\mathbf{x}^T\boldsymbol{\iota}_s(\omega)\boldsymbol{\iota}_s^T(\omega)\mathbf{x}=\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}$, the above constraint is equivalent to $\left|\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}-(D(\omega))^2\right| \le (\delta(\omega))^2\ \forall\omega\in B_p\cup B_s$. As the passband group delay response of the filter has to be positive, we have $\frac{\mathbf{x}^T\mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}}\ge 0\ \forall\omega\in B_p$. In other words, we have $-\frac{\mathbf{x}^T\mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}}\le 0\ \forall\omega\in B_p$. To minimize the maximum passband group delay of the filter subject to the positivity constraint on the passband group delay response of the filter as well as the specification on the maximum absolute difference between the designed magnitude square response and the desirable magnitude square response over both the passband and the stopband of the filter, the filter design problem is formulated as the following optimization problem:

Problem (P)

$$\begin{array}{ll} \min\limits_{\mathbf{x}} & f(\mathbf{x})=\max\limits_{\omega\in B_p}\frac{\mathbf{x}^T\mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}}, \\ \text{subject to} & g_1(\mathbf{x},\omega)=\left|\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}-(D(\omega))^2\right|-(\delta(\omega))^2\le 0 \quad \forall\omega\in B_p\cup B_s, \\ \text{and} & g_2(\mathbf{x},\omega)=-\frac{\mathbf{x}^T\mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}}\le 0 \quad \forall\omega\in B_p, \end{array}$$

where $f(\mathbf{x})$ is the cost function of the optimization problem, $g_1(\mathbf{x},\omega)$ is the one norm functional inequality constraint and $g_2(\mathbf{x},\omega)$ are the rational functional inequality constraint of the optimization problem.

2.2. **Solution method.** As Problem (**P**) is an NP hard functional inequality constrained optimization problem, there are oscillations when running conventional optimization algorithms. Hence, it is very challenge to find the global minimum of the optimization problem. To address this difficulty, the one norm functional inequality constraint of the optimization problem is approximated by a smooth function as follows [12] so that the oscillations could be avoided. Define

$$g_\sigma(\mathbf{x},\omega)\equiv\begin{cases}\left|\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}-(D(\omega))^2\right|-(\delta(\omega))^2 & \left|\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}-(D(\omega))^2\right|\ge\frac{\sigma}{2} \\ \frac{\left(\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}-(D(\omega))^2\right)^2}{\sigma}+\frac{\sigma}{4}-(\delta(\omega))^2 & \text{otherwise}\end{cases} \quad \forall\omega\in B_p\cup B_s.$$

It is worth noting that $g_\sigma(\mathbf{x},\omega)\approx g(\mathbf{x},\omega)\ \forall\omega\in B_p\cup B_s$ as $\sigma\to 0^+$. Hence, Problem (**P**) could be approximated by the following optimization problem:

Problem ($\mathbf{P}'_\sigma$)

$$\begin{array}{ll} \min\limits_{\mathbf{x}} & f(\mathbf{x})=\max\limits_{\omega\in B_p}\frac{\mathbf{x}^T\mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}}, \\ \text{subject to} & g_\sigma(\mathbf{x},\omega)\le 0 \quad \forall\omega\in B_p\cup B_s, \\ \text{and} & g_2(\mathbf{x},\omega)=-\frac{\mathbf{x}^T\mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T\mathbf{Q}_2(\omega)\mathbf{x}}\le 0 \quad \forall\omega\in B_p. \end{array}$$

Problem $(\mathbf{P}'_\sigma)$ is a nonconvex functional inequality constrained optimization problem. It is challenge to find the global minimum of the optimization problem. To address this difficulty, a modified filled function method is applied [13]. The algorithm is summarized as follows.

Algorithm

Step 1: Initialize a minimum improvement factor ε, an accepted error ε', an initial search point $\tilde{\mathbf{x}}_1$, a positive definite matrix $\mathbf{R}$ and an iteration index $k = 1$.

Step 2: Find a local minimum of the following optimization Problem $(\mathbf{P}_f)$ via the integration approach [14] based on the initial search point $\tilde{\mathbf{x}}_k$.

Problem $(\mathbf{P}_f)$

$$\min_{\mathbf{x}} \quad f(\mathbf{x}) = \max_{\omega \in B_p} \frac{\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}},$$

$$\text{subject to} \quad g_\sigma(\mathbf{x}, \omega) \le 0 \quad \forall \omega \in B_p \cup B_s,$$

$$g_2(\mathbf{x}, \omega) = -\frac{\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}} \le 0 \quad \forall \omega \in B_p,$$

$$\text{and} \quad g_3(\mathbf{x}) \equiv \max_{\omega \in B_p} \frac{\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}} - (1-\varepsilon) \max_{\omega \in B_p} \frac{(\tilde{\mathbf{x}}_k)^T \mathbf{Q}_1(\omega)\tilde{\mathbf{x}}_k}{(\tilde{\mathbf{x}}_k)^T \mathbf{Q}_2(\omega)\tilde{\mathbf{x}}_k} \le 0,$$

where $g_3(\mathbf{x}) \le 0$ is a discrete constraint we imposed. Denote the obtained local minimum as $\mathbf{x}_k^*$.

Step 3: Find a local minimum of the following optimization Problem $(\mathbf{P}_H)$ via the integration approach [14] based on the initial search point $\mathbf{x}_k^*$.

Problem $(\mathbf{P}_H)$

$$\min_{\mathbf{x}} \quad H(\mathbf{x}) \equiv \max_{\omega \in B_p} \frac{\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}} + \frac{1}{\left(\mathbf{x}-\mathbf{x}_k^*\right)^T \mathbf{R} \left(\mathbf{x}-\mathbf{x}_k^*\right)},$$

$$\text{subject to} \quad g_\sigma(\mathbf{x}, \omega) \le 0 \quad \forall \omega \in B_p \cup B_s,$$

$$g_2(\mathbf{x}, \omega) = -\frac{\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}} \le 0 \quad \forall \omega \in B_p,$$

$$\text{and} \quad g_4(\mathbf{x}) \equiv \max_{\omega \in B_p} \frac{\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}} - (1-\varepsilon) \max_{\omega \in B_p} \frac{\left(\mathbf{x}_k^*\right)^T \mathbf{Q}_1(\omega)\mathbf{x}_k^*}{\left(\mathbf{x}_k^*\right)^T \mathbf{Q}_2(\omega)\mathbf{x}_k^*} \le 0,$$

where $H(\mathbf{x})$ is a filled function we defined and $g_4(\mathbf{x}) \le 0$ is a discrete constraint we imposed. Denote the obtained local minimum as $\tilde{\mathbf{x}}_{k+1}$. Increment the value of k.

Step 4: Iterate Step 2 and Step 3 until $\left| \max_{\omega \in B_p} \frac{\left(\mathbf{x}_k^*\right)^T \mathbf{Q}_1(\omega)\mathbf{x}_k^*}{\left(\mathbf{x}_k^*\right)^T \mathbf{Q}_2(\omega)\mathbf{x}_k^*} - \max_{\omega \in B_p} \frac{\left(\mathbf{x}_{k-1}^*\right)^T \mathbf{Q}_1(\omega)\mathbf{x}_{k-1}^*}{\left(\mathbf{x}_{k-1}^*\right)^T \mathbf{Q}_2(\omega)\mathbf{x}_{k-1}^*} \right| \le \varepsilon'$. Take the final vector of $\mathbf{x}_k^*$ as the global minimum of the original optimization problem.

The working principle of the algorithm has been discussed in [13]. In this paper, an analytical bound on the computational complexity of the algorithm is derived. Suppose that the algorithm takes K iterations before the termination. As the constraint $g_3(\mathbf{x}) = \max_{\omega \in B_p} \frac{\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}} - (1-\varepsilon) \max_{\omega \in B_p} \frac{(\tilde{\mathbf{x}}_k)^T \mathbf{Q}_1(\omega)\tilde{\mathbf{x}}_k}{(\tilde{\mathbf{x}}_k)^T \mathbf{Q}_2(\omega)\tilde{\mathbf{x}}_k} \le 0$ is imposed on the Problem $(\mathbf{P}_f)$, a new local minimum of the Problem $(\mathbf{P}_f)$, which is $\mathbf{x}_k^*$, will not be located at $\tilde{\mathbf{x}}_k$, that is $\mathbf{x}_k^* \ne \tilde{\mathbf{x}}_k$, and the cost value evaluated at the new local minimum will guarantee to be lower than or equal to $1-\varepsilon$ multiplied to the cost value evaluated at $\tilde{\mathbf{x}}_k$, that is $f(\mathbf{x}_k^*) \le (1-\varepsilon) f(\tilde{\mathbf{x}}_k)$ for $k \le K$. Similarly, as the constraint $g_4(\mathbf{x}) = \max_{\omega \in B_p} \frac{\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}} - (1-\varepsilon) \max_{\omega \in B_p} \frac{\left(\mathbf{x}_k^*\right)^T \mathbf{Q}_1(\omega)\mathbf{x}_k^*}{\left(\mathbf{x}_k^*\right)^T \mathbf{Q}_2(\omega)\mathbf{x}_k^*} \le 0$ is imposed on the Problem $(\mathbf{P}_H)$, a new local minimum of the Problem $(\mathbf{P}_H)$, which is $\tilde{\mathbf{x}}_{k+1}$, will not be located at $\mathbf{x}_k^*$, that is $\tilde{\mathbf{x}}_{k+1} \ne \mathbf{x}_k^*$, and the cost value evaluated at the new local minimum will guarantee to be lower than or equal to $1-\varepsilon$ multiplied to the cost value evaluated at $\mathbf{x}_k^*$, that is $f(\tilde{\mathbf{x}}_{k+1}) \le (1-\varepsilon) f(\mathbf{x}_k^*)$ for $k \le K$. Hence, we have $f(\tilde{\mathbf{x}}_{k+1}) \le (1-\varepsilon) f(\mathbf{x}_k^*) \le (1-\varepsilon)^2 f(\tilde{\mathbf{x}}_k)$ for $k \le K$. This further implies that $\tilde{\mathbf{x}}_k$ for $k \le K$ will not be stuck at local minima of $f(\mathbf{x})$ because $0 < 1-\varepsilon < 1$. Also, we have $f(\tilde{\mathbf{x}}_k) \le (1-\varepsilon)^{2(k-1)} f(\tilde{\mathbf{x}}_1)$ for $k \le K$. Let the global minimum of the optimization problem be $\mathbf{x}^\bullet$, then we have $f(\mathbf{x}^\bullet) \le (1-\varepsilon)^{2(K-1)} f(\tilde{\mathbf{x}}_1)$ for $k \le K$. This implies that $K \le 1 - \frac{\log f(\tilde{\mathbf{x}}_1) - \log f(\mathbf{x}^\bullet)}{2\log(1-\varepsilon)}$ and the algorithm always converges. Let $\lceil z \rceil$ be the nearest integer of z such that $\lceil z \rceil \ge z$, then the computational complexity of the algorithm is bounded by that required for finding $2\left\lceil 1 - \frac{\log f(\tilde{\mathbf{x}}_1) - \log f(\mathbf{x}^\bullet)}{2\log(1-\varepsilon)} \right\rceil$ local minima of the optimization problem.

3. **Condition on Local Minimum of the Corresponding Unconstrained Optimization Problem.** Define $\tilde{f}(\mathbf{x},\omega) \equiv \frac{\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}}{\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}}$ $\forall \omega \in B_p$. Then, we have

$$\frac{\partial}{\partial \mathbf{x}} \tilde{f}(\mathbf{x},\omega) = \frac{\left(\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}\right)\left(\mathbf{Q}_1(\omega) + (\mathbf{Q}_1(\omega))^T\right)\mathbf{x} - \left(\mathbf{x}^T \mathbf{Q}_1(\omega)\mathbf{x}\right)\left(\mathbf{Q}_2(\omega) + (\mathbf{Q}_2(\omega))^T\right)\mathbf{x}}{\left(\mathbf{x}^T \mathbf{Q}_2(\omega)\mathbf{x}\right)^2} \quad \forall \omega \in B_p.$$

Denote $(\bar{\mathbf{x}},\bar{\omega})$ such that $\left.\frac{\partial}{\partial \mathbf{x}} \tilde{f}(\mathbf{x},\omega)\right|_{(\mathbf{x},\omega)=(\bar{\mathbf{x}},\bar{\omega})} = \mathbf{0}$. Denote $rank(\mathbf{Z})$ as the total number of linearly independent rows or the total number of linearly independent columns of the matrix $\mathbf{Z}$. Denote $\lambda(\bar{\mathbf{x}},\bar{\omega}) \equiv \frac{\bar{\mathbf{x}}^T \mathbf{Q}_2(\bar{\omega})\bar{\mathbf{x}}}{\bar{\mathbf{x}}^T \mathbf{Q}_1(\bar{\omega})\bar{\mathbf{x}}}$. Then, $\left.\frac{\partial}{\partial \mathbf{x}} \tilde{f}(\mathbf{x},\omega)\right|_{(\mathbf{x},\omega)=(\bar{\mathbf{x}},\bar{\omega})} = \mathbf{0}$ implies that

$$\left(\bar{\mathbf{x}}^T \mathbf{Q}_2(\bar{\omega})\bar{\mathbf{x}}\right)\left(\mathbf{Q}_1(\bar{\omega}) + (\mathbf{Q}_1(\bar{\omega}))^T\right)\bar{\mathbf{x}} = \left(\bar{\mathbf{x}}^T \mathbf{Q}_1(\bar{\omega})\bar{\mathbf{x}}\right)\left(\mathbf{Q}_2(\bar{\omega}) + (\mathbf{Q}_2(\bar{\omega}))^T\right)\bar{\mathbf{x}}.$$

This further implies that $\lambda(\bar{\mathbf{x}},\bar{\omega})\left(\mathbf{Q}_1(\bar{\omega}) + (\mathbf{Q}_1(\bar{\omega}))^T\right)\bar{\mathbf{x}} = \left(\mathbf{Q}_2(\bar{\omega}) + (\mathbf{Q}_2(\bar{\omega}))^T\right)\bar{\mathbf{x}}$. However, it is worth noting that $rank\left(\mathbf{Q}_1(\bar{\omega}) + (\mathbf{Q}_1(\bar{\omega}))^T\right) = 4$ and $rank\left(\mathbf{Q}_2(\bar{\omega}) + (\mathbf{Q}_2(\bar{\omega}))^T\right) = 2$ for $\bar{\omega}$ not equal to an integer multiple of π. Hence, in general, $\mathbf{Q}_1(\bar{\omega}) + (\mathbf{Q}_1(\bar{\omega}))^T$ and $\mathbf{Q}_2(\bar{\omega}) + (\mathbf{Q}_2(\bar{\omega}))^T$ are not invertible. Define $\mathbf{t}_1(\bar{\omega}) \in \Re[\omega]^{2\times N}$, $\mathbf{t}_2(\bar{\omega}) \in \Re[\omega]^{2\times N}$, $\mathbf{t}'(\bar{\omega}) \in \Re[\omega]^{2\times N}$, $\mathbf{A}_1(\bar{\omega}) \in \Re[\omega]^{(N-4)\times 2}$, $\mathbf{A}_2(\bar{\omega}) \in \Re[\omega]^{(N-4)\times 2}$, $\mathbf{B}_1(\bar{\omega}) \in \Re[\omega]^{2\times 2}$ and $\mathbf{B}_2(\bar{\omega}) \in \Re[\omega]^{(N-4)\times 2}$ such that $\mathbf{Q}_1(\bar{\omega}) + (\mathbf{Q}_1(\bar{\omega}))^T \equiv \begin{bmatrix} \mathbf{t}_1(\bar{\omega}) \\ \mathbf{t}_2(\bar{\omega}) \\ \mathbf{A}_1(\bar{\omega})\mathbf{t}_1(\bar{\omega}) + \mathbf{A}_2(\bar{\omega})\mathbf{t}_2(\bar{\omega}) \end{bmatrix}$ and $\mathbf{Q}_2(\bar{\omega}) + (\mathbf{Q}_2(\bar{\omega}))^T \equiv \begin{bmatrix} \mathbf{t}'(\bar{\omega}) \\ \mathbf{B}_1(\bar{\omega})\mathbf{t}'(\bar{\omega}) \\ \mathbf{B}_2(\bar{\omega})\mathbf{t}'(\bar{\omega}) \end{bmatrix}$. It is worth noting that $\mathbf{A}_1(\bar{\omega})$, $\mathbf{A}_2(\bar{\omega})$, $\mathbf{B}_1(\bar{\omega})$ and $\mathbf{B}_2(\bar{\omega})$ are uniquely defined based on $\mathbf{Q}_1(\bar{\omega}) + (\mathbf{Q}_1(\bar{\omega}))^T$ and $\mathbf{Q}_2(\bar{\omega}) + (\mathbf{Q}_2(\bar{\omega}))^T$. Define $\tilde{\mathbf{t}}(\bar{\omega}) \equiv \begin{bmatrix} \mathbf{t}_1(\bar{\omega}) \\ \mathbf{t}_2(\bar{\omega}) \end{bmatrix}$, $\mathbf{A}(\bar{\omega}) \equiv \begin{bmatrix} \mathbf{A}_1(\bar{\omega}) & \mathbf{A}_2(\bar{\omega}) \end{bmatrix}$ and $\mathbf{B}(\bar{\omega}) \equiv \begin{bmatrix} \mathbf{B}_1(\bar{\omega}) \\ \mathbf{B}_2(\bar{\omega}) \end{bmatrix}$. By extracting the submatrix out from the fourth row to the last row of $\mathbf{Q}_1(\bar{\omega}) + (\mathbf{Q}_1(\bar{\omega}))^T$ and denoting this submatrix as $\tilde{\mathbf{Q}}_1(\bar{\omega})$, then we have $\mathbf{A}(\bar{\omega})\tilde{\mathbf{t}}(\bar{\omega}) = \tilde{\mathbf{Q}}_1(\bar{\omega})$. This implies that $\mathbf{A}(\bar{\omega}) = \tilde{\mathbf{Q}}_1(\bar{\omega})\left(\tilde{\mathbf{t}}(\bar{\omega})\right)^T\left(\tilde{\mathbf{t}}(\bar{\omega})\left(\tilde{\mathbf{t}}(\bar{\omega})\right)^T\right)^{-1}$. Similarly, by extracting the submatrix out from the second row to the last row of $\mathbf{Q}_2(\bar{\omega}) + (\mathbf{Q}_2(\bar{\omega}))^T$ and denoting this submatrix as $\tilde{\mathbf{Q}}_2(\bar{\omega})$, then we have $\mathbf{B}(\bar{\omega})\mathbf{t}'(\bar{\omega}) = \tilde{\mathbf{Q}}_2(\bar{\omega})$. This implies that $\mathbf{B}(\bar{\omega}) = \tilde{\mathbf{Q}}_2(\bar{\omega})\left(\mathbf{t}'(\bar{\omega})\right)^T\left(\mathbf{t}'(\bar{\omega})\left(\mathbf{t}'(\bar{\omega})\right)^T\right)^{-1}$. Hence, $\lambda(\bar{\mathbf{x}},\bar{\omega})\left(\mathbf{Q}_1(\bar{\omega}) + (\mathbf{Q}_1(\bar{\omega}))^T\right)\bar{\mathbf{x}} = \big(\mathbf{Q}_2(\bar{\omega}) +$

$\left(\mathbf{Q}_2(\bar{\omega}))^T\right)\bar{\mathbf{x}}$ implies that

$$\lambda(\bar{\mathbf{x}},\bar{\omega})\begin{bmatrix}\mathbf{t}_1(\bar{\omega})\\ \mathbf{t}_2(\bar{\omega})\\ \mathbf{A}_1(\bar{\omega})\mathbf{t}_1(\bar{\omega})+\mathbf{A}_2(\bar{\omega})\mathbf{t}_2(\bar{\omega})\end{bmatrix}\bar{\mathbf{x}}=\begin{bmatrix}\mathbf{t}'(\bar{\omega})\\ \mathbf{B}_1(\bar{\omega})\mathbf{t}'(\bar{\omega})\\ \mathbf{B}_2(\bar{\omega})\mathbf{t}'(\bar{\omega})\end{bmatrix}\bar{\mathbf{x}}.$$

This further implies that $\lambda(\bar{\mathbf{x}},\bar{\omega})\mathbf{t}_1(\bar{\omega})\bar{\mathbf{x}}=\mathbf{t}'(\bar{\omega})\bar{\mathbf{x}}$, $\lambda(\bar{\mathbf{x}},\bar{\omega})\mathbf{t}_2(\bar{\omega})\bar{\mathbf{x}}=\mathbf{B}_1(\bar{\omega})\mathbf{t}'(\bar{\omega})\bar{\mathbf{x}}$ and

$$\lambda(\bar{\mathbf{x}},\bar{\omega})\left(\mathbf{A}_1(\bar{\omega})\mathbf{t}_1(\bar{\omega})+\mathbf{A}_2(\bar{\omega})\mathbf{t}_2(\bar{\omega})\right)\bar{\mathbf{x}}=\mathbf{B}_2(\bar{\omega})\mathbf{t}'(\bar{\omega})\bar{\mathbf{x}}.$$

In other words, we have $\mathbf{A}_1(\bar{\omega})\mathbf{t}'(\bar{\omega})\bar{\mathbf{x}}+\mathbf{A}_2(\bar{\omega})\mathbf{B}_1(\bar{\omega})\mathbf{t}'(\bar{\omega})\bar{\mathbf{x}}=\mathbf{B}_2(\bar{\omega})\mathbf{t}'(\bar{\omega})\bar{\mathbf{x}}$. Define $\mathbf{G}(\bar{\omega})\equiv\left(\mathbf{A}_1(\bar{\omega})+\mathbf{A}_2(\bar{\omega})\mathbf{B}_1(\bar{\omega})-\mathbf{B}_2(\bar{\omega})\right)\mathbf{t}'(\bar{\omega})$. Denote $\mathbf{G}_1(\bar{\omega})\in\Re[\omega]^{(N-4)\times 2}$, $\mathbf{G}_2(\bar{\omega})\in\Re[\omega]^{(N-4)\times(N-2)}$, $\bar{\mathbf{x}}_1\in\Re^{2\times 1}$ and $\bar{\mathbf{x}}_2\in\Re^{(N-2)\times 1}$ such that $\mathbf{G}(\bar{\omega})\equiv[\mathbf{G}_1(\bar{\omega})\ \mathbf{G}_2(\bar{\omega})]$ and $\bar{\mathbf{x}}\equiv\begin{bmatrix}\bar{\mathbf{x}}_1\\ \bar{\mathbf{x}}_2\end{bmatrix}$. Denote

$$\tilde{\mathbf{G}}(\bar{\omega})\equiv-\left((\mathbf{G}_1(\bar{\omega}))^T\mathbf{G}_1(\bar{\omega})\right)^{-1}(\mathbf{G}_1(\bar{\omega}))^T\mathbf{G}_2(\bar{\omega}).$$

Then, $\mathbf{A}_1(\bar{\omega})\mathbf{t}'(\bar{\omega})\bar{\mathbf{x}}+\mathbf{A}_2(\bar{\omega})\mathbf{B}_1(\bar{\omega})\mathbf{t}'(\bar{\omega})\bar{\mathbf{x}}=\mathbf{B}_2(\bar{\omega})\mathbf{t}'(\bar{\omega})\bar{\mathbf{x}}$ implies that $\mathbf{G}(\bar{\omega})\bar{\mathbf{x}}=\mathbf{0}$. This further implies that $\mathbf{G}_1(\bar{\omega})\bar{\mathbf{x}}_1+\mathbf{G}_2(\bar{\omega})\bar{\mathbf{x}}_2=\mathbf{0}$. In other words, we have

$$\bar{\mathbf{x}}_1=-\left((\mathbf{G}_1(\bar{\omega}))^T\mathbf{G}_1(\bar{\omega})\right)^{-1}(\mathbf{G}_1(\bar{\omega}))^T\mathbf{G}_2(\bar{\omega})\bar{\mathbf{x}}_2$$

or $\bar{\mathbf{x}}_1=\tilde{\mathbf{G}}(\bar{\omega})\bar{\mathbf{x}}_2$. That means, in order to satisfy $\frac{\partial}{\partial\mathbf{x}}\tilde{f}(\mathbf{x},\omega)\Big|_{(\mathbf{x},\omega)=(\bar{\mathbf{x}},\bar{\omega})}=\mathbf{0}$, there are two filter coefficients dependent on the other filter coefficients.

Similarly, as $\tilde{f}(\mathbf{x},\omega)=\frac{\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}qh(p)h(q)\cos(p-q)\omega}{\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}h(p)h(q)\cos(p-q)\omega}$ $\forall\omega\in B_p$, we have

$$\frac{\partial}{\partial\omega}\tilde{f}(\mathbf{x},\omega)=\frac{\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}qh(p)h(q)\cos(p-q)\omega\right)\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}(p-q)h(p)h(q)\sin(p-q)\omega\right)}{\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}h(p)h(q)\cos(p-q)\omega\right)^2}-\frac{\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}h(p)h(q)\cos(p-q)\omega\right)\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}q(p-q)h(p)h(q)\sin(p-q)\omega\right)}{\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}h(p)h(q)\cos(p-q)\omega\right)^2}$$

$\forall\omega\in B_p$. $\frac{\partial}{\partial\omega}\tilde{f}(\mathbf{x},\omega)\Big|_{(\mathbf{x},\omega)=([\bar{h}(0)\cdots\bar{h}(N-1)]^T,\bar{\omega})}=0$ implies that

$$\frac{\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}q\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right)\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right)}{\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right)^2}-\frac{\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right)\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}q(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right)}{\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right)^2}=0.$$

This further implies that

$$\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1} q\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right)\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right)$$
$$=\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right)\left(\sum_{p=0}^{N-1}\sum_{q=0}^{N-1} q(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right).$$

Hence, we have

$$\left(\sum_{p=0}^{1}\sum_{q=0}^{1} q\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}+\sum_{p=0}^{1}\sum_{q=2}^{N-1} q\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right.$$
$$\left.+\sum_{p=2}^{N-1}\sum_{q=0}^{1} q\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}+\sum_{p=2}^{N-1}\sum_{q=2}^{N-1} q\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right)$$
$$\left(\sum_{p=0}^{1}\sum_{q=0}^{1}(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}+\sum_{p=0}^{1}\sum_{q=2}^{N-1}(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right.$$
$$\left.+\sum_{p=2}^{N-1}\sum_{q=0}^{1}(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}+\sum_{p=2}^{N-1}\sum_{q=2}^{N-1}(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right)$$
$$=\left(\sum_{p=0}^{1}\sum_{q=0}^{1}\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}+\sum_{p=0}^{1}\sum_{q=2}^{N-1}\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right.$$
$$\left.+\sum_{p=2}^{N-1}\sum_{q=0}^{1}\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}+\sum_{p=2}^{N-1}\sum_{q=2}^{N-1}\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right)$$
$$\left(\sum_{p=0}^{1}\sum_{q=0}^{1} q(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}+\sum_{p=0}^{1}\sum_{q=2}^{N-1} q(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right.$$
$$\left.+\sum_{p=2}^{N-1}\sum_{q=0}^{1} q(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}+\sum_{p=2}^{N-1}\sum_{q=2}^{N-1} q(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right).$$

As $\bar{\mathbf{x}}_1=\tilde{\mathbf{G}}(\bar{\omega})\bar{\mathbf{x}}_2$, there exists $g_{i,k}(\bar{\omega})$ for $i=0,1$ and for $k=2,3,\cdots,N-1$ such that

$$\bar{h}(0)=\sum_{k=2}^{N-1} g_{0,k}(\bar{\omega})\bar{h}(k) \text{ and } \bar{h}(1)=\sum_{k=2}^{N-1} g_{1,k}(\bar{\omega})\bar{h}(k). \tag{1}$$

This implies that

$$\left(\sum_{p=0}^{1}\sum_{q=0}^{1} q\left(\sum_{k=2}^{N-1} g_{p,k}(\bar{\omega})\bar{h}(k)\right)\left(\sum_{k=2}^{N-1} g_{q,k}(\bar{\omega})\bar{h}(k)\right)\cos(p-q)\bar{\omega}\right.$$
$$+\sum_{p=0}^{1}\sum_{q=2}^{N-1} q\left(\sum_{k=2}^{N-1} g_{p,k}(\bar{\omega})\bar{h}(k)\right)\bar{h}(q)\cos(p-q)\bar{\omega}$$
$$+\sum_{p=2}^{N-1}\sum_{q=0}^{1} q\bar{h}(p)\left(\sum_{k=2}^{N-1} g_{q,k}(\bar{\omega})\bar{h}(k)\right)\cos(p-q)\bar{\omega}$$
$$\left.+\sum_{p=2}^{N-1}\sum_{q=2}^{N-1} q\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right)$$
$$\left(\sum_{p=0}^{1}\sum_{q=0}^{1}(p-q)\left(\sum_{k=2}^{N-1} g_{p,k}(\bar{\omega})\bar{h}(k)\right)\left(\sum_{k=2}^{N-1} g_{q,k}(\bar{\omega})\bar{h}(k)\right)\sin(p-q)\bar{\omega}\right.$$

$$
\begin{aligned}
&+\sum_{p=0}^{1}\sum_{q=2}^{N-1}(p-q)\left(\sum_{k=2}^{N-1}g_{p,k}(\bar{\omega})\bar{h}(k)\right)\bar{h}(q)\sin(p-q)\bar{\omega} \\
&+\sum_{p=2}^{N-1}\sum_{q=0}^{1}(p-q)\bar{h}(p)\left(\sum_{k=2}^{N-1}g_{q,k}(\bar{\omega})\bar{h}(k)\right)\sin(p-q)\bar{\omega} \\
&+\left.\sum_{p=2}^{N-1}\sum_{q=2}^{N-1}(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right) \\
=&\left(\sum_{p=0}^{1}\sum_{q=0}^{1}\left(\sum_{k=2}^{N-1}g_{p,k}(\bar{\omega})\bar{h}(k)\right)\left(\sum_{k=2}^{N-1}g_{q,k}(\bar{\omega})\bar{h}(k)\right)\cos(p-q)\bar{\omega}\right. \\
&+\sum_{p=0}^{1}\sum_{q=2}^{N-1}\left(\sum_{k=2}^{N-1}g_{p,k}(\bar{\omega})\bar{h}(k)\right)\bar{h}(q)\cos(p-q)\bar{\omega} \\
&+\sum_{p=2}^{N-1}\sum_{q=0}^{1}\bar{h}(p)\left(\sum_{k=2}^{N-1}g_{q,k}(\bar{\omega})\bar{h}(k)\right)\cos(p-q)\bar{\omega} \qquad (2)\\
&+\left.\sum_{p=2}^{N-1}\sum_{q=2}^{N-1}\bar{h}(p)\bar{h}(q)\cos(p-q)\bar{\omega}\right) \\
&\left(\sum_{p=0}^{1}\sum_{q=0}^{1}q(p-q)\left(\sum_{k=2}^{N-1}g_{p,k}(\bar{\omega})\bar{h}(k)\right)\left(\sum_{k=2}^{N-1}g_{q,k}(\bar{\omega})\bar{h}(k)\right)\sin(p-q)\bar{\omega}\right. \\
&+\sum_{p=0}^{1}\sum_{q=2}^{N-1}q(p-q)\left(\sum_{k=2}^{N-1}g_{p,k}(\bar{\omega})\bar{h}(k)\right)\bar{h}(q)\sin(p-q)\bar{\omega} \\
&+\sum_{p=2}^{N-1}\sum_{q=0}^{1}q(p-q)\bar{h}(p)\left(\sum_{k=2}^{N-1}g_{q,k}(\bar{\omega})\bar{h}(k)\right)\sin(p-q)\bar{\omega} \\
&+\left.\sum_{p=2}^{N-1}\sum_{q=2}^{N-1}q(p-q)\bar{h}(p)\bar{h}(q)\sin(p-q)\bar{\omega}\right).
\end{aligned}
$$

Now, we are ready for deriving a good initial condition for the global optimization algorithm. First, a filter with the least maximum ripple magnitude over both the passband and the stopband is designed via the semi-definite programming technique and the spectral factorization technique [3-8]. Then, by discarding the first two filter coefficients and retaining the rest of the filter coefficients of the filter, we have $\bar{h}(k)$ for $k = 2, 3, \cdots, N-1$. Next, $\bar{\omega}$ in (2) is solved numerically via computer aided design tools. Finally, the first two filter coefficients are computed using (1). The obtained filter coefficients are employed as the initial condition of the global optimization algorithm.

4. **Computer Numerical Simulation Results.** Since desired phase responses of nonlinear phase peak constrained FIR filters are imposed in existing designs, it is very difficult to have a fair comparison. We intend to compare our works to that presented in [11] because the works presented in [11] are the most related works to our works found in literature.

Both the length and the desired magnitude response of the filter are chosen the same as that in [11] in order to have a fair comparison, that is $N = 30$ and

$$D(\omega) = \begin{cases} 1 & |\omega| \le 0.12\pi \\ 0 & |\omega| \ge 0.24\pi \end{cases}.$$

As the optimization problem is to minimize the maximum passband group delay of the filter subject to the positivity constraint on the passband group delay response of the filter, the specification on the maximum absolute difference between the designed magnitude square response and the desirable magnitude square response over the passband of the filter is set exactly the same as that over the stopband of the filter. Since there are tradeoffs among the maximum absolute difference between the designed magnitude square response and the desirable magnitude square response, the length, the bandwidth and the center frequency of the filter, $\delta(\omega)$ is set to -34.5dB for $|\omega| \leq 0.12\pi$ and $|\omega| \geq 0.24\pi$, which is good enough for most applications. In order to have a good approximation between the quadratic NP hard functional inequality constrained optimization problem and the corresponding nonconvex functional inequality constrained optimization problem, σ should be small. $\sigma = 10^{-6}$ is chosen in this paper which are small enough for most applications. In order not to terminate our algorithm when the convergence of our algorithm is slow and to obtain a high accuracy of the obtained global minimum, both ε and ε' should be small. $\varepsilon = 10^{-6}$ and $\varepsilon' = 10^{-6}$ are chosen in this paper which are small enough for most applications. The initial condition $\tilde{\mathbf{x}}_1$ of the global optimization algorithm is obtained as discussed in Section 3. As $\mathbf{R}$ is a positive definite matrix, it controls the spread of the hill of $H(\mathbf{x})$ at $\mathbf{x}_k^*$. If $\mathbf{R}$ is a diagonal matrix with all diagonal elements being the same and positive, then large value of these diagonal elements will result to a wide spread of the hill of $H(\mathbf{x})$ at $\mathbf{x}_k^*$ and vice versa. Since local minima of nonlinear phase peak constrained FIR filters are usually located very close together, the spreads of the hills of $H(\mathbf{x})$ at $\mathbf{x}_k^*$ should be small and $\mathbf{R}$ is chosen as the diagonal matrix with all diagonal elements equal to 10^{-3}, which is small enough for most applications.

Based on the above chosen parameters, it only takes three iterations for the algorithm to terminate. Hence, our proposed method is very efficient. It can be seen from Figure 1 and Figure 2 that the maximum passband group delay as well as the square root of the maximum absolute difference between the designed magnitude square response and the desirable magnitude square response over the passband and the stopband of the filter designed via our proposed approach are 6.8778, -56.9425dB and -34.6062dB, respectively, in which the required constraints are all satisfied. Compared with the results obtained in [11], the maximum passband group delay as well as the square root of the

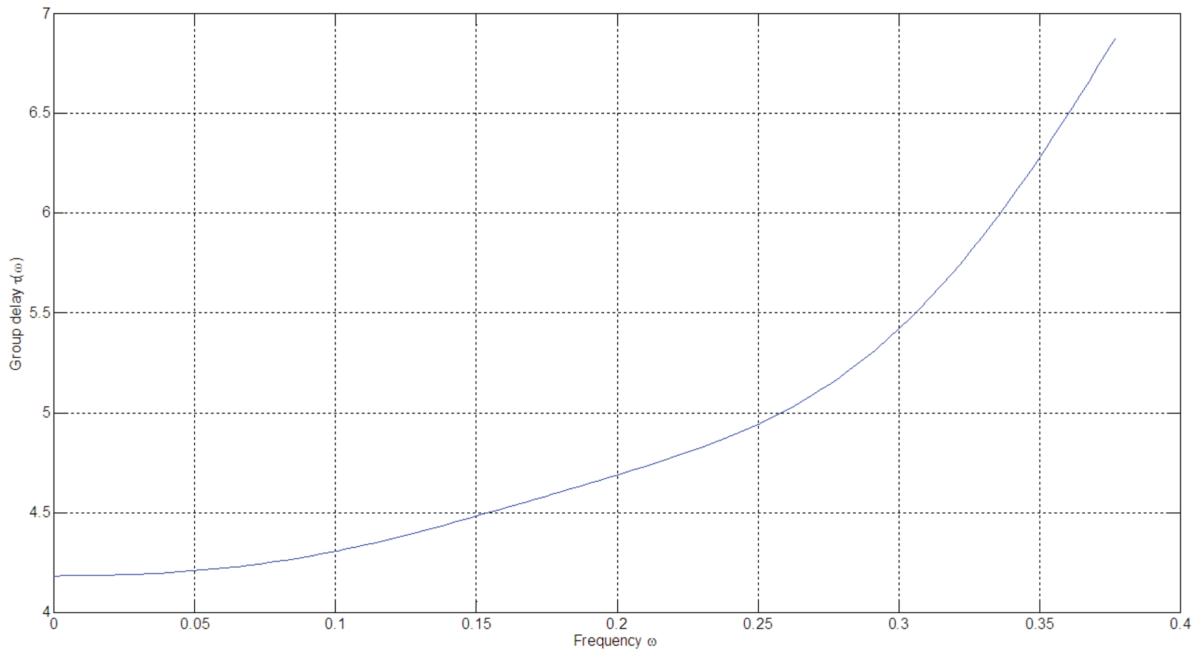

FIGURE 1. The passband group delay response of our designed nonlinear phase peak constrained FIR filter

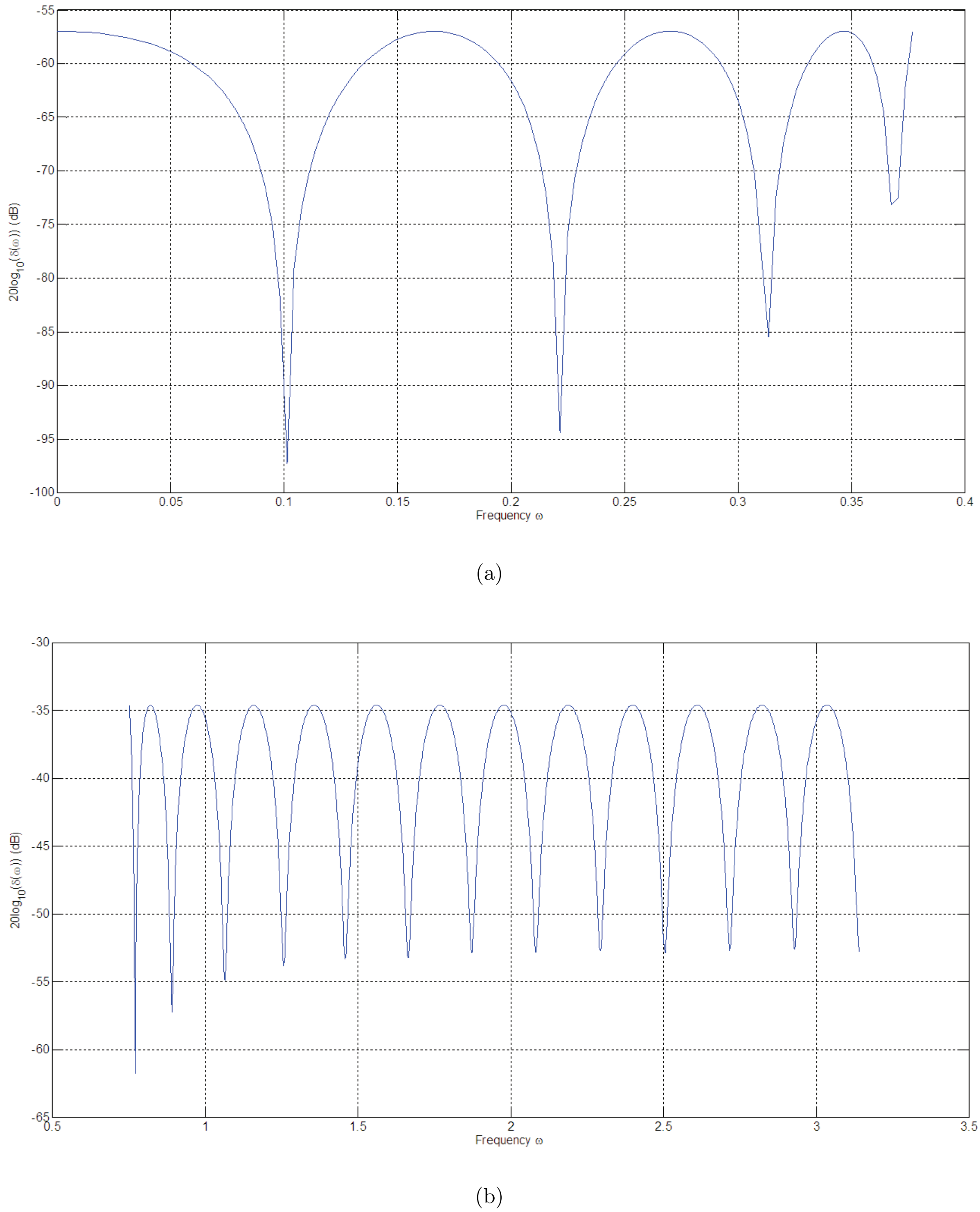

FIGURE 2. (a) The square root of the maximum absolute difference between the designed magnitude square response and the desirable magnitude square response over the passband; (b) that over the stopband of our designed nonlinear phase peak constrained FIR filter

maximum absolute difference between the designed magnitude square response and the desirable magnitude square response over the passband and the stopband of the filter are 12.43898, −26.7653dB and −44.7822dB, respectively. Although the performance on the square root of the maximum absolute difference between the designed magnitude square response and the desirable magnitude square response over the stopband of our designed filter is slightly worse than that of [11], both the maximum passband group delay and the square root of the maximum absolute difference between the designed magnitude square response and the desirable magnitude square response over the passband of our designed filter are significantly better than that of [11]. This is because our proposed algorithm guarantees to find the global minimum of the nonconvex optimization problem, in which the method discussed in [11] could not guarantee the obtained solution being the global minimum. Also, as the desired phase response was imposed in the design discussed in [11], the maximum passband group delay of the design discussed in [11] is actually not minimized.

5. **Conclusion.** This paper formulates a minimax passband group delay nonlinear phase peak constrained FIR filter design problem as a quadratic NP hard functional inequality constrained optimization problem. The one norm of the functional inequality constraint of the optimization problem is first approximated by a smooth function so that the quadratic NP hard functional inequality constrained optimization problem is converted to a nonconvex functional inequality constrained optimization problem. Then, a modified filled function method is applied for finding the global minimum of the nonconvex optimization problem. By employing a local minimum of the corresponding unconstrained optimization problem as the initial condition of our proposed global optimization algorithm, computer numerical simulation results show that our proposed method could efficiently and effectively design a minimax passband group delay nonlinear phase peak constrained FIR filter without imposing a desirable phase response.

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REFERENCES

[1] X. Lai, Y. Zuo, Y. Guo and D. Peng, A complex-error and phase-error constrained least-squares design of FIR filters with reduced group delay error, *Chinese Control and Decision Conference*, pp.1967-1972, 2009.

[2] X. Lai, J. Wang and Z. Xu, A new minimax design of two-dimensional FIR filters with reduced group delay, *The 7th Asian Control Conference*, pp.610-614, 2009.

[3] L. R. Rabiner, N. Y. Graham and H. D. Helms, Linear programming design of IIR digital filters with arbitrary magnitude function, *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol.ASSP-22, no.2, pp.117-123, 1974.

[4] Y. Kamp and C. J. Wellekens, Optimal design of minimum-phase FIR filters, *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol.ASSP-31, no.4, pp.922-926, 1983.

[5] H. Samueli, Linear programming design of digital data transmission filters with arbitrary magnitude specifications, *IEEE International Conference on Communications*, vol.2, pp.967-971, 1988.

[6] S.-P. Wu, S. Boyd and L. Vandenberghe, FIR filter design via semidefinite programming and spectral factorization, *The 35th Conference on Decision and Control*, pp.271-276, 1996.

[7] B. Alkire and L. Vandenberghe, Interior-point methods for magnitude filter design, *IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol.6, pp.3821-3824, 2001.

[8] T. N. Davidson, Z.-Q. Luo and J. F. Sturm, Linear matrix inequality formulation of spectral mask constraints with applications to FIR filter design, *IEEE Transactions on Signal Processing*, vol.50, no.11, pp.2702-2715, 2002.

[9] Z. Lin and Y. Liu, Design of complex FIR filters with reduced group error using semidefinite programming, *IEEE Signal Processing Letters*, vol.13, no.9, pp.529-532, 2006.

[10] Z. Lin and Y. Liu, Design of arbitrary complex coefficient WLS FIR filters with group delay constraints, *IEEE Transactions on Signal Processing*, vol.57, no.8, pp.3274-3279, 2009.

[11] G. Calvagno, G. M. Cortelazzo and G. A. Mian, A technique for multiple criterion approximation of FIR filters in magnitude and group delay, *IEEE Transactions on Signal Processing*, vol.43, no.2, pp.393-400, 1995.

[12] C. Y.-F. Ho, B. W.-K. Ling, Y.-Q. Liu, P. K.-S. Tam and K.-L. Teo, Optimal design of magnitude responses of rational infinite impulse response filters, *IEEE Transactions on Signal Processing*, vol.54, no.10, pp.4039-4046, 2006.

[13] C. Y.-F. Ho, B. W.-K. Ling, L. Benmesbah, T. C.-W. Kok, W.-C. Siu and K.-L. Teo, Two-channel linear phase FIR QMF bank minimax design via global nonconvex optimization programming, *IEEE Transactions on Signal Processing*, vol.58, no.8, pp.4436-4441, 2010.

[14] C. Y.-F. Ho, B. W.-K. Ling, Z.-W. Chi, M. Shikh-Bahaei, Y.-Q. Liu and K.-L. Teo, Design of near-allpass strictly stable minimal-phase real-valued rational IIR filters, *IEEE Transactions on Circuits and Systems – II: Express Briefs*, vol.55, no.8, pp.781-785, 2008.

[15] S.-J. Zhang and S.-S. Hu, Output feedback tracking control for a class of MIMO nonlinear minimum phase systems based on RBF neural networks, *International Journal of Innovative Computing, Information and Control*, vol.4, no.4, pp.803-812, 2008.

[16] K. Belarbi, A. Belhani and K. Fujimoto, Multivariable fuzzy logic controller based on a compensator of interactions and genetic tuning, *International Journal of Innovative Computing, Information and Control*, vol.2, no.6, pp.1207-1217, 2006.