Mathematical Analysis on the Uniqueness of Reverse Algorithm for Measuring Elastic-Plastic Properties by Sharp Indentation

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The reverse analysis provides a convenient method to determine four elastic-plastic parameters through an indentation curve such as Young's modulus E, hardness H, yield strength σ_y and strain hardening exponent n. In this paper, mathematical analysis on a reverse algorithm from Dao model (Dao *et al.*, *Acta Mater.*, 2001, **49**: 3899) was carried out, which thought that only when $20 \le E^* / \sigma_{0.033} \le 26$ and $0.3 < n \le 0.5$, the reverse algorithm would yield two solutions of n by dimensionless function Π_2. It is shown that, however, there are also two solutions of n when $20 \le E^* / \sigma_{0.033} \le 26$ and $0 \le n < 0.1$. A unique n can be obtained by dimensionless function Π_3 instead of Π_2 in these two ranges. E and H can be uniquely determined by a full indentation curve, and σ_y can be determined if n is unique. Furthermore, sensitivity analysis on obtaining n from dimensionless function Π_3 or Π_2 has been made.

KEY WORDS: Elastic-plastic properties; Sharp indentation; Reverse algorithm; Uniqueness; Sensitivity

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1. Introduction

With the rapid development of modern microelectronics and high integration of devices, mechanical properties of small-scale materials have attracted increasing attention. A good understanding on their mechanical behaviors becomes particularly indispensable in applications[1, 2]. The traditional mechanical testing can not be applicable due to small dimensions of materials. A lot of new methods, such as micro-tensile, micro-bending, micro-cantilever and instrumented indentation tests, have been developed[3–10]. Among these methods, instrumented indentation has attained an ever-increasing application due to its high resolution of displacement, real-time monitoring, and high data acquisition rate[6–10].

Figure 1 shows a typical indentation response by a sharp indenter on a metal. The loading curve can be described by the equation, $P = Ch^2$, where C is the loading curvature. The initial unloading slope $\left(\mathrm{d}P_\mathrm{u}/\mathrm{d}h\right)\big|_{h_\mathrm{m}}$ is the contact stiffness S, with P_u the unloading force and h_m the maximum displacement at the maximum load P_m. The term h_r represents the residual displacement after complete unloading. The total work is described as $W_\mathrm{t} = W_\mathrm{e} + W_\mathrm{p}$ with W_e the elastic work and W_p the plastic work. Young's modulus E and hardness H can be directly obtained from the indentation curve (see Fig. 1) by the Oliver-Pharr (O-P) method[7]. However, the O-P method can only deal with the sink-in phenomena, and if pile-up occurs, it may overestimate E and H[11, 12]. Furthermore, elastic-plastic properties of a material cannot completely be characterized by Young's modulus and hardness. The plastic behavior of a homogeneous and isotropic metal is usually described by a power-law stress-strain (σ-ε) relationship, which can be expressed as[13]:

$$\sigma = E\varepsilon \quad \text{for} \quad \sigma \leq \sigma_\mathrm{y} \quad \text{and} \quad \sigma = R\varepsilon^n \quad \text{for} \quad \sigma \geq \sigma_\mathrm{y} \tag{1}$$

where σ_y is the yield stress, $R = \sigma_\mathrm{y}(E/\sigma_\mathrm{y})^n$ is a strength coefficient, and n is the strain hardening exponent. When plastic deformation occurs, the total strain ε can be divided into two parts, *i.e.* $\varepsilon = \varepsilon_\mathrm{y} + \varepsilon_\mathrm{p}$, where ε_y is the yield strain at yield stress σ_y and ε_p is the effective strain beyond ε_y. So, in the case of $\sigma > \sigma_\mathrm{y}$, Eq. (1) can be represented as:

$$\sigma = \sigma_\mathrm{y}\left(1 + \frac{E}{\sigma_\mathrm{y}}\varepsilon_\mathrm{p}\right)^n \tag{2}$$

Here, it is worth noting that Poisson's ratio v is not an important factor in indentation experiments and for most engineering materials, $\nu \approx 0.3$[14]. Therefore, to fully determine elastic-plastic properties of a material, three independent parameters (*i.e.*, E, σ_y and n) should be known.

Although the indentation technique can date back to the work of Tabor[15] in 1951, the systemic study of obtaining three elastic-plastic parameters (E, σ_y and n) from a single indentation curve has been carried out in the recent decade. Based on the forward analysis for predicting the indentation response of a given set of elastic-plastic properties[16–18], Giannakopoulos *et al*.[9] introduced the reverse problem and proposed a comprehensive analytical framework. However, their work was developed by the small deformation finite element method, and in fact, deformation beneath indenter tip can reach 25−36% equivalent strain[19]. According to large deformation finite element results, Cheng and Cheng[20–22] derived a set of dimensionless functions that relate an indentation curve to elastic-plastic properties, but there was a lack of a full analytical framework to extract mechanical properties. Subsequently, Dao *et al*.[23] constructed universal forward and reverse analysis algorithms (referred to Dao model hereafter), which were verified by systematic experiments. However, there is still no consensus on whether elastic-plastic properties of a material can be uniquely determined by a single indentation curve. To analyze forward and reverse processes, Lu et al[24] considered about 9000 different combinations of elastic-plastic properties in seven representative methods. Chen et al[25] demonstrated the existence of "mystical materials", where materials with different elastic-plastic properties share almost the same P-h curves. Although many studies have been made on the uniqueness problem[22–26], there is not a suitable method to uniquely determine elastic-plastic properties of engineering materials by a simple indentation test. In this paper, uniqueness of the reverse algorithm in Dao model was studied by a mathematical method. The analysis was focused on dimensionless functions in the reverse procedure and their monotonic problems. Then, the sensitivity in prediction of n was discussed.

2. The reverse algorithm

Based on dimensionless analysis and large deformation finite element method, six dimensionless functions Π_1, Π_2, Π_3, Π_4, Π_5 and Π_6 were obtained, and then, a set of reverse analysis algorithm was established in Dao model[23]. The ranges of E, σ_y and n used in calculations and expressions of these dimensionless functions are given in Appendix A. Fig. 2 displays an adapted reverse analysis framework, which involves the following steps:

(1) Identify the ratio h_r / h_m from dimensionless function Π_5 according to the total work W_t and the plastic work W_p under an indentation curve.

(2) Using the contact stiffness S, the maximum load P_{m} and $h_{\mathrm{r}} / h_{\mathrm{m}}$, determine H and E^* from functions Π_4 and Π_6, where the reduced Young's modulus E^* is defined as, $E^* = [(1 - v^2)/E + (1 - v_{\mathrm{i}}^2)/E_{\mathrm{i}}]^{-1}$, with subscript '$i$' being the indenter.

(3) With E^* being determined and C directly observed from the loading curve, calculate $\sigma_{0.033}$ from function Π_1, where $\sigma_{0.033}$ is the representative stress defined at strain $\varepsilon_{\mathrm{r}} = \varepsilon_{\mathrm{p}} = 0.033$. By using $\sigma_{0.033}$, the dimensionless function Π_1 was normalized to be independent of the strain hardening exponent n.

(4) Obtain h_{m} from the loading curve. Using the values of E^*, $\sigma_{0.033}$ and S obtained above, calculate n from function Π_2.

(5) Using E^*, $\sigma_{0.033}$ and n, calculate σ_{y} from Eq. (2).

3. Uniqueness of reverse analysis

According to the mathematical expressions of Π_1, Π_4 and Π_5, it is obvious that they are monotonic functions of $E^*/\sigma_{0.033}$ ($20 \leq E^*/\sigma_{0.033} \leq 770$ in [23]) and $h_{\mathrm{r}} / h_{\mathrm{m}}$, respectively. Rearranging dimensionless functions of Π_4 and Π_6 (in Fig. 2), we have:

$$H = \frac{S^2}{P_{\mathrm{m}}}\left(\frac{\Pi_4}{\Pi_6}\right)^2 \tag{3}$$

$$E^* = \frac{\Pi_4}{P_{\mathrm{m}}}\left(\frac{S}{\Pi_6}\right)^2 \tag{4}$$

Because Π_6 is a constant[23], H and E^* can be taken as a monotonic function of $h_{\mathrm{r}} / h_{\mathrm{m}}$. Following the steps (1)–(3), H, E^* and $\sigma_{0.033}$ can be uniquely determined. If n can be uniquely determined, σ_{y} is also unique. Thus, the key point of this uniqueness problem is how to determine n.

The dimensionless function Π_2 (Eq. (A2)) can be rewritten as:

$$an^3 + bn^2 + cn + d - \Pi_2 = 0 \tag{5}$$

where coefficients a, b, c and d are functions of $\ln(E^*/\sigma_{0.033})$. Denoting $n = N - b/3a$, Eq. (5) becomes

$$N^3 + \left(-\frac{b^2}{3a^2} + \frac{c}{a}\right)N + \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d - \Pi_2}{a} = 0 \tag{6}$$

where $a \neq 0$ (*i.e.*, $E^*/\sigma_{0.033} \neq 50$). Similarly, introducing two parameters p and q, Eq. (6) can be rearranged as

$$N^3 + pN + q = 0 \tag{7}$$

where $p = -\frac{b^2}{3a^2} + \frac{c}{a}$, and $q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d - \Pi_2}{a}$.

According to the Cardan formula[27], there are three roots (N_1, N_2, N_3) of Eq. (7). Thus, the three roots of Eq. (5) can be obtained as follows:

$$\begin{aligned} n_1 &= N_1 - \frac{b}{3a} = \sqrt[3]{T+\sqrt{D}} + \sqrt[3]{T-\sqrt{D}} - \frac{b}{3a} \\ n_2 &= N_2 - \frac{b}{3a} = \omega_1\sqrt[3]{T+\sqrt{D}} + \omega_2\sqrt[3]{T-\sqrt{D}} - \frac{b}{3a} \\ n_3 &= N_3 - \frac{b}{3a} = \omega_2\sqrt[3]{T+\sqrt{D}} + \omega_1\sqrt[3]{T-\sqrt{D}} - \frac{b}{3a} \end{aligned} \tag{8}$$

where $\omega_1 = \left(-1+\sqrt{3}i\right)/2$, $\omega_2 = \left(-1-\sqrt{3}i\right)/2$, $T = -q/2$, and $D = (q/2)^2 + (p/3)^3$ with $i = \sqrt{-1}$. Based on the characteristic features of these roots[27], the following conclusions can be proved: (1) when $D > 0$, there are one real root and two conjugate imaginary roots; (2) when $D = 0$, there are three real roots, in which at least two are unequal; and (3) when $D < 0$, there are three unequal real roots.

According to Eq. (6), to discuss the unique problem of n by using dimensionless function Π_2, the range of $E^*/\sigma_{0.033}$ can be divided into three intervals such as, 20–50, 50, and 50–770.

Case 1: $20 \leq E^*/\sigma_{0.033} < 50$

A three-dimensional graphics of discriminant $D\left(E^*/\sigma_{0.033}, \Pi_2\right) = (q/2)^2 + (p/3)^3$ is shown in Fig. 3, from which two cross-sections (curves A and B in Fig. 4) were made with $D(E^*/\sigma_{0.033}, \Pi_2) = 0$. These two lines separate the area into three parts: two of them with $D > 0$ (*i.e.*, one real root), and the other with $D < 0$ (*i.e.*, three unequal real roots). Curves of dimensionless function $\Pi_2(E^*/\sigma_{0.033}, n)$ with $20 \leq E^*/\sigma_{0.033} < 50$ and n = 0, 0.1, 0.3, and 0.5 are also plotted in Fig. 4. Obviously, all the curves fall into an area of $D \leq 0$, implying that there are three real roots (n_1, n_2 and n_3) according to the Cardan formula[27]. To distinguish the analytical expression in Eq. (8) that can be used to calculate the value of n, calculations were carried out by taking the values of $E^*/\sigma_{0.033}$ and Π_2 from Fig. 4. Let us take the curve $n = 0$ for example. As shown in Fig. 5(a), the calculated values of n_1 exceed 0.5, and thus the expression of n_1 would give a wrong solution. In the case of $20 \leq E^*/\sigma_{0.033} \leq 26$, the expression of n_2 yields a reasonable prediction of n, but an unreasonable, negative value of n_2 if $26 < E^*/\sigma_{0.033} < 50$. It is worth noting that, if $20 \leq E^*/\sigma_{0.033} \leq 26$, Eq. (8) has two reasonable n, *i.e.* n_2 and n_3. That is, n cannot be uniquely determined. Figs. 5(c)–(d) show the results with $n = 0.1$, 0.3, and 0.5, respectively. The non-uniqueness situation in calculating n also exists in the case of $20 \leq E^*/\sigma_{0.033} \leq 26$ and $n = 0.5$. Thus, except for the cases of $20 \leq E^*/\sigma_{0.033} \leq 26$, $0 \leq n < 0.1$ and $0.3 < n \leq 0.5$, one real value of n can be uniquely determined with the expression of n_3 in Eq. (8).

Case 2: $E^*/\sigma_{0.033} = 50$

In the case of $E^*/\sigma_{0.033} = 50$, Eq. (5) can be rewritten as:

$$0.02357n^2 - 1.25279n + 5.34692 - \Pi_2 = 0 \tag{9}$$

The two solutions of Eq. (9) are

$$\begin{aligned} n_1 &= 21.2145\left(1.25279 - 0.30704\sqrt{11.301 + \Pi_2}\right) \\ n_2 &= 21.2145\left(1.25279 + 0.30704\sqrt{11.301 + \Pi_2}\right) \end{aligned} \tag{10}$$

Since the range of Π_2 is from 4.72642 to 5.34692, it is obvious that the reasonable solution is n_1.

Case 3: $50 < E^*/\sigma_{0.033} \leq 770$

A three-dimensional graphics of $D\left(E^*/\sigma_{0.033}, \Pi_2\right) = \left(q/2\right)^2 + \left(p/3\right)^3$ in the case of $50 < E^*/\sigma_{0.033} \leq 770$ is shown in Fig. 6. The surface of discriminant D is also sectioned by using a plane of $D(E^*/\sigma_{0.033}, \Pi_2) = 0$, however, there is no cross curve in the $E^*/\sigma_{0.033}$ - Π_2 plane. All the discriminant values $D > 0$, namely, the solution of n is unique by using the expression of n_1 in Eq. (8) in all the cases of $50 < E^*/\sigma_{0.033} \leq 770$.

Based on above discussion, non-uniqueness of n determined by Π_2 occurs in the case of $20 \leq E^*/\sigma_{0.033} \leq 26$, $0.3 < n \leq 0.5$ and $0 \leq n < 0.1$. However, the non-unique range in Dao model is just in the domain of $20 \leq E^*/\sigma_{0.033} \leq 26$, $0.3 < n \leq 0.5$[23]. Lan *et al.*[24] obtained the similar non-unique domain based on about 9000 different combinations of elastic-plastic properties (there are only 76 different combinations in Dao model), which is $20 \le E^*/\sigma_{0.033} \le 26$ and $0.37 < n \le 0.5$. Obviously, the range of n is less than that of Dao model. Compared with Dao and Lan's results, a new non-unique domain of $20 \le E^*/\sigma_{0.033} \le 26$ and $0 \le n < 0.1$ is obtained.

Dao *et al.*[23] found out that the single solution of n could be obtained by using dimensionless function Π_3 instead of Π_2. Similar to the analysis of Π_2, the mathematical expression of Π_3 (Eq. (A3)) can be rearranged as:

$$a'n^2 + b'n + c' - \Pi_3 = 0 \tag{11}$$

where coefficients a', b', c' and d' are functions of $\ln(\sigma_{0.033}/E^*)$. The roots of Eq. (11) can be expressed as

$$\begin{aligned} n_1^* &= \frac{-b' + \sqrt{\Delta}}{2a'} \\ n_2^* &= \frac{-b' - \sqrt{\Delta}}{2a'} \end{aligned} \tag{12}$$

where $\Delta = \Delta\left(\sigma_{0.033}/E^*, \Pi_3\right) = b'^2 - 4a'(c' - \Pi_3)$. A three-dimensional graphics of discriminant $\Delta\left(\sigma_{0.033}/E^*, \Pi_3\right)$ is plotted in Fig. 7, in which two curves A and B (see Fig. 8) are sectioned with $\Delta = 0$. They separate the area into three regions: two of them with $\Delta < 0$ and the other with $\Delta > 0$. It is worth noting that all the curves of dimensionless function Π_3 (see Fig. 10 (b) in [23]) fall into the region of $\Delta \ge 0$. That is, there are two real solutions of Eq. (12). As shown in Fig. 9, one real value of n can be uniquely determined with the expression of n_2^* for all the range of $\sigma_{0.033} / E^*$. Furthermore, if $\sigma_{0.033} / E^*$ is very small (especially less than 0.005), all the curves of n = 0, 0.1, 0.3 and 0.5 are too close to be distinguished. Thus, n is mainly determined by dimensionless function Π_2, and only in the case of $20 \le E^*/\sigma_{0.033} \le 26$, $0.3 < n \le 0.5$ and $0 \le n < 0.1$, dimensionless function Π_3 can be used.

4. Sensitivity analysis in the prediction of *n*

According to uniqueness analysis, the non-unique domains in determining n are quite inconsistent with that in Dao model. Therefore, it is necessary to conduct a particular sensitivity analysis in the prediction of n. The experimental errors cannot be avoided and small variations in experiments may influence largely the values of Π_2 and Π_3. Here, a $\pm 2\%$ variation of Π_2 or Π_3 is assumed in analyzing the corresponding variation of n.

As shown in Fig. 10, it is seen that variations of n in Eq. (8) are within 7% in the case of $50 < E^*/\sigma_{0.033} \leq 770$. If $26 < E^*/\sigma_{0.033} \leq 50$, n can be uniquely determined by n_3 in Eq. (8) based on Π_2. However, based on sensitivity analysis, there is an unacceptable variation (almost 100%) in the prediction of n. Hence, Π_3 is applied to predict n in the case of $26 < E^*/\sigma_{0.033} \leq 50$.

In the case of $20 \leq E^*/\sigma_{0.033} \leq 50$, a unique solution of n can be obtained using dimensionless function Π_3. Figure 11 illustrates the variations of n caused by a change of $\pm 2\%$ in Π_3 for $n_0 = 0.15$, 0.3, and 0.5, respectively. It is obvious that n seems insensitive to uncertainty in Π_3 and varies within $n_0 \pm 0.15$ if $n_0 > 0.15$. However, for low hardening materials ($n_0 \leq 0.15$), prediction values of n are sensitive to an error of $\pm 2\%$ in Π_3 and unreliable. This is not surprising because, when $20 \leq E^*/\sigma_{0.033} \leq 50$ and $n_0 \leq 0.15$, the curves of dimensional function $\Pi_3(\sigma_{0.033}/E^*)$ as shown in Fig. 8, are close to each other and also to the region of $\Delta < 0$ (no real root).

5. Conclusion

In this paper, six dimensionless functions in the reverse algorithm proposed in Dao model have been analyzed. The main results are summarized as follows:

(1) Young's modulus E and hardness H can be uniquely determined by an indentation curve, and if the strain hardening exponent n is known, the yield strength σ_y can also be uniquely determined.

(2) The non-uniqueness problem happens in calculation of the strain hardening exponent n. Besides the range of $E^*/\sigma_{0.033} \leq 26$ and $0.3 < n \leq 0.5$ mentioned in Dao model, non-uniqueness also exists in the case of $20 \leq E^*/\sigma_{0.033} \leq 26$ and $0 \leq n < 0.1$, which can be solved by using dimensionless function Π_3 instead of Π_2.

(3) Sensitivity analysis show that, in the case of $26 < E^*/\sigma_{0.033} \leq 50$, n is highly sensitive to Π_2, and in the case of $20 \leq E^*/\sigma_{0.033} \leq 50$ and $n \leq 0.15$, n is sensitive to Π_3.

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APPENDIX A

The six dimensionless functions (Π_1~Π_6) in Dao model are based on a parametric study of 76 cases, which cover mechanical properties of engineering metals with E from 10 to 210 GPa, σ_y from 30 to 3000 MPa, and n from 0 to 0.5. Their detailed expressions are listed below:

$$\Pi_1 = \frac{C}{\sigma_{0.033}} = -1.131\left[\ln\left(\frac{E^*}{\sigma_{0.033}}\right)\right]^3 + 13.635\left[\ln\left(\frac{E^*}{\sigma_{0.033}}\right)\right]^2 - 30.594\left[\ln\left(\frac{E^*}{\sigma_{0.033}}\right)\right] + 29.267 \quad \text{(A1)}$$

$$\begin{aligned}\Pi_2\left(\frac{E^*}{\sigma_{0.033}}, n\right) &= \frac{S}{E^* h_{\mathrm{m}}} \\ &= \left(-1.40557n^3 + 0.77526n^2 + 0.15830n - 0.06831\right)\left[\ln\left(\frac{E^*}{\sigma_{0.033}}\right)\right]^3 \\ &\quad + \left(17.93006n^3 - 9.22091n^2 - 2.37733n + 0.86295\right)\left[\ln\left(\frac{E^*}{\sigma_{0.033}}\right)\right]^2 \\ &\quad + \left(-79.99715n^3 + 40.5562n^2 + 9.00157n - 2.54543\right)\left[\ln\left(\frac{E^*}{\sigma_{0.033}}\right)\right] \\ &\quad + 122.65069n^3 - 63.88418n^2 - 9.58936n + 6.20045\end{aligned} \quad \text{(A2)}$$

$$\begin{aligned}\Pi_3\left(\frac{\sigma_{0.033}}{E^*}, n\right) &= \frac{h_{\mathrm{r}}}{h_{\mathrm{m}}} \\ &= \left(0.0101n^2 + 0.0017639n - 0.0040837\right)\left[\ln\left(\frac{\sigma_{0.033}}{E^*}\right)\right]^3 \\ &\quad + \left(0.14386n^2 + 0.018153n - 0.088198\right)\left[\ln\left(\frac{\sigma_{0.033}}{E^*}\right)\right]^2 \\ &\quad + \left(0.59505n^2 + 0.034074n - 0.65417\right)\left[\ln\left(\frac{\sigma_{0.033}}{E^*}\right)\right] \\ &\quad + 0.5818n^2 - 0.08846n - 0.6729\end{aligned} \quad \text{(A3)}$$

$$\Pi_4 = \frac{H}{E^*} \approx 0.268536\left(0.9952495 - \frac{h_{\mathrm{r}}}{h_{\mathrm{m}}}\right)^{1.1142735} \quad \text{(A4)}$$

$$\Pi_5 = \frac{W_{\mathrm{p}}}{W_{\mathrm{t}}} = 1.61217\left\{1.13111 - 1.74756^{\left[-1.49291\left(\frac{h_{\mathrm{r}}}{h_{\mathrm{m}}}\right)^{2.535334}\right]} - 0.075187\left(\frac{h_{\mathrm{r}}}{h_{\mathrm{m}}}\right)^{1.135826}\right\} \quad \text{(A5)}$$

$$\Pi_6 = \frac{S}{E^*}\sqrt{\frac{H}{P_{\mathrm{m}}}} \quad \text{(A6)}$$

Figure captions

Fig. 1 Schematic of a typical load-displacement curve of an elastic-plastic material by sharp indentation.

Fig. 2 Illustration of an adapted reverse analysis algorithm.

Fig. 3 Three-dimensional graphics of discriminant $D(E^*/\sigma_{0.033}, \Pi_2)$ if $20 \le E^*/\sigma_{0.033} < 50$.

Fig. 4 Curves sectioned from Fig. 3 by a plane of $D(E^*/\sigma_{0.033}, \Pi_2) = 0$ and dimensionless function $\Pi_2(E^*/\sigma_{0.033}, n)$ in the case of $20 \le E^*/\sigma_{0.033} < 50$ and $n = 0, 0.1, 0.3, 0.5$.

Fig. 5 Calculated results of n_1, n_2 and n_3 for curves with (a) $n = 0$, (b) $n = 0.1$, (c) $n = 0.3$, and (d) $n = 0.5$.

Fig. 6 Three-dimensional graphics of discriminant $D(E^*/\sigma_{0.033}, \Pi_2)$ if $50 < E^*/\sigma_{0.033} \le 770$.

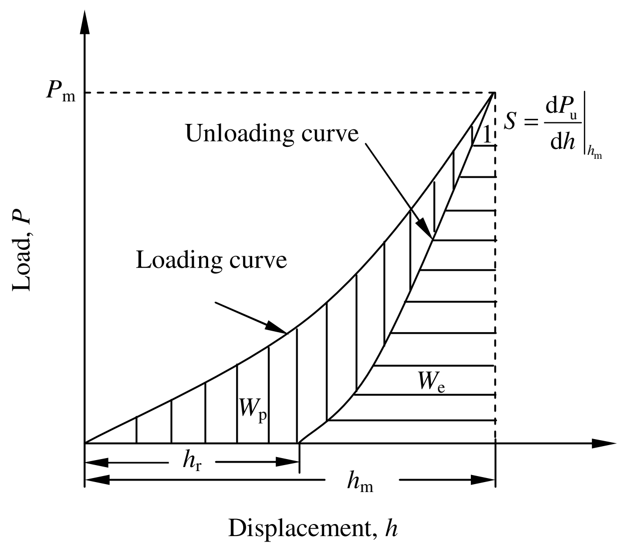
Fig. 7 Three-dimensional graphics of discriminant $\Delta\left(\sigma_{0.033}/E^*, \Pi_3\right)$.

Fig. 8 Curves sectioned from Fig. 7 and dimensionless function $\Pi_3(\sigma_{0.033}/E^*, n)$.

Fig. 9 Calculated results of n_1^* and n_2^* for curves with $n = 0, 0.1, 0.3$, and 0.5, respectively.

Fig. 10 Variations of n caused by a change of $\pm 2\%$ in Π_2 for $n_0 = 0, 0.1, 0.3$, and 0.5 in the case of $50 < E^*/\sigma_{0.033} \le 770$, where n_0 is the real value without input error.

Fig. 11 Variations of n caused by a change of $\pm 2\%$ in Π_3 for $n_0 = 0.15, 0.3$, and 0.5 in the case of $0.02 \le \sigma_{0.033}/E^* \le 0.05$ (*i.e.*, $20 \le E^*/\sigma_{0.033} \le 50$).



P_m
Unloading curve
$S = \left.\frac{dP_u}{dh}\right|_{h_m}$
1
Load, P
Loading curve
W_e
W_p
h_r
h_m
Displacement, h

Fig. 1

$C, W_p, W_t, P_m, h_m, S \longrightarrow E^*, H, \sigma_{0.033}, n, \sigma_y$

$$\frac{W_p}{W_t} = \Pi_5\left(\frac{h_r}{h_m}\right) \longrightarrow \text{solve for } \frac{h_r}{h_m}$$

$$\downarrow$$

$$\frac{S}{E^*}\sqrt{\frac{H}{P_m}} = \Pi_6 \text{ and } \frac{H}{E^*} = \Pi_4\left(\frac{h_r}{h_m}\right) \longrightarrow \text{solve for } H, E^*$$

$$\downarrow$$

$$C = \sigma_{0.033}\Pi_1\left(\frac{E^*}{\sigma_{0.033}}\right) \longrightarrow \text{solve for } \sigma_{0.033}$$

$$\downarrow$$

$$\frac{S}{E^* h_m} = \Pi_2\left(\frac{E^*}{\sigma_{0.033}}, n\right) \longrightarrow \text{solve for } n$$

$$\downarrow$$

$$\sigma_{0.033} = \sigma_y\left(1 + \frac{E}{\sigma_y}0.033\right)^n \longrightarrow \text{solve for } \sigma_y$$

Fig. 2

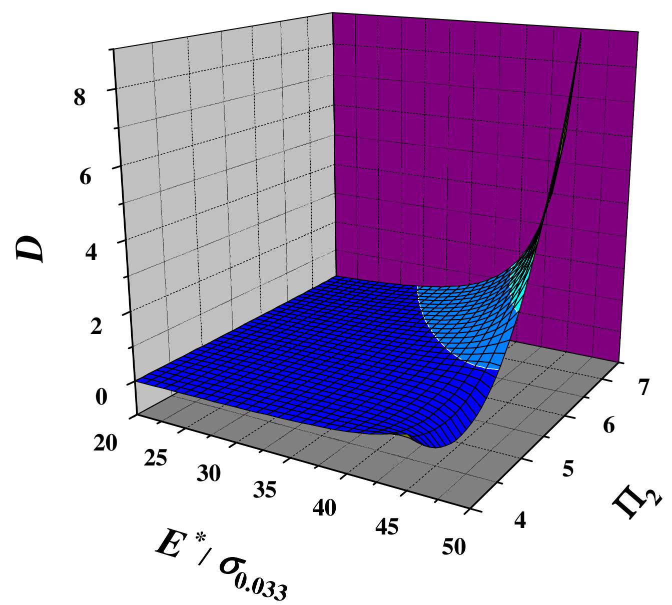
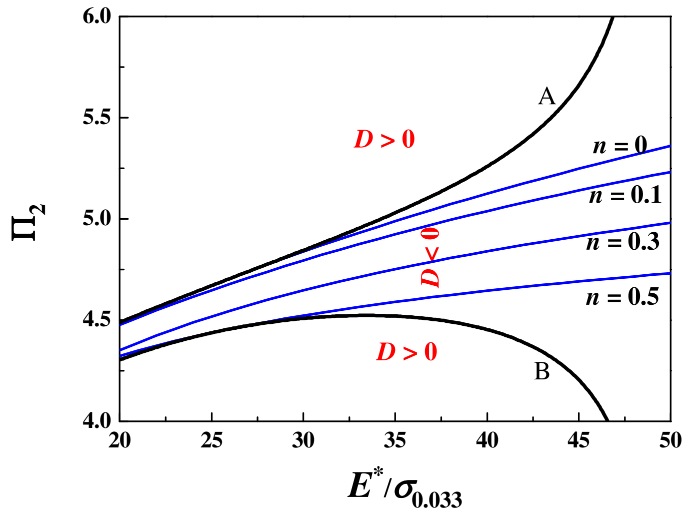


Fig. 3



6.0
5.5
5.0
4.5
4.0
A
B
D > 0
D < 0
D > 0
n = 0
n = 0.1
n = 0.3
n = 0.5
Π2
20
25
30
35
40
45
50
E*/σ0.033

Fig. 4

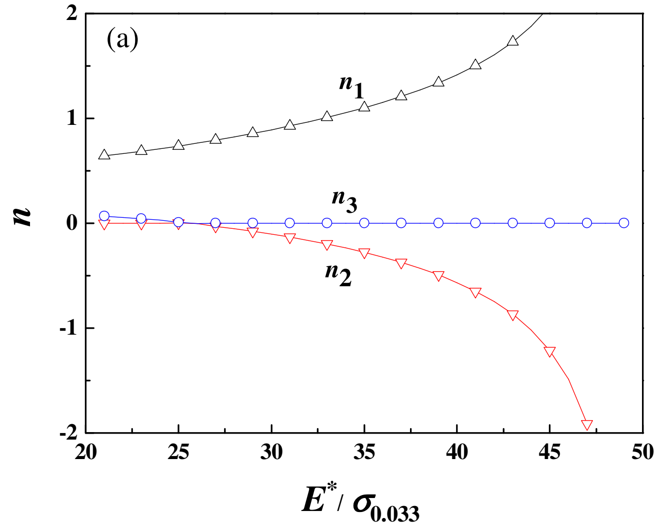


Fig. 5(a)

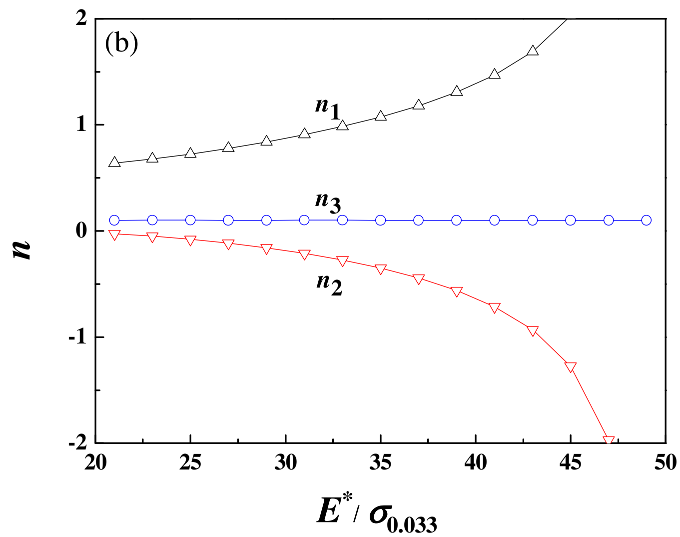


Fig. 5(b)

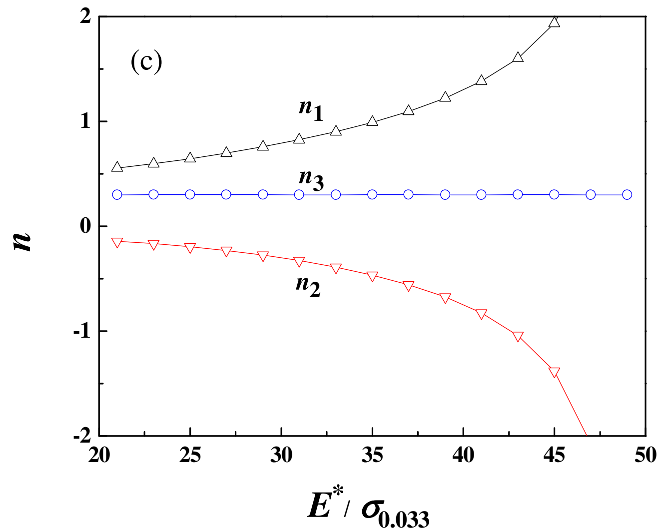


Fig. 5(c)

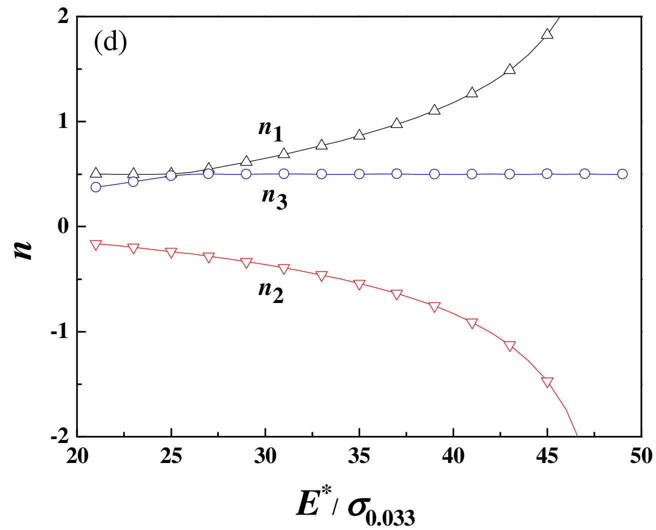


Fig. 5(d)

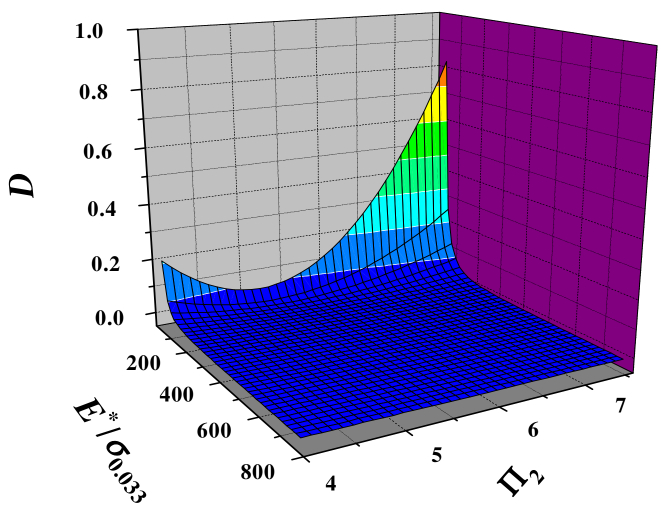
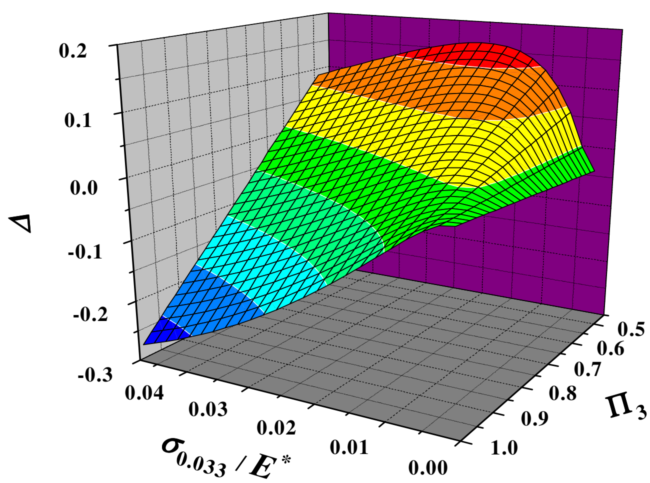
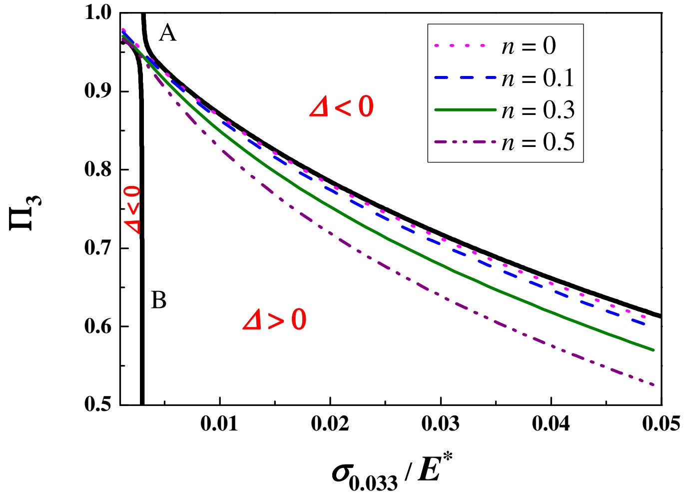


Fig. 6



0.2
0.1
0.0
-0.1
-0.2
-0.3
Δ
0.04
0.03
0.02
0.01
0.00
$\sigma_{0.033} / E^*$
0.5
0.6
0.7
0.8
0.9
1.0
Π_3

Fig. 7



1.0
0.9
0.8
0.7
0.6
0.5
0.01
0.02
0.03
0.04
0.05
Π_3
$\sigma_{0.033} / E^*$
A
B
$\Delta < 0$
$\Delta > 0$
$\Delta < 0$
$n = 0$
$n = 0.1$
$n = 0.3$
$n = 0.5$

Fig. 8

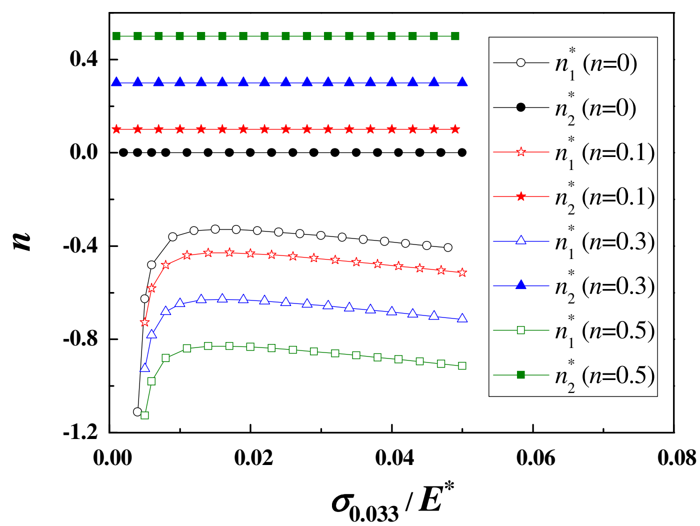
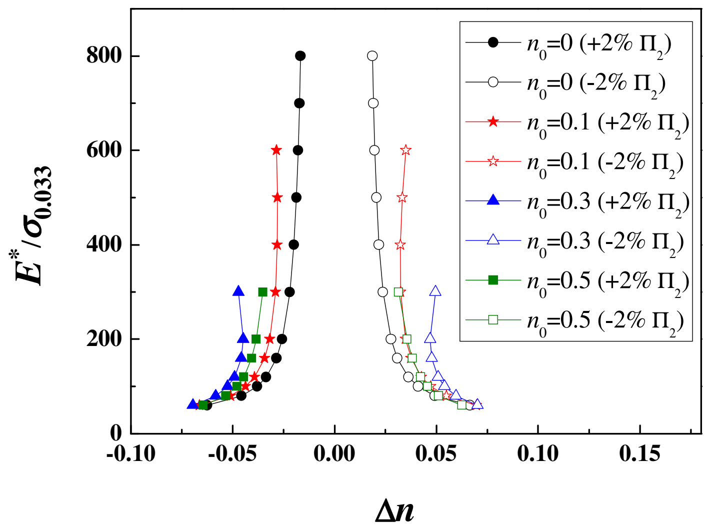
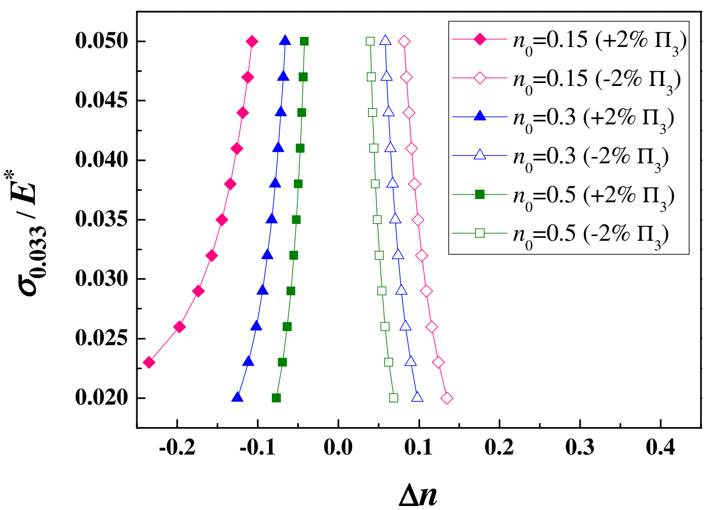

0.4
0.0
-0.4
-0.8
-1.2
n
0.00
0.02
0.04
0.06
0.08
$\sigma_{0.033} / E^*$
n_1^* (n=0)
n_2^* (n=0)
n_1^* (n=0.1)
n_2^* (n=0.1)
n_1^* (n=0.3)
n_2^* (n=0.3)
n_1^* (n=0.5)
n_2^* (n=0.5)

Fig. 9



n_0=0 (+2% Π_2)
n_0=0 (-2% Π_2)
n_0=0.1 (+2% Π_2)
n_0=0.1 (-2% Π_2)
n_0=0.3 (+2% Π_2)
n_0=0.3 (-2% Π_2)
n_0=0.5 (+2% Π_2)
n_0=0.5 (-2% Π_2)
$E^*/\sigma_{0.033}$
Δn

Fig. 10



n_0=0.15 (+2% Π_3)
n_0=0.15 (-2% Π_3)
n_0=0.3 (+2% Π_3)
n_0=0.3 (-2% Π_3)
n_0=0.5 (+2% Π_3)
n_0=0.5 (-2% Π_3)
$\sigma_{0.033}/E^*$
0.050
0.045
0.040
0.035
0.030
0.025
0.020
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
Δn

Fig. 11