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STABILITY ANALYSIS OF A CLASS OF DISCRETE SYSTEMS WITH IMPULSIVE EFFECTS

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Abstract. This paper investigates stability problems of nonlinear discrete systems with impulsive effects. The impulse is described by a nonlinear function of the state vector x. Employing a Lyapunov functional-based method, sufficient conditions for asymptotical stability of this class of discrete systems are derived. Finally, a numerical example is presented to illustrate the efficiency of our results.

Keywords. impulsive control, impulsive increments, discrete systems, stability analysis, asymptotical stability.

AMS (MOS) subject classification: 37C75,37B25,37N35.

1 Introduction

In recent decades, dynamical systems with impulsive effects (or called impulsive systems), usually described by impulsive differential equations, have attracted considerable attention since many evolution systems in the real world are characterized by the fact that they suffer a abrupt change of their state. This kind of systems exist in a great range of disciplines, including chaos control, chaos synchronization, neural networks, circuit systems and biological systems, and so on [1-2]. Stability is an essential problem for impulsive systems. So far, many contributions to stability properties of impulsive systems have been published in recent years, see for instance the books by V. Lak et al. [3] and by T. Yang [4] for more details.

On the other hand, stability analysis of discrete systems is of importance for both theoretical challenges and practical applications. Many results for robust stability of discrete systems have been obtained, see [5]-[8] and the references therein. However, to the best of our knowledge, stability problems for impulsive discrete systems are seldom investigated except [8]. Note that these stability criteria in [8] are based on linear impulsive increments rather than nonlinear impulsive increments, which severely limit their results' application range.

In order to derive more general useful results, we shall investigate stability criteria of an impulsive discrete system, whose impulses are in form of general nonlinear functions. Sufficient conditions for asymptotical stability of this kind of systems are established. Our results in this paper are not only highly effective but also can be used in many broad disciplines.

The rest of this paper is organized as follows. In section II, an impulsive discrete system and some stability definitions are introduced. In Section III, sufficient criteria of asymptotical stability for impulsive discrete systems with nonlinear impulsive increments are established. In Section IV, a numerical example is presented to demonstrate our main results. Concluding remarks are given in Section V.

2 Preliminaries and Problem Statement

In this section, after introducing an impulsive discrete system we present some preliminary concepts which will be used in this paper.

Consider the following impulsive discrete system:

$$\begin{aligned} x(m+1) &= f(x(m)), m \neq n_k, & \text{(1a)} \\ \Delta x(m) &= g_k(x(m)), m = n_k, & \text{(1b)} \\ x(0) &= x_0, m \in K. & \text{(1c)} \end{aligned}$$

where $x \in R^n$ is the state vector, $g_k \in R^n$ is the control input vector satisfying $g_k(x(m)) = x(m+1) - x(m)$, $f : R_+ \times R^n \to R^n$ and $g_k : R^n \to R^n$ are continuous functions, and $n_k < n_{k+1}$, $n_0 = 0$, with $n_k \to \infty$ as $k \to \infty$, $k \in K = \{0, 1, 2, \cdots\}$.

Remark 1 *Stability performance of an impulsive system depends on not only $f(x(m))$ but also $g_k(x(m))$. However, most of existing stability results only consider linear impulsive increments. For instance, in [8], the impulsive increment is a linear function of the state vector $x(m)$. In this paper, we shall adopt a general nonlinear function as our impulsive increment and derive sufficient conditions for asymptotical stability of the impulsive discrete system, which extend the results in [8].*

Now, we introduce some stability definitions of impulsive discrete system (1).

Definition 1 *System (1) is said to be stable if $\forall \varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that*

$$\|x(m)\| < \varepsilon, \forall m \in K, as \|x_0\| \leq \delta(\varepsilon). \tag{2}$$

Definition 2 *System (1) is said to be asymptotically stable if system (1) is stable and*

$$\lim_{m \to +\infty} \|x(m)\| = 0. \tag{3}$$

The following assumption is needed in this paper.

Assumption 1 *For impulsive instants* $\{n_k\}$,

$$n_k \in K \text{ and } n_k + 2 \leq n_{k+1}, k \in K, \tag{4}$$

hold.

Remark 2 *Assumption 1 ensures that the jumping impulses can not happen continuously.*

3 Main Results

In this section, we shall derive the sufficient conditions of asymptotical stability for the impulsive discrete system (1) with nonlinear impulsive increments $\{n_k, g_k(x)\}$, where $g_k(x), k \in K$, is a nonlinear function of x and k. Main results of this paper are presented in the following theorems.

Theorem 1 *Suppose that Assumption 1 holds,*

$$\|x(m) + g_k(x(m))\| \leq l_k \|x(m)\|, m = n_k, \forall k \in K, \tag{5}$$

and

$$\|f(x(m))\| \leq s \|x(m)\|, \forall m \in K. \tag{6}$$

Then, system (1) is asymptotically stable if $\forall m \geq 3$ *and there exists* $\alpha > 0$ *such that*

$$\ln s + \frac{2}{m-2} \sum_{i=0}^{\gamma} |\ln l_i| \leq -\alpha, \tag{7}$$

where $\gamma = \left[\frac{m}{2}\right]$. $[\rho]$ *denotes the largest integer which does not excess* ρ.

Proof 1 *For any* $m \in K$, *let* $m \in (n_k, n_{k+1}]$. *Since*

$$\begin{aligned} \|x(m)\| &= \|f(x(m-1))\| \\ &\leq s^{m-n_k-1} \|x(n_k+1)\| \\ &= s^{m-n_k-1} \|x(n_k) + g_k(x(n_k))\| \\ &\leq s^{m-n_k-1} l_k \|x(n_k)\|, \end{aligned}$$

we have

$$\|x(n_{k+1})\| \leq s^{n_{k+1}-n_k-1} l_k \|x(n_k)\|, k \in K. \tag{8}$$

By induction, we obtain

$$\|x(n_k)\| \leq s^{n_k-k} \prod_{i=0}^{k-1} l_i \|x_0\|, \quad k \in K. \tag{9}$$

Furthermore, it follows from (9) that

$$\|x(m)\| \leq s^{m-k-1}\prod_{i=0}^{k} l_i \|x_0\| \quad = e^{(m-k-1)\ln s + \sum_{i=0}^{k}\ln l_i} \|x_0\|. \tag{10}$$

Since $n_k \geq 2k + n_0 \geq 2k$ holds from (4), then $m \geq 2k$. Hence, it follows from (7) that

$$\begin{aligned}(m-k-1)\ln s + \sum_{i=0}^{k}\ln l_i \quad &\leq \quad (m-k-1)\left(\ln s + \frac{2}{m-2}\sum_{i=0}^{r}|\ln l_i|\right) \\ &\leq \quad \left(1-\frac{m}{2}\right)\alpha, \quad \forall\, m \geq 3.\end{aligned}$$

Therefore, it is clear that

$$\|x(m)\| \leq e^{\left(1-\frac{m}{2}\right)\alpha}\|x_0\|, \forall\, m \geq 3. \tag{11}$$

In accordance with Definition 1, system (1) is stable.

Furthermore, from (11) we can see

$$\lim_{m\to+\infty}\|x(m)\| = 0. \tag{12}$$

Therefore, the impulsive discrete system (1) is asymptotically stable. The proof is complete.

If all nonlinear functions $g_k(x)$ are identical at the impulsive instants n_k, i.e., $g_k(x) = g(x), \forall\, k \in K$, then we have the following results for asymptotical stability of the impulsive controlled discrete system (1).

Theorem 2 *Suppose that Assumption 1 holds,*

$$\|x(m) + g(x(m))\| \leq l\|x(m)\|, \quad m = n_k, \quad \forall\, k \in K, \tag{13}$$

and

$$\|f(x(m))\| \leq s\|x(m)\|, \forall\, m \in K. \tag{14}$$

Then, system (1) is asymptotically stable if

$$\ln s + 5|\ln l| < 0. \tag{15}$$

Proof 2 *For any $m \in K$, let $m \in (n_k, n_{k+1}]$, then*

$$\|x(m)\| \leq e^{(m-k-1)\ln s + (k+1)\ln l}\|x_0\|, \tag{16}$$

and

$$\begin{aligned}(m-k-1)\ln s + (k+1)\ln l \quad &\leq \quad (m-k-1)\left[\ln s + \left(1+\tfrac{4}{m-2}\right)|\ln l|\right] \\ &\leq \quad (m-k-1)(\ln s + 5|\ln l|), \quad m \geq 3.\end{aligned}$$

By (15), we obtain

$$(m-k-1)\ln s + (k+1)\ln l \le \left(\frac{m}{2}-1\right)(\ln s + 5\left|\ln l\right|) < 0, \quad m \ge 3. \quad (17)$$

Substituting (17) into (16) yields

$$\begin{aligned} \|x(m)\| &\le e^{(m-k-1)\ln s + (k+1)\ln l}\, \|x_0\| \\ &\le e^{\left(\frac{m}{2}-1\right)(\ln s + 5\left|\ln l\right|)}\, \|x_0\|, \quad m \ge 3. \end{aligned} \quad (18)$$

Thus, there exists $\bar{\sigma} > 0$ *such that the following inequality holds,*

$$\|x(m)\| \le e^{\bar{\sigma}}\, \|x_0\|, \quad \forall\, m \in K,$$

which shows that system (1) is stable. Moreover, by (18), we also see $\lim_{m\to+\infty} \|x(m)\| = 0$*. Therefore, system (1) is asymptotically stable.*

The proof is completed.

4 An Numerical Example

In this section, we present an example to illustrate our results obtained. Consider the impulsive discrete system (1) with the following specifications:

$$f(x(m)) = \left(\frac{x_1+x_2}{16}, -\frac{\sqrt{x_1x_3}}{8}, \frac{x_2+x_3}{16}\right)^T, \quad \forall m \in K, \quad x_0 = (\,-1.2, 0, 0.3)^T.$$

$$g_k(x(m)) = g_k(x)$$

$$= \left(\,-e^{-\frac{1}{2(k+1)}}\sqrt{x_2x_3}, -e^{-\frac{1}{2(k+1)}}\sqrt{x_1x_3}, -e^{-\frac{1}{2(k+1)}}\sqrt{x_1x_2}\right)^T,$$

and $n_k = 3k, k \in K$.

By using MATLAB software, we get $s = \frac{1}{8}$, $l_k = e^{-\frac{1}{2(k+1)}}$. Then, we can check

$$\ln s + \frac{2}{m-2}\sum_{i=0}^{r}\left|\ln l_i\right| < -\ln 8 + \frac{r+1}{m-2}$$

$$\le -\ln 8 + 2 < 0, \forall\, m \ge 3,$$

which satisfies the inequality (7). Then, according to Theorem 1, it can be shown that impulsive discrete system (1) is asymptotically stable.

5 Conclusion

In this paper, we investigated a stability problem for impulsive discrete systems with nonlinear impulsive increments. Sufficient conditions for asymptotical stability are derived by employing a Lyapunov-based method. Through a numerical example, we demonstrate our results' effectiveness and usefulness.

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