Department of Mathematics and Statistics

The Largest Eigenvalue of Nonnegative Tensors

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Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgement has been made.

Signature: ______________________________

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Abstract

Tensors are simply generalisation of matrices. Many properties of matrices have been generalised to tensors. Over the past few years, the spectral theory of tensors has been developed. The Perron-Frobenius Theorem and the minimax theorem are two examples of the property of nonnegative matrices which have been extended to nonnegative tensors. This leads to extension of the Collatz method for finding the largest eigenvalue of nonnegative matrices to nonnegative tensors. In this thesis, we study the methods for finding the largest eigenvalue of square tensors and rectangular tensors. We also study the convergence of the methods and show that the method for rectangular tensors is Q-linear convergence under weak irreducibility condition. We further generalise the method to nonnegative polynomial eigenvalue problems. The method is convergent for irreducible nonnegative polynomials. We explore the case for both homogeneous and nonhomogeneous polynomials. We also present a convergent method for solving the optimisation problem where the objective function is a nonnegative general polynomial with spherical constraint.

List of Publications

The following are publications arising from the thesis:

N. F. Ibrahim. An algorithm for the largest eigenvalue of nonhomogeneous nonnegative polynomials. Submitted to Numerical Algebra, Control and Optimization.

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Chapter 1

Introduction

1.1 Background

The theory of nonnegative matrices emerged in the early 20th century when Frobenius [17, 18, 19] extended some properties of positive square matrices which were introduced by Perron [48] to irreducible nonnegative square matrices. The Perron-Frobenius Theorem for matrices is the fundamental property of nonnegative matrices. Its applications are not only in mathematics but also in Leontief's Input-Output Economic Model [40], demography (Leslie population age distribution model)[40] and in the ranking of football teams [26]. Since then, the study of nonnegative matrices has become one of the most important fields in linear algebra.

Almost a century after the astonishing discovery of the Perron-Frobenius theorem, Chang *et al.* [8] generalised this theorem to nonnegative tensors. The term "tensor" is basically applied to data in three or more dimensions. It is also referred to as higher-order tensor, or as a multi-dimensional, multi-way or n-way array. A matrix is a tensor of order two. An article by Qi, *et al.* [52] presented a survey on tensors and their applications.

One of the vital components in the study of tensors is spectral theory. This research area has seen rapid development in the past few years. In 2012, Qi [50] presented another survey, this time on the spectral theory of tensors, and categorised the spectral theory of tensors to 18 research topics. The spectral theory of tensors has applications in a wide range of fields such as in measuring higher order connectivity in linked objects [36], medical resonance imaging [5, 54], higher-order Markov chains [43], positive definiteness of even-order multivariate forms in automatic control [44], and best rank one approximation in data analysis [34, 53].

1.2 Literature Review

1.2.1 Nonnegative Matrices

Since most spectral theories of nonnegative tensors are generalised from nonnegative matrices, it is reasonable that we look at nonnegative matrices first.
Matrix $\boldsymbol{A} = (a_{ij})$ is said to be nonnegative if $a_{ij} \geq 0$. A vector $\boldsymbol{x}$ is called the eigenvector of matrix $\boldsymbol{A}$ and a scalar λ is called the eigenvalue of matrix $\boldsymbol{A}$ if the following equation is satisfied

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x},$$

where $\boldsymbol{A}$ is a square matrix. The spectral radius of $\boldsymbol{A}$ is an eigenvalue of matrix $\boldsymbol{A}$.

Definition 1. [60] Let $\boldsymbol{A} = (a_{i,j})$ be an $n \times n$ complex matrix with eigenvalues λ_i, $1 \leq i \leq n$. Then

$$\rho(\boldsymbol{A}) \equiv \max_{1 \leq i \leq n} |\lambda_i| \tag{1.1}$$

is the spectral radius of the matrix $\boldsymbol{A}$.

In some eigenvalue problems, we want to find all the eigenvalues of matrices. For this kind of problem, we can use QR method [14, 15, 33, 21]. However, in some problems we only want to find the largest eigenvalue. For example in Input-Output Analysis (in Economic) [65], in order to prove the unique solution of a system, we need to find the largest eigenvalue of a nonnegative matrix. The methods that can be used for this purpose are the Collatz method [66, 10], power method [61, 21] and Arnoldi method [2, 21].

We focus on the Collatz method. In 1942, Collatz [10] wrote that if $\boldsymbol{A}$ is a real symmetric $n \times n$ square matrix and $\boldsymbol{u} = (u_i, ..., u_n)$ is a positive vector, then the matrix $\boldsymbol{A}$ has at least one eigenvalue between the interval

$$\min_{x_i>0} \frac{(\boldsymbol{Ax})_i}{x_i} \leq \lambda \leq \max_{x_i>0} \frac{(\boldsymbol{Ax})_i}{x_i}.$$

Later, in 1950, Wielandt [63] improved the result of Collatz and presented the following well known minimax theorem.

Theorem 1. [10] Let $\boldsymbol{A} = (a_{i,j})$ be an irreducible[1] nonnegative $n \times n$ matrix. Then,

$$\min_{\boldsymbol{x}\in\mathbb{R}_{>0}} \max_{x_i>0} \frac{(\boldsymbol{Ax})_i}{x_i} = \lambda_0 = \max_{\boldsymbol{x}\in\mathbb{R}_{>0}} \min_{x_i>0} \frac{(\boldsymbol{Ax})_i}{x_i}, \tag{1.2}$$

where λ_0 is the unique positive eigenvalue corresponding to the positive eigenvector.

Based on Theorem 1, the following Collatz method for finding the largest eigenvalue of irreducible nonnegative square matrices is produced [66]:

Theorem 2. [66] Let $\boldsymbol{A} \geq \boldsymbol{0}$ be an irreducible $n \times n$ matrix, and let $\boldsymbol{x}^{(1)}$ be an arbitrary column vector with n positive components. Defining

$$\boldsymbol{x}^{(k)} = \boldsymbol{A}\boldsymbol{x}^{(k-1)} = ... = \boldsymbol{A}^{(k)}\boldsymbol{x}^{(1)}, \quad k \geq 2, \tag{1.3}$$

let

$$\bar{\lambda}_k = \max_{1\leq i\leq n} \left(\frac{x_i^{(k+1)}}{x_i^{(k)}} \right) \quad \textit{and} \quad \underline{\lambda}_k = \min_{1\leq i\leq n} \left(\frac{x_i^{(k+1)}}{x_i^{(k)}} \right). \tag{1.4}$$

Then

$$\underline{\lambda}_1 \leq \underline{\lambda}_2 \leq ... \leq \rho(\boldsymbol{A}) \leq ... \leq \bar{\lambda}_2 \leq \bar{\lambda}_1. \tag{1.5}$$

This method is similar to the power method in terms of generating the sequence of $\boldsymbol{x}^{(k)}$. A comparison was made in [66] between the two methods. One of the differences is how the sequence $\lambda_{(k)}$ is determined. For the Collatz method, the spectral radius is bounded and squeezed. Initially, upper bound $\bar{\lambda}_1$ and lower bound $\underline{\lambda}_1$ are produced in first iteration. As the iterative process continued, $\bar{\lambda}_k \leq \bar{\lambda}_{k-1}$ and $\underline{\lambda}_{k-1} \leq \underline{\lambda}_k$. Eventually, we will get $\rho(\boldsymbol{A})$.
A matrix $\boldsymbol{A}$ is primitive if it is nonnegative and its kth power is positive for some natural number k. For primitive matrix $\boldsymbol{A}$, this method is convergent but we cannot say the same for power method.

[1] A square $n \times n$ matrix $\boldsymbol{A} = a_{ij}$ is called reducible if the indices $1, 2, ..., n$ can be divided into two disjoint nonempty sets $i_1, i_2, ..., i_\mu$ and $j_1, j_2, ..., j_\nu$, with $\mu + \nu = n$, such that $a_{i_\alpha j_\beta} = 0$ for $\alpha = 1, 2, ..., \mu$ and $\beta = 1, 2, ..., \nu$. A square matrix that is not reducible is said to be irreducible.

Theorem 3. [60] In Theorem 2 both the sequences $\{\underline{\lambda}_k\}$ and $\{\bar{\lambda}_k\}$ converge to $\rho(\boldsymbol{A})$, from an arbitrary initial positive vector $\boldsymbol{x}^{(0)}$, if and only if the irreducible nonnegative matrix $\boldsymbol{A}$ is primitive.

For an irreducible matrix, this algorithm is not guaranteed to be convergent. Fortunately, we can fix this by shifting the diagonal of the matrices as suggested in [66].

Theorem 4. [66] If the nonnegative matrix $\boldsymbol{A}$ is an $n \times n$ irreducible matrix then the matrix $\epsilon\boldsymbol{I} + \boldsymbol{A}$, where $\epsilon > 0$, is primitive.

If we apply the Collatz method to the matrix $\boldsymbol{B} = \epsilon\boldsymbol{I} + \boldsymbol{A}$, it is guaranteed to be convergent for the irreducible matrix $\boldsymbol{A}$. If λ_0 is the largest eigenvalue of matrix $\boldsymbol{B}$, the largest eigenvalue of matrix $\boldsymbol{A}$ is $\lambda_0 - \epsilon$ and both matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same associated eigenvector.

Another version of this method was introduced in [66] by applying the Collatz method to the matrix $(\epsilon\boldsymbol{I} - \boldsymbol{A})^{-1}$. This alternative was claimed to have a superior rate compared to the original Collatz method if ϵ is chosen sufficiently close to the spectral radius of $\boldsymbol{A}$. The matrix $(\epsilon\boldsymbol{I} - \boldsymbol{A})^{-1}$ is primitive hence it is convergent. This version is similar to the inverse power method; however, it is better because it is proven to be convergent for any irreducible nonnegative matrix $\boldsymbol{A}$.

However, the Collatz method is not guaranteed to be convergent for all reducible matrices. In order to overcome this problem, Wood and O'Neill [66] suggested adding the following matrix:

$$\boldsymbol{E} = \begin{pmatrix} 0 & \varepsilon & 0 \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & & \ddots & \varepsilon \\ \varepsilon & 0 & \cdots & 0 \end{pmatrix}$$

with small $\varepsilon > 0$ to the reducible matrix $\boldsymbol{A}$. Now we have an irreducible matrix $\boldsymbol{A} + \boldsymbol{E}$ and this matrix can be applied to the Collatz method. The effects of the perturbation of matrix $\boldsymbol{A}$ were discussed in [66] and it can be concluded that if the value of ε is appropriately small, the largest eigenvalue of $\boldsymbol{A}$ is not greatly affected.

In [64, 66], a comparison was made between the methods for finding the largest eigenvalue of a nonnegative matrix. The Collatz method [66] was compared to the power method [21], Arnoldi method [2], Orthogonal Iteration [21] and Simultaneous Iteration [59]. We focus in detail here on the power method.

Power Method

Step 1: Choose initial vector $\boldsymbol{x}^{(0)}$ and set $k = 0$.

Step 2: Calculate

$$\boldsymbol{x}^{(k+1)} = \frac{\boldsymbol{A}\boldsymbol{x}^{(k)}}{\|\boldsymbol{A}\boldsymbol{x}^{(k)}\|_2}.$$

Step 3: Set k=k+1 and go to Step 2.

In [66], two ways were used to determine the largest eigenvalue using the power method:

1. Compute $\lambda^{(k)} = \boldsymbol{u}^T\boldsymbol{A}\boldsymbol{x}^{(k)}/\boldsymbol{u}^T\boldsymbol{x}^{(k)}$, where $\boldsymbol{u}$ is chosen such that $\boldsymbol{u}^T\boldsymbol{x}^{(0)} \neq \boldsymbol{0}$.
2. The differences of two corresponding nonzero components of $\boldsymbol{x}^{(k+1)}$ and $\boldsymbol{x}^{(k)}$.

Wood and O'Neill [66] tested the power method using some 2×2, 3×3 and 4×4 irreducible matrices. They showed the cases where power method performs poorly. Depending on the convergent criterion and how the $\lambda^{(k)}$ is obtained, the power method does not converge, converges very slowly or converges to the incorrect value for some of the test matrices. In the experiment, the test matrix is set as $\boldsymbol{A}$. When the the Collatz method is applied to the matrix $(\epsilon\boldsymbol{I} - \boldsymbol{A})^{-1}$, this method converges.

Both the power method and the Collatz method have the rate of convergence λ_1/λ_0, where λ_0 is the largest eigenvalue and λ_1 is a subdominant eigenvalue of matrix $\boldsymbol{A}$. However if we apply the Collatz method to the matrix $(\epsilon\boldsymbol{I} - \boldsymbol{A})^{-1}$, the convergence rate is $|(\epsilon - \lambda_0)/(\epsilon - \lambda_1)|$ and obviously, this is the best rate of convergence if ϵ is selected close enough to λ_0 [66].

Another test by Wood and O'Neill [64] showed that the Collatz method has a lower number of flops (count floating-point operations) compared to the Arnoldi method, Orthogonal Iteration, Simultaneous Iteration, and the eig and eigs functions in MATLAB which use the Implicitly Restarted Arnoldi Algorithm and QR method, respectively. The test matrix used for these comparisons is from a Markov model of a random walk on a $(k+1) \times (k+1)$ triangular grid which usually denotes $\boldsymbol{Mark}(k+1)$. The irreducible matrix $\boldsymbol{Mark}(25)$ which contains 351-dimension and 123201 nonzero entries is used. All the methods are tested using matrix $\boldsymbol{Mark}(25) + \boldsymbol{I}_{351}$. The Arnoldi method, eigs and eig are also tested with matrix $\boldsymbol{Mark}(25)$. In terms of the number of flops, the Collatz method is superior than all these methods.

However, when tested for very large matrices, matrix $\boldsymbol{Mark}(50)$ with dimensions 1326 and matrix $\boldsymbol{Mark}(100)$ with dimensions 5151, Collatz methods perform about the same as the other methods. For the case when the Collatz method converges slowly, Wood and O'Neill [64] suggested the Hybrid method, that is, to perform several steps of the Collatz method to the matrix A then set the ϵ as the latest upper bound of the spectral radius, $\bar{\lambda}_{(k)}$, and apply $(\epsilon \boldsymbol{I} - \boldsymbol{A})^{-1}$ to the Collatz method. However, for very large matrices, the Hybrid method performed less effectively compared to the Collatz method. One of the advantages of the Collatz method is that, when $(\epsilon \boldsymbol{I} - \boldsymbol{A})^{-1}$ is applied, it is guaranteed to be convergent for an irreducible matrix $\boldsymbol{A}$. For methods such as the Arnoldi and eig function in MATLAB, they have the advantage of providing the other eigenvalues of the matrix.

1.2.2 Nonnegative Tensors

Nonnegative tensors are usually referred to as nonnegative square tensors unless defined differently. The definitions of the eigenvalue and eigenvector of the tensor were first introduced by Qi [49] and Lim [35] independently. Qi [49] studied the real and complex eigenvalue and eigenvector of the symmetric tensor. The definitions were then generalised by Chang *et al.* [6]. Not long after that, Chang *et al.* extended the Perron-Frobenius Theorem to nonnegative square tensors [8]. With the introduction of the eigenvalue and eigenvector of the tensor, and then the generalisation of the Perron-Frobenius Theorem to nonnegative square tensors, the research for eigenvalue problems of tensors was triggered. Yang and Yang [67, 68] further generalised the other results of the Perron-Frobenius Theorem and other properties of matrices to square tensors including the minimax theorem.

Nonnegative square tensors can be classified as strictly nonnegative tensors [25], weakly irreducible nonnegative tensors [16], weakly primitive tensors [16], irreducible nonnegative tensors [8], primitive tensors [9], weakly positive tensors [75] and essentially positive tensors [47]. In order to clearly see the relationship between these classes, Hu *et al.* [25] illustrated a diagram showing the connections between the classes of tensors.

Consequently, with the existance of the Perron-Frobenius Theorem for nonnegative square tensors and the minimax theorem, Ng *et al.* [43] presented an iterative method for calculating the largest eigenvalue and the associated eigenvector for nonnegative square tensors. This method is extended from the Collatz method [10, 64, 60] for finding the largest eigenvalue of matrices, which has some features that are similar to the power method [60]. Because of the similarity, the Ng-Qi-Zhou method and its updates and extension are sometimes called power algorithm or power method in the literatures. The numerical results in [43] show that the Ng-Qi-Zhou method is efficient however not always convergent, for irreducible tensors. Later, this method was proven to be convergent for primitive nonnegative square tensors in [9]. In [16], the convergence of the method under weakly primitive square tensors was established. An irreducible tensor is primitive but not vice versa. Zhang and Qi [74] showed that the method has a linear convergence rate for essentially positive tensors. An essentially positive tensor is primitive but the reverse is not valid.

The method of [43] was improved in [38] and was proven to be convergent for an irreducible nonnegative tensor. This improved method resembled a version of the Collatz method by Wood and O'Neill [66] for nonnegative matrices. Zhang *et al.* [75] established the linear convergence of the improved method for weakly positive tensors and Zhou *et al.* [78] established the Q-linear convergence of the improved method under weak irreducibility condition. A new discovery was made recently in [79]; some spectral properties of symmetric nonnegative tensors were studied and one of the finding was that the minimax theorem does not require the weakly irreducible condition, just the symmetric alone.

Other methods that can be used for finding the spectral radius of nonnegative square tensors were presented by Hu *et al.* [25] and Yang [70]. Hu *et al.* [25] proposed a modified version of [16] and proved that the modified method was R-linear convergent. Hu *et al.* also gave a convergent algorithm for a general nonnegative square tensor. Meanwhile, Yang presented an extension of the smoothing method for finding the largest eigenvalue of a nonnegative matrix [72].

Besides the class of square tensors, there are also rectangular tensors. The class of real rectangular tensors can be found in the strong ellipticity condition problem in solid mechanics [28, 29, 56, 58, 71] and the entanglement problem in quantum physics [11, 13]. Qi *et al.* [51] introduced the definition of M-eigenvalues of rectangular tensors and Wang *et al.* [62] proposed a method to compute the largest M-eigenvalue of a fourth-order tensor.

The study of the properties of singular values of non-square tensors can be found in [35, 7]. In 2005, Lim [35] introduced the singular values of non-square tensors and extended the Perron-Frobenius theorem to singular values of non-square tensors. Later, the Perron-Frobenius theorem for singular values of nonnegative rectangular tensors was given in [7]. As a result, an algorithm for finding the largest singular value of a nonnegative rectangular tensor was also proposed in [7]. Yang and Yang [69] established the convergence of the algorithm for nonnegative primitive rectangular tensors. The method was updated in [76] and this modified method was shown to be convergent for any irreducible nonnegative rectangular tensor. Another modified version of the algorithm was given by Zhang in [73]. Discussions of other methods for rectangular tensors can be found in [23, 37].

The Perron-Frobenius theorem also has been extended to homogeneous and monotone functions [20], nonnegative multilinear forms [16] and nonnegative polynomial maps [16].

1.3 Overview of the Thesis

In Chapter 2, we focus on finding the largest eigenvalue of irreducible nonnegative square tensors. The Perron-Frobenius Theorem for tensors, which is an extension of the Perron-Frobenius Theorem for matrices, provides the path to suitable methods of finding the largest eigenvalue of irreducible nonnegative square tensors. We study the iterative methods by [43, 38] under irreducibility condition, primitivity condition, and also weak irreducibility condition. We also show the application of the method for testing the positive definiteness of a class of multivariate forms and the numerical results.

In Chapter 3, we consider nonnegative rectangular tensors and study the iterative methods by [7, 76] for finding the largest singular value of irreducible nonnegative rectangular tensors. These methods are similar to the methods for square tensors [43, 38]. The method of [76] is convergent under weakly irreducible condition and we show that the method is Q-linear convergent under weakly irreducible condition.

Chapter 4 is devoted to the eigenvalue problem of nonnegative polynomial. The problems considered in the previous chapters were related to homogeneous polynomials. In this chapter, we not only focus on the eigenvalue problem of nonnegative homogeneous polynomials, but also of nonnegative nonhomogeneous polynomials. We present a convergent iterative method for finding the largest eigenvalue of nonhomogeneous nonnegative polynomials. We also expand the concept of primitivity to polynomials.

Chapter 5 is concerned with the optimisation problem whereby the objective function is a nonnegative polynomial and the constraint is spherical. We consider both homogeneous and nonhomogeneous nonnegative polynomials and present a convergent method to solve the problem whereby the objective function is an irreducible nonnegative polynomial.

Finally in Chapter 6, we conclude this thesis.

Chapter 2

Nonnegative Square Tensors

2.1 Introduction

In this chapter, we study some properties of irreducible nonnegative square tensors and the methods for finding the largest eigenvalue of irreducible nonnegative square tensors. We focus in particular on the methods in [38, 43] and the convergence analysis. Throughout this chapter, when the term "nonnegative tensor" is used, it refers to nonnegative square tensors.

We denote $\mathbb{R}_+ = [0, \infty)$ as the set of nonnegative numbers, $\mathbb{R}_{>0} = (0, \infty)$ as the set of positive numbers, $\mathbb{R}^n_+$ as the cone of nonnegative vectors and $\mathbb{R}^n_{>0}$ as the cone of positive vectors.

2.2 Nonnegative Matrices

We start this chapter with some properties related to nonnegative square matrices which are useful in this chapter.

Let $\boldsymbol{A}$ be an $n \times n$ nonnegative matrix. The graph associated to $\boldsymbol{A} = (a_{ij})$, $G(\boldsymbol{A})$, is the directed graph with vertices $1, 2, ..., n$ and an edge from i to j if and only if $a_{ij} \neq 0$ [60] (p.19, [60]). A directed graph is strongly connected if there is a directed path between any two distinct vertices (p.20, [60]).

Theorem 5. (p.20,[60]) An $n \times n$ complex matrix $\boldsymbol{A}$ is irreducible if and only if its directed graph $G(\boldsymbol{A})$ is strongly connected.

Theorem 6. (p.51,[60]) Let $\boldsymbol{A}$ be an irreducible matrix, with $G(\boldsymbol{A})$ as the associated directed graph. If the greatest common divisor (g.c.d.) of the lengths of its closed paths is equal to one, then $\boldsymbol{A}$ is primitive.

The converse of Theorem 6 also holds.

We state here the Perron-Frobenius Theorem for nonnegative matrices.

Theorem 7. (Chapter 2,[60]) If $\boldsymbol{A}$ is an irreducible nonnegative square matrix, then

(1) $\rho(\boldsymbol{A}) > 0$ is an eigenvalue;

(2) there exists a nonnegative vector $\boldsymbol{x}_0 > 0$, i.e. all components of $\boldsymbol{x}_0$ are positive, such that $\boldsymbol{A}\boldsymbol{x}_0 = \rho(\boldsymbol{A})\boldsymbol{x}_0$;

(3) (uniqueness) if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \rho(\boldsymbol{A})$;

(4) $\rho(\boldsymbol{A})$ is a simple eigenvalue of $\boldsymbol{A}$;

(5) if λ is an eigenvalue of $\boldsymbol{A}$, then $|\lambda| \leq \rho(\boldsymbol{A})$.

Furthermore, if a nonnegative matrix $\boldsymbol{M}$ is primitive, then

$$\rho(\boldsymbol{M}) > |\lambda|, \quad \forall \lambda \in \sigma(\boldsymbol{M}) \setminus \{\rho(\boldsymbol{M})\}, \tag{2.1}$$

where $\sigma(\boldsymbol{M})$ is the spectrum of $\boldsymbol{M}$.

Corollary 1. [41] An irreducible matrix with a nonzero main diagonal is primitive.

Spectral norm of a matrix is defined as follows.

Definition 2. (p.9,[60]) If $\boldsymbol{A} = (a_{i,j})$ is an $n \times n$ complex matrix, then

$$||\boldsymbol{A}|| = \sup_{\boldsymbol{x} \neq 0} \frac{||\boldsymbol{A}\boldsymbol{x}||}{||\boldsymbol{x}||}$$

is the spectral norm of the matrix $\boldsymbol{A}$ where $|| \cdot ||$ denotes a vector norm on the vector space $\mathbb{R}^n$.

Proposition 1. [21, 24] The spectral radius of an $n \times n$ matrix $\boldsymbol{A}$ is characterized by the equality

$$\rho(\boldsymbol{A}) = \inf_{||\cdot|| \in N} ||\boldsymbol{A}||$$

where N denotes the set of all possible spectral norms of $\boldsymbol{A}$. For any $\epsilon > 0$, there exists a spectral norm $||\cdot||_\epsilon \in N$ such that

$$||\boldsymbol{A}|| \leq \rho(\boldsymbol{A}) + \epsilon.$$

2.3 Nonnegative Tensors

Let $\mathbb{R}$ be the real number field. We consider an m-order n-dimensional tensor $\mathcal{A}$ consisting of n^m entries in $\mathbb{R}$,

$$\mathcal{A} = (a_{i_1 \cdots i_m}), \ \ a_{i_1 \cdots i_m} \in \mathbb{R}, \ \ 1 \leq i_1 \cdots i_m \leq n. \tag{2.2}$$

If $a_{i_1 \cdots i_m} \geq 0$, $\mathcal{A}$ is called nonnegative real square tensor and if $a_{i_1 \cdots i_m} > 0$, $\mathcal{A}$ is called positive real square tensor. When $m = 2$, tensor $\mathcal{A}$ is reduced to a square matrix. Let $\mathcal{A}\boldsymbol{x}^{m-1}$ be a vector in $\mathbb{R}^n$ such that

$$\left(\mathcal{A}\boldsymbol{x}^{m-1}\right)_i = \sum_{i_2, \cdots, i_n = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \quad i = 1, 2, \cdots, n. \tag{2.3}$$

We use the definition of eigenvalue and eigenvector of tensor which was given in [8].

Definition 3. [8] Let $\mathcal{A}$ be an m-order n-dimensional tensor and $\mathbb{C}$ be the set of all complex numbers. Assume that $\mathcal{A}\boldsymbol{x}^{m-1}$ is not identical to zero. We say $(\lambda, \boldsymbol{x}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{\mathbf{0}\})$ is an eigenvalue-eigenvector of $\mathcal{A}$ if

$$\mathcal{A}\boldsymbol{x}^{m-1} = \lambda \boldsymbol{x}^{[m-1]}, \tag{2.4}$$

where, $\boldsymbol{x}^{[\alpha]} = [x_1^\alpha, x_2^\alpha, ..., x_n^\alpha]^T$.

The definition of spectral radius of tensor was given in [67].

Definition 4. [67] Let $\mathcal{A}$ be an m-order n-dimensional real tensor.

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \ is \ an \ eigenvalue \ of \ \mathcal{A}\}. \tag{2.5}$$

We call $\rho(\mathcal{A})$ the spectral radius of $\mathcal{A}$.

The associated graph of an m-order n-dimensional nonnegative tensor $\mathcal{A}$, is the directed graph $\mathcal{G}(\mathcal{A})$, with vertices $1, 2, ..., n$ and an edge from i to j if and only if $a_{ii_2\cdots i_m} \neq 0$ for some $i_l = j$, $l = 2, 3, ..., m$. This definition can be found in [78].

Definition 5. [16] An m-order n-dimensional square tensor $\mathcal{A}$ is called weakly irreducible if $\mathcal{G}(\mathcal{A})$ is strongly connected. If $\mathcal{G}(\mathcal{A})$ is strongly connected and the greatest common divisor (gcd) of the lengths of its circuits is equal to one, then $\mathcal{A}$ is called weakly primitive.

Definition 6. [8] An m-order n-dimensional tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$\mathcal{A}_{i_1 i_2 \ldots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I. \tag{2.6}$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ *irreducible.*

Proposition 2. [16] If nonnegative tensor $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is weakly irreducible. For matrix (when $m = 2$), $\mathcal{A}$ is irreducible if and only if $\mathcal{A}$ is weakly irreducible.

Let $\mathcal{I}$ be the m-order n-dimensional unit tensor whose entries are

$$\mathcal{I}_{i_1 i_2 \ldots i_m} = \begin{cases} 1 & \text{if } i_1 = i_2 = \ldots = i_m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Proposition 3. [25] If nonnegative tensor $\mathcal{A}$ is weakly irreducible, then $\mathcal{A} + \mathcal{I}$ is weakly primitive.

Define the sequence $\{\mathcal{A}^{(k)}\boldsymbol{x}\}$ for any vector $\boldsymbol{x} \in \mathbb{R}^n_+$ as:

$$\begin{aligned}
\mathcal{A}^{(1)}\boldsymbol{x} &= \mathcal{A}\left(\boldsymbol{x}\right)^{m-1}, \ \boldsymbol{z}^{(1)} = \left(\mathcal{A}^{(1)}\boldsymbol{x}\right)^{\left[\frac{1}{m-1}\right]}, \\
\mathcal{A}^{(2)}\boldsymbol{x} &= \mathcal{A}\left(\boldsymbol{z}^{(1)}\right)^{m-1}, \ \boldsymbol{z}^{(2)} = \left(\mathcal{A}^{(2)}\boldsymbol{x}\right)^{\left[\frac{1}{m-1}\right]}, \\
&\vdots \\
\mathcal{A}^{(k)}\boldsymbol{x} &= \mathcal{A}\left(\boldsymbol{z}^{(k-1)}\right)^{m-1}, \ \boldsymbol{z}^{(k)} = \left(\mathcal{A}^{(k)}\boldsymbol{x}\right)^{\left[\frac{1}{m-1}\right]}, \ k \geq 2.
\end{aligned}$$

Definition 7. [9] A nonnegative tensor $\mathcal{A}$ is primitive if there exists a positive integer k such that $\mathcal{A}^{(k)}\boldsymbol{x} \in \mathbb{R}^n_+$ for any nonzero $\boldsymbol{x} \in \mathbb{R}^n_{>0}$. Furthermore, we call the least value of such k the primitive degree.

Definition 8. [47] A nonnegative m-order n-dimensional tensor $\mathcal{A}$ is essentially positive if $\mathcal{A}\boldsymbol{x}^{m-1} \in \mathbb{R}^n_{>0}$ for any nonzero $\boldsymbol{x} \in \mathbb{R}^n_+$.

Positive tensors and essentially positive tensors are primitive.

Definition 9. [9] For any vector $\boldsymbol{x} \in \mathbb{R}^n_+$, we define $\boldsymbol{x}^{[1/(m-1)]} = ((x_1^{1/(m-1)}), ..., (x_1^{1/(m-1)}))$. Let $\mathcal{A}$ be nonnegative m-order n-dimensional tensor. We define its associated nonlinear map $\boldsymbol{T}_{\mathcal{A}} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by $\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x} = (\mathcal{A}\boldsymbol{x}^{m-1})^{[1/(m-1)]}$. The polynomial $\boldsymbol{T}_{\mathcal{A}}$ possesses the following immediate properties:

(i) (Positively 1-homogeneous)For $t \geq 0$, we have $\boldsymbol{T}_{\mathcal{A}}(t\boldsymbol{x}) = t\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}$.

(ii) (Increasing) If $\boldsymbol{x} \leq \boldsymbol{y}$, then $\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x} \leq \boldsymbol{T}_{\mathcal{A}}\boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+$.

(iii) The map $\boldsymbol{x} \longmapsto \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}$ is continuous and bounded. Namely, there exists $C = C_{\mathcal{A}} > 0$ such that $||\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}|| \leq C||\boldsymbol{x}||$ for $\boldsymbol{x} \in \mathbb{R}^n_+$, where $||\cdot||$ denotes the standard Euclidean norm on $\mathbb{R}^n$.

2.4 Perron-Frobenius Theorem for Tensors

Perron-Frobenius Theorem is important in the study of spectral radius of nonnegative square tensors. The theorem was generalised to nonnegative square tensors by [8] and continued by [67, 68]. The following is Perron-Frobenius weak version for tensors:

Theorem 8. [8] If $\mathcal{A}$ is a nonnegative tensor of order m dimension n, then there exist $\lambda_0 \geq 0$ and a nonnegative vector $\boldsymbol{x}_0 \neq \boldsymbol{0}$ such that

$$\mathcal{A}\boldsymbol{x}^{m-1} = \lambda_0 \boldsymbol{x}_0^{[m-1]}. \tag{2.8}$$

In the theorem above, λ_0 is a real number and $\boldsymbol{x}_0$ is a real vector. The theorem below is generalised Perron-Frobenius Theorem for nonnegative tensors strong version:

Theorem 9. [8] If $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n, then there exist $\lambda_0 > 0$ and $\boldsymbol{x}_0 > \boldsymbol{0}, \boldsymbol{x}_0 \in \mathbb{R}^n$ such that

$$\mathcal{A}\boldsymbol{x}_0^{m-1} = \lambda_0 \boldsymbol{x}_0^{[m-1]}. \tag{2.9}$$

Moreover, if λ is an eigenvalue with nonnegative eigenvector, then $\lambda = \lambda_0$. If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq \lambda_0$.

Perron-Frobenius Theorem for matrices states that the spectral radius is simple. However for tensor, this theorem does not guarantee the simplicity of the λ_0. The Geometric multiplicity of eigenvalue of tensor can be defined as follows:

Definition 10. [8] Let λ be an eigenvalue of tensor $\mathcal{A}$ of order m and dimension n. It has geometric multiplicity q if the maximum number of linearly independent eigenvectors corresponding to λ equals q. If $q = 1$, then λ is called geometrically simple.

The following minimax theorem for irreducible nonnegative tensors was extended from Collatz minimax theorem for irreducible nonnegative matrices.

Theorem 10. [8] Assume that $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n. Then

$$\min_{\boldsymbol{x}\in\mathbb{R}_{>0}} \max_{x_i>0} \frac{(\mathcal{A}\boldsymbol{x}^{m-1})_i}{x_i^{m-1}} = \lambda_0 = \max_{\boldsymbol{x}\in\mathbb{R}_{>0}} \min_{x_i>0} \frac{(\mathcal{A}\boldsymbol{x}^{m-1})_i}{x_i^{m-1}}, \tag{2.10}$$

where λ_0 is the unique positive eigenvalue corresponding to the positive eigenvector.

We also have the following lemmas:

Lemma 1. [8] If an m-order n-dimensional nonnegative tensor $\mathcal{A}$ is irreducible, then

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} > 0 \quad \forall 1 \le i \le n. \tag{2.11}$$

Lemma 2. [43] If an m-order n-dimensional nonnegative tensor $\mathcal{A}$ is irreducible, then for any positive vector $\boldsymbol{x} > \mathbf{0}$, $\boldsymbol{x} \in \mathbb{R}^n$, $\mathcal{A}\boldsymbol{x}^{m-1}$ is a positive vector; i.e.,

$$\mathcal{A}\boldsymbol{x}^{m-1} > 0. \tag{2.12}$$

Lemma 3. [43] Suppose that $\mathcal{A}$ is a nonnegative tensor of order m and dimension n, $\boldsymbol{x}$ and $\boldsymbol{y}$ are two nonnegative column vectors, and t is a positive number. Then, we have the following.

(i) If $\boldsymbol{x} \ge \boldsymbol{y}$, then $\mathcal{A}\boldsymbol{x}^{m-1} \ge \mathcal{A}\boldsymbol{y}^{m-1}$;

(ii) $\mathcal{A}(t\boldsymbol{x})^{m-1} = t^{m-1}\mathcal{A}(\boldsymbol{x})^{m-1}$.

From the lemmas above, [43] presented the following theorem. An iterative method for finding a lower bound and upper bound of the largest eigenvalue of an irreducible nonnegative tensor was derived from this result. The algorithm will be given in the next section.

Theorem 11. [43] Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n and let $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ be an arbitrary positive vector. Let $\boldsymbol{y}^{(0)} = \mathcal{A}(\boldsymbol{x}^{(0)})^{m-1}$. Define

$$\begin{aligned}
\boldsymbol{x}^{(1)} &= \frac{(\boldsymbol{y}^{(0)})^{[\frac{1}{m-1}]}}{\|(\boldsymbol{y}^{(0)})^{[\frac{1}{m-1}]}\|}, \quad \boldsymbol{y}^{(1)} = \mathcal{A}(\boldsymbol{x}^{(1)})^{m-1}, \\
\boldsymbol{x}^{(2)} &= \frac{(\boldsymbol{y}^{(1)})^{[\frac{1}{m-1}]}}{\|(\boldsymbol{y}^{(1)})^{[\frac{1}{m-1}]}\|}, \quad \boldsymbol{y}^{(2)} = \mathcal{A}(\boldsymbol{x}^{(2)})^{m-1}, \\
&\vdots \\
\boldsymbol{x}^{(k+1)} &= \frac{(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}}{\|(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}\|}, \quad \boldsymbol{y}^{(k+1)} = \mathcal{A}(\boldsymbol{x}^{(k+1)})^{m-1}, \quad k \geq 2 \qquad (2.13) \\
&\vdots
\end{aligned}$$

and let

$$\underline{\lambda}_k = min_{x_i^{(k)}>0} \frac{(\mathcal{A}(\boldsymbol{x}^{(k)})^{m-1})_i}{(x_i^{(k)})^{m-1}}, \quad \bar{\lambda}_k = max_{x_i^{(k)}>0} \frac{(\mathcal{A}(\boldsymbol{x}^{(k)})^{m-1})_i}{(x_i^{(k)})^{m-1}}, \quad k = 1, 2, ... \qquad (2.14)$$

Assume that λ_0 is the unique positive eigenvalue corresponding to a nonnegative eigenvector. Then,

$$\underline{\lambda}_1 \leq \underline{\lambda}_2 \leq ... \leq \lambda_0 \leq ... \leq \bar{\lambda}_2 \leq \bar{\lambda}_1. \qquad (2.15)$$

Theorem 12. [9] A primitive nonnegative tensor $\mathcal{A}$ is irreducible.

An example that shows the converse is false was given in [9].

Theorem 13. [9] If $\mathcal{A}$ is an irreducible nonnegative m-order n-dimensional tensor with $a_{ii...i} > 0, i = 1, 2, ..., n$, then $\mathcal{A}$ is primitive.

Corollary 2. If $\mathcal{A}$ is an essentially positive tensor of m-order and n-dimensional, then $\mathcal{A}$ is primitive.

Corollary 3. [49] Suppose that $\mathcal{B} = a(\mathcal{A} + bI)$, where a and b are two real numbers. Then $\lambda_{\mathcal{B}}$ is an eigenvalue of $\mathcal{B}$ if and only if $\lambda_{\mathcal{B}} = a(\lambda + b)$ and λ is an eigenvalue of $\mathcal{A}$. In this case, they have the same eigenvectors.

By Theorem 9, Theorem 13 and Corollary 8, we have the following theorem:

Theorem 14. [38] Suppose $\mathcal{A}$ is an irreducible nonnegative tensor. For any $\rho > 0$, let $\mathcal{B} = \mathcal{A} + \rho\mathcal{I}$. Then, we have

(i) $\mathcal{B}$ is primitive;

(ii) If λ is the largest eigenvalue of $\mathcal{B}$, then $\lambda - \rho$ is the largest eigenvalue of $\mathcal{A}$.

Theorem 15. [9] Let $\mathcal{A}$ be primitive. If λ is an eigenvalue with $|\lambda| = \rho(\mathcal{A})$, then $\lambda = \rho(\mathcal{A})$, i.e., cyclic index, $k = 1$.

2.5 Algorithm

Based on Theorem 11, we have the following algorithm for calculating the largest eigenvalue of an irreducible nonnegative tensor. This method is extension of a Collatz method [66].

Algorithm 1. [43]

Step 0: Choose $\boldsymbol{x}^{(0)} > \mathbf{0}$, $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. Set $k = 0$

Step 1: Compute

$$\begin{aligned} \boldsymbol{y}^{(k)} &= \mathcal{A}\left(\boldsymbol{x}^{(k)}\right)^{m-1}, \\ \underline{\lambda}_k &= \min_{x_i^{(k)}>0} \frac{\left(\boldsymbol{y}^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}}, \\ \bar{\lambda}_k &= \max_{x_i^{(k)}>0} \frac{\left(\boldsymbol{y}^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}}. \end{aligned}$$

Step 2: If $\bar{\lambda}_k = \underline{\lambda}_k$, then let $\lambda = \bar{\lambda}_k$ and stop. Otherwise, compute

$$\boldsymbol{x}^{(k+1)} = \frac{\left(\boldsymbol{y}^{(k)}\right)^{\left[\frac{1}{m-1}\right]}}{\left\|\left(\boldsymbol{y}^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right\|},$$

replace k by $k+1$ and go to Step 1.

For Algorithm 1, by Theorem 11, we have the theorem below:

Theorem 16. [43] Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n. Assume that λ_0 is the unique positive eigenvalue corresponding to a nonnegative eigenvector. Then, Algorithm 1 produces the value of λ_0 in a finite number of steps or generates two convergent sequences $\{\underline{\lambda}_k\}$ and $\{\bar{\lambda}_k\}$. Furthermore, let $\underline{\lambda} = \lim_{k\to+\infty} \underline{\lambda}_k$ and $\bar{\lambda} = \lim_{k\to+\infty} \bar{\lambda}_k$. Then, $\underline{\lambda}$ and $\bar{\lambda}$ are a lower bound and an upper bound, respectively, of λ_0. If $\underline{\lambda} = \bar{\lambda}$, then $\lambda_0 = \underline{\lambda} = \bar{\lambda}$.

We can say the limit exists because $\underline{\lambda}$ is a monotonically increasing sequence and has an upper bound. Hence it implies $\{\boldsymbol{x}^{(k)}\}$ converges to a vector $\boldsymbol{x}$. However, Algorithm 1 only produce a convergent sequence but it doesn't prove that $\{\bar{\lambda}_k\}$ and $\{\underline{\lambda}_k\}$ converge to λ_0 if tensor $\mathcal{A}$ is irreducible. If tensor $\mathcal{A}$ is primitive, Algorithm 1 is convergent [9]. Recall that $\mathcal{T}_{\mathcal{A}}\boldsymbol{x} = (\mathcal{A}\boldsymbol{x}^{m-1})^{\frac{1}{(m-1)}}$.

Proposition 4. [9] For the notation used in Algorithm 1, the following statements hold:

(i) For all $k \in \mathbb{N}$, $\underline{\lambda}_k \le \underline{\lambda}_{k+1}$ and $\bar{\lambda}_k \ge \bar{\lambda}_{k+1}$.

(ii) If $\mathcal{A}$ is irreducible, then $\underline{\lambda}_{k+1} \nearrow \underline{\lambda}$, $\bar{\lambda}_{k+1} \searrow \bar{\lambda}$, and $\underline{\lambda} \le \rho(\mathcal{A}) \le \bar{\lambda}$.

(iii) There exists a subsequence $\boldsymbol{x}^{(k_j)} \longrightarrow \boldsymbol{x}^*$ with $||\boldsymbol{x}^*|| = 1$.

(iv) $(\underline{\lambda}_k)^{\frac{1}{m-1}}\boldsymbol{x}^{(k)} \le (\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]} = \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^{(k)} \le (\bar{\lambda}_k)^{\frac{1}{(m-1)}}\boldsymbol{x}^{(k)}$; hence, $\underline{\lambda}^{\frac{1}{(m-1)}}\boldsymbol{x}^* \le \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^* \le \bar{\lambda}^{\frac{1}{(m-1)}}\boldsymbol{x}^*$.

(v) For all $k \in \mathbb{N}$, there exists $1 \le i_0 \le n$ such that $(\boldsymbol{T}_{\mathcal{A}}^{k+1}\boldsymbol{x}^*)_{i_0} = \underline{\lambda}^{\frac{1}{(m-1)}}(\boldsymbol{T}_{\mathcal{A}}^{k}\boldsymbol{x}^*)_{i_0}$.

Proof. [9]

(i) From the Algorithm 1, by the definition of λ_k, for each $k = 1, 2, ...$

$$\mathcal{A}(\boldsymbol{x}^{(k)})^{m-1} \ge \lambda_k(\boldsymbol{x}^{(k)})^{[m-1]},$$

and since we choose initial $\boldsymbol{x}^{(0)} > \boldsymbol{0}$,

$$\mathcal{A}(\boldsymbol{x}^{(k)})^{m-1} \ge \underline{\lambda}_k(\boldsymbol{x}^{(k)})^{[m-1]} > 0.$$

We have from (2.13), $\boldsymbol{y}^{(k)} = \mathcal{A}(\boldsymbol{x}^{(k)})^{m-1}$.

$$(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]} = (\mathcal{A}(\boldsymbol{x}^{(k)})^{m-1})^{[\frac{1}{m-1}]} \ge (\underline{\lambda}_k(\boldsymbol{x}^{(k)})^{[m-1]})^{[\frac{1}{m-1}]} = \underline{\lambda}_k^{[\frac{1}{m-1}]}\boldsymbol{x}^{(k)} > 0.$$

Also from (2.13),

$$\boldsymbol{x}^{(k+1)} = \frac{(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}}{||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||} \geq \frac{\underline{\lambda}_k^{[\frac{1}{m-1}]}\boldsymbol{x}^{(k)}}{||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||} > 0. \tag{2.16}$$

By Lemma 3,

$$\begin{aligned}
\mathcal{A}(\boldsymbol{x}^{(k+1)})^{m-1} &\geq \mathcal{A}\left(\frac{\underline{\lambda}_k^{[\frac{1}{m-1}]}\boldsymbol{x}^{(k)}}{||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||}\right)^{m-1} = \frac{\underline{\lambda}_k\mathcal{A}(\boldsymbol{x}^{(k)})^{m-1}}{||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||^{[m-1]}}, \\
&= \frac{\underline{\lambda}_k\boldsymbol{y}^{(k)}}{||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||^{[m-1]}} = \frac{\underline{\lambda}_k\left((\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}\right)^{[m-1]}}{||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||^{[m-1]}}, \\
&= \underline{\lambda}_k(\boldsymbol{x}^{(k+1)})^{[m-1]}.
\end{aligned}$$

Thus we have $\mathcal{A}(\boldsymbol{x}^{(k+1)})^{m-1} \geq \underline{\lambda}_k(\boldsymbol{x}^{(k+1)})^{[m-1]}$, that is, for each $i = 1, 2, ..., n$,

$$\underline{\lambda}_k \leq \frac{\left(\mathcal{A}(\boldsymbol{x}^{(k+1)})^{m-1}\right)_i}{(x_i^{(k+1)})^{[m-1]}}.$$

We know that $\underline{\lambda}_{k+1} = \left(\mathcal{A}(\boldsymbol{x}^{(k+1)})^{m-1}\right)_i/(x_i^{(k+1)})^{[m-1]}$, then we can conclude

$$\underline{\lambda}_k \leq \underline{\lambda}_{k+1}.$$

Similarly, we can show $\bar{\lambda}_{k+1} \leq \bar{\lambda}_k$.

(ii) We have proved that $\underline{\lambda}_k \leq \underline{\lambda}_{k+1}$ and $\bar{\lambda}_{k+1} \leq \bar{\lambda}_k$. By Theorem 10,

$$\begin{aligned}
\underline{\lambda}_k &\leq \lambda_0 \leq \bar{\lambda}_k, \\
\underline{\lambda}_k \leq \underline{\lambda}_{k+1} \leq ... \leq \underline{\lambda} &\leq \lambda_0 \leq \bar{\lambda} \leq ... \leq \bar{\lambda}_{k+1} \leq \bar{\lambda}_k, \\
\underline{\lambda} &\leq \lambda_0 = \rho(\mathcal{A}) \leq \bar{\lambda}.
\end{aligned}$$

(iii) Since each of $\boldsymbol{x}^{(k)}$ is a unit vector, then the sequence $\{\boldsymbol{x}^{(k)}\}$ is bounded. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

(iv) We have from Algorithm 1

$$\begin{aligned}
\boldsymbol{y}^{(k)} = \mathcal{A}(\boldsymbol{x}^{(k)})^{m-1} &\geq \lambda_k(\boldsymbol{x}^{(k)})^{[m-1]}, \\
\left(\boldsymbol{y}^{(k)}\right)^{[\frac{1}{m-1}]} = \left(\mathcal{A}(\boldsymbol{x}^{(k)})^{m-1}\right)^{[\frac{1}{m-1}]} &\geq \lambda_k^{[\frac{1}{m-1}]}(\boldsymbol{x}^{(k)}).
\end{aligned}$$

From Definition 9, $\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^{(k)} = \left(\mathcal{A}(\boldsymbol{x}^{(k)})^{m-1}\right)^{[\frac{1}{m-1}]}$,

$$\underline{\lambda}_k^{[\frac{1}{m-1}]}(\boldsymbol{x}^{(k)}) \leq \left(\boldsymbol{y}^{(k)}\right)^{[\frac{1}{m-1}]} = \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^{(k)},$$

and using the same argument, we can get

$$\bar{\lambda}_k^{[\frac{1}{m-1}]}(\boldsymbol{x}^{(k)}) \geq \left(\boldsymbol{y}^{(k)}\right)^{[\frac{1}{m-1}]} = \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^{(k)}.$$

As $\boldsymbol{x}^{(k)} \to \boldsymbol{x}^*$,

$$\underline{\lambda}_k^{[\frac{1}{m-1}]}(\boldsymbol{x}^*) \leq \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^* \leq \bar{\lambda}_k^{[\frac{1}{m-1}]}(\boldsymbol{x}^*).$$

(v) We prove this statement by contradiction. Suppose that there is a positive integer k such that

$$\boldsymbol{T}_{\mathcal{A}}^{k+1}\boldsymbol{x}^\star > \underline{\lambda}^{[\frac{1}{m-1}]}\boldsymbol{T}_{\mathcal{A}}^{k}\boldsymbol{x}^\star.$$

Then there exists $\boldsymbol{x}^{(p)}$ close enough to $\boldsymbol{x}^\star$ such that

$$\boldsymbol{T}_{\mathcal{A}}^{k+1}\boldsymbol{x}^{(p)} > \underline{\lambda}^{[\frac{1}{m-1}]}\boldsymbol{T}_{\mathcal{A}}^{k}\boldsymbol{x}^{(p)}.$$

Now we consider for all i,

$$\frac{(\boldsymbol{T}_{\mathcal{A}}^{k+1}\boldsymbol{x}^{(p)})_i}{(\boldsymbol{T}_{\mathcal{A}}^{k}\boldsymbol{x}^{(p)})_i} > \underline{\lambda}^{[\frac{1}{m-1}]}.$$

However, we can show that

$$\frac{(\boldsymbol{T}_{\mathcal{A}}^{k+1}\boldsymbol{x}^{(p)})_i}{(\boldsymbol{T}_{\mathcal{A}}^{k}\boldsymbol{x}^{(p)})_i} = \left(\frac{y_i^{(p+k)}}{(x_i^{(p+k)})^{[m-1]}}\right)^{[\frac{1}{m-1}]}. \tag{2.17}$$

From Algorithm 1, $\frac{y_i^{(k)}}{(x_i^{(k)})^{[m-1]}} \geq \underline{\lambda}_k$. This means that there exist some i_0 such that

$$\frac{(\boldsymbol{T}_{\mathcal{A}}^{k+1}\boldsymbol{x}^{(p)})_{i_0}}{(\boldsymbol{T}_{\mathcal{A}}^{k}\boldsymbol{x}^{(p)})_{i_0}} = \left(\underline{\lambda}_{(p+k)}\right)^{[\frac{1}{m-1}]}.$$

This contradicts the assumption $\boldsymbol{T}_{\mathcal{A}}^{k+1}\boldsymbol{x}^\star > \underline{\lambda}^{[\frac{1}{m-1}]}\boldsymbol{T}_{\mathcal{A}}^{k}\boldsymbol{x}^\star$.

To show (2.17) in details; from Algorithm 1, at k-th iteration, we have

$$\begin{aligned}
\boldsymbol{y}^{(k)} &= \mathcal{A}(\boldsymbol{x}^{(k)})^{m-1}, \\
\boldsymbol{x}^{(k+1)} &= \frac{(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}}{||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||}, \\
\boldsymbol{y}^{(k+1)} &= \mathcal{A}(\boldsymbol{x}^{(k+1)})^{m-1}, \\
&\vdots
\end{aligned}$$

From the above equation,

$$\begin{aligned}\boldsymbol{x}^{(k+1)} &= \frac{(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}}{||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||},\\ ||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||\boldsymbol{x}^{(k+1)} &= (\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}. \end{aligned} \tag{2.18}$$

Now we put (2.18) as $\boldsymbol{x}$ into $\mathcal{A}\boldsymbol{x}^{m-1}$ and we will have

$$\begin{aligned}\mathcal{A}(||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||\boldsymbol{x}^{(k+1)})^{m-1} &= \mathcal{A}((\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]})^{m-1},\\ ||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||^{[m-1]}\mathcal{A}(\boldsymbol{x}^{(k+1)})^{m-1} &= \mathcal{A}((\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]})^{m-1},\\ ||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||^{[m-1]}\boldsymbol{y}^{(k+1)} &= \mathcal{A}((\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]})^{m-1}. \end{aligned} \tag{2.19}$$

We also have $\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^{(p)} = [\mathcal{A}(\boldsymbol{x}^{(p)})^{m-1}]^{[\frac{1}{m-1}]} = (\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}$ and

$$\begin{aligned}\boldsymbol{T}^2_{\mathcal{A}}\boldsymbol{x}^{(p)} &= \boldsymbol{T}_{\mathcal{A}}(\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^{(p)}),\\ &= \boldsymbol{T}_{\mathcal{A}}((\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}),\\ &= (\mathcal{A}((\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]})^{m-1})^{[\frac{1}{m-1}]},\\ &= (||(\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}||^{[m-1]}\boldsymbol{y}^{(p+1)})^{[\frac{1}{m-1}]},\\ &= ||(\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}||(\boldsymbol{y}^{(p+1)})^{[\frac{1}{m-1}]}. \end{aligned}$$

Using the same argument as above,

$$\begin{aligned}\boldsymbol{T}^3_{\mathcal{A}}\boldsymbol{x}^{(p)} &= \boldsymbol{T}_{\mathcal{A}}(\boldsymbol{T}^2_{\mathcal{A}}\boldsymbol{x}^{(p)}),\\ &= ||(\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}||\ ||(\boldsymbol{y}^{(p+1)})^{[\frac{1}{m-1}]}||(\boldsymbol{y}^{(p+2)})^{[\frac{1}{m-1}]}, \end{aligned}$$

and

$$\boldsymbol{T}^k_{\mathcal{A}}\boldsymbol{x}^{(p)} = ||(\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}||\ ||(\boldsymbol{y}^{(p+1)})^{[\frac{1}{m-1}]}||\cdots||(\boldsymbol{y}^{(p+k-2)})^{[\frac{1}{m-1}]}||(\boldsymbol{y}^{(p+k-1)})^{[\frac{1}{m-1}]}, \tag{2.20}$$

$$\begin{aligned}\boldsymbol{T}^{k+1}_{\mathcal{A}}\boldsymbol{x}^{(p)} &= ||(\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}||\ ||(\boldsymbol{y}^{(p+1)})^{[\frac{1}{m-1}]}||\cdots\\ &\quad ||(\boldsymbol{y}^{(p+k-2)})^{[\frac{1}{m-1}]}||\ ||(\boldsymbol{y}^{(p+k-1)})^{[\frac{1}{m-1}]}||(\boldsymbol{y}^{(p+k)})^{[\frac{1}{m-1}]}. \end{aligned}$$

From Algorithm 1, we have $\boldsymbol{x}^{(k+1)} = (\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}/||(\boldsymbol{y}^{(k)})^{[\frac{1}{m-1}]}||$. Shift the indices so that $\boldsymbol{x}^{(k)} = (\boldsymbol{y}^{(k-1)})^{[\frac{1}{m-1}]}/||(\boldsymbol{y}^{(k-1)})^{[\frac{1}{m-1}]}||$. Then $\boldsymbol{x}^{(p+k)} = (\boldsymbol{y}^{(p+k-1)})^{[\frac{1}{m-1}]}/||(\boldsymbol{y}^{(p+k-1)})^{[\frac{1}{m-1}]}||$ and

$$(\boldsymbol{y}^{(p+k-1)})^{[\frac{1}{m-1}]} = ||(\boldsymbol{y}^{(p+k-1)})^{[\frac{1}{m-1}]}||\boldsymbol{x}^{(p+k)}. \qquad (2.21)$$

By (2.21), equation (2.20) becomes

$$\boldsymbol{T}_{\mathcal{A}}^k \boldsymbol{x}^{(p)} = ||(\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}|| \; ||(\boldsymbol{y}^{(p+1)})^{[\frac{1}{m-1}]}||...||(\boldsymbol{y}^{(p+k-2)})^{[\frac{1}{m-1}]}|| \; ||(\boldsymbol{y}^{(p+k-1)})^{[\frac{1}{m-1}]}||\boldsymbol{x}^{(p+k)}.$$

Thus we have the ratio

$$\begin{aligned}
&\frac{(\boldsymbol{T}_{\mathcal{A}}^{k+1}\boldsymbol{x}^{(p)})_i}{(\boldsymbol{T}_{\mathcal{A}}^{k}\boldsymbol{x}^{(p)})_i} \\
= &\frac{||(\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}|| \; ||(\boldsymbol{y}^{(p+1)})^{[\frac{1}{m-1}]}||...||(\boldsymbol{y}^{(p+k-2)})^{[\frac{1}{m-1}]}|| \; ||(\boldsymbol{y}^{(p+k-1)})^{[\frac{1}{m-1}]}||(\boldsymbol{y}^{(p+k)})_i^{[\frac{1}{m-1}]}}{||(\boldsymbol{y}^{(p)})^{[\frac{1}{m-1}]}|| \; ||(\boldsymbol{y}^{(p+1)})^{[\frac{1}{m-1}]}||...||(\boldsymbol{y}^{(p+k-2)})^{[\frac{1}{m-1}]}|| \; ||(\boldsymbol{y}^{(p+k-1)})^{[\frac{1}{m-1}]}||x_i^{(p+k)}} \\
= &\frac{(\boldsymbol{y}^{(p+k)})_i^{[\frac{1}{m-1}]}}{x_i^{(p+k)}} \\
= &\left(\frac{(\boldsymbol{y}^{(p+k)})_i}{(\boldsymbol{x}^{(p+k)})_i^{[m-1]}}\right)^{[\frac{1}{m-1}]}.
\end{aligned}$$

□

Chang *et al.* [9] gave the following results for primitive tensors. These results show that the Algorithm 1 is convergent for primitive tensors.

Let $\boldsymbol{X}$ be a Banach space.

Definition 11. [9] A mapping $\boldsymbol{T} : \boldsymbol{X} \to \boldsymbol{X}$ is called strongly increasing if $\boldsymbol{x} \le \boldsymbol{y}$ and $\boldsymbol{x} \ne \boldsymbol{y}$ imply $\boldsymbol{Tx} < \boldsymbol{Ty}$.

Theorem 17. [9] Let $\mathcal{A}$ be a nonnegative tensor of order m and dimension n. $\mathcal{A}$ is primitive if and only if there exists $r \in \mathbb{N}$ such that $\boldsymbol{T}_{\mathcal{A}}^r$ is strongly increasing.

Theorem 18. [9] If $\mathcal{A}$ is primitive, then $\underline{\lambda} = \rho(\mathcal{A}) = \bar{\lambda}$ and $\boldsymbol{x}^{(k)} \to \boldsymbol{x}^*$, i.e., $\boldsymbol{x}^{(k)}$ converges to the positive eigenvector with respect to $\rho(\mathcal{A})$.

Proof. [9] For the first step, we are going to show that $\underline{\lambda} = \rho(\mathcal{A}) = \bar{\lambda}$. We have $\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x} = (\mathcal{A}\boldsymbol{x}^{m-1})^{[\frac{1}{m-1}]}$ and we know that $(\underline{\lambda})^{\frac{1}{m-1}}\boldsymbol{x}^* \le \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^*$. Suppose $(\underline{\lambda})^{\frac{1}{m-1}}\boldsymbol{x}^* \ne \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^*$. By Theorem 17, since $\mathcal{A}$ is primitive, there exists $r \in \mathbb{N}$ such that $\boldsymbol{T}_{\mathcal{A}}^r$ is strongly increasing. By Definition 11, $(\underline{\lambda})^{\frac{1}{m-1}}\boldsymbol{x}^* \le \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^*$ and $(\underline{\lambda})^{\frac{1}{m-1}}\boldsymbol{x}^* \ne \boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^*$ imply $\boldsymbol{T}_{\mathcal{A}}^r((\underline{\lambda})^{\frac{1}{m-1}}\boldsymbol{x}^*) < \boldsymbol{T}_{\mathcal{A}}^r(\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}^*)$. From Definition 9,

$$\begin{aligned}
\boldsymbol{T}_{\mathcal{A}}^2\boldsymbol{x} &= \boldsymbol{T}_{\mathcal{A}}(\boldsymbol{T}_{\mathcal{A}}\boldsymbol{x}), \\
&\vdots \\
\boldsymbol{T}_{\mathcal{A}}^r\boldsymbol{x} &= \boldsymbol{T}_{\mathcal{A}}(\boldsymbol{T}_{\mathcal{A}}^{r-1}\boldsymbol{x}).
\end{aligned}$$

Thus we have $(\underline{\lambda})^{\frac{1}{m-1}} \boldsymbol{T}^r_{\mathcal{A}} \boldsymbol{x}^* < \boldsymbol{T}^{r+1}_{\mathcal{A}} \boldsymbol{x}^*$. However this contradicts statement (5) of Proposition 4 which states for all $k \in \mathbb{N}$, there exists $1 \leq i_0 \leq n$ such that $(\boldsymbol{T}^{k+1}_{\mathcal{A}} \boldsymbol{x}^*)_{i_0} = \underline{\lambda}^{\frac{1}{(m-1)}} (\boldsymbol{T}^k_{\mathcal{A}} \boldsymbol{x}^*)_{i_0}$. Hence we must have $(\underline{\lambda})^{\frac{1}{m-1}} \boldsymbol{x}^* = \boldsymbol{T}_{\mathcal{A}} \boldsymbol{x}^*$. Similarly, we can show $(\bar{\lambda})^{\frac{1}{m-1}} \boldsymbol{x}^* = \boldsymbol{T}_{\mathcal{A}} \boldsymbol{x}^*$. Then,

$$\begin{aligned} (\underline{\lambda})^{\frac{1}{m-1}} \boldsymbol{x}^* &= \boldsymbol{T}_{\mathcal{A}} \boldsymbol{x}^* = (\bar{\lambda})^{\frac{1}{m-1}} \boldsymbol{x}^*, \\ \underline{\lambda} &= \bar{\lambda}. \end{aligned}$$

Thus $\rho(\mathcal{A}) = \underline{\lambda} = \bar{\lambda}$.

For the next step, we are going to prove $\boldsymbol{x}^{(k)} \to \boldsymbol{x}^*$ by contradiction. Suppose there exists $\{r_k\}$ such that $\boldsymbol{x}^{(r_k)} \to \boldsymbol{y}^*$ with $||\boldsymbol{y}^*|| = 1$ and $\boldsymbol{y}^* \neq \boldsymbol{x}^*$. Using the same approach as previous step,

$$\underline{\lambda}^{\frac{1}{m-1}} \boldsymbol{y}^* = \boldsymbol{T}_{\mathcal{A}} \boldsymbol{y}^* = \boldsymbol{y}^* \bar{\lambda}^{\frac{1}{m-1}} \boldsymbol{y}^*,$$
$$\underline{\lambda} = \bar{\lambda} = \rho(\mathcal{A}).$$

Hence

$$\begin{aligned} \rho(\mathcal{A})^{\frac{1}{m-1}} \boldsymbol{y}^* &= \boldsymbol{T}_{\mathcal{A}} \boldsymbol{y}^*, \\ \rho(\mathcal{A})^{\frac{1}{m-1}} \boldsymbol{y}^* &= (\mathcal{A}(\boldsymbol{y}^*)^{m-1})^{[\frac{1}{m-1}]}, \\ \rho(\mathcal{A})(\boldsymbol{y}^*)^{[m-1]} &= \mathcal{A}(\boldsymbol{y}^*)^{m-1}, \end{aligned}$$

which means by the definition of eigenvector and eigenvalue of tensors, $\boldsymbol{y}^*$ is the associated eigenvector of $\rho(\mathcal{A})$. However, by the uniqueness of the positive eigenvector with eigenvalue $\rho(\mathcal{A})$ in Perron-Frobenius Theorem, it means $\boldsymbol{y}^* = \boldsymbol{x}^*$.

□

Consequently, the following result was produced.

Corollary 4. [9] If $\mathcal{A}$ is an essentially positive tensor of order m and dimension n, then the Algorithm 1 converges.

Later, Algorithm 1 was modified so that it is convergent if $\mathcal{A}$ is an irreducible nonnegative tensor [38]. The modified algorithm is as follows:

Algorithm 2. [38]

Step 0: Choose $\boldsymbol{x}^{(1)} > \mathbf{0}$, $\boldsymbol{x}^{(1)} \in \mathbb{R}^n$ and $\rho > 0$. Let $\mathcal{B} = \mathcal{A} + \rho\mathcal{I}$, and set $k := 1$.

Step 1: Compute

$$\begin{aligned} \boldsymbol{y}^{(k)} &= \mathcal{B}\left(\boldsymbol{x}^{(k)}\right)^{m-1}, \\ \underline{\lambda}_k &= \min_{x_i^{(k)}>0} \frac{\left(\boldsymbol{y}^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}}, \\ \bar{\lambda}_k &= \max_{x_i^{(k)}>0} \frac{\left(\boldsymbol{y}^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}}. \end{aligned}$$

Step 2: If $\bar{\lambda}_k = \underline{\lambda}_k$, then let $\lambda = \bar{\lambda}_k$ and stop. Otherwise, compute

$$\boldsymbol{x}^{(k+1)} = \frac{\left(\boldsymbol{y}^{(k)}\right)^{\left[\frac{1}{m-1}\right]}}{\left\|\left(\boldsymbol{y}^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right\|},$$

replace k by $k+1$ and go to Step 1.

Theorem 19. [38] Suppose $\mathcal{A}$ is an irreducible nonnegative tensor. Let $\mathcal{B} = \mathcal{A} + \rho\mathcal{I}$, where $\rho > 0$. Assume that λ is the largest eigenvalue of $\mathcal{B}$. Then, Algorithm 2 produces a value of λ in a finite number of steps, or generates two sequences $\{\underline{\lambda}_k\}$ and $\{\bar{\lambda}_k\}$ which converge to λ. Furthermore, $\lambda - \rho$ is the largest eigenvalue of $\mathcal{A}$.

Algorithm 1 may not converge for some irreducible nonnegative tensors, but Algorithm 2 always converges for irreducible nonnegative tensors.

2.6 Convergence Rate

In this section, we further study the convergence rate of Algorithm 2. For simplicity, we can take $\rho = 1$. Thus we have $\mathcal{B} = \mathcal{A} + \mathcal{I}$. Under weak irreducibility condition, Algorithm 2 is Q-linear convergent [78].

Theorem 20. [78] Suppose nonnegative tensor $\mathcal{A}$ is weakly irreducible. Let $\mathcal{B} = \mathcal{A} + \mathcal{I}$ and assume that λ is the largest eigenvalue of $\mathcal{B}$. Then, Algorithm 2 produces a value of λ and a corresponding eigenvector $\boldsymbol{u}$ in a finite number of steps, or generates three convergent sequences $\{\underline{\lambda}_k\}, \{\bar{\lambda}_k\}$ and $\{\boldsymbol{x}^{(k)}\}$ such that

$\lim_{k\to\infty} \underline{\lambda}_k = \lim_{k\to\infty} \bar{\lambda}_k = \lambda$ and $\lim_{k\to\infty} \boldsymbol{x}^{(k)} = \boldsymbol{u}$. Furthermore, $\lambda - 1$ is the largest eigenvalue of $\mathcal{A}$ associated with the eigenvector $\boldsymbol{u}$.

If we let

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{x}) &= \mathcal{B}\boldsymbol{x}^{m-1}, \\ \boldsymbol{G}(\boldsymbol{x}) &= \boldsymbol{F}(\boldsymbol{x})^{[\frac{1}{m-1}]}, \\ \boldsymbol{H}(\boldsymbol{x}) &= \frac{\boldsymbol{G}(\boldsymbol{x})}{\phi(\boldsymbol{G}(\boldsymbol{x}))}, \end{aligned} \tag{2.22}$$

where $\phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ is defined by

$$\phi(\boldsymbol{x}) = ||\boldsymbol{x}||_1 = \Sigma_{i=1}^n x_i, \tag{2.23}$$

for any nonnegative $\boldsymbol{x} \in \mathbb{R}^n$. We can see that sequence $\{\boldsymbol{x}^{(k)}\}$ in Theorem 20 is generated by

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{H}(\boldsymbol{x}^{(k)}), \quad k = 1, 2, ... \tag{2.24}$$

and $\phi(\boldsymbol{x}^{(k)}) = 1$ for all $k = 1, 2, ...$

Lemma 4. [78] Let $\mathcal{A}$ and $\mathcal{B}$ as in Theorem 20. For any $\boldsymbol{x} \in \mathbb{R}^n_{>0}$, $\boldsymbol{F}'(\boldsymbol{x})$, the Jacobian of $\boldsymbol{F}$ at $\boldsymbol{x}$, is a primitive matrix.

Proof. [78] Let $\boldsymbol{x} \in \mathbb{R}^n_{>0}$ and $\boldsymbol{M} = \boldsymbol{F}'(x)$. We know that $\boldsymbol{F}(\boldsymbol{x}) = \mathcal{B}\boldsymbol{x}^{m-1}$. Since $x_i > 0$ for all $i = 1, 2, ..., n$, $\boldsymbol{M}$ depends on $\mathcal{B}$. The matrix $M_{i,j} > 0$ for any $i, j = 1, 2, ..., n$ if and only if $b_{ii_2...i_m} \neq 0$ for some $i_l = j$, $l = 2, 3, ..., m$. From definition of graph of tensor, $\mathcal{G}(\boldsymbol{M}) = \mathcal{G}(\mathcal{B})$. Since $\mathcal{B}$ is primitive and by Theorem 12, $\mathcal{B}$ is irreducible. Since $\mathcal{B}$ is irreducible then $\mathcal{B}$ is weakly irreducible. By Definition 5, $\mathcal{G}(\mathcal{B})$ is strongly connected since $\mathcal{B}$ is weakly irreducible. Hence graph of matrix $\boldsymbol{M}$ is also strongly connected and therefore $\boldsymbol{M}$ is irreducible. $\mathcal{B} = \mathcal{A} + \mathcal{I}$ which means $b_{ii...i} \neq 0$ for $i = 1, 2, ..., n$. Thus we have $M_{ii} \neq 0$. Since $\boldsymbol{M}$ is an irreducible matrix with nonzero main diagonal , it is primitive. □

Lemma 5. [78] Let $\mathcal{A}$, $\mathcal{B}$, λ and $\boldsymbol{u}$ be as in Theorem 20, and let $\boldsymbol{H}'(\boldsymbol{u})$ be the Jacobian of the function $\boldsymbol{H}$ at $\boldsymbol{u}$. Then,

$$\rho(\boldsymbol{H}'(\boldsymbol{u})) < 1. \tag{2.25}$$

Proof. [78] Let λ be the largest eigenvalue of $\mathcal{B}$ and $\boldsymbol{u}$ be the corresponding eigenvector. We have $\boldsymbol{H}(\boldsymbol{u}) = G(\boldsymbol{u})/\phi(\boldsymbol{G}(\boldsymbol{u}))$. We want to show

$$\rho(\boldsymbol{H}'(\boldsymbol{u})) = \rho(\frac{\boldsymbol{G}'(\boldsymbol{u})\phi(\boldsymbol{G}(\boldsymbol{u})) - \boldsymbol{G}(\boldsymbol{u})\phi'(\boldsymbol{G}(\boldsymbol{u}))}{\phi^2(\boldsymbol{G}(\boldsymbol{u}))}) < 1.$$

We have $\boldsymbol{F}(\boldsymbol{u}) = \mathcal{B}\boldsymbol{u}^{m-1} = \lambda\boldsymbol{u}^{[m-1]}$ and $\phi(\boldsymbol{u}) = 1$. Hence, from (2.22), $\boldsymbol{G}(\boldsymbol{u}) = (\boldsymbol{F}(\boldsymbol{u}))^{[\frac{1}{m-1}]} = \lambda^{[\frac{1}{m-1}]}\boldsymbol{u}$. Let $\lambda_1 = \lambda^{[\frac{1}{m-1}]}$ so we have

$$\boldsymbol{G}(\boldsymbol{u}) = \lambda_1\boldsymbol{u}. \tag{2.26}$$

Now we compute $\boldsymbol{G}'(\boldsymbol{u})$, the Jacobian of $\boldsymbol{G}$ at $\boldsymbol{u}$.

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{u}) &= (\boldsymbol{F}(\boldsymbol{u}))^{[\frac{1}{m-1}]}, \\ &= \begin{bmatrix} (F_1(\boldsymbol{u}))^{[\frac{1}{m-1}]} \\ (F_2(\boldsymbol{u}))^{[\frac{1}{m-1}]} \\ \vdots \\ (F_n(\boldsymbol{u}))^{[\frac{1}{m-1}]} \end{bmatrix},\end{aligned}$$

$$\nabla((F_1(\boldsymbol{u}))^{[\frac{1}{m-1}]}) = \frac{1}{m-1}(F_1(\boldsymbol{u}))^{[\frac{2-m}{m-1}]}\nabla F_1(\boldsymbol{u}).$$

By the same method, we can get

$$\nabla((F_i(\boldsymbol{u}))^{[\frac{1}{m-1}]}) = \frac{1}{m-1}(F_i(\boldsymbol{u}))^{[\frac{2-m}{m-1}]}\nabla F_i(\boldsymbol{u}) \quad for \quad i = 1, 2, ..., n.$$

Thus the Jacobian of $\boldsymbol{G}$ at $\boldsymbol{u}$,

$$\begin{aligned}
\boldsymbol{G}'(\boldsymbol{u}) &= \nabla((\boldsymbol{F}(\boldsymbol{u}))^{[\frac{1}{m-1}]}) = \begin{pmatrix} \nabla((F_1(\boldsymbol{u}))^{[\frac{1}{m-1}]}) \\ \nabla((F_2(\boldsymbol{u}))^{[\frac{1}{m-1}]}) \\ \vdots \\ \nabla((F_n(\boldsymbol{u}))^{[\frac{1}{m-1}]}) \end{pmatrix} \\
&= \begin{bmatrix} \frac{1}{m-1}(F_1(\boldsymbol{u}))^{[\frac{2-m}{m-1}]}\nabla F_1(\boldsymbol{u}) \\ \frac{1}{m-1}(F_2(\boldsymbol{u}))^{[\frac{2-m}{m-1}]}\nabla F_2(\boldsymbol{u}) \\ \vdots \\ \frac{1}{m-1}(F_n(\boldsymbol{u}))^{[\frac{2-m}{m-1}]}\nabla F_n(\boldsymbol{u}) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{m-1}(F_1(\boldsymbol{u}))^{[\frac{2-m}{m-1}]} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{m-1}(F_n(\boldsymbol{u}))^{[\frac{2-m}{m-1}]} \end{bmatrix} \begin{bmatrix} \nabla F_1(\boldsymbol{u}) \\ \nabla F_2(\boldsymbol{u}) \\ \vdots \\ \nabla F_n(\boldsymbol{u}) \end{bmatrix} \\
&= diag(\frac{1}{m-1}(F(\boldsymbol{u}))^{[\frac{2-m}{m-1}]})F'(\boldsymbol{u}) \\
&= diag(\frac{1}{m-1}(\lambda \boldsymbol{u}^{[m-1}]))^{[\frac{2-m}{m-1}]})\boldsymbol{F}'(\boldsymbol{u}) \\
&= \frac{1}{m-1}diag((\lambda \boldsymbol{u}^{[m-1]})^{[\frac{2-m}{m-1}]})\boldsymbol{F}'(\boldsymbol{u}) \\
&= \frac{1}{m-1}diag((\lambda^{\frac{1}{[m-1]}}\boldsymbol{u})^{[2-m]})\boldsymbol{F}'(\boldsymbol{u}), \\
&= \frac{1}{m-1}diag((\lambda_1 \boldsymbol{u})^{[2-m]})\boldsymbol{F}'(\boldsymbol{u}).
\end{aligned}$$

Notice that $\frac{1}{m-1}diag((\lambda_1 \boldsymbol{u})^{[2-m]})$ is a constant with $\lambda_1 > 0$ and $\boldsymbol{u}$ is a positive vector. Thus $\mathcal{G}(\boldsymbol{G}'(\boldsymbol{u})) = \mathcal{G}(\boldsymbol{F}'(\boldsymbol{u}))$. By Lemma 4, $\boldsymbol{G}'(\boldsymbol{u})$ is a primitive matrix.

Since $\boldsymbol{G}'(\boldsymbol{u})$ is a primitive matrix, by Theorem 7 , the eigenvalues $v_1, v_2, ..., v_n$ of $\boldsymbol{G}'(\boldsymbol{u})$ can be ordered in such a way that

$$v_1 = \rho(\boldsymbol{G}'(\boldsymbol{u})) > |v_2| \geq |v_3| \geq ... \geq |v_n|. \tag{2.27}$$

For all $t > 1$, we expand $\boldsymbol{G}(t\boldsymbol{u})$ about $\boldsymbol{u}$ using Taylor's Series,

$$
\begin{aligned}
t\lambda_1 \boldsymbol{u} &= \boldsymbol{G}(t\boldsymbol{u}) \\
&= \boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{G}'(\boldsymbol{u})(t\boldsymbol{u} - \boldsymbol{u}) + o(||t\boldsymbol{u} - \boldsymbol{u}||) \\
&= \boldsymbol{G}(\boldsymbol{u}) + (t-1)\,\boldsymbol{G}'(\boldsymbol{u})\boldsymbol{u} + o(||(t-1)\boldsymbol{u}||) \\
&= \boldsymbol{G}(\boldsymbol{u}) + (t-1)\,\boldsymbol{G}'(\boldsymbol{u})\boldsymbol{u} + o(t-1) \\
&= \lambda_1 \boldsymbol{u} + (t-1)\,\boldsymbol{G}'(\boldsymbol{u})\boldsymbol{u} + o(t-1), \\
t\lambda_1 \boldsymbol{u} - \lambda_1 \boldsymbol{u} &= (t-1)\,\boldsymbol{G}'(\boldsymbol{u})\boldsymbol{u} + o(t-1), \\
(t-1)\lambda_1 \boldsymbol{u} &= (t-1)\,\boldsymbol{G}'(\boldsymbol{u})\boldsymbol{u} + o(t-1),
\end{aligned}
$$

which implies $\boldsymbol{G}'(\boldsymbol{u})\boldsymbol{u} = \lambda_1 \boldsymbol{u}$. Since $\boldsymbol{G}'(\boldsymbol{u})$ is primitive and $\boldsymbol{u} > \boldsymbol{0}$, by Theorem 7, $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{G}'(\boldsymbol{u})$ associated with the largest eigenvalue $\lambda_1 = v_1$.

We have from (2.23) and (2.26),

$$
\begin{aligned}
\phi(\boldsymbol{G}(\boldsymbol{u})) &= \phi(\lambda_1 \;\boldsymbol{u}) \\
&= \sum_{i=1}^{n} (\lambda_1 u_i) \\
&= \lambda_1 \sum_{i=1}^{n} (u_i) \\
&= \lambda_1 \phi(\boldsymbol{u}) \\
&= \lambda_1 (1) \\
&= \lambda_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\phi(\boldsymbol{G}(\boldsymbol{u})) &= G_1(\boldsymbol{u}) + G_2(\boldsymbol{u}) + ... + G_n(\boldsymbol{u}), \\
\phi'(\boldsymbol{G}(\boldsymbol{u})) &= G_1'(\boldsymbol{u}) + G_2'(\boldsymbol{u}) + ... + G_n'(\boldsymbol{u}) \\
&= \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} G_1'(\boldsymbol{u}) \\ G_2'(\boldsymbol{u}) \\ \vdots \\ G_n'(\boldsymbol{u}) \end{pmatrix} \\
&= \boldsymbol{e} G'(\boldsymbol{u}),
\end{aligned}
$$

where $\boldsymbol{e}$ is the n-dimensional row vector of all ones.

Thus, from (2.22),

$$\begin{aligned}
\boldsymbol{H}(\boldsymbol{u}) &= \frac{\boldsymbol{G}(\boldsymbol{u})}{\phi(\boldsymbol{G}(\boldsymbol{u}))},\\
\boldsymbol{H}'(\boldsymbol{u}) &= \frac{\boldsymbol{G}'(\boldsymbol{u})\phi(\boldsymbol{G}(\boldsymbol{u})) - \boldsymbol{G}(\boldsymbol{u})\phi'(\boldsymbol{G}(\boldsymbol{u}))}{\phi^2(\boldsymbol{G}(\boldsymbol{u}))}\\
&= \frac{\boldsymbol{G}'(\boldsymbol{u})\lambda_1 - \boldsymbol{G}(\boldsymbol{u})\phi'(\boldsymbol{G}(\boldsymbol{u}))}{\lambda_1^2}\\
&= \frac{\boldsymbol{G}'(\boldsymbol{u})\lambda_1 - \boldsymbol{G}(\boldsymbol{u})\boldsymbol{e}\,\boldsymbol{G}'(\boldsymbol{u})}{\lambda_1^2}\\
&= \frac{\boldsymbol{G}'(\boldsymbol{u})}{\lambda_1} - \frac{\boldsymbol{G}(\boldsymbol{u})\boldsymbol{e}\,\boldsymbol{G}'(\boldsymbol{u})}{\lambda_1^2}\\
&= \frac{\boldsymbol{G}'(\boldsymbol{u})}{\lambda_1} - \frac{\lambda_1\boldsymbol{u}\boldsymbol{e}\,\boldsymbol{G}'(\boldsymbol{u})}{\lambda_1^2}\\
&= \frac{\boldsymbol{G}'(\boldsymbol{u}) - \boldsymbol{u}\boldsymbol{e}\boldsymbol{G}'(\boldsymbol{u})}{\lambda_1}.
\end{aligned}$$

Let $\boldsymbol{M} = \boldsymbol{G}'(\boldsymbol{u})$ and $\boldsymbol{Q} = \boldsymbol{M} - \boldsymbol{ueM}$. Now we have $\boldsymbol{H}'(\boldsymbol{u}) = (\boldsymbol{Q}/\lambda_1)$. In this proof, we want to show

$$\rho(\boldsymbol{H}'(\boldsymbol{u})) = \rho(\frac{\boldsymbol{Q}}{\lambda_1}) < 1.$$

First, we show that the spectral radius of $\boldsymbol{Q}$ is equal to $|v_2|$. In order to do this, it is enough to show that the spectrum of $\boldsymbol{Q}$ is $\sigma(\boldsymbol{Q}) = \{0, v_2, v_3, ..., v_n\}$.

Since

$$\begin{aligned}
\phi(\boldsymbol{u}) &= 1\\
&= u_1 + u_2 + ... + u_n\\
&= \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}\\
&= \boldsymbol{eu},
\end{aligned}$$

hence $\boldsymbol{eu} = 1$, therefore

$$\begin{aligned}
\boldsymbol{Q} &= \boldsymbol{M} - \boldsymbol{ueM}, \\
\boldsymbol{Q}^T\boldsymbol{e}^T &= (\boldsymbol{M} - \boldsymbol{ueM})^T\boldsymbol{e}^T \\
&= \boldsymbol{Me}^T - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{u}^T\boldsymbol{e}^T \\
&= \boldsymbol{Me}^T - \boldsymbol{M}^T\boldsymbol{e}^T(\boldsymbol{eu})^T \\
&= \boldsymbol{Me}^T - \boldsymbol{M}^T\boldsymbol{e}^T(1) \\
&= 0.
\end{aligned}$$

Now we can say $\boldsymbol{e}^T$ is an eigenvector of $\boldsymbol{Q}^T$ associated with the eigenvalue 0.

We consider two cases for M^T.

Case 1: The matrix $\boldsymbol{M}^T = \boldsymbol{G}'(\boldsymbol{u})^T$ is diagonizable, which means, $\boldsymbol{M}^T$ is semisimple. For $i = 2, 3, ..., n$, let assume $\boldsymbol{M}^T\boldsymbol{w}^i = v_i\boldsymbol{w}^i$, that is $\boldsymbol{w}^i$ is an eigenvector of $\boldsymbol{M}^T$ associated with the eigenvalue v_i, and the set $\{\boldsymbol{w}^i : i = 2, 3, ..., n\}$ is linearly independent. Thus, for $i = 2, 3, ..., n$,

$$v_i\boldsymbol{u}^T\boldsymbol{w}^i = \boldsymbol{u}^T v_i\boldsymbol{w}^i = \boldsymbol{u}^T\boldsymbol{M}^T\boldsymbol{w}^i.$$

We have previously $\boldsymbol{G}'(\boldsymbol{u})\boldsymbol{u} = \boldsymbol{M}\boldsymbol{u} = \lambda_1\boldsymbol{u}$. So

$$\begin{aligned}
(\boldsymbol{M}\ \boldsymbol{u})^T &= (\lambda_1\boldsymbol{u})^T, \\
\boldsymbol{u}^T\boldsymbol{M}^T &= \lambda_1\boldsymbol{u}^T. \qquad (2.28)
\end{aligned}$$

Hence,

$$\begin{aligned}
v_i\boldsymbol{u}^T\boldsymbol{w}^i = \boldsymbol{u}^T\boldsymbol{M}^T\boldsymbol{w}^i &= \lambda_1\boldsymbol{u}^T\boldsymbol{w}^i, \\
(v_i - \lambda_1)\boldsymbol{u}^T\boldsymbol{w}^i &= 0.
\end{aligned}$$

It is either $v_i = \lambda_1$ or $\boldsymbol{u}^T\boldsymbol{w}^i = 0$ for $i = 2, 3, ..., n$. However, $v_i \neq \lambda_1$ for $i = 2, 3, ..., n$ by (2.27). So we must have $\boldsymbol{u}^T\boldsymbol{w}^i = 0$.

Now we have

$$\begin{aligned}
\boldsymbol{Q}^T\boldsymbol{w}^i &= (\boldsymbol{M} - \boldsymbol{ueM})^T\boldsymbol{w}^i \\
&= (\boldsymbol{M}^T - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{u}^T)\boldsymbol{w}^i \\
&= \boldsymbol{M}^T\boldsymbol{w}^i - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{u}^T\boldsymbol{w}^i \\
&= \boldsymbol{M}^T\boldsymbol{w}^i - \boldsymbol{0}.
\end{aligned}$$

Since we assume $\boldsymbol{M}^T \boldsymbol{w}^i = v_i \boldsymbol{w}^i$, we have $\boldsymbol{Q}^T \boldsymbol{w}^i = v_i \boldsymbol{w}^i$, $\boldsymbol{w}^i$ is an eigenvector of $\boldsymbol{Q}^T$ associated with the eigenvalue v_i for $i = 2, 3, ..., n$. Now we prove the set $\{\boldsymbol{e}^T, \boldsymbol{w}^i, i = 2, 3, ..., n\}$, which is eigenvectors of $\boldsymbol{Q}$ is linearly independent. Assume

$$\alpha_1 \boldsymbol{e}^T + \alpha_2 \boldsymbol{w}^2 + ... + \alpha_n \boldsymbol{w}^n = \boldsymbol{0}, \tag{2.29}$$

and $v_i \neq 0$ for $i = 2, 3, ..., p$ and $v_j = 0$ for $j = p+1, ..., n$. We have

$$\begin{aligned} \boldsymbol{Q}^T \boldsymbol{e}^T &= 0\boldsymbol{e}^T, \\ \boldsymbol{Q}^T \boldsymbol{w}^i &= v_i \boldsymbol{w}^i, \quad i = 2, 3, ..., n. \end{aligned}$$

Therefore

$$\begin{aligned} \boldsymbol{Q}^T \boldsymbol{e}^T + \boldsymbol{Q}^T \boldsymbol{w}^2 + ... + \boldsymbol{Q}^T \boldsymbol{w}^n &= 0\boldsymbol{e}^T + v_2 \boldsymbol{w}^2 + ... + v_p \boldsymbol{w}^p, \\ \alpha_1 \boldsymbol{Q}^T \boldsymbol{e}^T + \alpha_2 \boldsymbol{Q}^T \boldsymbol{w}^2 + ... + \alpha_n \boldsymbol{Q}^T \boldsymbol{w}^n &= \alpha_2 v_2 \boldsymbol{w}^2 + ... + \alpha_p v_p \boldsymbol{w}^p. \end{aligned} \tag{2.30}$$

From (2.29),

$$\begin{gathered} \boldsymbol{Q}^T(\alpha_1 \boldsymbol{e}^T + \alpha_2 \boldsymbol{w}^2 + ... + \alpha_n \boldsymbol{w}^n) = \boldsymbol{0}, \\ \alpha_1 \boldsymbol{Q}^T \boldsymbol{e}^T + \alpha_2 \boldsymbol{Q}^T \boldsymbol{w}^2 + ... + \alpha_n \boldsymbol{Q}^T \boldsymbol{w}^n = \boldsymbol{0}. \end{gathered}$$

Hence (2.30) became,

$$\boldsymbol{0} = \alpha_2 v_2 \boldsymbol{w}^2 + ... + \alpha_p v_p \boldsymbol{w}^p.$$

Since set $\{\boldsymbol{w}^i, i = 2, 3, ..., n\}$ is linearly independent, we obtain $\alpha_2 = \alpha_3 = ... = \alpha_p = 0$. Hence, by (2.29),

$$\alpha_1 \boldsymbol{e}^T + \alpha_{p+1} \boldsymbol{w}^{p+1} + ... + \alpha_n \boldsymbol{w}^n = \boldsymbol{0}, \tag{2.31}$$

$$\begin{gathered} \boldsymbol{M}^T(\alpha_1 \boldsymbol{e}^T + \alpha_{p+1} \boldsymbol{w}^{p+1} + ... + \alpha_n \boldsymbol{w}^n) = \boldsymbol{0}, \\ \alpha_1 \boldsymbol{M}^T \boldsymbol{e}^T + \alpha_{p+1} \boldsymbol{M}^T \boldsymbol{w}^{p+1} + ... + \alpha_n \boldsymbol{M}^T \boldsymbol{w}^n = \boldsymbol{0}. \end{gathered}$$

Since $\boldsymbol{M}^T \boldsymbol{w}^i = v_i \boldsymbol{w}^i$ and $v_j = 0$ for $j = p+1, ..., n$,

$$\alpha_1 \boldsymbol{M}^T \boldsymbol{e}^T + \alpha_{p+1} v_{p+1} \boldsymbol{w}^{p+1} + ... + \alpha_n v_n \boldsymbol{w}^n = \boldsymbol{0}, \tag{2.32}$$

$$\alpha_1 \boldsymbol{M}^T \boldsymbol{e}^T = \boldsymbol{0}. \tag{2.33}$$

We have $\alpha_1 = 0$ since $\boldsymbol{M}^T \boldsymbol{e}^T > \boldsymbol{0}$ because $\boldsymbol{M}$ is diagonalizable. Hence by (2.31),

$$\alpha_{p+1}\boldsymbol{w}^{p+1} + ... + \alpha_n \boldsymbol{w}^n = 0. \tag{2.34}$$

The set $\{\boldsymbol{w}^i, i = p+1, ..., n\}$ is linearly independent, hence we get $\alpha_{p+1} = ... = \alpha_n = 0$. Now we have $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ which implies that the set $\{\boldsymbol{e}^T, \boldsymbol{w}^i, i = 2, 3, ..., n\}$ is linearly independent. Therefore the spectrum of $\boldsymbol{Q}$ is $\sigma(\boldsymbol{Q}) = \{0, v_2, v_3, ..., v_n\}$.

Case 2: The matrix $\boldsymbol{M}^T$ is not diagonalizable or defective. Defective matrix has less than n distinct eigenvalues. Suppose $\boldsymbol{M}^T$ has distinct eigenvalues $v_1 = \lambda_1, v_2, ..., v_q, \quad q < n$, and these eigenvalues can be ordered as follow.

$$v_1 = \lambda_1 > |v_2| \geq |v_3| \geq ... \geq |v_q|. \tag{2.35}$$

Then, $\boldsymbol{M}^T$ has the form $\boldsymbol{M}^T = \boldsymbol{XJX}^{-1}$, where $\boldsymbol{J} = diag\{\boldsymbol{J}_1, \boldsymbol{J}_2, ..., \boldsymbol{J}_q\}$ is in canonical form. Let the square matrices $\boldsymbol{J}_i$ where $i = 1, 2, ..., q$ be the Jordan blocks with various sizes in the form of

$$\boldsymbol{J}_i = \begin{pmatrix} v_i & 1 & & & \\ & v_i & 1 & & \\ & & v_i & \ddots & \\ & & & \ddots & 1 \\ & & & & v_i \end{pmatrix},$$

where v_i is an eigenvalue of $\boldsymbol{M}^T$. Let $\boldsymbol{J}_1 = [\lambda_1]$ and $\boldsymbol{X}_i$ is the i-th column vector of $\boldsymbol{X}$, $i = 1, 2, ..., n$. For each Jordan block $\boldsymbol{J}_i$, $i = 2, ..., q$, we assume the size of $\boldsymbol{J}_i$ is l_i. We have $\boldsymbol{M}^T = \boldsymbol{XJX}^{-1}$, hence

$$\begin{aligned} &\boldsymbol{M}^T\boldsymbol{X} = \boldsymbol{XJ}, \\ &\boldsymbol{M}^T \begin{pmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \boldsymbol{X}_3 & \boldsymbol{X}_4 & \cdots \end{pmatrix} = \\ &\begin{pmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \boldsymbol{X}_3 & \boldsymbol{X}_4 & \cdots \end{pmatrix} \begin{pmatrix} \lambda_1 & 1 & & & & \\ & v_2 & 1 & & & \\ & & v_2 & \ddots & & \\ & & & v_2 & 1 & \\ & & & & \ddots & \end{pmatrix}. \end{aligned}$$

Since we consider the $\boldsymbol{J}_i$ where $i = 2, ..., q$,

$$\begin{aligned}
\boldsymbol{M}^T\boldsymbol{X}_2 &= \boldsymbol{X}_1 + v_2\boldsymbol{X}_2, \\
\boldsymbol{M}^T\boldsymbol{X}_3 &= \boldsymbol{X}_2 + v_2\boldsymbol{X}_3, \\
\boldsymbol{M}^T\boldsymbol{X}_4 &= \boldsymbol{X}_3 + v_2\boldsymbol{X}_4, \\
&\vdots \\
\boldsymbol{M}^T\boldsymbol{X}_{l_2+1} &= \boldsymbol{X}_{l_2} + v_2\boldsymbol{X}_{l_2+1}, \\
\boldsymbol{M}^T\boldsymbol{X}_{l_2+2} &= v_3\boldsymbol{X}_{l_2+2}, \\
\boldsymbol{M}^T\boldsymbol{X}_{l_2+3} &= \boldsymbol{X}_{l_2+1} + v_3\boldsymbol{X}_{l_2+2}, \\
&\vdots
\end{aligned}$$

Same as in Case 1, since we have $\boldsymbol{M}^T\boldsymbol{X}_2 = v_2\boldsymbol{X}_2$ and (2.28), then

$$\begin{aligned}
v_2\boldsymbol{u}^T\boldsymbol{X}_2 &= \boldsymbol{u}^T v_2\boldsymbol{X}_2 = \boldsymbol{u}^T\boldsymbol{M}^T\boldsymbol{X}_2 = \lambda_1\boldsymbol{u}^T\boldsymbol{X}_2, \\
(v_2 - \lambda_1)\boldsymbol{u}^T\boldsymbol{X}_2 &= 0.
\end{aligned}$$

By (2.35), $v_2 \neq \lambda_1$. Thus, $\boldsymbol{u}^T\boldsymbol{X}_2 = 0$. Hence,

$$\begin{aligned}
\boldsymbol{Q}^T &= (\boldsymbol{M} - \boldsymbol{ueM})^T, \\
\boldsymbol{Q}^T\boldsymbol{X}_2 &= (\boldsymbol{M} - \boldsymbol{ueM})^T\boldsymbol{X}_2 \\
&= (\boldsymbol{M}^T - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{u}^T)\boldsymbol{X}_2 \\
&= \boldsymbol{M}^T\boldsymbol{X}_2 - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{u}^T\boldsymbol{X}_2 \\
&= \boldsymbol{M}^T\boldsymbol{X}_2 - 0 \\
&= v_2\boldsymbol{X}_2,
\end{aligned}$$

which means $\boldsymbol{X}_2$ is an eigenvector of $\boldsymbol{Q}^T$ associated with eigenvalue v_2. Since $\boldsymbol{M}^T\boldsymbol{X}_3 = v_2\boldsymbol{X}_3 + \boldsymbol{X}_2$,

$$\begin{aligned}
v_2\boldsymbol{u}^T\boldsymbol{X}_3 &= \boldsymbol{u}^T v_2\boldsymbol{X}_3 \\
&= \boldsymbol{u}^T(\boldsymbol{M}^T\boldsymbol{X}_3 - \boldsymbol{X}_2) \\
&= \boldsymbol{u}^T\boldsymbol{M}^T\boldsymbol{X}_3 - \boldsymbol{u}^T\boldsymbol{X}_2 \\
&= \boldsymbol{u}^T\boldsymbol{M}^T\boldsymbol{X}_3 - 0,
\end{aligned}$$

and by (2.28), $v_2\boldsymbol{u}^T\boldsymbol{X}_3 = (\lambda_1\boldsymbol{u}^T)\boldsymbol{X}_3$. Now we get $(v_2 - \lambda_1)\boldsymbol{u}^T\boldsymbol{X}_3 = 0$. We have

$\boldsymbol{u}^T\boldsymbol{X}_3 = 0$ since $v_2 \neq \lambda_1$ by (2.35). Hence,

$$\begin{aligned}
\boldsymbol{Q}^T &= (\boldsymbol{M} - \boldsymbol{ueM})^T, \\
\boldsymbol{Q}^T\boldsymbol{X}_3 &= (\boldsymbol{M} - \boldsymbol{ueM})^T\boldsymbol{X}_3 \\
&= (\boldsymbol{M}^T - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{u}^T)\boldsymbol{X}_3 \\
&= \boldsymbol{M}^T\boldsymbol{X}_3 - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{u}^T\boldsymbol{X}_3 \\
&= \boldsymbol{M}^T\boldsymbol{X}_3 - 0 \\
&= v_2\boldsymbol{X}_3 + \boldsymbol{X}_2.
\end{aligned}$$

Similarly, $v_2\boldsymbol{u}^T\boldsymbol{X}_4 = \boldsymbol{u}^T v_2\boldsymbol{X}_4 = \boldsymbol{u}^T(\boldsymbol{M}^T\boldsymbol{X}_4 - \boldsymbol{X}_3) = \boldsymbol{u}^T\boldsymbol{M}^T\boldsymbol{X}_4 - \boldsymbol{u}^T\boldsymbol{X}_3$. By $\boldsymbol{u}^T\boldsymbol{X}_3 = 0$ and (2.28,) we have $v_2\boldsymbol{u}^T\boldsymbol{X}_4 = u^T\boldsymbol{M}^T\boldsymbol{X}_4 = (\lambda_1\boldsymbol{u}^T)\boldsymbol{X}_4$. Thus $(v_2 - \lambda_1)\boldsymbol{u}^T\boldsymbol{X}_4 = 0$. Since (2.35), $\boldsymbol{u}^T\boldsymbol{X}_4 = 0$. Then,

$$\begin{aligned}
\boldsymbol{Q}^T &= (\boldsymbol{M} - \boldsymbol{ueM})^T, \\
\boldsymbol{Q}^T\ \boldsymbol{X}_4 &= (\boldsymbol{M} - \boldsymbol{ueM})^T\boldsymbol{X}_4 \\
&= (\boldsymbol{M}^T - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{u}^T)\boldsymbol{X}_4 \\
&= \boldsymbol{M}^T\boldsymbol{X}_4 - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{u}^T\boldsymbol{X}_4 \\
&= \boldsymbol{M}^T\boldsymbol{X}_4 - 0 \\
&= v_2\boldsymbol{X}_4 + \mathbf{X}_3.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\boldsymbol{Q}^T\boldsymbol{X}_2 &= v_2\boldsymbol{X}_2, \\
\boldsymbol{Q}^T\boldsymbol{X}_3 &= v_2\boldsymbol{X}_3 + \boldsymbol{X}_2, \\
\boldsymbol{Q}^T\boldsymbol{X}_4 &= v_2\boldsymbol{X}_4 + \boldsymbol{X}_2, \\
&\vdots \\
\boldsymbol{Q}^T\boldsymbol{X}_{l_2+1} &= \boldsymbol{X}_{l_2} + v_2\boldsymbol{X}_{l_2+1}, \\
\boldsymbol{Q}^T\boldsymbol{X}_{l_2+2} &= v_3\boldsymbol{X}_{l_2+2}, \\
\boldsymbol{Q}^T\boldsymbol{X}_{l_2+3} &= \boldsymbol{X}_{l_2+1} + v_3\boldsymbol{X}_{l_2+2}, \\
&\vdots
\end{aligned}$$

Using the same argument as in Case 1, we can show that the set $\{\boldsymbol{e}^T, \boldsymbol{X}_i, i = 2, 3, ..., n\}$ is linearly independent. Let $\boldsymbol{Y} = [\boldsymbol{e}^T, \boldsymbol{X}_i, i = 2, 3, ..., n]$. Therefore,

$\boldsymbol{Q}^T\boldsymbol{Y} = \boldsymbol{Y}diag\{[0], J_2, ..., J_q\}$. Hence the spectrum of $\boldsymbol{Q}$ is $\sigma(\boldsymbol{Q}) = \sigma(\boldsymbol{Q}^T) = \{0, v_2, v_3, ..., v_q\}$. This means the spectral radius of $\boldsymbol{Q}$, $\rho(\boldsymbol{Q}) = |v_2|$ and

$$\rho(\boldsymbol{H}'(\boldsymbol{u})) = \rho(\frac{\boldsymbol{Q}}{\lambda_1}) = \frac{|v_2|}{\lambda_1} < 1,$$

since $\lambda_1 > |v_2|$. □

Now by Lemma 4 and Lemma 5, we can show that Algorithm 2 is Q-linear convergent.

Theorem 21. [78] Let $\mathcal{A}$, $\mathcal{B}$, $\{\boldsymbol{x}^{(k)}\}$ and $\boldsymbol{u}$ as in Theorem 20. Then the convergence rate of sequence $\{\boldsymbol{x}^{(k)}\}$ is Q-linear , i.e., there exists a vector norm $||\cdot||$ such that

$$\limsup_{k\to\infty} \frac{||\boldsymbol{x}^{(k+1)} - \boldsymbol{u}||}{||\boldsymbol{x}^{(k)} - \boldsymbol{u}||} < 1. \tag{2.36}$$

Proof. [78] By Proposition 1, there exist an $\epsilon > 0$ and a spectral norm $||\cdot||_\epsilon \in \mathbb{N}$ such that

$$||\boldsymbol{H}'(\boldsymbol{u})||_\epsilon \leq \rho(\boldsymbol{H}'(\boldsymbol{u})) + \epsilon.$$

Then, by Lemma 5,

$$||\boldsymbol{H}'(\boldsymbol{u})||_\epsilon \leq \rho(\boldsymbol{H}'(\boldsymbol{u})) + \epsilon < 1 \tag{2.37}$$

From (2.24), we have $\boldsymbol{x}^{k+1} = \boldsymbol{H}(\boldsymbol{x}^k)$ for $k = 1, 2, ...$ and $\boldsymbol{u} = \boldsymbol{H}(\boldsymbol{u})$, therefore,

$$\boldsymbol{x}^{(k+1)} - \boldsymbol{u} = \boldsymbol{H}(\boldsymbol{x}^{(k)}) - \boldsymbol{H}(\boldsymbol{u}). \tag{2.38}$$

Expand $\boldsymbol{x}^k$ at $\boldsymbol{x}$ using Taylor expansion,

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{x}^{(k)}) &= \boldsymbol{H}(\boldsymbol{u}) + \boldsymbol{H}'(\boldsymbol{u})(\boldsymbol{x}^{(k)} - \boldsymbol{u}) + o(||\boldsymbol{x}^{(k)} - \boldsymbol{u}||_\epsilon), \\ \boldsymbol{H}(\boldsymbol{x}^{(k)}) - \boldsymbol{H}(\boldsymbol{u}) &= \boldsymbol{H}'(\boldsymbol{u})(\boldsymbol{x}^{(k)} - \boldsymbol{u}) + o(||\boldsymbol{x}^{(k)} - \boldsymbol{u}||_\epsilon). \end{aligned} \tag{2.39}$$

Now we have

$$\begin{aligned} \boldsymbol{x}^{(k+1)} - \boldsymbol{u} = \boldsymbol{H}(\boldsymbol{x}^{(k)}) - \boldsymbol{H}(\boldsymbol{u}) &= \boldsymbol{H}'(\boldsymbol{u})(\boldsymbol{x}^{(k)} - \boldsymbol{u}) + o(||\boldsymbol{x}^{(k)} - \boldsymbol{u}||_\epsilon), \\ \frac{\boldsymbol{x}^{(k+1)} - \boldsymbol{u}}{\boldsymbol{x}^{(k)} - \boldsymbol{u}} &= \boldsymbol{H}'(\boldsymbol{u}), \\ \frac{||\boldsymbol{x}^{(k+1)} - \boldsymbol{u}||_\epsilon}{||\boldsymbol{x}^{(k)} - \boldsymbol{u}||_\epsilon} &= ||\boldsymbol{H}'(\boldsymbol{u})||_\epsilon. \end{aligned}$$

By (2.37),

$$\limsup_{k\to\infty} \frac{||\boldsymbol{x}^{k+1} - \boldsymbol{u}||_\epsilon}{||\boldsymbol{x}^{(k)} - \boldsymbol{u}||_\epsilon} < 1,$$

hence the convergence rate of the sequence $\{\boldsymbol{x}^{(k)}\}$ is linear. □

2.7 Application

Algorithm 2 can be used for testing positive definiteness of a class of multivariate forms. We denote $f(\boldsymbol{x})$, a homogeneous polynomial of mth degree and n variables as

$$f(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^m = \sum_{i_1,i_2,...,m=1}^{n} a_{i_1,i_2,...,m} x_{i_1} x_{i_2} ... x_{i_m}. \tag{2.40}$$

The polynomial $f(\boldsymbol{x})$ is called positive definite if

$$f(\boldsymbol{x}) > 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{x} \neq \mathbf{0}. \tag{2.41}$$

In this case, m must be even. We call $\mathcal{A}$ supersymmetric if its entries $a_{i_1,i_2,...,m}$ are invariant under any permutation of their indices $\{i_1, i_2, ..., m\}$. The supersymmetric tensor $\mathcal{A}$ is called positive definite if $f(\boldsymbol{x})$ is positive definite. For an even order real supersymmetric tensor $\mathcal{A}$, the eigenvalues exist [49]. Furthermore, $\mathcal{A}$ is positive definite if and only if all of its real eigenvalues are positive. Therefore, the smallest real eigenvalue of $\mathcal{A}$ determines the positive definiteness of tensor $\mathcal{A}$. If the smallest eigenvalue is positive, then $\mathcal{A}$ is positive definite.

Theorem 22. [49] The eigenvalues of supersymmetric tensor $\mathcal{A}$ lie in the following n disks:

$$|\lambda - a_{i\, i...i}| \leq \sum \{|a_{i\, i_2...i_m}| : i_2, ..., i_m = 1, 2, ..., n, I_{i\, i_2...i_m} = 0\},$$

for $i = 1, 2, ..., n$.

Theorem 23. [49] Suppose that $\mathcal{B} = a(\mathcal{A} + b\mathcal{I})$, where $\mathcal{A}$ is a supersymmetric tensor, and a and b are two real numbers. Then $\mu = a(\lambda + b)$ is an eigenvalue of $\mathcal{B}$ if and only if λ is an eigenvalue of $\mathcal{A}$.

For any supersymmetric tensor $\mathcal{A}$, let

$$L_{\mathcal{A}} = \min\{a_{i\,i\ldots i} - C_i : i = 1, ..., n\}, \tag{2.42}$$

$$U_{\mathcal{A}} = \max\{a_{i\,i\ldots i} + C_i : i = 1, ..., n\}, \tag{2.43}$$

where

$$C_i = \sum\{|a_{i\,i_2\ldots i_m}| : i_2, ..., i_m = 1, ..., n, I_{i\,i_2\ldots i_m} = 0\}, i = 1, 2, ..., n.$$

By Theorem 22, $L_{\mathcal{A}}$ and $U_{\mathcal{A}}$ are respectively the lower and upper bounds of real eigenvalues of $\mathcal{A}$. For Algorithm 3 that will be introduced in this section, we only consider the following type of real supersymmetric tensor $\mathcal{A}$ satisfying

$$a_{i_1 i_2\ldots i_m} = \begin{cases} > 0 & \text{if } i_1 = i_2 = \ldots = i_m, \\ \leq 0 & \text{otherwise.} \end{cases} \tag{2.44}$$

The tensor $\mathcal{A}$ reduces to a matrix when $m = 2$. The study of this form for matrix can be found in [60].

For a real supersymmetric tensor $\mathcal{A}$ satisfying (2.44), let

$$\mathcal{C} = U_{\mathcal{A}}\mathcal{I} - \mathcal{A}, \tag{2.45}$$

where $U_{\mathcal{A}}$ is defined in (2.43). Obviously, $\mathcal{C}$ is a nonnegative tensor. Now let λ be the largest eigenvalue of $\mathcal{C}$.

$$\begin{aligned} \mathcal{C}\boldsymbol{x}^{m-1} &= \lambda \boldsymbol{x}^{[m-1]}, \\ (U_{\mathcal{A}}\mathcal{I} - \mathcal{A})\boldsymbol{x}^{m-1} &= \lambda \boldsymbol{x}^{[m-1]}. \end{aligned}$$

For any eigenvalue $\lambda_{\mathcal{A}}$ of $\mathcal{A}$,

$$\begin{aligned} \mathcal{A}\boldsymbol{x}_{\mathcal{A}}^{m-1} &= \lambda_{\mathcal{A}}\boldsymbol{x}_{\mathcal{A}}^{[m-1]}, \\ U_{\mathcal{A}}\boldsymbol{x}_{\mathcal{A}}^{m-1} - \mathcal{A}\boldsymbol{x}_{\mathcal{A}}^{m-1} &= -\lambda_{\mathcal{A}}\boldsymbol{x}_{\mathcal{A}}^{[m-1]} + U_{\mathcal{A}}\boldsymbol{x}_{\mathcal{A}}^{m-1}, \\ (U_{\mathcal{A}}\mathcal{I} - \mathcal{A})\boldsymbol{x}_{\mathcal{A}}^{m-1} &= (U_{\mathcal{A}} - \lambda_{\mathcal{A}})\boldsymbol{x}_{\mathcal{A}}^{[m-1]}. \end{aligned}$$

Therefore we have

$$\lambda \geq U_{\mathcal{A}} - \lambda_{\mathcal{A}},$$

$$\lambda_{\mathcal{A}} \geq U_{\mathcal{A}} - \lambda.$$

We know that $\lambda_{\mathcal{A}}$ is any eigenvalue of $\mathcal{A}$, hence $(U_{\mathcal{A}} - \lambda)$ is the smallest eigenvalue of $\mathcal{A}$ could be.

A method was suggested in [38] for computing the smallest eigenvalue of the tensor $\mathcal{A}$ which satisfies (2.44). Tensor $\mathcal{A}$ is positive definite if the smallest eigenvalue is positive. Otherwise, $\mathcal{A}$ is not positive definite.

Algorithm 3. [38]

Step 0: If $a_{ii\ldots i} \leq 0$ for some $1 \leq i \leq n$, then $\mathcal{A}$ is not positive definite. Compute the upper bound of real eigenvalues, $U_{\mathcal{A}}$ by the formula (2.43) and let $\mathcal{C} = U_{\mathcal{A}}\mathcal{I} - \mathcal{A}$.

Step 1: By using Algorithm 2, compute λ, the largest eigenvalue of $\mathcal{C}$.

Step 2: Let $\mu = U_{\mathcal{A}} - \lambda$. If $\mu > 0$ then $\mathcal{A}$ is positive definite. Otherwise, $\mathcal{A}$ is not positive definite.

It was shown in [38] that Algorithm 3 performed well and is promising. Three problems were generated randomly and tested to show the efficiency of Algorithm 3.

Problem 1. [38] $f(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^m$ is defined as

$$f(\boldsymbol{x}) = -\left(\sum_{i=1}^{n} b_i x_i\right)^{\frac{m}{2}} \left(\sum_{i=1}^{n} c_i x_i\right)^{\frac{m}{2}} + \sum_{i=1}^{n} a_i x_i^m,$$

where, b_i and c_i are random numbers in $[0, 1]$ for all $i = 1, 2, ..., n$, $a_i = (b_i c_i)^{\frac{m}{2}} + A_d$ for $i = 1, 2, ..., n$, and A_d is a positive number. Obviously, $a_{ii\ldots i} = A_d$ for all $i = 1, 2, ..., n$.

Problem 2. [38] $f(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^m$ is defined as

$$f(\boldsymbol{x}) = -\left(\sum_{i=1}^{n} b_i x_i\right)^{\frac{m}{2}} \left(\sum_{i=1}^{n} x_i^{\frac{m}{2}}\right) + \sum_{i=1}^{n} a_i x_i^m,$$

where, b_i is a random number in $[0, 1]$ for all $i = 1, 2, ..., n$, $a_i = b_i^{\frac{m}{2}} + A_d$ for all $i = 1, 2, ..., n$, and A_d is a positive number. For this problem, $a_{ii\ldots i} = A_d$ for all $i = 1, 2, ..., n$.

Problem 3. [38] $f(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^m$ is defined as

$$f(\boldsymbol{x}) = -\left(\sum_{i=1}^{n} b_i x_i^{\frac{m}{2}}\right)^{\frac{m}{2}} + \sum_{i=1}^{n} a_i x_i^m,$$

where, b_i and a_i are generated in the same way as in Problem 2.

The following tables are the numerical results of the test. In these tables, n is the dimension and m is the order of the randomly generated problem, meanwhile A_d is a parameter. For each (m, n, A_d), a total of 100 test problem was generated for Problem 1, 2 and 3.. The **Yes** column shows the number of problems which are positive definite and the **No** column shows the number of problems which are not positive definite. Based on the numerical results, we can say that Algorithm 3 is efficient.

2.8 Conclusion

In this chapter, we discussed some properties of nonnegative real square tensors and the methods for finding the largest eigenvalue of an irreducible tensor. The improved method, Algorithm 2 is convergent for irreducible tensor and has Q-linear rate of convergence under weak irreducible condition. We also provided an example of the application which is to determine the positive definiteness of a class of multivariate form. There are a few other methods in literature for finding the spectral radius of nonnegative square tensors. Interested reader can find them in [25, 70].

Problem			Problem 1			Problem 2			Problem 3		
m	n	A_d	Yes	No	CPU(s)	Yes	No	CPU(s)	Yes	No	CPU(s)
4	20	10	0	100	0.0017	0	100	0.0033	100	0	0.0037
4	20	10^2	0	100	0.0020	43	57	0.0033	100	0	0.0036
4	20	10^3	96	4	0.0020	100	0	0.0034	100	0	0.0037
4	20	10^4	100	0	0.0016	100	0	0.0034	100	0	0.0036
4	20	10^5	100	0	0.0020	100	0	0.0031	100	0	0.0039
4	40	10	0	100	0.0022	0	100	0.0042	7	93	0.0044
4	40	10^2	0	100	0.0020	0	100	0.0047	100	0	0.0045
4	40	10^3	0	100	0.0023	100	0	0.0045	100	0	0.0044
4	40	10^4	100	0	0.0022	100	0	0.0045	100	0	0.0045
4	40	10^5	100	0	0.0023	100	0	0.0042	100	0	0.0044
4	60	10	0	100	0.0030	0	100	0.0053	0	100	0.0050
4	60	10^2	0	100	0.0027	0	100	0.0063	100	0	0.0047
4	60	10^3	0	100	0.0030	69	31	0.0045	100	0	0.0052
4	60	10^4	1	99	0.0023	100	0	0.0052	100	0	0.0047
4	60	10^5	100	0	0.0025	100	0	0.0053	100	0	0.0053
4	80	10	0	100	0.0030	0	100	0.0055	0	100	0.0056
4	80	10^2	0	100	0.0028	0	100	0.0061	100	0	0.0056
4	80	10^3	0	100	0.0031	0	100	0.0058	100	0	0.0052
4	80	10^4	0	100	0.0025	100	0	0.0059	100	0	0.0058
4	80	10^5	100	0	0.0034	100	0	0.0056	100	0	0.0059
4	100	10	0	100	0.0031	0	100	0.0067	0	100	0.0063
4	100	10^2	0	100	0.0033	0	100	0.0077	100	0	0.0064
4	100	10^3	0	100	0.0034	0	100	0.0067	100	0	0.0063
4	100	10^4	0	100	0.0036	100	0	0.0066	100	0	0.0058
4	100	10^5	100	0	0.0036	100	0	0.0066	100	0	0.0064
6	20	10	0	100	0.0019	0	100	0.0039	99	1	0.0041
6	20	10^2	0	100	0.0017	0	100	0.0041	100	0	0.0037
6	20	10^3	0	100	0.0020	35	65	0.0042	100	0	0.0041
6	20	10^4	1	99	0.0020	100	0	0.0041	100	0	0.0036
6	20	10^5	95	5	0.0023	100	0	0.0041	100	0	0.0041
6	40	10	0	100	0.0023	0	100	0.0053	6	94	0.0042
6	40	10^2	0	100	0.0027	0	100	0.0050	100	0	0.0047
6	40	10^3	0	100	0.0025	0	100	0.0052	100	0	0.0045
6	40	10^4	0	100	0.0027	66	34	0.0053	100	0	0.0042
6	40	10^5	0	100	0.0025	100	0	0.0070	100	0	0.0039

Table 2.1: Output of Algorithm 3 for Problems 1-3 [38].

Problem			Problem 1			Problem 2			Problem 3		
m	n	A_d	Yes	No	CPU(s)	Yes	No	CPU(s)	Yes	No	CPU(s)
6	60	10	0	100	0.0028	0	100	0.0064	0	100	0.0047
6	60	10^2	0	100	0.0034	0	100	0.0061	100	0	0.0041
6	60	10^3	0	100	0.0028	0	100	0.0059	100	0	0.0047
6	60	10^4	0	100	0.0027	0	100	0.0063	100	0	0.0048
6	60	10^5	0	100	0.0028	100	0	0.0061	100	0	0.0048
6	80	10	0	100	0.0030	0	100	0.0072	0	100	0.0056
6	80	10^2	0	100	0.0031	0	100	0.0067	100	0	0.0052
6	80	10^3	0	100	0.0030	0	100	0.0070	100	0	0.0055
6	80	10^4	0	100	0.0037	0	100	0.0067	100	0	0.0053
6	80	10^5	0	100	0.0033	98	2	0.0070	100	0	0.0127
6	100	10	0	100	0.0036	0	100	0.0078	0	100	0.0055
6	100	10^2	0	100	0.0036	0	100	0.0081	100	0	0.0055
6	100	10^3	0	100	0.0034	0	100	0.0081	100	0	0.0055
6	100	10^4	0	100	0.0037	0	100	0.0081	100	0	0.0053
6	100	10^5	0	100	0.0037	4	96	0.0080	100	0	0.0058
8	20	10	0	100	0.0019	0	100	0.0044	100	0	0.0033
8	20	10^2	0	100	0.0017	0	100	0.0039	100	0	0.0034
8	20	10^3	0	100	0.0020	0	100	0.0041	100	0	0.0034
8	20	10^4	0	100	0.0019	47	53	0.0037	100	0	0.0033
8	20	10^5	0	100	0.0022	100	0	0.0039	100	0	0.0034
8	40	10	0	100	0.0025	0	100	0.0056	6	94	0.0034
8	40	10^2	0	100	0.0028	0	100	0.0053	100	0	0.0031
8	40	10^3	0	100	0.0025	0	100	0.0053	100	0	0.0033
8	40	10^4	0	100	0.0027	0	100	0.0053	100	0	0.0031
8	40	10^5	0	100	0.0023	8	92	0.0048	100	0	0.0036
8	60	10	0	100	0.0033	0	100	0.0064	0	100	0.0036
8	60	10^2	0	100	0.0041	0	100	0.0067	100	0	0.0034
8	60	10^3	0	100	0.0031	0	100	0.0066	100	0	0.0034
8	60	10^4	0	100	0.0031	0	100	0.0066	100	0	0.0036
8	60	10^5	0	100	0.0034	0	100	0.0061	100	0	0.0036
8	80	10	0	100	0.0037	0	100	0.0080	0	100	0.0037
8	80	10^2	0	100	0.0036	0	100	0.0078	100	0	0.0039
8	80	10^3	0	100	0.0036	0	100	0.0078	100	0	0.0041
8	80	10^4	0	100	0.0036	0	100	0.0080	100	0	0.0037
8	80	10^5	0	100	0.0045	0	100	0.0072	100	0	0.0033

Table 2.2: Output of Algorithm 3 for Problems 1-3 [38].

Problem			Problem 1			Problem 2			Problem 3		
m	n	A_d	Yes	No	CPU(s)	Yes	No	CPU(s)	Yes	No	CPU(s)
8	100	10	0	100	0.0042	0	100	0.0089	0	100	0.0042
8	100	10^2	0	100	0.0039	0	100	0.0091	100	0	0.0041
8	100	10^3	0	100	0.0041	0	100	0.0089	100	0	0.0041
8	100	10^4	0	100	0.0044	0	100	0.0088	100	0	0.0044
8	100	10^5	0	100	0.0042	0	100	0.0088	100	0	0.0042
10	20	10	0	100	0.0016	0	100	0.0041	99	1	0.0023
10	20	10^2	0	100	0.0022	0	100	0.0045	100	0	0.0030
10	20	10^3	0	100	0.0019	0	100	0.0042	100	0	0.0025
10	20	10^4	0	100	0.0019	0	100	0.0041	100	0	0.0023
10	20	10^5	0	100	0.0020	44	56	0.0042	100	0	0.0025
10	40	10	0	100	0.0025	0	100	0.0056	11	89	0.0017
10	40	10^2	0	100	0.0025	0	100	0.0055	100	0	0.0020
10	40	10^3	0	100	0.0027	0	100	0.0052	100	0	0.0022
10	40	10^4	0	100	0.0028	0	100	0.0053	100	0	0.0022
10	40	10^5	0	100	0.0028	0	100	0.0055	100	0	0.0022
10	60	10	0	100	0.0033	0	100	0.0066	0	100	0.0006
10	60	10^2	0	100	0.0034	0	100	0.0067	100	0	0.0006
10	60	10^3	0	100	0.0036	0	100	0.0069	100	0	0.0005
10	60	10^4	0	100	0.0042	0	100	0.0066	100	0	0.0005
10	60	10^5	0	100	0.0034	0	100	0.0067	100	0	0.0005
10	80	10	0	100	0.0036	0	100	0.0078	0	100	0.0008
10	80	10^2	0	100	0.0041	0	100	0.0075	100	0	0.0006
10	80	10^3	0	100	0.0041	0	100	0.0078	100	0	0.0009
10	80	10^4	0	100	0.0034	0	100	0.0080	100	0	0.0006
10	80	10^5	0	100	0.0039	0	100	0.0080	100	0	0.0006
10	100	10	0	100	0.0047	0	100	0.0089	0	100	0.0006
10	100	10^2	0	100	0.0047	0	100	0.0086	100	0	0.0008
10	100	10^3	0	100	0.0047	0	100	0.0091	100	0	0.0005
10	100	10^4	0	100	0.0042	0	100	0.0089	100	0	0.0009
10	100	10^5	0	100	0.0044	0	100	0.0091	100	0	0.0008

Table 2.3: Output of Algorithm 3 for Problems 1-3 [38].

Chapter 3

Nonnegative Rectangular Tensors

3.1 Introduction

The class of real rectangular tensor was introduced in [7] and can be found in the strong ellipticity condition problem in solid mechanics [28, 29, 56, 71] and the entanglement problem in quantum physics [11, 13]. In this chapter, we study some of the properties of real rectangular tensors and the methods for finding the largest singular value of rectangular tensors. Most of the properties are generalisations of those for real square tensors.

3.2 Nonnegative Rectangular Tensor

Let p, q, m and n be positive integers, and $m, n \geq 2$. We call $\mathcal{A}$ a real (p, q)-th order $(m \times n)$ dimensional rectangular tensor, where

$$\mathcal{A} = (a_{i_1 \cdots i_p j_1 \cdots j_q}), \quad a_{i_1 \cdots i_p j_1 \cdots j_q} \in \mathbb{R}, \quad 1 \leq i_1, \cdots, i_p \leq m \quad 1 \leq j_1, \cdots, j_q \leq n. \tag{3.1}$$

If $a_{i_1 \cdots i_p j_1 \cdots j_q} \geq 0$, $\mathcal{A}$ is called a nonnegative real rectangular tensor. If $a_{i_1 \cdots i_p j_1 \cdots j_q} > 0$, $\mathcal{A}$ is called a positive real rectangular tensor. If $m = n$, $\mathcal{A}$ is a square tensor, which we have discussed in previous chapter. When $p = q = 1$, a rectangular tensor is reduced to an $m \times n$ rectangular matrix. Let $(\mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q) \in \mathbb{R}^m$, where

$$\left(\mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q\right)_i = \sum_{i_2, \cdots, i_p = 1}^{m} \sum_{j_1, \cdots, j_q = 1}^{n} \mathcal{A}_{i i_2 \cdots i_p j_1, \cdots, j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}, \quad i = 1, 2, ..., m,$$

and let $(\mathcal{A}\boldsymbol{x}^p\boldsymbol{y}^{q-1}) \in \mathbb{R}^n$, where

$$\left(\mathcal{A}\boldsymbol{x}^p y^{q-1}\right)_j = \sum_{i_1,\cdots,i_p=1}^{m} \sum_{j_2,\cdots,j_q=1}^{n} \mathcal{A}_{i_1\cdots i_p j j_2,\cdots,j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q}, \quad j = 1,2,...,n.$$

For rectangular tensor, we set $M = p+q$ and $N = m+n$. Let

$$\begin{aligned} \mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q &= \lambda \boldsymbol{x}^{[M-1]}, \\ \mathcal{A}\boldsymbol{x}^p\boldsymbol{y}^{q-1} &= \lambda \boldsymbol{y}^{[M-1]}. \end{aligned} \tag{3.2}$$

Let $\mathbb{C}$ be the set of all complex numbers. We call $\lambda \in \mathbb{C}$ a singular value of $\mathcal{A}$, $\boldsymbol{x} \in \mathbb{C}^m \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{y} \in \mathbb{C}^n \setminus \{\boldsymbol{0}\}$ are left and right eigenvectors of $\mathcal{A}$, associated with the singular value λ if $\lambda, \boldsymbol{x}$ and $\boldsymbol{y}$ satisfy equation (3.2). The spectral radius of $\mathcal{A}$ is defined as $\rho(\mathcal{A}) = \{max|\lambda| : \lambda \text{ is a singular value of } \mathcal{A}\}$.

For any $j = 1,2,...,n$, let $\mathcal{A}_{\bullet,j} = (a_{i_1\cdots i_p,j\cdots j})$ be a p-th order m-dimensional square tensor. For any $i = 1,2,...,m$, let $\mathcal{A}_{i,\bullet} = (a_{i\cdots i,j_1,\cdots,j_q})$ be a q-th order n-dimensional square tensor.

Definition 12. [7, 76] A nonnegative rectangular tensor $\mathcal{A}$ is irreducible if all the square tensors $\mathcal{A}_{\bullet,j}, j = 1,...,n$, and $\mathcal{A}_{i,\bullet}, i = 1,...,m$, are irreducible.

Let $\mathcal{A}$ be a (p,q)th order $(m \times n)$ dimensional nonnegative rectangular tensor. The graph $\mathcal{G}(\mathcal{A}) = (V, E(\mathcal{A}))$ is the associated graph of tensor $\mathcal{A}$. The vertex set is $V = \cup_{j=1}^{p} V_j + \cup_{j=p+1}^{M} V_j$, with $V_j = \{1,2,...,m\}$ for $j = 1,2,...,p$ and $V_j = \{1,2,...,n\}$ for $j = p+1,...,M$, $M = p+q$. An edge $(i_k, i_l) \in V_k \times V_l$ exists if and only if $a_{i_1\cdots i_p j_1\cdots j_q} > 0$ for some $M-2$ indices $\{i_1,\cdots,i_p,j_1,\cdots,j_q\}\setminus\{i_k,i_l\}$. A tensor $\mathcal{A}$ is called weakly irreducible if the graph $\mathcal{G}(\mathcal{A})$ is connected [16].

Lemma 6. [7] If $\mathcal{A}$ is irreducible, then all the tensors $\mathcal{A}_{\bullet,j}$, $j = 1,...,n$, and $\mathcal{A}_{i,\bullet}$, $i = 1,...,m$, do not have eigenvalue 0.

Lemma 7. [7] If $\mathcal{A}$ is irreducible, then for any $(\boldsymbol{x},\boldsymbol{y}) \in (\mathbb{R}^m\setminus\{\boldsymbol{0}\}) \times (\mathbb{R}^m\setminus\{\boldsymbol{0}\})$, $\mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q \neq \boldsymbol{0}$ and $\mathcal{A}\boldsymbol{x}^p\boldsymbol{y}^{q-1} \neq \boldsymbol{0}$.

Lemma 8. [7] Let $\mathcal{A}$ be nonnegative and irreducible, and let $(\lambda,(\boldsymbol{x},\boldsymbol{y})) \in \mathbb{R}_+ \times (\mathbb{R}^m_{>0} \times \mathbb{R}^n_{>0})$ be a solution of (3.2). If $(\mu,(\boldsymbol{u},\boldsymbol{v})) \in \mathbb{R}_+ \times ((\mathbb{R}^m_+\setminus\{\boldsymbol{0}\}) \times (\mathbb{R}^n_+\setminus\{\boldsymbol{0}\}))$ satisfies

$$\mathcal{A}\boldsymbol{u}^{p-1}\boldsymbol{v}^q \geq (or \leq)\mu\boldsymbol{u}^{[M-1]}, \quad and \quad \mathcal{A}\boldsymbol{u}^p\boldsymbol{v}^{q-1} \geq (or \leq)\mu\boldsymbol{v}^{[M-1]},$$

then $\mu \leq (or \geq)\lambda$.

The following is the Perron-Frobenius theorem for nonnegative rectangular tensor which was extended in [7].

Theorem 24. [7] [35] Let $\mathcal{A}$ be an irreducible nonnegative rectangular tensor of order (p,q) and dimension $(m \times n)$, then there exist $\lambda_0 > 0$ and $\boldsymbol{x}_0 \in \mathbb{R}^m_{>0}$ and $\boldsymbol{y}_0 \in \mathbb{R}^n_{>0}$ such that

$$\begin{aligned} \mathcal{A}\boldsymbol{x}_0^{p-1}\boldsymbol{y}_0^q &= \lambda_0 \boldsymbol{x}_0^{[M-1]}, \\ \mathcal{A}\boldsymbol{x}_0^{p}\boldsymbol{y}_0^{q-1} &= \lambda_0 \boldsymbol{y}_0^{[M-1]}. \end{aligned} \tag{3.3}$$

Moreover, if λ is a singular value with strong positive left and right eigenvectors, then $\lambda = \lambda_0$. The strong positive left and right eigenvectors are unique up to a multiplicative constant. $|\lambda| \leq \lambda_0$ for all singular values λ of $\mathcal{A}$.

The minimax theorem also was extended in [7] to nonnegative rectangular tensors.

Theorem 25. [7] Assume that $\mathcal{A}$ is an irreducible nonnegative rectangular tensor of order (p,q) and dimension $(m \times n)$, then

$$\begin{aligned} \lambda_0 &= \min_{(\boldsymbol{x},\boldsymbol{y})\in(\mathbb{R}^m_+\setminus\{\boldsymbol{0}\})\times(\mathbb{R}^n_+\setminus\{\boldsymbol{0}\})} \max_{i,j}\left(\frac{(\mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q)_i}{\boldsymbol{x}_i^{M-1}}, \frac{(\mathcal{A}\boldsymbol{x}^{p}\boldsymbol{y}^{q-1})_i}{\boldsymbol{y}_j^{M-1}}\right) \\ &= \min_{(\boldsymbol{x},\boldsymbol{y})\in(\mathbb{R}^m_+\setminus\{\boldsymbol{0}\})\times(\mathbb{R}^n_+\setminus\{\boldsymbol{0}\})} \min_{i,j}\left(\frac{(\mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q)_i}{\boldsymbol{x}_i^{M-1}}, \frac{(\mathcal{A}\boldsymbol{x}^{p}\boldsymbol{y}^{q-1})_i}{\boldsymbol{y}_j^{M-1}}\right), \end{aligned}$$

where λ_0 is the unique positive singular value corresponding to strongly positive left and right eigenvectors.

3.3 Algorithms

An algorithm for finding the largest singular value of irreducible nonnegative rectangular tensor was proposed in [7]. This algorithm is an extension of the Algorithm 1 for finding the largest eigenvalue of an irreducible nonnegative square tensor in the previous chapter. We will study this algorithm and one of its update.

For any two vectors $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^n, \boldsymbol{x} \geq \boldsymbol{y}$ means that $\boldsymbol{x} - \boldsymbol{y} \in \mathbb{R}^n_+$ and $\boldsymbol{x} > \boldsymbol{y}$ means $\boldsymbol{x} - \boldsymbol{y} \in \mathbb{R}^n_{>0}$.

Lemma 9. [7] Suppose that $\mathcal{A}$ is a nonnegative rectangular tensor of order (p,q) and dimension $(m \times n)$, $\boldsymbol{x} \in \mathbb{R}^m_+, \bar{\boldsymbol{x}} \in \mathbb{R}^m_+, \boldsymbol{y} \in \mathbb{R}^n_+$, and $\bar{\boldsymbol{y}} \in \mathbb{R}^n_+$ are four nonnegative column vectors, and t is a positive number. Then, we have

(i) If $\boldsymbol{x} \geq \bar{\boldsymbol{x}} \geq \mathbf{0}$ and $\boldsymbol{y} \geq \bar{\boldsymbol{y}} \geq \mathbf{0}$, then $\mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q \geq \mathcal{A}\bar{x}^{p-1}\bar{\boldsymbol{y}}^q$ and $\mathcal{A}\boldsymbol{x}^p\boldsymbol{y}^{q-1} \geq \mathcal{A}\bar{\boldsymbol{x}}^p\bar{\boldsymbol{y}}^{q-1}$.

(ii) $\mathcal{A}(t\boldsymbol{x})^{p-1}(t\boldsymbol{y})^q = t^{M-1}\mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q$ and $\mathcal{A}(t\boldsymbol{x})^p(t\boldsymbol{y})^{q-1} = t^{M-1}\mathcal{A}\boldsymbol{x}^p\boldsymbol{y}^{q-1}$.

Lemma 10. [7] Suppose that $\mathcal{A}$ is a nonnegative irreducible rectangular tensor of order (p, q) and dimension $(m \times n)$. Then, for any two strongly positive vectors $\boldsymbol{x} \in \mathbb{R}^m_{>0}$ and $\boldsymbol{y} > \mathbf{0}, \boldsymbol{x} \in \mathbb{R}^n_{>0}$, $\mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q > \mathbf{0}$ and $\mathcal{A}\boldsymbol{x}^p\boldsymbol{y}^{q-1} > \mathbf{0}$, which means $\mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q$ and $\mathcal{A}\boldsymbol{x}^p\boldsymbol{y}^{q-1}$ are strongly positive vectors.

The Theorem 26 below will be the foundation for Algorithm 4.

Theorem 26. [7] Suppose that $\mathcal{A}$ is a nonnegative irreducible rectangular tensor of order (p, q) and dimension $(m \times n)$. Let $\boldsymbol{x}^{(0)} \in \mathbb{R}^m_{>0}$ and $\boldsymbol{y}^{(0)} \in \mathbb{R}^n_{>0}$ are two arbitrary strongly positive vectors. Let $\boldsymbol{\xi}^{(0)} = \mathcal{A}(\boldsymbol{x}^{(0)})^{p-1}(\boldsymbol{y}^{(0)})^q$ and $\boldsymbol{\eta}^{(0)} = \mathcal{A}(\boldsymbol{x}^{(0)})^p(\boldsymbol{y}^{(0)})^{q-1}$. Define

$$
\begin{aligned}
\boldsymbol{x}^{(1)} &= \frac{\left(\boldsymbol{\xi}^{(0)}\right)^{\left[\frac{1}{M-1}\right]}}{\|\ (\boldsymbol{\xi}^{(0)}, \boldsymbol{\eta}^{(0)})^{\left[\frac{1}{M-1}\right]}\ \|}, \\
\boldsymbol{y}^{(1)} &= \frac{\left(\boldsymbol{\eta}^{(0)}\right)^{\left[\frac{1}{M-1}\right]}}{\|\ (\boldsymbol{\xi}^{(0)}, \boldsymbol{\eta}^{(0)})^{\left[\frac{1}{M-1}\right]}\ \|}, \\
\boldsymbol{\xi}^{(1)} &= \mathcal{A}(\boldsymbol{x}^{(1)})^{p-1}(\boldsymbol{y}^{(1)})^q, \\
\boldsymbol{\eta}^{(1)} &= \mathcal{A}(\boldsymbol{x}^{(1)})^p(\boldsymbol{y}^{(1)})^{q-1}, \\
&\vdots \\
\boldsymbol{x}^{(k+1)} &= \frac{\left(\boldsymbol{\xi}^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\|\ (\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})^{\left[\frac{1}{M-1}\right]}\ \|}, \\
\boldsymbol{y}^{(k+1)} &= \frac{\left(\boldsymbol{\eta}^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\|\ (\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})^{\left[\frac{1}{M-1}\right]}\ \|}, \\
\boldsymbol{\xi}^{(k+1)} &= \mathcal{A}(\boldsymbol{x}^{(k+1)})^{p-1}(\boldsymbol{y}^{(k+1)})^q, \\
\boldsymbol{\eta}^{(k+1)} &= \mathcal{A}(\boldsymbol{x}^{(k+1)})^p(\boldsymbol{y}^{(k+1)})^{q-1}, \\
&\vdots
\end{aligned}
$$

for $k \geq 1$ and let

$$
\begin{aligned}
\underline{\lambda}_k &= \min_{\boldsymbol{x}_i^{(k)}>0, \boldsymbol{y}_j^{(k)}>0} \left(\frac{\boldsymbol{\xi}_i^{(k)}}{(\boldsymbol{x}_i^{(k)})^{M-1}}, \frac{\boldsymbol{\eta}_j^{(k)}}{(\boldsymbol{y}_j^{(k)})^{M-1}} \right), \\
\bar{\lambda}_{k+1} &= \max_{\boldsymbol{x}_i^{(k)}>0, \boldsymbol{y}_j^{(k)}>0} \left(\frac{\boldsymbol{\xi}_i^{(k)}}{(\boldsymbol{x}_i^{(k)})^{M-1}}, \frac{\boldsymbol{\eta}_j^{(k)}}{(\boldsymbol{y}_j^{(k)})^{M-1}} \right),
\end{aligned}
$$

for $k = 1, 2, \ldots$. Assume that λ_0 is the unique positive singular value of $\mathcal{A}$. Then,

$$\underline{\lambda}_1 \le \underline{\lambda}_2 \le \cdots \le \lambda_0 \le \cdots \bar{\lambda}_2 \le \bar{\lambda}_1. \tag{3.4}$$

We state here the algorithm for calculating the largest singular value of nonnegative irreducible rectangular tensor [7].

Algorithm 4. [7]

Step 0: Choose $\boldsymbol{x}^{(0)} \in \mathbb{R}^m_+, \boldsymbol{x}^{(0)} \ne \mathbf{0}$, and $\boldsymbol{y}^{(0)} \in \mathbb{R}^n_+, \boldsymbol{y}^{(0)} \ne \mathbf{0}$. Let $\boldsymbol{\xi}^{(0)} = \mathcal{A}(\boldsymbol{x}^{(0)})^{p-1}(\boldsymbol{y}^{(0)})^q$ and $\boldsymbol{\eta}^{(0)} = \mathcal{A}(\boldsymbol{x}^{(0)})^p(\boldsymbol{y}^{(0)})^{q-1}$. Set $k = 0$.

Step 1: Compute

$$\begin{aligned}
\boldsymbol{x}^{(k+1)} &= \frac{\left(\boldsymbol{\xi}^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})^{\left[\frac{1}{M-1}\right]}\|}, \\
\boldsymbol{y}^{(k+1)} &= \frac{\left(\boldsymbol{\eta}^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})^{\left[\frac{1}{M-1}\right]}\|}, \\
\boldsymbol{\xi}^{(k+1)} &= \mathcal{A}(\boldsymbol{x}^{(k+1)})^{p-1}(\boldsymbol{y}^{(k+1)})^q, \\
\boldsymbol{\eta}^{(k+1)} &= \mathcal{A}(\boldsymbol{x}^{(k+1)})^p(\boldsymbol{y}^{(k+1)})^{q-1}.
\end{aligned}$$

Let

$$\begin{aligned}
\underline{\lambda}_{k+1} &= \min_{x_i^{(k+1)}>0, y_j^{(k+1)}>0} \left(\frac{\boldsymbol{\xi}_i^{(k+1)}}{(x_i^{(k+1)})^{M-1}}, \frac{\boldsymbol{\eta}_j^{(k+1)}}{(y_j^{(k+1)})^{M-1}} \right), \\
\bar{\lambda}_{k+1} &= \max_{x_i^{(k+1)}>0, y_j^{(k+1)}>0} \left(\frac{\boldsymbol{\xi}_i^{(k+1)}}{(x_i^{(k+1)})^{M-1}}, \frac{\boldsymbol{\eta}_j^{(k+1)}}{(y_j^{(k+1)})^{M-1}} \right).
\end{aligned}$$

Step 2: If $\bar{\lambda}_{k+1} = \underline{\lambda}_{k+1}$, then stop. Otherwise, replace k by $k+1$ and go to Step 1.

Theorem 27. [7] Suppose that $\mathcal{A}$ is a nonnegative irreducible rectangular tensor of order (p, q) and dimension $(m \times n)$. Assume that λ_0 is the unique positive singular value of $\mathcal{A}$. Then, Algorithm 4 produces the value of λ_0 in a finite number of steps, or generates two convergent sequences $\{\underline{\lambda}_k\}$ and $\{\bar{\lambda}_k\}$. Furthermore, let $\underline{\lambda} = \lim_{k\to+\infty} \underline{\lambda}_k$ and $\bar{\lambda} = \lim_{k\to+\infty} \bar{\lambda}_k$. Then, $\underline{\lambda}$ and $\bar{\lambda}$ are lower bound and upper bound of λ_0, respectively. If $\underline{\lambda} = \bar{\lambda}$, then $\lambda_0 = \underline{\lambda} = \bar{\lambda}$.

However, Theorem 27 only states that Algorithm 4 produces two convergent sequences $\{\underline{\lambda}_k\}$ and $\{\bar{\lambda}_k\}$. It does not show that $\{\underline{\lambda}_k\}$ and $\{\bar{\lambda}_k\}$ converge to λ_0. A modified version of Algorithm 4 was presented in [76] and it was proven that the modified algorithm converged for any nonnegative irreducible rectangular tensor.

For a rectangular tensor $\mathcal{A}$, let $\rho > 0, \boldsymbol{x} \in \mathbb{R}_+^m$, $\boldsymbol{y} \in \mathbb{R}_+^n$ and

$$\begin{aligned} \mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) &= \mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q + \rho \boldsymbol{x}^{[M-1]}, & (3.5)\\ \mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) &= \mathcal{A}\boldsymbol{x}^{p}\boldsymbol{y}^{q-1} + \rho \boldsymbol{y}^{[M-1]}. & (3.6) \end{aligned}$$

By Theorem 24 and Theorem 25 , we can get the following result [76]:

Theorem 28. [76] If $\mathcal{A}$ is a irreducible nonnegative rectangular tensor of order (p, q) and dimension $(m \times n)$, then there exist $\mu_0 > 0, \boldsymbol{x}_0 \in \mathbb{R}_{>0}^m$ and $\boldsymbol{y}_0 \in \mathbb{R}_{>0}^n$ such that

$$\begin{aligned} \mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}_0, \boldsymbol{y}_0) &= \mu_0 \boldsymbol{x}_0^{[M-1]},\\ \mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}_0, \boldsymbol{y}_0) &= \mu_0 \boldsymbol{y}_0^{[M-1]}. \end{aligned} \qquad (3.7)$$

Moreover, μ_0 satisfies the following equalities:

$$\begin{aligned} \mu_0 &= \min_{(\boldsymbol{x},\boldsymbol{y}) \in (\mathbb{R}_+^m \setminus \{\boldsymbol{0}\}) \times (\mathbb{R}_+^n \setminus \{\boldsymbol{0}\})} \max_{i,j} \left(\frac{\mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})_i}{x_i^{[M-1]}}, \frac{\mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})_j}{y_j^{[M-1]}} \right)\\ &= \max_{(\boldsymbol{x},\boldsymbol{y}) \in (\mathbb{R}_+^m \setminus \{\boldsymbol{0}\}) \times (\mathbb{R}_+^n \setminus \{\boldsymbol{0}\})} \min_{i,j} \left(\frac{\mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})_i}{x_i^{[M-1]}}, \frac{\mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})_j}{y_j^{[M-1]}} \right), \end{aligned}$$

and $\mu_0 - \rho$ is the largest singular value of $\mathcal{A}$.

The polynomials (3.7) are monotone and homogeneous.

Lemma 11. [76] For any $\boldsymbol{x}, \bar{\boldsymbol{x}} \in \mathbb{R}_+^m, \boldsymbol{y}, \bar{\boldsymbol{y}} \in \mathbb{R}_+^n$ and $t > 0$, we have the following results:

(i) If $\boldsymbol{x} \geq \bar{\boldsymbol{x}}$ and $\boldsymbol{y} \geq \bar{\boldsymbol{y}}$, then $\mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) \geq \mathcal{B}_{\boldsymbol{x}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ and $\mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) \geq \mathcal{B}_{\boldsymbol{y}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$. Furthermore, if $x_i > \bar{x}_i$ for some $1 \leq i \leq m$, then $\mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})_i > \mathcal{B}_{\boldsymbol{x}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})_i$. Similarly, if $y_j > \bar{y}_j$ for some $i \leq j \leq n$, then $\mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})_j > \mathcal{B}_{\boldsymbol{y}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})_j$.

(ii) $\mathcal{B}_{\boldsymbol{x}}(t\boldsymbol{x}, t\boldsymbol{y}) = t^{M-1}\mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})$ and $\mathcal{B}_{\boldsymbol{y}}(t\boldsymbol{x}, t\boldsymbol{y}) = t^{M-1}\mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})$

Lemma 12. [76] For any $\boldsymbol{x} \in \mathbb{R}_+^m, \boldsymbol{y}, \in \mathbb{R}_+^n$ and $\rho > 0$, $\mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})$ and $\mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})$ are strongly positive vectors.

For all vectors $\boldsymbol{x} \in \mathbb{R}^m_+ \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{y} \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}$, define the sequences $\{\mathcal{B}^{(k)}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})\}$ and $\{\mathcal{B}^{(k)}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})\}$ as

$$
\begin{aligned}
\mathcal{B}^{(1)}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) &= \mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}), \quad \mathcal{B}^{(1)}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}), \\
a^{(1)} &= \left(\mathcal{B}^{(1)}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})\right)^{\left[\frac{1}{M-1}\right]}, b^{(1)} = \left(\mathcal{B}^{(1)}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})\right)^{\left[\frac{1}{M-1}\right]}, \\
\mathcal{B}^{(2)}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) &= \mathcal{B}_{\boldsymbol{x}}(a^{(1)}, b^{(1)}), \quad \mathcal{B}^{(2)}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{B}_{\boldsymbol{y}}(a^{(1)}, b^{(1)}) \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
a^{(k)} &= \left(\mathcal{B}^{(k-1)}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})\right)^{\left[\frac{1}{M-1}\right]}, b^{(k)} = \left(\mathcal{B}^{(k-1)}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})\right)^{\left[\frac{1}{M-1}\right]}, \ k \geq 1, \quad (3.8) \\
\mathcal{B}^{(k+1)}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) &= \mathcal{B}_{\boldsymbol{x}}(a^{(k)}, b^{(k)}), \quad \mathcal{B}^{(k+1)}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{B}_{\boldsymbol{y}}(a^{(k)}, b^{(k)}), \ k \geq 1
\end{aligned}
$$

The following useful results are given by [76]:

Theorem 29. [76] Suppose that $\mathcal{A}$ is an irreducible nonnegative (p, q)th order $(m \times n)$ dimensional rectangular tensor. Then there exists a positive integer s such that $\mathcal{B}^{(s)}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^m_{>0}$ and $\mathcal{B}^{(s)}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n_{>0}$ for any $\boldsymbol{x} \in \mathbb{R}^m_+ \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{y} \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}$.

Theorem 30. [76] Let $\mathcal{A}$ be an irreducible nonnegative (p, q)th order $(m \times n)$ dimensional rectangular tensor. Suppose $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathbb{R}^m_+ \setminus \{\boldsymbol{0}\}$, $\boldsymbol{x}^1 \geq \boldsymbol{x}^2$, and $\boldsymbol{y}^1, \boldsymbol{y}^2 \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}$, $\boldsymbol{y}^1 \geq \boldsymbol{y}^2$. If $x^1_{i_0} < x^2_{i_0}$ for some $1 \leq i_0 \leq m$, or $y^1_{j_0} < y^2_{j_0}$ for some $1 \leq j_0 \leq n$, then there exists a positive integer s such that $\mathcal{B}^{(s)}_{\boldsymbol{x}}(\boldsymbol{x}^1, \boldsymbol{y}^1) < \mathcal{B}^{(s)}_{\boldsymbol{x}}(\boldsymbol{x}^2, \boldsymbol{y}^2)$ and $\mathcal{B}^{(s)}_{\boldsymbol{y}}(\boldsymbol{x}^1, \boldsymbol{y}^1) < \mathcal{B}^{(s)}_{y}(\boldsymbol{x}^2, \boldsymbol{y}^2)$.

We state here the modified algorithm for finding μ_0, the largest singular value of an irreducible rectangular tensor.

Algorithm 5. [76]

Step 0: Choose $\rho > 0$, $\boldsymbol{x}^{(1)} > \boldsymbol{0}$, and $\boldsymbol{y}^{(1)} > \boldsymbol{0}$. Set $k := 1$.

Step 1: Compute

$$\begin{aligned}\boldsymbol{\xi}^{(k)} &= \mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}),\\ \boldsymbol{\eta}^{(k)} &= \mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}).\end{aligned}$$

Let

$$\begin{aligned}\underline{\mu}_k &= \min_{x_i^{(k)}>0, y_j^{(k)}>0}\left(\frac{\xi_i^{(k)}}{(x_i^{(k)})^{M-1}}, \frac{\eta_j^{(k)}}{(y_j^{(k)})^{M-1}}\right),\\ \bar{\mu}_{k+1} &= \max_{x_i^{(k)}>0, y_j^{(k)}>0}\left(\frac{\xi_i^{(k)}}{(x_i^{(k)})^{M-1}}, \frac{\eta_j^{(k)}}{(y_j^{(k)})^{M-1}}\right).\end{aligned}$$

Step 2: If $\bar{\mu}_k = \underline{\mu}_k$, then stop. Otherwise, compute

$$\begin{aligned}\boldsymbol{x}^{(k+1)} &= \frac{\left(\boldsymbol{\xi}^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})^{\left[\frac{1}{M-1}\right]}\|},\\ \boldsymbol{y}^{(k+1)} &= \frac{\left(\boldsymbol{\eta}^{(k)}\right)^{\left[\frac{1}{M-1}\right]}}{\|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})^{\left[\frac{1}{M-1}\right]}\|},\end{aligned}$$

replace k by $k+1$ and go to Step 1.

This algorithm is proven to be convergent [76]. First, we give some results which are important in proving the convergence of Algorithm 5.

Lemma 13. [76] Suppose $\{\boldsymbol{x}^{(k)}\}$, $\{\boldsymbol{y}^{(k)}\}$, $\{\boldsymbol{\xi}^{(k)}\}$, $\{\boldsymbol{\eta}^{(k)}\}$ are the sequences produced by Algorithm 5. Then,

(i) For any $k \geq 1, \boldsymbol{x}^{(k)} > 0$, $\boldsymbol{y}^{(k)} > 0$,$\boldsymbol{\xi}^{(k)} > 0$,$\boldsymbol{\eta}^{(k)} > 0$.

$$\begin{aligned}\boldsymbol{x}^{(k+1)^{[M-1]}} &= \frac{\left(\boldsymbol{\xi}^{(k)}\right)}{\|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})\|},\\ \boldsymbol{y}^{(k+1)^{[M-1]}} &= \frac{\left(\boldsymbol{\eta}^{(k)}\right)}{\|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})\|}\end{aligned}$$

(ii) For any positive integer s,

$$\begin{aligned}\mathcal{B}_{\boldsymbol{x}}^{(s)}(\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}) &= \|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})\| \cdots \|(\boldsymbol{\xi}^{(k+s-2)}, \boldsymbol{\eta}^{(k+s-2)})\|\boldsymbol{\xi}^{(k+s-1)},\\ \mathcal{B}_{\boldsymbol{y}}^{(s)}(\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}) &= \|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})\| \cdots \|(\boldsymbol{\xi}^{(k+s-2)}, \boldsymbol{\eta}^{(k+s-2)})\|\boldsymbol{\eta}^{(k+s-1)},\\ \mathcal{B}_{\boldsymbol{x}}^{(s)}(\boldsymbol{e}^{(k)}, \boldsymbol{f}^{(k)}) &= \|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})\| \cdots \|(\boldsymbol{\xi}^{(k+s-1)}, \boldsymbol{\eta}^{(k+s-1)})\|\boldsymbol{\xi}^{(k+s)},\\ \mathcal{B}_{\boldsymbol{y}}^{(s)}(\boldsymbol{e}^{(k)}, \boldsymbol{f}^{(k)}) &= \|(\boldsymbol{\xi}^{(k)}, \boldsymbol{\eta}^{(k)})\| \cdots \|(\boldsymbol{\xi}^{(k+s-1)}, \boldsymbol{\eta}^{(k+s-1)})\|\boldsymbol{\eta}^{(k+s)},\end{aligned}$$

where $\boldsymbol{e}^{(k)} = \boldsymbol{\xi}^{(k)\left[\frac{1}{M-1}\right]}$, $\boldsymbol{f}^{(k)} = \boldsymbol{\eta}^{(k)\left[\frac{1}{M-1}\right]}$, and $\mathcal{B}_{\boldsymbol{x}}^{(s)}$ and $\mathcal{B}_{\boldsymbol{y}}^{(s)}$ are defined in (3.8).

Theorem 31. [76] Assume that $(\mu_0, \boldsymbol{x}_0, \boldsymbol{y}_0)$ is a solution of (24). Then,

$$\rho < \underline{\mu}_1 \leq \underline{\mu}_2 \leq \cdots \leq \underline{\mu}_0 \leq \cdots \leq \bar{\mu}_2 \leq \bar{\mu}_1.$$

We can see that the sequence $\{\underline{\mu}_k\}$ is monotonically increasing. Since the sequence has an upper bound, the limit exists. Similarly, the sequence $\{\bar{\mu}_k\}$ is monotonically decreasing. Since it has a lower bound, the limit also exists. Suppose $\underline{\mu} = \lim_{k\to\infty} \underline{\mu}_k$ and $\bar{\mu} = \lim_{k\to\infty} \bar{\mu}_k$, then $\rho < \underline{\mu} \leq \mu_0 \leq \bar{\mu}$ by Theorem 31.

Theorem 32. [76] Let $\{\boldsymbol{x}^{(k)}\}$, $\{\boldsymbol{y}^{(k)}\}$, $\{\boldsymbol{\xi}^{(k)}\}$, $\{\boldsymbol{\eta}^{(k)}\}$ be the sequences produced by Algorithm 5. Then,

(i) $\{\boldsymbol{x}^{(k)}\}$ and $\{\boldsymbol{y}^{(k)}\}$ have convergent subsequences which converges to $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, respectively. Moreover, $\boldsymbol{x}^* \in \mathbb{R}^m_+ \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{y}^* \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}$

(ii) $\mathcal{B}_{\boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{y}^*) \geq \underline{\mu}(\boldsymbol{x}^*)^{[M-1]}$ and $\mathcal{B}_{\boldsymbol{y}}(\boldsymbol{x}^*, \boldsymbol{y}^*) \geq \underline{\mu}(\boldsymbol{y}^*)^{[M-1]}$

(iii) $\underline{\mu} = \bar{\mu}$.

We have the following convergence result of Algorithm 5.

Theorem 33. [76] Suppose that a nonnegative (p, q)-th order $(m \times n)$ dimensional rectangular tensor $\mathcal{A}$ is irreducible. Assume that $(\mu_0, \boldsymbol{x}_0, \boldsymbol{y}_0)$ is a solution of (24). Then, Algorithm 5 produces the value of μ_0 in a finite number of steps, or generates two convergent sequences $\{\underline{\mu}_k\}$ and $\{\bar{\mu}_k\}$, both of which converge to μ_0. Furthermore, $\mu_0 - \rho$ is the largest singular value of $\mathcal{A}$.

Now we show that Algorithm 5 converges under weak irreducibility condition. This is the main contribution of this chapter.

Theorem 34. Suppose that a nonnegative (p, q)-th order $(m \times n)$ dimensional rectangular tensor $\mathcal{A}$ is weakly irreducible. Assume that $(\mu_0, \boldsymbol{x}_0, \boldsymbol{y}_0)$ is a solution of (24). Then, Algorithm 5 produces the value of μ_0 in a finite number of steps, or generates two convergent sequences $\{\underline{\mu}_k\}$ and $\{\bar{\mu}_k\}$, both of which converge to μ_0. Furthermore, $\mu_0 - \rho$ is the largest singular value of $\mathcal{A}$.

Define the polynomial map $\boldsymbol{P} = (P_1, ..., P_N)^T : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ by

$$\boldsymbol{P}(\boldsymbol{z}) = \begin{pmatrix} \mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q \\ \mathcal{A}\boldsymbol{x}^p\boldsymbol{y}^{q-1} \end{pmatrix}, \tag{3.9}$$

with $N = m+n$, $\boldsymbol{z} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$. Let P_i be a polynomial of degree $d_i \geq 1$, and the coefficient of each monomial in P_i is nonnegative.

The associated graph of $\boldsymbol{P}$ is the directed graph $\mathcal{G}(\boldsymbol{P}) = (V, E(\boldsymbol{P}))$, where $V = \{1, 2, ..., N\}$ and $(i, j) \in E(\boldsymbol{P})$ if the coefficient of variable z_j appears in the expression of P_i, or if this expression contains a monomial with degree less than d_i.

Definition 13. [16] Let $\boldsymbol{P} = (P_1, ..., P_N)^T : \mathbb{R}^N \to \mathbb{R}^N$ be a polynomial map, where each P_i a homogeneous polynomial of degree $d \geq 1$ with nonnegative coefficients. We say $\boldsymbol{P}$ is weakly irreducible if $\mathcal{G}(\boldsymbol{P})$ is strongly connected. If the directed graph $\mathcal{G}(\boldsymbol{P})$ is strongly connected and the great common divisor (g.c.d.) of the lengths of its circuits is equal to one then we say $\boldsymbol{P}$ is weakly primitive.

Another way to check the g.c.d. of the lengths of a graph is by observing the diagonal of its associated matrix. An irreducible matrix has a nonzero main diagonal entry if and only if the associated directed graph has a loop, a closed path of length one.

We can show $\boldsymbol{P}$ is weakly primitive by proving the associated matrix of its graph is primitive. Let $\boldsymbol{M}(\mathcal{G}(\boldsymbol{P}))$ be the associated matrix of graph $\mathcal{G}(\boldsymbol{P})$. We say $\boldsymbol{M}(\mathcal{G}(\boldsymbol{P}))$ is primitive if the graph $\mathcal{G}(\boldsymbol{P})$ is strongly connected and its g.c.d. of the lengths is equal to one.

Definition 14. We say $\mathcal{A}$ is weakly irreducible if $\boldsymbol{P}$ is weakly irreducible.

Define

$$\mathcal{B}(\boldsymbol{z}) = \begin{pmatrix} \mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^q + \rho\boldsymbol{x}^{[M-1]} \\ \mathcal{A}\boldsymbol{x}^p\boldsymbol{y}^{q-1} + \rho\boldsymbol{y}^{[M-1]} \end{pmatrix}, \tag{3.10}$$

and let

$$\mathcal{I}(\boldsymbol{z}) = \begin{pmatrix} \rho\boldsymbol{x}^{[M-1]} \\ \rho\boldsymbol{y}^{[M-1]} \end{pmatrix}. \tag{3.11}$$

Now $\mathcal{B}(\boldsymbol{z}) = \boldsymbol{P}(\boldsymbol{z}) + \mathcal{I}(\boldsymbol{z})$.

Lemma 14. If $\mathcal{A}$ is a weakly irreducible nonnegative (p,q)-th order $(m \times n)$ dimensional rectangular tensor then $\mathcal{B}(\boldsymbol{z})$ is weakly primitive.

Proof. Since $\mathcal{A}$ is weakly irreducible then $\boldsymbol{P}(\boldsymbol{z})$ is a weakly irreducible polynomial. By Definition 13, the graph of $\boldsymbol{P}(\boldsymbol{z})$, $\mathcal{G}(\boldsymbol{P}(\boldsymbol{z}))$ is strongly connected. By Theorem 5, the matrix of $\mathcal{G}(\boldsymbol{P}(\boldsymbol{z}))$ is irreducible. We know that $\mathcal{G}(\mathcal{I}(\boldsymbol{z}))$, graph of $\mathcal{I}(\boldsymbol{z})$ is a graph with self-loop at each vertices. Then, the matrix of $\mathcal{G}(\mathcal{I}(\boldsymbol{z}))$ is a diagonal matrix. Hence, by Corollary 1, the matrix of $\mathcal{G}(\mathcal{B}(\boldsymbol{z}))$ is primitive. By Theorem 6, $\mathcal{G}(\mathcal{B}(\boldsymbol{z}))$ is strongly connected and has g.c.d. equals to 1. This implies $\mathcal{B}(\boldsymbol{z})$ is weakly primitive by Definition 13. □

Now we can prove Theorem 34.

Proof. By Lemma 14 and Corollary 5.1 [16], Algorithm 5 converges. □

3.4 Rate of convergence

In this section, we show that Algorithm 5 is Q-linear convergent when $\mathcal{A}$ is a nonnegative weakly irreducible rectangular tensor of order (p,q) and dimension $(m \times n)$. We use the same argument as in [78].

Define

$$\boldsymbol{F}(\boldsymbol{z}) = \mathcal{B}(\boldsymbol{z}) = \begin{pmatrix} \mathcal{A}\boldsymbol{x}^{p-1}\boldsymbol{y}^{q} + \rho\boldsymbol{x}^{[M-1]} \\ \mathcal{A}\boldsymbol{x}^{p}\boldsymbol{y}^{q-1} + \rho\boldsymbol{y}^{[M-1]} \end{pmatrix}, \tag{3.12}$$

$$\boldsymbol{G}(\boldsymbol{z}) = \boldsymbol{F}(\boldsymbol{z})^{[\frac{1}{M-1}]}, \tag{3.13}$$

$$\boldsymbol{H}(\boldsymbol{z}) = \frac{\boldsymbol{G}(z)}{\phi(\boldsymbol{G}(\boldsymbol{z}))}, \tag{3.14}$$

where $\phi : \mathbb{R}_+^N \to \mathbb{R}_+$ is defined by

$$\phi(\boldsymbol{z}) = ||\boldsymbol{z}||_1 = \sum_{i=1}^{N} z_i, \tag{3.15}$$

for any nonnegative $\boldsymbol{z} \in \mathbb{R}_+^N$. We can see that the sequence $\{\boldsymbol{z}^{(k)}\}$ in Algorithm 5 is generated by

$$\boldsymbol{z}^{(k+1)} = \boldsymbol{H}(\boldsymbol{z}^{(k)}), \quad k = 1, 2, ..., \tag{3.16}$$

and $\phi(\boldsymbol{z}^{(k)}) = 1$ for all $k = 1, 2, ...$

Lemma 15. Let $\mathcal{A}$, μ_0, $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ be as in Theorem 34, and let $\boldsymbol{H}'(\boldsymbol{z}_0)$ be the Jacobian of the function $\boldsymbol{H}$ at $\boldsymbol{z}_0$. Then,

$$\rho(\boldsymbol{H}'(\boldsymbol{z}_0)) < 1.$$

Proof. Let μ_0 be the largest singular value of $\mathcal{B}$ and $\boldsymbol{z}_0$ be the corresponding eigenvector. We have $\boldsymbol{H}(\boldsymbol{z}_0) = \boldsymbol{G}(\boldsymbol{z}_0)/\phi(\boldsymbol{G}(\boldsymbol{z}_0))$. We want to show

$$\rho(\boldsymbol{H}'(\boldsymbol{z}_0)) = \rho(\frac{\boldsymbol{G}'(\boldsymbol{z}_0)\phi(\boldsymbol{G}(\boldsymbol{z}_0)) - \boldsymbol{G}(\boldsymbol{z}_0)\phi'(\boldsymbol{G}(\boldsymbol{z}_0))}{\phi^2(\boldsymbol{G}(\boldsymbol{z}_0))}) < 1.$$

We have $\boldsymbol{F}(\boldsymbol{z}_0) = \mathcal{B}(\boldsymbol{z}_0) = \mu_0 \boldsymbol{z}_0^{[M-1]}$ and $\phi(\boldsymbol{z}_0) = 1$. Hence, $\boldsymbol{G}(\boldsymbol{z}_0) = (\boldsymbol{F}(\boldsymbol{z}_0))^{[\frac{1}{M-1}]} = \mu_0^{[\frac{1}{M-1}]}\boldsymbol{z}_0$. Let $\mu_1 = \mu_0^{[\frac{1}{M-1}]}$ so we have

$$\boldsymbol{G}(\boldsymbol{z}_0) = \mu_1 \boldsymbol{z}_0. \tag{3.17}$$

Now we compute $\boldsymbol{G}'(\boldsymbol{z}_0)$, the Jacobian of $\boldsymbol{G}$ at $\boldsymbol{z}_0$. Let

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{z}_0) &= (\boldsymbol{F}(\boldsymbol{z}_0))^{[\frac{1}{M-1}]} \\ &= \begin{bmatrix} (F_1(\boldsymbol{z}_0))^{[\frac{1}{M-1}]} \\ (F_2(\boldsymbol{z}_0))^{[\frac{1}{M-1}]} \\ \vdots \\ (F_N(\boldsymbol{z}_0))^{[\frac{1}{M-1}]} \end{bmatrix}, \end{aligned}$$

$$\nabla((F_1(\boldsymbol{z}_0))^{[\frac{1}{M-1}]}) = \frac{1}{M-1}(F_1(\boldsymbol{z}_0))^{[\frac{2-M}{M-1}]}\nabla F_1(\boldsymbol{z}_0).$$

By the same method, we can get

$$\nabla((F_i(\boldsymbol{z}_0))^{[\frac{1}{M-1}]}) = \frac{1}{M-1}(F_i(\boldsymbol{z}_0))^{[\frac{2-M}{M-1}]}\nabla F_i(\boldsymbol{z}_0) \quad for \quad i = 1, 2, ..., N.$$

Thus the Jacobian of $\boldsymbol{G}$ at $\boldsymbol{z}_0$,

$$\begin{aligned}
\boldsymbol{G}'(\boldsymbol{z}_0) &= \nabla((\boldsymbol{F}(\boldsymbol{z}_0))^{[\frac{1}{M-1}]}) = \begin{pmatrix} \nabla((F_1(\boldsymbol{z}_0))^{[\frac{1}{M-1}]}) \\ \nabla((F_2(\boldsymbol{z}_0))^{[\frac{1}{M-1}]}) \\ \vdots \\ \nabla((F_N(\boldsymbol{z}_0))^{[\frac{1}{M-1}]}) \end{pmatrix} \\
&= \begin{bmatrix} \frac{1}{M-1}(F_1(\boldsymbol{z}_0))^{[\frac{2-M}{M-1}]}\nabla F_1(\boldsymbol{z}_0) \\ \frac{1}{M-1}(F_2(\boldsymbol{z}_0))^{[\frac{2-M}{M-1}]}\nabla F_2(\boldsymbol{z}_0) \\ \vdots \\ \frac{1}{M-1}(F_N(\boldsymbol{z}_0))^{[\frac{2-M}{M-1}]}\nabla F_N(\boldsymbol{z}_0) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{M-1}(F_1(\boldsymbol{z}_0))^{[\frac{2-M}{M-1}]} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{M-1}(F_N(\boldsymbol{z}_0))^{[\frac{2-M}{M-1}]} \end{bmatrix} \begin{bmatrix} \nabla F_1(\boldsymbol{z}_0) \\ \nabla F_2(\boldsymbol{z}_0) \\ \vdots \\ \nabla F_N(\boldsymbol{z}_0) \end{bmatrix} \\
&= diag(\frac{1}{M-1}(\boldsymbol{F}(\boldsymbol{z}_0))^{[\frac{2-M}{M-1}]})\boldsymbol{F}'(\boldsymbol{z}_0) \\
&= diag(\frac{1}{M-1}(\mu_0 \boldsymbol{z}_0^{[M-1]})^{[\frac{2-M}{M-1}]})\boldsymbol{F}'(\boldsymbol{z}_0) \\
&= \frac{1}{M-1} diag((\mu_0 \boldsymbol{z}_0^{[M-1]})^{[\frac{2-M}{M-1}]})\boldsymbol{F}'(\boldsymbol{z}_0) \\
&= \frac{1}{M-1} diag((\mu_0^{\frac{1}{[M-1]}} \boldsymbol{z}_0)^{[2-M]})\boldsymbol{F}'(\boldsymbol{z}_0) \\
&= \frac{1}{M-1} diag((\mu_1 \boldsymbol{z}_0)^{[2-M]})\boldsymbol{F}'(\boldsymbol{z}_0),
\end{aligned}$$

where $\frac{1}{M-1}diag((\mu_1\boldsymbol{z}_0)^{[2-M]})$ is a constant with $\mu_1 > 0$ and $\boldsymbol{z}_0$ is a positive vector. Thus $\mathcal{G}(\boldsymbol{G}'(\boldsymbol{z}_0)) = \mathcal{G}(\boldsymbol{F}'(\boldsymbol{z}_0))$. By Theorem 29, $\boldsymbol{G}'(\boldsymbol{z}_0)$ is a primitive matrix. Since $\boldsymbol{G}'(\boldsymbol{z}_0)$ is a primitive matrix, the eigenvalues $v_1, v_2, ..., v_N$ of $\boldsymbol{G}'(\boldsymbol{z}_0)$ can be ordered as follows.

$$v_1 = \rho(\boldsymbol{G}'(\boldsymbol{u})) > |v_2| \geq |v_3| \geq ... \geq |v_N|. \tag{3.18}$$

Note that $v_1 = \mu_1$. For all $t > 1$, we expand $\boldsymbol{G}(t\boldsymbol{z}_0)$ about $\boldsymbol{z}_0$ using the Taylor's Series,

$$\begin{aligned}
t\mu_1 \boldsymbol{z}_0 &= \boldsymbol{G}(t\boldsymbol{z}_0) \\
&= \boldsymbol{G}(\boldsymbol{z}_0) + \boldsymbol{G}'(\boldsymbol{z}_0)(t\boldsymbol{z}_0 - \boldsymbol{z}_0) + o(||t\boldsymbol{z}_0 - \boldsymbol{z}_0||) \\
&= \boldsymbol{G}(\boldsymbol{z}_0) + (t-1)\,\boldsymbol{G}'(\boldsymbol{z}_0)\boldsymbol{z}_0 + o(||(t-1)\boldsymbol{z}_0|| \\
&= \mu_1 \boldsymbol{z}_0 + (t-1)\,\boldsymbol{G}'(\boldsymbol{z}_0)\boldsymbol{z}_0 + o((t-1)||\boldsymbol{z}_0||), \\
&= \mu_1 \boldsymbol{z}_0 + (t-1)\,\boldsymbol{G}'(\boldsymbol{z}_0)\boldsymbol{z}_0 + o(t-1), \\
t\mu_1 \boldsymbol{z}_0 - \mu_1 \boldsymbol{z}_0 &= (t-1)\,\boldsymbol{G}'(\boldsymbol{z}_0)\boldsymbol{z}_0 + o(t-1), \\
(t-1)\mu_1 \boldsymbol{z}_0 &= (t-1)\,\boldsymbol{G}'(\boldsymbol{z}_0)\boldsymbol{z}_0 + o(t-1),
\end{aligned}$$

which implies $\boldsymbol{G}'(\boldsymbol{z}_0)\boldsymbol{z}_0 = \mu_1 \boldsymbol{z}_0$. Since $\boldsymbol{G}'(\boldsymbol{z}_0)$ is a primitive matrix and $\boldsymbol{z}_0 > \boldsymbol{0}$, by Theorem 7, $\boldsymbol{z}_0$ is an eigenvector of $\boldsymbol{G}'(\boldsymbol{z}_0)$ associated with the largest eigenvalue $\mu_1 = v_1$. From (3.15) and (3.18),

$$\begin{aligned}
\phi(\boldsymbol{G}(\boldsymbol{z}_0)) &= \phi(\mu_1 \boldsymbol{z}_0) \\
&= \sum_{i=1}^{N} (\mu_1 (\boldsymbol{z}_0)_i) \\
&= \mu_1 \sum_{i=1}^{N} ((\boldsymbol{z}_0)_i) \\
&= \mu_1 \phi(\boldsymbol{z}_0) \\
&= \mu_1 (1) \\
&= \mu_1.
\end{aligned}$$

We also have

$$\begin{aligned}
\phi(\boldsymbol{G}(\boldsymbol{z}_0)) &= G_1(\boldsymbol{z}_0) + G_2(\boldsymbol{z}_0) + ... + G_N(\boldsymbol{z}_0), \\
\phi'(\boldsymbol{G}(\boldsymbol{z}_0)) &= G_1'(\boldsymbol{z}_0) + G_2'(\boldsymbol{z}_0) + ... + G_N'(\boldsymbol{z}_0) \\
&= \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} G_1'(\boldsymbol{z}_0) \\ G_2'(\boldsymbol{z}_0) \\ \vdots \\ G_N'(\boldsymbol{z}_0) \end{pmatrix} \\
&= \boldsymbol{e}\,\boldsymbol{G}'(\boldsymbol{z}_0),
\end{aligned}$$

where $\boldsymbol{e}$ is the row vector of all ones with N-dimension. Thus, from (3.14),

$$\begin{aligned}
\boldsymbol{H}(\boldsymbol{z}_0) &= \frac{\boldsymbol{G}(\boldsymbol{z}_0)}{\phi(\boldsymbol{G}(\boldsymbol{z}_0))}, \\
\boldsymbol{H}'(\boldsymbol{z}_0) &= \frac{\boldsymbol{G}'(\boldsymbol{z}_0)\phi(\boldsymbol{G}(\boldsymbol{z}_0)) - \boldsymbol{G}(\boldsymbol{z}_0)\phi'(\boldsymbol{G}(\boldsymbol{z}_0))}{\phi^2(\boldsymbol{G}(\boldsymbol{z}_0))} \\
&= \frac{\boldsymbol{G}'(\boldsymbol{z}_0)\mu_1 - \boldsymbol{G}(\boldsymbol{z}_0)\phi'(\boldsymbol{G}(\boldsymbol{z}_0))}{\mu_1^2} \\
&= \frac{\boldsymbol{G}'(\boldsymbol{z}_0)\mu_1 - \boldsymbol{G}(\boldsymbol{z}_0)\boldsymbol{e}\,\boldsymbol{G}'(\boldsymbol{z}_0)}{\mu_1^2} \\
&= \frac{\boldsymbol{G}'(\boldsymbol{z}_0)}{\mu_1} - \frac{\boldsymbol{G}(\boldsymbol{z}_0)\boldsymbol{e}\ \boldsymbol{G}'(\boldsymbol{z}_0)}{\mu_1^2} \\
&= \frac{\boldsymbol{G}'(\boldsymbol{z}_0)}{\mu_1} - \frac{\mu_1\boldsymbol{z}_0\boldsymbol{e}\,\boldsymbol{G}'(\boldsymbol{z}_0)}{\mu_1^2} \\
&= \frac{\boldsymbol{G}'(\boldsymbol{z}_0) - \boldsymbol{z}_0\boldsymbol{e}\ \boldsymbol{G}'(\boldsymbol{z}_0)}{\mu_1}.
\end{aligned}$$

Let $\boldsymbol{M} = \boldsymbol{G}'(\boldsymbol{z}_0)$ and $\boldsymbol{Q} = \boldsymbol{M} - \boldsymbol{z}_0\boldsymbol{e}\boldsymbol{M}$. Now we have $\boldsymbol{H}'(\boldsymbol{z}_0) = (\boldsymbol{Q}/\mu_1)$. In this proof, we want to show

$$\rho(\boldsymbol{H}'(\boldsymbol{z}_0)) = \rho(\frac{\boldsymbol{Q}}{\mu_1}) < 1.$$

In order to do that, first, show that the spectral radius of $\boldsymbol{Q}$ is equal to $|\upsilon_2|$. However, it is enough to prove that the spectrum of $\boldsymbol{Q}$ is $\sigma(\boldsymbol{Q}) = \{0, \upsilon_2, \upsilon_3, ..., \upsilon_N\}$. We have

$$\begin{aligned}
\phi(\boldsymbol{z}_0) &= 1 \\
&= (z_0)_1 + (z_0)_2 + ... + (z_0)_N \\
&= \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} (z_0)_1 \\ (z_0)_2 \\ \vdots \\ (z_0)_N \end{pmatrix} \\
&= \boldsymbol{e}\boldsymbol{z}_0,
\end{aligned}$$

hence $\boldsymbol{e}\boldsymbol{z}_0 = 1$, therefore

$$\begin{aligned}
\boldsymbol{Q} &= \boldsymbol{M} - \boldsymbol{z}_0\boldsymbol{e}\boldsymbol{M}, \\
\boldsymbol{Q}^T\boldsymbol{e}^T &= (\boldsymbol{M} - \boldsymbol{z}_0\boldsymbol{e}\boldsymbol{M})^T\boldsymbol{e}^T \\
&= \boldsymbol{M}\boldsymbol{e}^T - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{z}_0^T\boldsymbol{e}^T \\
&= \boldsymbol{M}\boldsymbol{e}^T - \boldsymbol{M}^T\boldsymbol{e}^T(\boldsymbol{e}\boldsymbol{z}_0)^T \\
&= \boldsymbol{M}\boldsymbol{e}^T - \boldsymbol{M}^T\boldsymbol{e}^T(1) \\
&= \boldsymbol{0}.
\end{aligned}$$

Now we can say $\boldsymbol{e}^T$ is an eigenvector of $\boldsymbol{Q}^T$ associated with the eigenvalue 0.

We consider two cases for $\boldsymbol{M}^T$.

Case 1: The matrix $\boldsymbol{M}^T = \boldsymbol{G}'(\boldsymbol{z}_0)^T$ is diagonizable, which means, $\boldsymbol{M}^T$ is semisimple. For $i = 2, 3, ..., N$, we suppose $\boldsymbol{M}^T\boldsymbol{w}^i = v_i\boldsymbol{w}^i$, that is $\boldsymbol{w}^i$ is an eigenvector of $\boldsymbol{M}^T$ associated with the eigenvalue v_i. We also assume the set of eigenvector $\{\boldsymbol{w}^i : i = 2, 3, ..., N\}$ is linearly independent. Therefore, for $i = 2, 3, ..., N$,

$$v_i\boldsymbol{z}_0^T\boldsymbol{w}^i = \boldsymbol{z}_0^T v_i\boldsymbol{w}^i = \boldsymbol{z}_0^T\boldsymbol{M}^T\boldsymbol{w}^i.$$

We have previously $\boldsymbol{G}'(\boldsymbol{z}_0)\boldsymbol{z}_0 = \boldsymbol{M}\boldsymbol{z}_0 = \mu_1\boldsymbol{z}_0$. So,

$$\begin{aligned}
(\boldsymbol{M}\boldsymbol{z}_0)^T &= (\mu_1\boldsymbol{z}_0)^T, \\
\boldsymbol{z}_0^T\boldsymbol{M}^T &= \mu_1\boldsymbol{z}_0^T. \qquad (3.19)
\end{aligned}$$

Hence,

$$\begin{aligned}
v_i\boldsymbol{z}_0^T\boldsymbol{w}^i = \boldsymbol{z}_0^T\boldsymbol{M}^T\boldsymbol{w}^i &= \mu_1\boldsymbol{z}_0^T\boldsymbol{w}^i, \\
(v_i - \mu_1)\boldsymbol{z}_0^T\boldsymbol{w}^i &= 0.
\end{aligned}$$

It is either $v_i = \mu_1$ or $\boldsymbol{z}_0^T\boldsymbol{w}^i = 0$ for $i = 2, 3, ..., N$. However, $v_i \neq \mu_1$ for $i = 2, 3, ..., N$ by (3.18). So we must have $\boldsymbol{z}_0^T\boldsymbol{w}^i = 0$.
Now we have

$$\begin{aligned}
\boldsymbol{Q}^T\boldsymbol{w}^i &= (\boldsymbol{M} - \boldsymbol{z}_0\boldsymbol{e}\boldsymbol{M})^T\boldsymbol{w}^i \\
&= (\boldsymbol{M}^T - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{z}_0^T)\boldsymbol{w}^i \\
&= \boldsymbol{M}^T\boldsymbol{w}^i - \boldsymbol{M}^T\boldsymbol{e}^T\boldsymbol{z}_0^T\boldsymbol{w}^i \\
&= \boldsymbol{M}^T\boldsymbol{w}^i - \boldsymbol{0}.
\end{aligned}$$

Since we assume $\boldsymbol{M}^T\boldsymbol{w}^i = v_i\boldsymbol{w}^i$, we have $\boldsymbol{Q}^T\boldsymbol{w}^i = v_i\boldsymbol{w}^i$. The vector $\boldsymbol{w}^i$ is an eigenvector of $\boldsymbol{Q}^T$ associated with the eigenvalue v_i for $i = 2, 3, ..., N$. Now we prove the set $\{\boldsymbol{e}^T, \boldsymbol{w}^i, i = 2, 3, ..., N\}$, which is the set of eigenvectors of $\boldsymbol{Q}$ is linearly independent. We assume

$$\alpha_1\boldsymbol{e}^T + \alpha_2\boldsymbol{w}^2 + ... + \alpha_n\boldsymbol{w}^N = \boldsymbol{0}, \tag{3.20}$$

and $v_i \neq 0$ for $i = 2, 3, ..., p$ and $v_j = 0$ for $j = p+1, ..., N$. We have

$$\begin{aligned}\boldsymbol{Q}^T\boldsymbol{e}^T &= 0\boldsymbol{e}^T,\\ \boldsymbol{Q}^T\boldsymbol{w}^i &= v_i\boldsymbol{w}^i, \quad i = 2, 3, ..., N.\end{aligned}$$

Therefore,

$$\begin{aligned}\boldsymbol{Q}^T\boldsymbol{e}^T + \boldsymbol{Q}^T\boldsymbol{w}^2 + ... + \boldsymbol{Q}^T\boldsymbol{w}^N &= 0\boldsymbol{e}^T + v_2\boldsymbol{w}^2 + ... + v_p\boldsymbol{w}^p,\\ \alpha_1\boldsymbol{Q}^T\boldsymbol{e}^T + \alpha_2\boldsymbol{Q}^T\boldsymbol{w}^2 + ... + \alpha_N\boldsymbol{Q}^T\boldsymbol{w}^N &= \alpha_2 v_2\boldsymbol{w}^2 + ... + \alpha_p v_p\boldsymbol{w}^p. \quad (3.21)\end{aligned}$$

From (3.20),

$$\begin{gathered}\boldsymbol{Q}^T(\alpha_1\boldsymbol{e}^T + \alpha_2\boldsymbol{w}^2 + ... + \alpha_n\boldsymbol{w}^N) = \boldsymbol{0},\\ \alpha_1\boldsymbol{Q}^T\boldsymbol{e}^T + \alpha_2\boldsymbol{Q}^T\boldsymbol{w}^2 + ... + \alpha_N\boldsymbol{Q}^T\boldsymbol{w}^N = \boldsymbol{0}.\end{gathered}$$

Hence the right side of (3.21) became,

$$\boldsymbol{0} = \alpha_2 v_2\boldsymbol{w}^2 + ... + \alpha_p v_p\boldsymbol{w}^p.$$

Since set $\{\boldsymbol{w}^i, i = 2, 3, ..., N\}$ is linearly independent, $\alpha_2 = \alpha_3 = ... = \alpha_p = 0$. By (3.20),

$$\begin{gathered}\alpha_1\boldsymbol{e}^T + \alpha_{p+1}\boldsymbol{w}^{p+1} + ... + \alpha_N\boldsymbol{w}^N = \boldsymbol{0}, \quad (3.22)\\ \boldsymbol{M}^T(\alpha_1\boldsymbol{e}^T + \alpha_{p+1}\boldsymbol{w}^{p+1} + ... + \alpha_N\boldsymbol{w}^N) = \boldsymbol{0},\\ \alpha_1\boldsymbol{M}^T\boldsymbol{e}^T + \alpha_{p+1}\boldsymbol{M}^T\boldsymbol{w}^{p+1} + ... + \alpha_N\boldsymbol{M}^T\boldsymbol{w}^N = \boldsymbol{0}.\end{gathered}$$

Since $\boldsymbol{M}^T\boldsymbol{w}^i = v_i\boldsymbol{w}^i$ and $v_j = 0$ for $j = p+1, ..., N$,

$$\begin{aligned}\alpha_1\boldsymbol{M}^T\boldsymbol{e}^T + \alpha_{p+1}v_{p+1}\boldsymbol{w}^{p+1} + ... + \alpha_N v_N\boldsymbol{w}^N &= \boldsymbol{0}, \quad (3.23)\\ \alpha_1 M^T e^T &= 0. \quad (3.24)\end{aligned}$$

We have $\alpha_1 = 0$ since $\boldsymbol{M}^T \boldsymbol{e}^T > \boldsymbol{0}$ because $\boldsymbol{M}$ is diagonalizable. Hence, by (3.22),

$$\alpha_{p+1}\boldsymbol{w}^{p+1} + ... + \alpha_N \boldsymbol{w}^N = \boldsymbol{0}. \tag{3.25}$$

We know that the set $\{\boldsymbol{w}^i, i = p+1, ..., N\}$ is linearly independent, therefore $\alpha_{p+1} = ... = \alpha_N = 0$. Now we have, $\alpha_1 = \alpha_2 = ... = \alpha_N = 0$ which means the set of eigenvectors of $\boldsymbol{Q}$, $\{\boldsymbol{e}^T, \boldsymbol{w}^i, i = 2, 3, ..., N\}$ is linearly independent. Hence the spectrum of $\boldsymbol{Q}$ is $\sigma(\boldsymbol{Q}) = \{0, v_2, v_3, ..., v_N\}$.

Case 2: The matrix $\boldsymbol{M}^T$ is not diagonalizable or defective. A defective $N \times N$ matrix has less than N distinct eigenvalues. Suppose $\boldsymbol{M}^T$ has $q < N$ distinct eigenvalues $v_1 = \mu_1, v_2, ..., v_q$, and these eigenvalues can be ordered as follows.

$$v_1 = \mu_1 > |v_2| \geq |v_3| \geq ... \geq |v_q|. \tag{3.26}$$

Then, $\boldsymbol{M}^T$ has the form $\boldsymbol{M}^T = \boldsymbol{X}\boldsymbol{J}\boldsymbol{X}^{-1}$, where $\boldsymbol{J} = diag\{\boldsymbol{J}_1, \boldsymbol{J}_2, ..., \boldsymbol{J}_q\}$ is in canonical form. Let the square matrices $\boldsymbol{J}_i$, $i = 1, 2, ..., q$ be the Jordan blocks with various sizes in the form of

$$\boldsymbol{J}_i = \begin{pmatrix} v_i & 1 & & & \\ & v_i & 1 & & \\ & & v_i & \ddots & \\ & & & \ddots & 1 \\ & & & & v_i \end{pmatrix},$$

where v_i is an eigenvalue of $\boldsymbol{M}^T$. Set $\boldsymbol{J}_1 = [\mu_1]$ and $\boldsymbol{X}_i$ is the i-th column vector of $\boldsymbol{X}$, $i = 1, 2, ..., N$. Let l_i be the size of $\boldsymbol{J}_i$ for each Jordan block J_i, where $i = 1, 2, ..., q$. We have $\boldsymbol{M}^T = \boldsymbol{X}\boldsymbol{J}\boldsymbol{X}^{-1}$, hence

$$\begin{aligned} &\boldsymbol{M}^T\boldsymbol{X} = \boldsymbol{X}\boldsymbol{J}, \\ &\boldsymbol{M}^T \begin{pmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \boldsymbol{X}_3 & \boldsymbol{X}_4 & \cdots \end{pmatrix} = \\ &\begin{pmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \boldsymbol{X}_3 & \boldsymbol{X}_4 & \cdots \end{pmatrix} \begin{pmatrix} \mu_1 & 1 & & & \\ & v_2 & 1 & & \\ & & v_2 & \ddots & \\ & & & v_2 & 1 \\ & & & & \ddots \end{pmatrix}. \end{aligned}$$

Since we consider $\boldsymbol{J}_i$, where $i = 1, 2, ..., q$,

$$\begin{aligned}
\boldsymbol{M}^T \boldsymbol{X}_2 &= \boldsymbol{X}_1 + v_2 \boldsymbol{X}_2, \\
\boldsymbol{M}^T \boldsymbol{X}_3 &= \boldsymbol{X}_2 + v_2 \boldsymbol{X}_3, \\
\boldsymbol{M}^T \boldsymbol{X}_4 &= \boldsymbol{X}_3 + v_2 \boldsymbol{X}_4, \\
&\vdots \\
\boldsymbol{M}^T \boldsymbol{X}_{l_2+1} &= \boldsymbol{X}_{l_2} + v_2 \boldsymbol{X}_{l_2+1}, \\
\boldsymbol{M}^T \boldsymbol{X}_{l_2+2} &= v_3 \boldsymbol{X}_{l_2+2}, \\
\boldsymbol{M}^T \boldsymbol{X}_{l_2+3} &= \boldsymbol{X}_{l_2+1} + v_3 \boldsymbol{X}_{l_2+2}, \\
&\vdots
\end{aligned}$$

Same as in Case 1, since we have $\boldsymbol{M}^T \boldsymbol{X}_2 = v_2 \boldsymbol{X}_2$ and (3.19), then,

$$\begin{aligned}
v_2 \boldsymbol{z}_0^T \boldsymbol{X}_2 &= \boldsymbol{z}_0^T v_2 \boldsymbol{X}_2 = \boldsymbol{z}_0^T \boldsymbol{M}^T \boldsymbol{X}_2 = \mu_1 \boldsymbol{z}_0^T \boldsymbol{X}_2, \\
(v_2 - \mu_1) \boldsymbol{z}_0^T \boldsymbol{X}_2 &= 0.
\end{aligned}$$

By (3.26), $v_2 \neq \mu_1$. Thus, $\boldsymbol{z}_0^T \boldsymbol{X}_2 = 0$. Hence,

$$\begin{aligned}
\boldsymbol{Q}^T &= (\boldsymbol{M} - \boldsymbol{z}_0 \boldsymbol{e} \boldsymbol{M})^T, \\
\boldsymbol{Q}^T \boldsymbol{X}_2 &= (\boldsymbol{M} - \boldsymbol{z}_0 \boldsymbol{e} \boldsymbol{M})^T \boldsymbol{X}_2 \\
&= (\boldsymbol{M}^T - \boldsymbol{M}^T \boldsymbol{e}^T \boldsymbol{z}_0^T) \boldsymbol{X}_2 \\
&= \boldsymbol{M}^T \boldsymbol{X}_2 - \boldsymbol{M}^T \boldsymbol{e}^T \boldsymbol{z}_0^T \boldsymbol{X}_2 \\
&= \boldsymbol{M}^T \boldsymbol{X}_2 - \boldsymbol{0} \\
&= v_2 \boldsymbol{X}_2,
\end{aligned}$$

which means $\boldsymbol{X}_2$ is an eigenvector of $\boldsymbol{Q}^T$ associated with eigenvalue v_2. Since $\boldsymbol{M}^T \boldsymbol{X}_3 = v_2 \boldsymbol{X}_3 + \boldsymbol{X}_2$,

$$\begin{aligned}
v_2 \boldsymbol{z}_0^T \boldsymbol{X}_3 &= \boldsymbol{z}_0^T v_2 \boldsymbol{X}_3 \\
&= \boldsymbol{z}_0^T (\boldsymbol{M}^T \boldsymbol{X}_3 - \boldsymbol{X}_2) \\
&= \boldsymbol{z}_0^T \boldsymbol{M}^T \boldsymbol{X}_3 - \boldsymbol{z}_0^T \boldsymbol{X}_2 \\
&= \boldsymbol{z}_0^T \boldsymbol{M}^T \boldsymbol{X}_3 - 0,
\end{aligned}$$

and by (3.19), $v_2 \boldsymbol{z}_0^T \boldsymbol{X}_3 = (\mu_1 \boldsymbol{z}_0^T)\boldsymbol{X}_3$. Now we get $(v_2 - \mu_1)\boldsymbol{z}_0^T \boldsymbol{X}_3 = 0$. We have $\boldsymbol{z}_0^T \boldsymbol{X}_3 = 0$ since $v_2 \neq \mu_1$ by (3.26). Hence,

$$\begin{aligned}
\boldsymbol{Q}^T &= (\boldsymbol{M} - \boldsymbol{z}_0 \boldsymbol{e} \boldsymbol{M})^T, \\
\boldsymbol{Q}^T \boldsymbol{X}_3 &= (\boldsymbol{M} - \boldsymbol{z}_0 \boldsymbol{e} \boldsymbol{M})^T \boldsymbol{X}_3 \\
&= (\boldsymbol{M}^T - \boldsymbol{M}^T \boldsymbol{e}^T \boldsymbol{z}_0^T)\boldsymbol{X}_3 \\
&= \boldsymbol{M}^T \boldsymbol{X}_3 - \boldsymbol{M}^T \boldsymbol{e}^T \boldsymbol{z}_0^T \boldsymbol{X}_3 \\
&= \boldsymbol{M}^T \boldsymbol{X}_3 - \boldsymbol{0} \\
&= v_2 \boldsymbol{X}_3 + \boldsymbol{X}_2.
\end{aligned}$$

Similarly, $v_2 \boldsymbol{z}_0^T \boldsymbol{X}_4 = \boldsymbol{z}_0^T v_2 \boldsymbol{X}_4 = \boldsymbol{z}_0^T(\boldsymbol{M}^T \boldsymbol{X}_4 - \boldsymbol{X}_3) = \boldsymbol{z}_0^T \boldsymbol{M}^T \boldsymbol{X}_4 - \boldsymbol{z}_0^T \boldsymbol{X}_3$. But we have shown $\boldsymbol{z}_0^T \boldsymbol{X}_3 = 0$ and we also have (3.19). Therefore $v_2 \boldsymbol{z}_0^T \boldsymbol{X}_4 = \boldsymbol{z}_0^T \boldsymbol{M}^T \boldsymbol{X}_4 = (\mu_1 \boldsymbol{z}_0^T)\boldsymbol{X}_4$. Thus $(v_2 - \mu_1)\boldsymbol{z}_0^T \boldsymbol{X}_4 = 0$. By (3.26), $\boldsymbol{z}_0^T \boldsymbol{X}_4 = 0$. Then,

$$\begin{aligned}
\boldsymbol{Q}^T &= (\boldsymbol{M} - \boldsymbol{z}_0 \boldsymbol{e} \boldsymbol{M})^T, \\
\boldsymbol{Q}^T \boldsymbol{X}_4 &= (\boldsymbol{M} - \boldsymbol{z}_0 \boldsymbol{e} \boldsymbol{M})^T \boldsymbol{X}_4 \\
&= (\boldsymbol{M}^T - \boldsymbol{M}^T \boldsymbol{e}^T \boldsymbol{u}^T)\boldsymbol{X}_4 \\
&= \boldsymbol{M}^T \boldsymbol{X}_4 - \boldsymbol{M}^T \boldsymbol{e}^T \boldsymbol{z}_0^T \boldsymbol{X}_4 \\
&= \boldsymbol{M}^T \boldsymbol{X}_4 - \boldsymbol{0} \\
&= v_2 \boldsymbol{X}_4 + \boldsymbol{X}_3.
\end{aligned}$$

Hence,

$$\begin{aligned}
\boldsymbol{Q}^T\boldsymbol{X}_2 &= v_2\boldsymbol{X}_2,\\
\boldsymbol{Q}^T\boldsymbol{X}_3 &= v_2\boldsymbol{X}_3+\boldsymbol{X}_2,\\
\boldsymbol{Q}^T\boldsymbol{X}_4 &= v_2\boldsymbol{X}_4+\boldsymbol{X}_2,\\
&\vdots\\
\boldsymbol{Q}^T\boldsymbol{X}_{l_2+1} &= \boldsymbol{X}_{l_2}+v_2\boldsymbol{X}_{l_2+1},\\
\boldsymbol{Q}^T\boldsymbol{X}_{l_2+2} &= v_3\boldsymbol{X}_{l_2+2},\\
\boldsymbol{Q}^T\boldsymbol{X}_{l_2+3} &= \boldsymbol{X}_{l_2+1}+v_3\boldsymbol{X}_{l_2+2},\\
&\vdots
\end{aligned}$$

Same as in Case 1, we show that the set $\{\boldsymbol{e}^T, \boldsymbol{X}_i, i = 2, 3, ..., N\}$ is linearly independent. Let $\boldsymbol{Y} = [\boldsymbol{e}^T, \boldsymbol{X}_i, i = 2, 3, ..., N]$. Therefore, $\boldsymbol{Q}^T\boldsymbol{Y} = \boldsymbol{Y}diag\{[0], \boldsymbol{J}_2, ..., \boldsymbol{J}_q\}$. Now we have the spectrum of $\boldsymbol{Q}$, $\sigma(\boldsymbol{Q}) = \sigma(\boldsymbol{Q}^T) = \{0, v_2, v_3, ..., v_q\}$, which means, we have the spectral radius of $\boldsymbol{Q}$, $\rho(\boldsymbol{Q}) = |v_2|$. Hence,

$$\rho(\boldsymbol{H}'(\boldsymbol{z}_0)) = \rho(\frac{\boldsymbol{Q}}{\mu_1}) = \frac{|v_2|}{\mu_1} < 1,$$

since $\mu_1 > |v_2|$. □

Now we can prove that Algorithm 5 is Q-linear convergent if $\mathcal{A}$ is a weakly irreducible nonnegative rectangular tensor.

Theorem 35. Let $\mathcal{A}$ and $\{\boldsymbol{z}_0^{(k)}\}$ be as in Theorem 34. Then the convergence rate of sequence $\{\boldsymbol{z}_0^{(k)}\}$ is Q-linear , i.e., there exists a vector norm $||\cdot||$ such that

$$\limsup_{k\to\infty}\frac{||\boldsymbol{z}^{(k+1)}-\boldsymbol{z}_0||}{||\boldsymbol{z}^{(k)}-\boldsymbol{z}_0||} < 1. \tag{3.27}$$

Proof. By Proposition 1, there exist an $\epsilon > 0$ and a spectral norm $||\cdot||_\epsilon \in \mathbb{N}$ such that

$$||\boldsymbol{H}'(\boldsymbol{z}_0)||_\epsilon \le \rho(\boldsymbol{H}'(\boldsymbol{z}_0)) + \epsilon.$$

Then, by Lemma 15,

$$||\boldsymbol{H}'(\boldsymbol{z}_0)||_\epsilon \le \rho(\boldsymbol{H}'(\boldsymbol{z}_0)) + \epsilon < 1. \tag{3.28}$$

Therefore, by equation (3.16), $\boldsymbol{z}^{k+1} = \boldsymbol{H}(\boldsymbol{z}^k)$ for $k = 1, 2, ...$ and $\boldsymbol{z}_0 = \boldsymbol{H}(\boldsymbol{z}_0)$. Hence,

$$\boldsymbol{z}^{(k+1)} - \boldsymbol{z}_0 = \boldsymbol{H}(\boldsymbol{z}^{(k)}) - \boldsymbol{H}(\boldsymbol{z}_0). \tag{3.29}$$

Expand $\boldsymbol{z}^k$ at $\boldsymbol{z}_0$ using Taylor expansion,

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{z}^{(k)}) &= \boldsymbol{H}(\boldsymbol{z}_0) + \boldsymbol{H}'(\boldsymbol{z}_0)(\boldsymbol{z}^{(k)} - \boldsymbol{z}_0) + o(||\boldsymbol{z}^{(k)} - \boldsymbol{z}_0||_\epsilon), \\ \boldsymbol{H}(\boldsymbol{z}^{(k)}) - \boldsymbol{H}(\boldsymbol{z}_0) &= \boldsymbol{H}'(\boldsymbol{z}_0)(\boldsymbol{z}^{(k)} - \boldsymbol{z}_0) + o(||\boldsymbol{z}^{(k)} - \boldsymbol{z}_0||_\epsilon), \end{aligned} \tag{3.30}$$

and we have

$$\begin{aligned} \boldsymbol{z}^{(k+1)} - \boldsymbol{z}_0 &= \boldsymbol{H}(\boldsymbol{z}^{(k)}) - \boldsymbol{H}(\boldsymbol{z}_0) = \boldsymbol{H}'(\boldsymbol{z}_0)(\boldsymbol{z}^{(k)} - \boldsymbol{z}_0) + o(||\boldsymbol{z}^{(k)} - \boldsymbol{z}_0||_\epsilon), \\ \frac{\boldsymbol{z}^{(k+1)} - \boldsymbol{z}_0}{\boldsymbol{z}^{(k)} - \boldsymbol{z}_0} &= \boldsymbol{H}'(\boldsymbol{z}_0), \\ \frac{||\boldsymbol{z}^{(k+1)} - \boldsymbol{z}_0||_\epsilon}{||\boldsymbol{z}^{(k)} - \boldsymbol{z}_0||_\epsilon} &= ||\boldsymbol{H}'(\boldsymbol{z}_0)||_\epsilon. \end{aligned}$$

From equation (3.28), we obtain

$$\limsup_{k\to\infty} \frac{||\boldsymbol{z}^{k+1} - \boldsymbol{z}_0||_\epsilon}{||\boldsymbol{z}^{(k)} - \boldsymbol{z}_0||_\epsilon} < 1,$$

therefore, the sequence $\{\boldsymbol{z}^{(k)}\}$ has the convergence rate of Q-linear. □

3.5 Conclusion

In this chapter, we studied the rectangular tensors and its largest singular value. Algorithms for finding the largest singular value of rectangular tensors were also discussed. We also proved that the modified algorithm as proposed in [76] was convergent under weak irreducibility condition and has Q-linear rate of convergence. The study of rectangular tensor is relatively new. The latest development, a variant of Algorithm 4 has emerged and it was presented in [73] and proven to be convergent under some assumption.

Chapter 4

Nonnegative Polynomial Eigenvalue Problem

4.1 Introduction

In this chapter, we consider the polynomial eigenvalue problem [4], which has a wide range of applications such as in higher-order Markov chains [43], spectral hypergraph theory [36], medical resonance imaging [54, 5], positive definiteness of even-order multivariate forms in automatical control [44], best-rank one approximation in data analysis [34, 30, 53], population biology [46], mathematical economics [8, 42], discrete event systems [3, 22], idempotent analysis [12, 32], stochastic control [31] and game theory [57].

In Chapter 2, we studied a method for computing the largest eigenvalue of a nonnegative square tensor and its version which is convergent for irreducible tensors. This method was an extension of the Collatz method for finding the largest spectral radius of an irreducible nonnegative matrix [10, 60, 64], which we have discussed in Chapter 1. The methods in Chapter 2 also were extended to irreducible nonnegative rectangular tensors which were discussed in Chapter 3.

However, all the methods in the previous chapters are for homogeneous polynomials, it is unknown if these methods can be used to solve the eigenvalue problem of nonhomogeneous polynomials. In this chapter, we propose an algorithm for finding the largest eigenvalue of a nonhomogeneous nonnegative polynomial.

4.2 Nonnegative Polynomials

Let $\boldsymbol{P} = (P_1, ..., P_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map

$$P_i(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{R}^n_+} a_{i\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}, \quad i = 1, ..., n, \tag{4.1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, $a_{i\boldsymbol{\alpha}} = a_{i\alpha_1 \cdots \alpha_n}$, $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and the degree of each monomial is $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + ... + \alpha_n$. For each P_i, if d_i is the maximum degree of its monomials, then d_i is the degree of P_i . We call $\boldsymbol{P}$ a nonnegative polynomial if $a_{i\boldsymbol{\alpha}} \geq 0$ for each $a_{i\boldsymbol{\alpha}}$. The associated graph of $\boldsymbol{P}$ is the directed graph $\mathcal{G}(\boldsymbol{P}) = (V, E(\boldsymbol{P}))$, where $V = \{1, 2, ..., n\}$ and $(i, j) \in E(\boldsymbol{P})$ if the variable x_j appears in the expression of P_i or if P_i contains a monomial with degree less than d_i.

Definition 15. [16] $\boldsymbol{P}$ is weakly irreducible if the graph $\mathcal{G}(\boldsymbol{P})$ is strongly connected. If the graph $\mathcal{G}(\boldsymbol{P})$ is strongly connected and the greatest common divider (g.c.d.) of the length of the circuits is equal to one, $\boldsymbol{P}$ is weakly primitive.

Let $I \subset \{1, 2, ..., n\}$ where *part* $Q_I = \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_i > 0, i \in I\}$. The polynomial map $\boldsymbol{P}$ is irreducible if there is no part of $\mathbb{R}^n_+$ that is invariant by $\boldsymbol{P}$, except parts $Q_\emptyset$ and $Q_{\{1,2,...,n\}}$. Part Q_I is invariant by $\boldsymbol{P}$ if for all $\boldsymbol{x} \in Q_I, \boldsymbol{P}(\boldsymbol{x}) \in Q_I$.

Let $\delta = \max(d_1, ..., d_n)$. We call a number $\lambda \in \mathbb{C}$ an eigenvalue of $\boldsymbol{P}$ and a nonzero vector $\boldsymbol{x} \in \mathbb{C}^n$ the associated eigenvector if they satisfy the following equations:

$$P_i(x) = \lambda x_i^\delta, \quad \forall i = 1, ..., n. \tag{4.2}$$

Now we recall some previous results of eigenvalue and eigenvector of homogeneous and monotone maps on $\mathbb{R}^n_+$. A polynomial map $\boldsymbol{F} : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is said to be homogeneous and monotone if;

1. (Homogeneous) $\forall t \in \mathbb{R}_{>0}$ and $\forall \boldsymbol{x} \in \mathbb{R}^n_{>0}$, $\boldsymbol{F}(t\boldsymbol{x}) = t\boldsymbol{F}(\boldsymbol{x})$, and
2. (Monotone) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_{>0}$, $\boldsymbol{x} \leq \boldsymbol{y} \Rightarrow \boldsymbol{F}(\boldsymbol{x}) \leq \boldsymbol{F}(\boldsymbol{y})$.

For better understanding, we give a simple example of a nonhomogeneous polynomial,

$$\check{\boldsymbol{P}}(\boldsymbol{x}) = \begin{pmatrix} 2x_1x_2 + x_3x_4 \\ 5x_2x_4 + 3x_1x_3x_4 \\ x_4 + x_3^4 \\ x_1 \end{pmatrix}.$$

In the above polynomial,

$$\begin{aligned}
\check{P}_1(\boldsymbol{x}) &= 2x_1x_2 + x_3x_4, \quad a_{11100} = 2, \quad a_{10011} = 1, \quad d_1 = 2, \\
\check{P}_2(\boldsymbol{x}) &= 5x_2x_4 + 3x_1x_3x_4, \quad a_{20101} = 5, \quad a_{21011} = 3, \quad d_2 = 3, \\
\check{P}_3(\boldsymbol{x}) &= x_4 + x_3^4, \quad a_{30001} = 1, \quad a_{30040} = 1, \quad d_3 = 4 \\
\check{P}_4(\boldsymbol{x}) &= x_1, \quad a_{41000} = 1, \quad d_4 = 1.
\end{aligned}$$

Notice that $\check{\boldsymbol{P}}(\boldsymbol{x})$ is nonhomogeneous.

For $\boldsymbol{x} > 0$ and $\lambda > 0$, we call $(\boldsymbol{x}, \lambda)$ an eigenvector-eigenvalue pair of $\boldsymbol{F}$ if $\boldsymbol{F}(\boldsymbol{x}) = \lambda \boldsymbol{x}$. For $c \in (0, \infty)$ and $\Gamma \subseteq \{1, 2, ..., n\}$, we define $\bar{\boldsymbol{u}}_\Gamma = (\bar{u}_1, \bar{u}_2 \cdots, \bar{u}_n)^T > \boldsymbol{0}$ by

$$\bar{u}_i = \left\{ \begin{array}{lll} c, & if & i \in \Gamma, \\ 1, & if & i \notin \Gamma. \end{array} \right. \tag{4.3}$$

The associated graph of $\boldsymbol{F}$ is defined as the directed graph $\mathcal{G}(\boldsymbol{F})$ with vertices $\{1, 2, ..., n\}$ and an edge from i to j if and only if

$$\lim_{c \to \infty} F_i(\bar{u}_{\{j\}}) = \infty. \tag{4.4}$$

Theorem 36. [20] Let $\boldsymbol{F} : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be homogeneous and monotone. If $\mathcal{G}(\boldsymbol{F})$ is strongly connected then $\boldsymbol{F}$ has an eigenvector in $\mathbb{R}^n_{>0}$.

Theorem 36 provides a sufficient condition for the existence of a positive eigenvector of $\boldsymbol{F}$ and the Theorem 37 below gives a sufficient condition for the uniqueness of a positive eigenvector of $\boldsymbol{F}$. We let $\boldsymbol{u}$ be a positive eigenvector of $\boldsymbol{F}$ and $D\boldsymbol{F}(\boldsymbol{u}) = \left(\frac{\partial F_i}{\partial x_j}(\boldsymbol{u}) \right)_{i=1,j=1}^{n}$. Since we assume $\boldsymbol{F}$ is monotone, obviously $D\boldsymbol{F}(\boldsymbol{u})$ is a nonnegative square matrix.

Theorem 37. [45] Let $\boldsymbol{F} : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be homogeneous and monotone. Assume that $\boldsymbol{u} > 0$ is an eigenvector of $\boldsymbol{F}$. Suppose that $\boldsymbol{F}$ is C^1 in some open neighborhood of $\boldsymbol{u}$. Assume that matrix $\boldsymbol{A} = D\boldsymbol{F}(\boldsymbol{u})$ is either nilpotent or has a positive spectral radius ρ which is a simple root of the characteristic polynomial of $\boldsymbol{A}$. Then $\boldsymbol{u}$ is a unique eigenvector of $\boldsymbol{F}$ in $\mathbb{R}^n_{>0}$.

When $\boldsymbol{A}$ is primitive, we have the following theorem.

Theorem 38. [45] Let $\boldsymbol{F} : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be homogeneous and monotone. Assume that $\boldsymbol{u} > \boldsymbol{0}$ is an eigenvector of $\boldsymbol{F}$. Suppose that $\boldsymbol{F}$ is C^1 in some open neighborhood of $\boldsymbol{u}$. Assume that $\boldsymbol{A} = D\boldsymbol{F}(\boldsymbol{u})$ is primitive. Let $\boldsymbol{\psi} >\neq \boldsymbol{0}$ and assume that $\boldsymbol{\psi}^T\boldsymbol{u} = 1$. Then $\boldsymbol{u} > \boldsymbol{0}$ is the unique eigenvector of $\boldsymbol{F}$ in $\mathbb{R}^n_{>0}$ satisfying the condition $\boldsymbol{\psi}^T\boldsymbol{u} = 1$. Define $\boldsymbol{G} : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ as

$$\boldsymbol{G}(\boldsymbol{x}) = \frac{1}{\boldsymbol{\psi}^T\boldsymbol{F}(\boldsymbol{x})}\boldsymbol{F}(\boldsymbol{x}). \tag{4.5}$$

Then $lim_{\kappa\to\infty}\boldsymbol{G}^{\circ\kappa}(\boldsymbol{x}) = \boldsymbol{u}$ for each $\boldsymbol{x} \in \mathbb{R}^n_{>0}$.

We state here the Perron-Freobenius theorem for polynomial map for reference.

Theorem 39. [16] Let $\boldsymbol{P} = (P_1, ..., P_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map, where each P_i is a polynomial of degree $d_i \geq 1$ with nonnegative coefficients. Let $\delta_1, \cdots, \delta_n \in (0, \infty)$ be given and assume that $\delta_i \geq d_i, i \in \{1, 2, ..., n\}$. Consider the system

$$P_i(\boldsymbol{x}) = \lambda x_i^{\delta_i}, \quad i \in \{1, 2, ..., n\}, \quad \boldsymbol{x} \geq \boldsymbol{0}. \tag{4.6}$$

Assume that $\boldsymbol{P}$ is weakly irreducible. Then for each $a, p > 0$ there exists a unique positive vector $\boldsymbol{x} > 0$, depending on a, p, satisfying (4.6) and the condition $||\boldsymbol{x}||_p = a$. Suppose furthermore that $\boldsymbol{P}$ is irreducible. Then the system (4.6) has a unique solution, depending on a, p, satisfying $||\boldsymbol{x}||_p = a$, and all the coordinates of this solution are positive.

Proof. [16] We can write $\boldsymbol{P}$ as

$$P_i(x) = \sum_{\boldsymbol{\alpha}\in\mathbb{R}^n_+} a_{i\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}, \tag{4.7}$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $a_{i\boldsymbol{\alpha}} \geq 0$. Set $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + ... + \alpha_n$.

Now let $\delta = \max(\delta_1, ..., \delta_n)$ and let $\boldsymbol{F} = (F_1, ..., F_n)^T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be the following homogeneous monotone map

$$F_i(\boldsymbol{x}) = \left(\sum_{\boldsymbol{\alpha}\in\mathbb{R}^n_+} a_{i\boldsymbol{\alpha}} x_i^{\delta-\delta_i} \left(\frac{||\boldsymbol{x}||_p}{a} \right)^{\delta_i - |\boldsymbol{\alpha}|} \boldsymbol{x}^{\alpha} \right)^{\frac{1}{\delta}}, \qquad i \in \{1, 2, ..., n\}. \tag{4.8}$$

For every variable x_k effectively appears in the expression of P_i, there exists a directed edge from i to k in $\mathcal{G}(\boldsymbol{P})$ and also in $\mathcal{G}(\boldsymbol{F})$. The term $||\boldsymbol{x}||_p$ in F_i ensures there exists a directed edge from i to every $k \in \{1, 2, ..., n\}$ when monomial $a_{i\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}$ of degree $|\boldsymbol{\alpha}| < \delta_i$ appears in the expression. Because of the term $x_i^{\delta-\delta_i}$ in F_i, there exists a directed edge from i to i in $\mathcal{G}(\boldsymbol{F})$ if $\delta > \delta_i$. Therefore we can say $\mathcal{G}(\boldsymbol{F})$ is identical to $\mathcal{G}(\boldsymbol{P})$ except $\mathcal{G}(\boldsymbol{F})$ will have more loops.

Hence, since $\boldsymbol{P}$ is weakly irreducible, by definition, $\mathcal{G}(\boldsymbol{P})$ is strongly connected. Since $\mathcal{G}(\boldsymbol{F})$ is identical to $\mathcal{G}(\boldsymbol{P})$, $\mathcal{G}(\boldsymbol{F})$ is also strongly connected. By Theorem 36, $\boldsymbol{F}$ has an eigenvector in $\mathbb{R}^n_{>0}$. Consider $\boldsymbol{F}(\boldsymbol{x}) = \mu\boldsymbol{x}$. Since a_{ij} is nonnegative, for $\boldsymbol{x} > \boldsymbol{0}$, the associated eigenvalue $\mu > 0$. We may normalize $\boldsymbol{x}$ so that $||\boldsymbol{x}||_p = a$ since $\boldsymbol{F}$ is positively homogeneous. Therefore, the equation (4.8) becomes

$$[F_i(\boldsymbol{x})]^\delta = \sum_{\boldsymbol{\alpha}\in\mathbb{R}^n_+} a_{i\boldsymbol{\alpha}} x_i^{\delta-\delta_i}\boldsymbol{x}^{\boldsymbol{\alpha}} = \mu^\delta x_i^\delta. \tag{4.9}$$

For $\boldsymbol{x} > \boldsymbol{0}$, when we multiply both sides by $x_i^{\delta_i-\delta}$,

$$\sum_{\boldsymbol{\alpha}\in\mathbb{R}^n_+} a_{i\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}} = \mu^\delta x_i^{\delta_i} = P_i(\boldsymbol{x}), \quad \lambda = \mu^\delta. \tag{4.10}$$

This shows that $\boldsymbol{x}$ satisfies equation (4.6) . Any solution of (4.6) which satisfies condition $||\boldsymbol{x}||_p = a$ is also an eigenvector of $\boldsymbol{F}$. Consider any solution $\boldsymbol{x} > \boldsymbol{0}$. Let $\boldsymbol{A} = D\boldsymbol{F}(\boldsymbol{x})$. The directed graph of matrix $\boldsymbol{A}$, $\mathcal{G}(\boldsymbol{A})$ coincides with $\mathcal{G}(\boldsymbol{F})$ and as mentioned before, $\mathcal{G}(\boldsymbol{F})$ coincides with $\mathcal{G}(pP)$. Since $\mathcal{G}(\boldsymbol{P})$ is strongly connected, $\mathcal{G}(\boldsymbol{A})$ is also strongly connected. This also implies $\boldsymbol{A}$ is irreducible. By Perron-Frobenius theorem for matrix, there is positive spectral radius of $\boldsymbol{A}$. According to Theorem 37, $\boldsymbol{F}$ has a unique positive eigenvector up to a multiplicative constant. Then, the system (4.6) also has a unique solution $\boldsymbol{x} > \boldsymbol{0}$.

Now we prove the second part of the theorem. Let $\boldsymbol{x} \geq\neq \boldsymbol{0}$ be a solution of (4.6) such that $||\boldsymbol{x}||_p = a$, and let $I = \{i \in \{1, ..., n\} | x_i \neq 0\}$. By the definition of irreducible polynomial, if $\boldsymbol{P}$ is irreducible then $Part\ Q_I$ is invariant by $\boldsymbol{P}$ where $I = \{1, ..., n\}$. This implies (4.6) only has positive solutions. In the first part we have proved that (4.6) has a unique positive solution. Hence, (4.6) has a unique solution which is positive. $\square$

4.3 Homogeneous Nonnegative Polynomials

For homogeneous polynomials with nonnegative coefficients, we have the following results.

Corollary 5. [16] Let $\boldsymbol{P} = (P_1, ..., P_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map, where each P_i is a homogeneous polynomial of degree $d \geq 1$ with nonnegative coefficients. Assume that $\boldsymbol{P}$ is weakly irreducible. Then, the unique scalar λ such that there is a positive vector $\boldsymbol{u}$ with $P_i(\boldsymbol{u}) = \lambda \boldsymbol{u}_i^d$ for all $i \in \{1, 2, ..., n\}$ satisfies

$$\inf_{\boldsymbol{x} \in \mathbb{R}^n_{>0}} \max_{i \in \{1,2,...,n\}} \frac{P_i(\boldsymbol{x})}{x_i^d} = \lambda = \sup_{\boldsymbol{x} \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}} \min_{i \in \{1,2,...,n\}, x_i \neq 0} \frac{P_i(\boldsymbol{x})}{x_i^d}. \tag{4.11}$$

Corollary 6. [16] Let $\boldsymbol{P}$, d and λ be as in Corollary 5. If $\nu \in \mathbb{C}$ and $\boldsymbol{v} = (v_1, ..., v_n)^T \in \mathbb{C}^n \setminus \{0\}$ such that $P_i(\boldsymbol{v}) = \nu v_i^d$, for all $i \in \{1, 2, ..., n\}$, then $|\nu| \leq \lambda$.

We can find the vector $\boldsymbol{x}$ in (4.6) using a simple power type algorithm when $\boldsymbol{P}(\boldsymbol{x})$ is homogeneous. Assume that each polynomial P_i is homogeneous of degree d with nonnegative coefficients.

Algorithm 6.

Step 0: Choose $\boldsymbol{x}^{(1)} \in \mathbb{R}^n_{>0}$. Set $k = 1$.

Step 1: Compute

$$\begin{aligned} P_i^{(k)} &= \sum_{\boldsymbol{\alpha} \in \mathbb{R}^n_+} a_{i\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}, \quad i = 1, 2, ..., n, \\ \underline{\lambda}_k &= \min_{i \in \{1,2,...,n\}, x_i^{(k)} > 0} \frac{P_i^{(k)}}{(x_i^{(k)})^d}, \\ \bar{\lambda}_k &= \max_{x_i^{(k)} > 0} \frac{P_i^{(k)}}{(x_i^{(k)})^d}. \end{aligned}$$

Step 2: If $\bar{\lambda}_k = \underline{\lambda}_k$, then let $\lambda = \bar{\lambda}_k$ and stop. Otherwise, compute

$$\boldsymbol{x}^{(k+1)} = \frac{\left(\boldsymbol{P}^{(k)}\right)^{[\frac{1}{d}]}}{\left\| \left(\boldsymbol{P}^{(k)}\right)^{[\frac{1}{d}]} \right\|_1},$$

replace k by $k + 1$ and go to Step 1.

When $\boldsymbol{P}$ is weakly primitive, the Algorithm 6 converges to a positive vector [16].

Corollary 7. [16] Let $\boldsymbol{P}$ and d be as in Corollary 5, and assume in addition that $\boldsymbol{P}$ is weakly primitive. Then, the sequence $\{\boldsymbol{x}^{(k)}\}$ produced by the Algorithm 6 converges to the unique vector $\boldsymbol{u} \in \mathbb{R}^n_{>0}$ satisfying $P_i(\boldsymbol{u}) = \lambda u_i^d$, for $i \in \{1, 2, ..., n\}$, and $\boldsymbol{\psi}^T \boldsymbol{u} = 1$.

Corollary 8. [16] Let $\boldsymbol{P}$, d, $\boldsymbol{u}$ and λ be as in Corollary 7, $\boldsymbol{F}$ as in (4.8), let $\boldsymbol{M} := \boldsymbol{F}'(\boldsymbol{u})$, and let r denote the maximal modulus of the eigenvalues of $\boldsymbol{M}$ distinct from λ. Then, the sequence $\{\boldsymbol{x}^{(k)}\}$ produced by the Algorithm 6 satisfies

$$\lim_{k \longrightarrow \infty} \sup ||\boldsymbol{x}^{(k)} - \boldsymbol{u}||^{1/k} \leq \lambda^{-1} r. \tag{4.12}$$

4.4 Nonhomogeneous Nonnegative Polynomials

We can also use power method to compute the largest eigenvalue of nonhomogeneous nonnegative polynomials. From the proof of Theorem 39, it is clear that any eigenvalue of homogeneous map $\boldsymbol{F}$ subject to $||\boldsymbol{x}||_p = a$ is also an eigenvalue of nonhomogeneous nonnegative polynomial $\boldsymbol{P}$. Since the character of power method will normalize the sequence $\{\boldsymbol{x}^{(k)}\}$ such that $||\boldsymbol{x}^{(k)}||_p = 1$, we can directly use the method to find the largest eigenvalue of nonhomogeneous nonnegative polynomials.

Let $\boldsymbol{P} = (P_1, ..., P_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a nonhomogeneous polynomial map, where each P_i is a polynomial of degree $d_i \geq 1$ with nonnegative coefficients. Let $\delta = \max(d_1, ..., d_n)$. We define $\lambda \in \mathbb{C}$ as an eigenvalue of $\boldsymbol{P}$ and a nonzero vector $\boldsymbol{x} \in \mathbb{C}^n$ is the associated eigenvector as follows:

$$\begin{aligned} P_i(\boldsymbol{x}) &= \lambda x_i^{\delta}, \quad \forall i = 1, ..., n, \\ ||\boldsymbol{x}||_p &= a, \quad p, a > 0. \end{aligned} \tag{4.13}$$

We need the constraints of $||\boldsymbol{x}||_p = a$, where $p, a > 0$ since the solutions of $\boldsymbol{P}$ are in the sphere. We give the minimax theorem when $\boldsymbol{P}$ is nonhomogeneous.

Corollary 9. Let $\boldsymbol{P} = (P_1, ..., P_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map, where each P_i is a polynomial of degree $d_i \geq 1$ with nonnegative coefficients. Let

$\delta = \max(d_1, ..., d_n)$ and $||\boldsymbol{x}||_p$ be the ℓ_p-norm where $||\boldsymbol{x}||_p = a$ for $a, p > 0$. Assume that $\boldsymbol{P}$ is weakly irreducible. Then, the unique scalar λ such that there is a positive vector $\boldsymbol{u}$ with $P_i(\boldsymbol{u}) = \lambda u_i^{d_i}$ for all $i \in \{1, 2, ..., n\}$ satisfies

$$\begin{aligned} \lambda &= \inf_{\boldsymbol{x}\in\mathbb{R}^n_{>0}, ||\boldsymbol{x}||_p = a} \max_{i\in\{1,2,...,n\}} \frac{P_i(\boldsymbol{x})}{x_i^\delta} \\ &= \sup_{\boldsymbol{x}\in\mathbb{R}^n_{+}\setminus\{\boldsymbol{0}\}, ||\boldsymbol{x}||_p = a} \min_{i\in\{1,2,...,n\}, x_i\neq 0} \frac{P_i(\boldsymbol{x})}{x_i^\delta}. \end{aligned} \tag{4.14}$$

Proof. Let $\boldsymbol{F} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a homogeneous monotone map, and $\rho(\boldsymbol{F})$ be the largest eigenvalue of $\boldsymbol{F}$.

$$\mu_*(\boldsymbol{x}) = \min_{i\in\{1,2,...,n\}, x_i\neq 0} \frac{F_i(\boldsymbol{x})}{x_i}, \quad \mu^*(\boldsymbol{x}) = \max_{i\in\{1,2,...,n\}} \frac{F_i(\boldsymbol{x})}{x_i}.$$

$$\begin{aligned} r_* &= \sup_{\boldsymbol{x}\in\mathbb{R}^n_+\setminus\{\boldsymbol{0}\}} \mu_*(\boldsymbol{x}) = \sup_{\boldsymbol{x}\in\mathbb{R}^n_+\setminus\{\boldsymbol{0}\}} \min_{i\in\{1,2,...,n\}, x_i\neq 0} \frac{F_i(\boldsymbol{x})}{x_i}. \\ r^* &= \inf_{\boldsymbol{x}\in\mathbb{R}^n_{>0}} \mu^*(\boldsymbol{x}) = \inf_{\boldsymbol{x}\in\mathbb{R}^n_{>0}} \max_{i\in\{1,2,...,n\}} \frac{F_i(\boldsymbol{x})}{x_i}. \end{aligned}$$

By Theorem 3.1 [45],

$$\rho(\boldsymbol{F}) = r^*. \tag{4.15}$$

Then, by Lemma 2.8 [1], we can deduce that

$$r_* = r^*. \tag{4.16}$$

Define $\boldsymbol{F}$ as in the proof of Theorem 39, so that $F_i(\boldsymbol{x}) = [P_i(\boldsymbol{x})]^{\frac{1}{\delta}}$. By Theorem 39, $\boldsymbol{F}$ has a positive eigenvector. Let μ be the associated eigenvalue. By definition of $\rho(\boldsymbol{F})$ and r^*, $\rho(\boldsymbol{F}) \geq \mu \geq r^*$. From (4.15), we can say $\mu = r^*$. Hence,

$$\begin{aligned} \mu &= r^* = \inf_{\boldsymbol{x}\in\mathbb{R}^n_{>0}, ||\boldsymbol{x}||_p = a} \max_{i\in\{1,2,...,n\}} \frac{F_i(\boldsymbol{x})}{x_i} = \inf_{\boldsymbol{x}\in\mathbb{R}^n_{>0}, ||\boldsymbol{x}||_p = a} \max_{i\in\{1,2,...,n\}} \frac{P_i(\boldsymbol{x})^{\frac{1}{\delta}}}{x_i}, \\ \lambda &= \mu^\delta = \inf_{\boldsymbol{x}\in\mathbb{R}^n_{>0}, ||\boldsymbol{x}||_p = a} \max_{i\in\{1,2,...,n\}} \frac{P_i(\boldsymbol{x})}{x_i^\delta}. \end{aligned}$$

We prove another expression in (4.14) using the same reasoning. By definition of r^* and r_*, $r_* \geq \mu \geq r^*$. By (4.16), $r_* = \mu$. Now we can conclude

$$\begin{aligned} \mu &= r_* = \sup_{\boldsymbol{x}\in\mathbb{R}^n_+\setminus\{\boldsymbol{0}\}, ||\boldsymbol{x}||_p = a} \min_{i\in\{1,2,...,n\}, x_i\neq 0} \frac{F_i(\boldsymbol{x})}{x_i} \\ &= \sup_{\boldsymbol{x}\in\mathbb{R}^n_+\setminus\{\boldsymbol{0}\}, ||\boldsymbol{x}||_p = a} \min_{i\in\{1,2,...,n\}, x_i\neq 0} \frac{P_i(\boldsymbol{x})^{\frac{1}{\delta}}}{x_i}. \\ \lambda &= \mu^\delta = \sup_{\boldsymbol{x}\in\mathbb{R}^n_+\setminus\{\boldsymbol{0}\}, ||\boldsymbol{x}||_p = a} \min_{i\in\{1,2,...,n\}, x_i\neq 0} \frac{P_i(\boldsymbol{x})}{x_i^\delta}. \end{aligned}$$

□

Corollary 10. Let $\boldsymbol{P} = (P_1, ..., P_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map with P_i being a polynomial of degree $d_i \geq 1$ and the coefficient of each monomial in P_i nonnegative. Let $\delta = \max(d_1, ..., d_n)$. Suppose that $\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x}) + \boldsymbol{x}^{[\delta]}$. Then $\lambda + 1$ is an eigenvalue of $\boldsymbol{Q}(\boldsymbol{x})$ if and only if λ is an eigenvalue of $\boldsymbol{P}(\boldsymbol{x})$. In this case, they have the same eigenvector. If $\boldsymbol{P}$ is weakly irreducible, then $\boldsymbol{Q}$ is weakly primitive.

Proof. Let λ be an eigenvalue of $\boldsymbol{P}$ and $\boldsymbol{x}$ be the associated eigenvector. By definition,

$$\boldsymbol{P}(\boldsymbol{x}) = \lambda \boldsymbol{x}^{[\delta]}.$$

$$\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x}) + \boldsymbol{x}^{[\delta]} = \lambda \boldsymbol{x}^{[\delta]} + \boldsymbol{x}^{[\delta]} = (\lambda + 1)\boldsymbol{x}^{[\delta]}.$$

By definition, $\lambda + 1$ is an eigenvalue of $\boldsymbol{Q}(\boldsymbol{x})$ associated with eigenvector $\boldsymbol{x}$.
For the second part, let $\boldsymbol{P}$ be a weakly irreducible polynomial. Then, the graph of $\boldsymbol{P}$, $\mathcal{G}(\boldsymbol{P})$ is strongly connected. The term $\boldsymbol{x}^{[\delta]}$ in $\boldsymbol{Q}$ contributes the loops for each vertices in the graph of $\boldsymbol{Q}$, $\mathcal{G}(\boldsymbol{Q})$. Hence, $\mathcal{G}(\boldsymbol{Q})$ is strongly connected and contains loops at each vertices. By Definition 15, $\boldsymbol{Q}$ is weakly primitive. □

Theorem 40. Let $\boldsymbol{P} = (P_1, ..., P_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be an irreducible polynomial map with P_i is a polynomial of degree $d_i \geq 1$ and the coefficient of each monomial in P_i is nonnegative. Let $\delta = \max(d_1, ..., d_n)$ and $\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x}) + \boldsymbol{x}^{[\delta]}$. Let $\boldsymbol{x}^{(0)} \in \mathbb{R}^n_{>0}$ be an arbitrary vector.

Define

$$\boldsymbol{x}^{(1)} = \frac{(\boldsymbol{Q}(\boldsymbol{x}^{(0)}))^{[\frac{1}{\delta}]}}{\|(\boldsymbol{Q}(\boldsymbol{x}^{(0)}))^{[\frac{1}{\delta}]}\|_p}, \boldsymbol{x}^{(2)} = \frac{(\boldsymbol{Q}(\boldsymbol{x}^{(1)}))^{[\frac{1}{\delta}]}}{\|(\boldsymbol{Q}(\boldsymbol{x}^{(1)}))^{[\frac{1}{\delta}]}\|_p}, ..., \boldsymbol{x}^{(k+1)} = \frac{(\boldsymbol{Q}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}}{\|(\boldsymbol{Q}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}\|_p}, \quad k \geq 2.$$

Let

$$\underline{\lambda}_k = \min_{x_i^{(k)} > 0} \frac{Q_i(\boldsymbol{x}^{(k)})}{(x_i^{(k)})^\delta}, \quad \bar{\lambda}_k = \max_{x_i^{(k)} > 0} \frac{Q_i(\boldsymbol{x}^{(k)})}{(x_i^{(k)})^\delta}, \quad k = 1, 2, ... \ .$$

Assume that λ_0 is the unique positive eigenvalue of $\boldsymbol{Q}$ corresponding to a non-negative eigenvector. Then,

$$\underline{\lambda}_1 \leq \underline{\lambda}_2 \leq ... \leq \lambda_0 \leq ... \leq \bar{\lambda}_2 \leq \bar{\lambda}_1.$$

Proof. Obviously, by Corollary 9, for $k = 1, 2, ...,$ $\underline{\lambda}_k \leq \lambda_0 \leq \bar{\lambda}_k$. Now we only need to prove that for any $k \geq 1$, $\underline{\lambda}_k \leq \underline{\lambda}_{k+1}$ and $\bar{\lambda}_{k+1} \leq \bar{\lambda}_k$. Let $\boldsymbol{F}$ be as in the proof of Theorem 39 and

$$\underline{\mu}_k = \min \frac{F_i(\boldsymbol{x}^{(k)})}{x_i^{(k)}} = \min \frac{Q_i(\boldsymbol{x}^{(k)})^{\frac{1}{\delta}}}{x_i^{(k)}} = \underline{\lambda}_k^{\frac{1}{\delta}}.$$

Any solution of $\boldsymbol{F}$ is also a solution of $\boldsymbol{Q}$. We have

$$\underline{\mu}_k \leq \frac{F_i(x^{(k)})}{x_i^{(k)}},$$
$$0 \leq \underline{\mu}_k \boldsymbol{x}^{(k)} \leq \boldsymbol{F}(\boldsymbol{x}^{(k)}).$$

Now we have

$$\begin{aligned} \boldsymbol{x}^{(k+1)} &= \frac{\boldsymbol{Q}(\boldsymbol{x}^{(k)})^{\frac{1}{\delta}}}{\|\boldsymbol{Q}(\boldsymbol{x}^{(k)})^{\frac{1}{\delta}}\|_p} \\ &= \frac{\boldsymbol{F}(\boldsymbol{x}^{(k)})}{\|\boldsymbol{F}(\boldsymbol{x}^{(k)})\|_p} \geq \frac{\underline{\mu}_k \boldsymbol{x}^{(k)}}{\|\boldsymbol{F}(\boldsymbol{x}^{(k)})\|_p} > 0, \end{aligned}$$

and since we $\boldsymbol{F}$ is monotone and homogeneous,

$$\boldsymbol{F}(\boldsymbol{x}^{(k+1)}) \geq \boldsymbol{F}\left(\frac{\underline{\mu}_k \boldsymbol{x}^{(k)}}{\|\boldsymbol{F}(\boldsymbol{x}^{(k)})\|_p} \right) = \underline{\mu}_k \left(\frac{\boldsymbol{F}(\boldsymbol{x}^{(k)})}{\|\boldsymbol{F}(\boldsymbol{x}^{(k)})\|_p} \right) = \underline{\mu}_k \boldsymbol{x}^{(k+1)},$$

which means, for each $i = 1, 2, ..., n$, $\frac{F_i(\boldsymbol{x}^{(k+1)})}{x_i^{(k+1)}} \geq \underline{\mu}_k$. This implies $\underline{\mu}_k \geq \underline{\mu}_{k+1}$ and

$$\underline{\lambda}_{k+1} = (\underline{\mu}_{k+1})^{\delta} \geq (\underline{\mu}_k)^{\delta} = \underline{\lambda}_k.$$

Similarly, let,

$$\bar{\mu}_k = \max \frac{F_i(\boldsymbol{x}^{(k)})}{x_i^{(k)}} = \max \frac{Q_i(\boldsymbol{x}^{(k)})^{\frac{1}{\delta}}}{x_i^{(k)}} = \bar{\lambda}_k^{\frac{1}{\delta}},$$

and

$$\bar{\mu}_k \boldsymbol{x}^{(k)} \geq \boldsymbol{F}(\boldsymbol{x}^{(k)}).$$

Then, we have

$$\boldsymbol{x}^{(k+1)} = \frac{\boldsymbol{Q}(\boldsymbol{x}^{(k)})^{\frac{1}{\delta}}}{\|\boldsymbol{Q}(\boldsymbol{x}^{(k)})^{\frac{1}{\delta}}\|_p} = \frac{\boldsymbol{F}(\boldsymbol{x}^{(k)})}{\|\boldsymbol{F}(\boldsymbol{x}^{(k)})\|_p} \leq \frac{\bar{\mu}_k \boldsymbol{x}^{(k)}}{\|\boldsymbol{F}(\boldsymbol{x}^{(k)})\|_p},$$

and

$$\boldsymbol{F}(\boldsymbol{x}^{(k+1)}) \leq \boldsymbol{F}\left(\frac{\bar{\mu}_k \boldsymbol{x}^{(k)}}{\|\boldsymbol{F}(\boldsymbol{x}^{(k)})\|_p} \right) = \bar{\mu}_k \left(\frac{\boldsymbol{F}(\boldsymbol{x}^{(k)})}{\|\boldsymbol{F}(\boldsymbol{x}^{(k)})\|_p} \right) = \bar{\mu}_k \boldsymbol{x}^{(k+1)},$$

which means, for each $i = 1, 2, ..., n$, $\frac{F_i(\boldsymbol{x}^{(k+1)})}{x_i^{(k+1)}} \leq \bar{\mu}_k$. Hence, $\bar{\mu}_{k+1} \leq \bar{\mu}_k$ and $\bar{\lambda}_{k+1} = \bar{\mu}_{k+1}^{\delta} \leq \bar{\mu}_k^{\delta} = \bar{\lambda}_k$. □

4.5 Primitive Polynomials

In this section, we give some results that are important to prove that our proposed algorithm in the next section is convergent. These results are obtained by following the same argument as in [9]. First, we state some useful definitions.

We define the map $\boldsymbol{T_P x} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ as

$$\boldsymbol{T_P x} = (\boldsymbol{P}(\boldsymbol{x}))^{\frac{1}{\delta}}.$$

Let $\boldsymbol{X}$ be a Banach space. A map $\boldsymbol{T} : \boldsymbol{X} \to \boldsymbol{X}$ is called strongly positive if $\boldsymbol{0} \leq \boldsymbol{x}$ and $\boldsymbol{x} \neq \boldsymbol{0}$ imply $\boldsymbol{0} < \boldsymbol{Tx}$. It is called strongly increasing if $\boldsymbol{x} \leq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$ imply $\boldsymbol{Tx} < \boldsymbol{Ty}$.

Definition 16. An irreducible nonnegative polynomial $\boldsymbol{P}$ is primitive if $\boldsymbol{T_P}$ does not have an invariant set S on the boundary of $\mathbb{R}^n_+$, $\{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\}$ except the trivial invariant set $\{\boldsymbol{0}\}$.

Lemma 16. Let nonnegative polynomial $\boldsymbol{P}$ be irreducible, then

$$\sum_{\boldsymbol{\alpha} \in \mathbb{R}^n_+} a_{i\boldsymbol{\alpha}} > 0, \;\; for \;\; all \;\; 1 \leq i \leq n.$$

Proof. We know that $\boldsymbol{P}$ is irreducible. Therefore, $\boldsymbol{P}$ is also weakly irreducible. By definition, if $\boldsymbol{P}$ is weakly irreducible, $\mathcal{G}(\boldsymbol{P})$ is strongly connected. As mentioned at the beginning of this chapter, a polynomial has the general form of $P_i(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{R}^n_+} a_{i\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$, $1 \leq i \leq n$. We prove this lemma by contradiction. Suppose there exists i_0 such that $\sum_{\boldsymbol{\alpha} \in \mathbb{R}^n_+} a_{i_0\boldsymbol{\alpha}} = 0$. This implies $P_{i_0} = 0$. By the definition of graph of $\boldsymbol{P}$, that means there is no edge from i_0 to other vertices. This contradicts the definition of strongly connected. It means $\sum_{\boldsymbol{\alpha} \in \mathbb{R}^n_+} a_{i\boldsymbol{\alpha}} > 0$, for all $1 \leq i \leq n$. □

Lemma 17. Let nonnegative polynomial $\boldsymbol{P}$ be irreducible. If $\boldsymbol{0} \leq \boldsymbol{x} < \boldsymbol{y}$, then $\boldsymbol{T_P x} < \boldsymbol{T_P y}$.

Proof. We know that since $\boldsymbol{0} \leq \boldsymbol{x} < \boldsymbol{y}$, there exist $\epsilon > 0$ such that $y_i - x_i \geq \epsilon$ for all $i \in \{1, 2, ..., n\}$.

We also know that

$$\begin{aligned}
\boldsymbol{T_P x} &< \boldsymbol{T_P y}, \\
(\boldsymbol{P}(\boldsymbol{x}))^{[1/\delta]} &< (\boldsymbol{P}(\boldsymbol{y}))^{[1/\delta]}, \\
\boldsymbol{P}(\boldsymbol{x}) &< \boldsymbol{P}(\boldsymbol{y}), \\
\boldsymbol{0} &< \boldsymbol{P}(\boldsymbol{y}) - \boldsymbol{P}(\boldsymbol{x}).
\end{aligned}$$

Hence, it is enough to show that $\boldsymbol{P}(\boldsymbol{y}) - \boldsymbol{P}(\boldsymbol{x}) > \boldsymbol{0}$. First, observe that

$$y_1^{\alpha_1}(x_2^{\alpha_2}...x_{n-1}^{\alpha_{n-1}} - y_2^{\alpha_2}...y_{n-1}^{\alpha_{n-1}})x_n^{\alpha_n} \leq 0,$$
$$y_1^{\alpha_1}x_2^{\alpha_2}...x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n} - y_1^{\alpha_1}y_2^{\alpha_2}...y_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n} \leq 0.$$

Thus,

$$\begin{aligned}
\boldsymbol{P}(\boldsymbol{y}) - \boldsymbol{P}(\boldsymbol{x}) &= \sum a_{i\boldsymbol{\alpha}}(y_1^{\alpha_1}...y_n^{\alpha_n} - x_1^{\alpha_1}...x_n^{\alpha_n}), \\
&\geq \sum a_{i\boldsymbol{\alpha}}(y_1^{\alpha_1}...y_n^{\alpha_n} - x_1^{\alpha_1}...x_n^{\alpha_n} \\
&\quad + y_1^{\alpha_1}x_2^{\alpha_2}...x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n} - y_1^{\alpha_1}y_2^{\alpha_2}...y_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}) \\
&\geq \sum a_{i\boldsymbol{\alpha}}(y_1^{\alpha_1}...y_n^{\alpha_n} - y_1^{\alpha_1}y_2^{\alpha_2}...y_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n} \\
&\quad + y_1^{\alpha_1}x_2^{\alpha_2}...x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n} - x_1^{\alpha_1}...x_n^{\alpha_n}) \\
&\geq \sum a_{i\boldsymbol{\alpha}}(y_1^{\alpha_1}...y_{n-1}^{\alpha_{n-1}}(y_n^{\alpha_n} - x_n^{\alpha_n}) + (y_1^{\alpha_1} - x_1^{\alpha_1})x_2^{\alpha_2}...x_n^{\alpha_n}) \\
&\geq \sum a_{i\boldsymbol{\alpha}}y_1^{\alpha_1}...y_{n-1}^{\alpha_{n-1}}(y_n^{\alpha_n} - x_n^{\alpha_n}) \\
&\geq \sum a_{i\boldsymbol{\alpha}}(y_1^{\alpha_1} - x_1^{\alpha_1})...(y_n^{\alpha_n} - x_n^{\alpha_n}) \\
&\geq \epsilon^n \sum a_{i\boldsymbol{\alpha}}, \qquad for\ all\ i \in \{1, 2, ..., n\}.
\end{aligned}$$

By Lemma 16, $\sum a_{i\boldsymbol{\alpha}} > 0$, hence $\boldsymbol{P}(\boldsymbol{y}) - \boldsymbol{P}(\boldsymbol{x}) > \boldsymbol{0}$. □

Corollary 11. If the nonnegative polynomial $\boldsymbol{P}$ is irreducible, then for $\boldsymbol{x} > \boldsymbol{0}$, $\boldsymbol{T_P x} > \boldsymbol{0}$.

Proof. Assume $\boldsymbol{x} > \boldsymbol{0}$. Lemma 16 states that $\sum a_{i\boldsymbol{\alpha}} > 0$, for all $1 \leq i \leq n$. Hence $\boldsymbol{P}(\boldsymbol{x}) > \boldsymbol{0}$ and $\boldsymbol{T_P}(\boldsymbol{x}) = (\boldsymbol{P}(\boldsymbol{x}))^{1/\delta} > \boldsymbol{0}$. □

Theorem 41. Let $\boldsymbol{P}$ be a nonnegative polynomial. Then $\boldsymbol{P}$ is primitive if and only if there exists $r \in \mathbb{N}$ such that $\boldsymbol{T_P^r}(\mathbb{R}_+^n \setminus \{\boldsymbol{0}\}) \subset \mathbb{R}_{>0}^n$ (i.e. for $\boldsymbol{x} \geq \boldsymbol{0}$ *and* $\boldsymbol{x} \neq \boldsymbol{0}$, $\boldsymbol{T_P^r x} > \boldsymbol{0}$).

Proof. (⇐) First we will show that $\boldsymbol{P}$ is irreducible by contradiction. Suppose $\boldsymbol{P}$ is reducible. Then exists nonempty proper subset $I \subset \{1, ..., n\}$, where $Q_I = \{\boldsymbol{x} \in \mathbb{R}^n_+ | x_i > 0,\ i \in I\}$ and $\boldsymbol{P}(\boldsymbol{x}) \in Q_I$. This means $P_i = 0$ for all $i \notin I$. Let $\boldsymbol{x}^{(J)} = \boldsymbol{T}^{(J)}_{\boldsymbol{P}}\boldsymbol{x}$ for $J \in \mathbb{N}$, we have $x^{(J)}_i = 0$ for all $i \notin I$. This contradicts assumption $\boldsymbol{T}^r_{\boldsymbol{P}}\boldsymbol{x} > \boldsymbol{0}$. Hence, $\boldsymbol{P}$ must be irreducible.

Now we show that trivial invariant set $\{\boldsymbol{0}\}$ is the only invariant set of $\boldsymbol{T_P}$ in $\{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\}$. Suppose $S \subset \{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\}$ be a $\boldsymbol{T_P}$-invariant set, then $\boldsymbol{T}^r_{\boldsymbol{P}}S = S \subset \{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\}$. Earlier we have assumed for $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{x} \neq \boldsymbol{0}$, $\boldsymbol{T}^r_{\boldsymbol{P}}\boldsymbol{x} > 0$. Hence $S = \{\boldsymbol{0}\}$ and this implies $\boldsymbol{P}$ is primitive.

(⇒) Let $\boldsymbol{x} > 0$. By Corollary 11, $\boldsymbol{T}^r_{\boldsymbol{P}}\boldsymbol{x} > \boldsymbol{0}$ for all $r \geq 1$.

Now we show there exist $r \in \mathbb{N}$ such that $\boldsymbol{T}^r_{\boldsymbol{P}}(\mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}) \subset \mathbb{R}^n_{>0}$. It is suffice to show $\boldsymbol{T}^r_{\boldsymbol{P}}(\{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\} \cap \partial B_1(\boldsymbol{0})) \subset \mathbb{R}^n_{>0}$, where $\partial B_1(\boldsymbol{0})$ is the unit sphere centered at the origin. By definition of primitive polynomial, for $\boldsymbol{x} \in \{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\} \cap \partial B_1(\boldsymbol{0})$, the set $S(\boldsymbol{x}) = \{\boldsymbol{x}, \boldsymbol{T_P}\boldsymbol{x}, \boldsymbol{T}^2_{\boldsymbol{P}}\boldsymbol{x}, ...\}$ cannot be in $\{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\}$. This means for $\boldsymbol{x} \in \{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\} \cap \partial B_1(\boldsymbol{0})$, there exist a natural number associated to that particular $\boldsymbol{x}$, $r(\boldsymbol{x})$, such that $\boldsymbol{T}^{r(\boldsymbol{x})}_{\boldsymbol{P}} x \in \mathbb{R}^n_{>0}$. Since the mapping $\boldsymbol{x} \to \boldsymbol{T}^k_{\boldsymbol{P}}\boldsymbol{x}$ is continuous for all $k \in \mathbb{N}$, there exist a neighborhood $U(\boldsymbol{x})$ of $\boldsymbol{x}$ such that $\boldsymbol{T}^{r(\boldsymbol{x})}_{\boldsymbol{P}}(U(\boldsymbol{x})) \subset \mathbb{R}^n_{>0}$. Since $\{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\} \cap \partial B_1(\boldsymbol{0})$ is compact, there exist finite covering $\{U(\boldsymbol{x}_1), ..., U(\boldsymbol{x}_q)\}$ of $\{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\} \cap \partial B_1(\boldsymbol{0})$. This means for each $\boldsymbol{x}_1, ..., \boldsymbol{x}_q \in \{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\} \cap \partial B_1(\boldsymbol{0})$, there exist $r(\boldsymbol{x}_1), ..., r(\boldsymbol{x}_q)$ such that $\boldsymbol{T}^{r(\boldsymbol{x}_1)}_{\boldsymbol{P}}(U(\boldsymbol{x}_1)), ..., \boldsymbol{T}^{r(\boldsymbol{x}_q)}_{\boldsymbol{P}}(U(\boldsymbol{x}_q)) \subset \mathbb{R}^n_{>0}$. Let $r = \max\{r(\boldsymbol{x}_1), ..., r(\boldsymbol{x}_q)\}$. By Corollary 11, $\boldsymbol{T}^r_{\boldsymbol{P}}(\{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\} \cap \partial B_1(\boldsymbol{0})) \subset \mathbb{R}^n_{>0}$. □

For any index subset $I \subset \{1, 2, ..., n\}$, let $D_I = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n_+ | x_i = 0$ for all $i \in I$, $x_j \neq 0$ for all $j \notin I\}$.

Lemma 18. Let nonnegative polynomial $\boldsymbol{P}$ be irreducible. Suppose there exists $\boldsymbol{z} \in \{\mathbb{R}^n_+ \setminus \mathbb{R}^n_{>0}\} \setminus \{\boldsymbol{0}\}$ and $r \in \mathbb{N}$ such that $\boldsymbol{T}^r_{\boldsymbol{P}}\boldsymbol{z} \in \mathbb{R}^n_{>0}$ and assume r is the least positive integer such that $\boldsymbol{T}^r_{\boldsymbol{P}}\boldsymbol{z} \in \mathbb{R}^n_{>0}$. Then there exist proper nonempty index subsets $\{I_J \subset \{1, 2, ..., n\} | 0 \leq J \leq r-1\}$ and set $K_J = \{(\alpha_1 ... \alpha_n) |\ \textit{for all } i \in I_J, \alpha_i = 0\}$ such that

$$\sum_{(\alpha_1 ... \alpha_n) \in K_J} a_{i\alpha_1 ... \alpha_n} > 0 \ \textit{ for all } i \notin I_{J+1} \ \textit{ where } I_r = \emptyset.$$

Proof. Since $\boldsymbol{z} \in \{\{\mathbb{R}_+^n \backslash \mathbb{R}_{>0}^n\} \backslash \{\mathbf{0}\}\}$, there exists a proper index subset $I_0 \neq \emptyset$ such that $\boldsymbol{z} \in D_{I_0}$. Let $\boldsymbol{z}^{(J)} = \boldsymbol{T}_{\boldsymbol{P}}^J \boldsymbol{z},\ J = 1, ..., r-1$. By the definition of r, $\boldsymbol{z}^{(J)} \notin \mathbb{R}_{>0}^n$, so define $I_J = \{i | z_i^{(J)} = 0\}$ and $K_J = \{(\alpha_1 ... \alpha_n) |\ for\ all\ i \in I_J, \alpha_i = 0\}$. Thus for all $k \notin I_{J+1}$,

$$\begin{aligned} \boldsymbol{z}_k^{(J+1)} &= (\sum a_{k\alpha_1...\alpha_n} (z_1^{(J)})^{\alpha_1} ... (z_n^{(J)})^{\alpha_n})^{[\frac{1}{\delta}]} \\ &\leq \sum_{(\alpha_1...\alpha_n) \in K_J} a_{k\alpha_1...\alpha_n} (z_1^{(J)})^{\alpha_1} ... (z_n^{(J)})^{\alpha_n} \\ &\leq ||\boldsymbol{z}^{(J)}||^{[\delta]} \sum_{(\alpha_1...\alpha_n) \in K_J} a_{k\alpha_1...\alpha_n}. \end{aligned}$$

Hence $||\boldsymbol{z}^{(J)}||^{[\delta]} \sum_{(\alpha_1...\alpha_n) \in K_J} a_{k\alpha_1...\alpha_n} \geq z_k^{(J+1)} > 0$ for $k \notin I_{J+1}$. Since $||\boldsymbol{z}^{(J)}||^{[\delta]} > 0$, $\sum_{(\alpha_1...\alpha_n) \in K_J} a_{i\alpha_1...\alpha_n} > 0$ for all $i \notin I_{J+1}$, where $I_r = \emptyset$. □

Theorem 42. Let $\boldsymbol{P}$ be a nonnegative polynomial. The polynomial $\boldsymbol{P}$ is primitive if and only if there exists $r \in \mathbb{N}$ such that $\boldsymbol{T}_{\boldsymbol{P}}^r$ is strongly increasing.

Proof. ($\Rightarrow$) Assume $\mathbf{0} \leq \boldsymbol{x} < \boldsymbol{y}$. By Lemma 17, $\boldsymbol{T}_{\boldsymbol{P}}\boldsymbol{x} < \boldsymbol{T}_{\boldsymbol{P}}\boldsymbol{y}$. Apply Lemma 17 repeatedly,

$$\begin{aligned} \boldsymbol{T}_{\boldsymbol{P}}(\boldsymbol{T}_{\boldsymbol{P}}\boldsymbol{x}) &< \boldsymbol{T}_{\boldsymbol{P}}(\boldsymbol{T}_{\boldsymbol{P}}\boldsymbol{y}), \\ \boldsymbol{T}_{\boldsymbol{P}}^2\boldsymbol{x} &< \boldsymbol{T}_{\boldsymbol{P}}^2\boldsymbol{y}, \\ &\vdots \\ \boldsymbol{T}_{\boldsymbol{P}}^r\boldsymbol{x} &< \boldsymbol{T}_{\boldsymbol{P}}^r\boldsymbol{y}. \end{aligned}$$

Now we assume $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x} \in D_{I_0}$ for some proper nonempty subset I_0. Apply Lemma 18 to $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x}$, we obtain a sequence of index subsets $\{I_J | 0 \leq J \leq r\}$, where $\{I_J | 0 \leq J \leq r-1\}$ are proper nonempty subsets such that $\sum_{(\alpha_1...\alpha_n) \in K_J} a_{i\alpha_1...\alpha_n} > 0$ for all $i \notin I_{J+1},\ J = 0, 1, ..., r-1$ and $I_r = \emptyset$.

Let $\boldsymbol{\xi}^{(J)} = \boldsymbol{T}_{\boldsymbol{P}}^J \boldsymbol{x}$ and $\boldsymbol{\eta}^{(J)} = \boldsymbol{T}_{\boldsymbol{P}}^J \boldsymbol{y}$ for $J = 0, 1, ..., r$. We claim

$$\xi_k^{(J)} < \eta_k^{(J)}\ \ for\ all\ \ k \notin I_J,\ \ J = 0, 1, ..., r, \tag{4.17}$$

and we are going to prove this by induction. At $J = 0$, $\boldsymbol{\xi}^{(0)} = \boldsymbol{x}$ and $\boldsymbol{\eta}^{(0)} = \boldsymbol{y}$. By assumption $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x} \in D_{I_0}$, $\xi_k^{(0)} < \eta_k^{(0)}$ for all $k \notin I_0$.

Suppose that (4.17) holds for $J \geq 1$. Let $\epsilon_J > 0$ be such that $\eta_k^{(J)} - \xi_k^{(J)} \geq \epsilon_J$, for all $k \notin I_J$. Now we want to show $\eta_k^{(J+1)} > \xi_k^{(J+1)}$ for all $k \notin I_{J+1}$. In order to do this, we use the same argument as in the proof of Lemma 17,

$$\begin{aligned}\eta_k^{(J+1)} - \xi_k^{(J+1)} &= \sum a_{k\boldsymbol{\alpha}}((\eta_1^{(J)})^{\alpha_1}...(\eta_n^{(J)})^{\alpha_n} - (\xi_1^{(J)})^{\alpha_1}...(\xi_n^{(J)})^{\alpha_n}) \\ &\geq \sum a_{k\boldsymbol{\alpha}}((\eta_1^{(J)})^{\alpha_1} - (\xi_1^{(J)})^{\alpha_1})...((\eta_n^{(J)})^{\alpha_n} - (\xi_n^{(J)})^{\alpha_n}) \\ &\geq \epsilon_J^n \sum_{(\alpha_1...\alpha_n)\in K_J} a_{k\boldsymbol{\alpha}} > 0 \ \textit{for all} \ k \notin I_{J+1}.\end{aligned}$$

So now we have $\eta_k^{(J+1)} > \xi_k^{(J+1)}$ for all $k \notin I_{J+1}$. Thus (4.17) holds for $J = 0, 1, ..., r$. Since I_r is an empty set, this concludes $\eta_i^{(r)} > \xi_i^{(r)}$ for all i, which means $\boldsymbol{T}_{\boldsymbol{P}}^r\boldsymbol{y} > \boldsymbol{T}_{\boldsymbol{P}}^r\boldsymbol{x}$, $\boldsymbol{T}_{\boldsymbol{P}}^r$ is strongly increasing.

($\Leftarrow$) Suppose that there exists $r \in \mathbb{N}$ such that $\boldsymbol{T}_{\boldsymbol{P}}^r$ is strongly increasing, which means for $\boldsymbol{x} \leq \boldsymbol{y} \in \mathbb{R}_+^n$ and $\boldsymbol{x} \neq \boldsymbol{y}$ implies $\boldsymbol{T}_{\boldsymbol{P}}^r\boldsymbol{x} < \boldsymbol{T}_{\boldsymbol{P}}^r\boldsymbol{y}$. By the definition of strongly increasing, for any $\mathbf{0} \leq \boldsymbol{z}$ and $\mathbf{0} \neq \boldsymbol{z}$ implies $\mathbf{0} = \boldsymbol{T}_{\boldsymbol{P}}^r(\mathbf{0}) < \boldsymbol{T}_{\boldsymbol{P}}^r\boldsymbol{z}$. This means for $\boldsymbol{z} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, $\boldsymbol{T}_{\boldsymbol{P}}^r\boldsymbol{z} \in \mathbb{R}_{>0}^n$. By Theorem 41 , $\boldsymbol{P}$ is primitive. □

4.6 Algorithm

Based on Theorem 40, we present the following algorithm for finding the largest eigenvalue of a nonhomogeneous nonnegative polynomial $\boldsymbol{P}$.

Algorithm 7.

Step 0: Choose $\boldsymbol{x}^{(1)} \in \mathbb{R}_{>0}^n$. Let $\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x}) + \boldsymbol{x}^{[\delta]}$ and let $k = 1$.

Step 1: Compute

$$\begin{aligned}Q_i(\boldsymbol{x}^{(k)}) &= \sum_{\boldsymbol{\alpha}\in\mathbb{R}_+^n} a_{i\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}} + \boldsymbol{x}^{[\delta]}, \quad i = 1, 2, ..., n, \\ \underline{\lambda}_k &= \min_{i\in\{1,2,...,n\}, x_i^{(k)}>0} \frac{Q_i(\boldsymbol{x}^{(k)})}{(x_i^{(k)})^\delta}, \\ \bar{\lambda}_k &= \max_{x_i^{(k)}>0} \frac{Q_i(\boldsymbol{x}^{(k)})}{(x_i^{(k)})^\delta}.\end{aligned}$$

Step 2: If $\bar{\lambda}_k = \underline{\lambda}_k$, then let $\lambda = \bar{\lambda}_k$ and stop. Otherwise, compute

$$\boldsymbol{x}^{(k+1)} = \frac{\left(\boldsymbol{Q}(\boldsymbol{x}^{(k)})\right)^{[\frac{1}{\delta}]}}{\left\|\left(\boldsymbol{Q}(\boldsymbol{x}^{(k)})\right)^{[\frac{1}{\delta}]}\right\|_p},$$

replace k by $k+1$ and go to Step 1.

We define

$$
\begin{aligned}
\boldsymbol{T_Q x} &= \Big(\boldsymbol{Q}(\boldsymbol{x}) \Big)^{[\frac{1}{\delta}]}, \\
\boldsymbol{T}_{\boldsymbol{Q}}^2 \boldsymbol{x} &= \boldsymbol{T}_{\boldsymbol{Q}}(\boldsymbol{T}_{\boldsymbol{Q}} \boldsymbol{x}), \\
\boldsymbol{T}_{\boldsymbol{Q}}^3 \boldsymbol{x} &= \boldsymbol{T}_{\boldsymbol{Q}}(\boldsymbol{T}_{\boldsymbol{Q}}^2 \boldsymbol{x}), \\
&\vdots \\
\boldsymbol{T}_{\boldsymbol{Q}}^r \boldsymbol{x} &= \boldsymbol{T}_{\boldsymbol{Q}}(\boldsymbol{T}_{\boldsymbol{Q}}^{r-1} \boldsymbol{x}).
\end{aligned}
$$

We obtain the following results by using the same argument as Proposition 5.1 [9].

Proposition 5. For the notation used in Algorithm 7, the following statements hold:

1. If $\boldsymbol{Q}$ is irreducible, then $\underline{\lambda}_{k+1} \nearrow \underline{\lambda}$, $\bar{\lambda}_{k+1} \searrow \bar{\lambda}$, and $\underline{\lambda} \leq \bar{\lambda}$.

2. There exists a subsequence $\boldsymbol{x}^{(k_j)} \longrightarrow \boldsymbol{x}^*$ with $||\boldsymbol{x}^*|| = 1$.

3. $(\underline{\lambda}_k)^{\frac{1}{\delta}} \boldsymbol{x}^{(k)} \leq \boldsymbol{T}_{\boldsymbol{Q}} \boldsymbol{x}^{(k)} \leq (\bar{\lambda}_k)^{\frac{1}{\delta}} \boldsymbol{x}^{(k)}$; hence, $\underline{\lambda}^{\frac{1}{\delta}} \boldsymbol{x}^* \leq \boldsymbol{T}_{\boldsymbol{Q}} \boldsymbol{x}^* \leq \bar{\lambda}^{\frac{1}{\delta}} \boldsymbol{x}^*$.

4. For all $k \in \mathbb{N}$, there exists $1 \leq i_0 \leq n$ such that $(\boldsymbol{T}_{\boldsymbol{Q}}^{k+1} \boldsymbol{x}^*)_{i_0} = \underline{\lambda}^{\frac{1}{\delta}} (\boldsymbol{T}_{\boldsymbol{Q}}^k \boldsymbol{x}^*)_{i_0}$.

Proof. First, we prove Statement 1. By Theorem 40, $\underline{\lambda}_{k+1} \to \underline{\lambda}$, $\bar{\lambda}_{k+1} \to \bar{\lambda}$ and $\underline{\lambda} \leq \bar{\lambda}$.

Now, we prove the second statement. At each iteration, $\boldsymbol{x}^{(k)}$ is a normalized vector, $||\boldsymbol{x}^{(k)}|| = 1$, hence $\{\boldsymbol{x}^{(k)}\}$ is bounded. Every bounded sequence has a convergent subsequence that converges to a point in the same set; $\boldsymbol{x}^{(k_j)} \longrightarrow \boldsymbol{x}^*$ with $||\boldsymbol{x}^*|| = 1$.

For the next statement, from Algorithm 7,

$$
\begin{aligned}
\boldsymbol{Q}(\boldsymbol{x}^{(k)}) &\geq \underline{\lambda}_k (\boldsymbol{x}^{(k)})^{[\delta]}, \\
(\boldsymbol{Q}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]} &\geq \underline{\lambda}_k^{\frac{1}{\delta}} (\boldsymbol{x}^{(k)}), \\
\boldsymbol{T}_{\boldsymbol{Q}} \boldsymbol{x}^{(k)} &\geq \underline{\lambda}_k^{\frac{1}{\delta}} (\boldsymbol{x}^{(k)}),
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{Q}(\boldsymbol{x}^{(k)}) &\leq \bar{\lambda}_k (\boldsymbol{x}^{(k)})^{[\delta]}, \\
(\boldsymbol{Q}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]} &\leq \bar{\lambda}_k^{\frac{1}{\delta}} (\boldsymbol{x}^{(k)}), \\
\boldsymbol{T}_{\boldsymbol{Q}} \boldsymbol{x}^{(k)} &\leq \bar{\lambda}_k^{\frac{1}{\delta}} (\boldsymbol{x}^{(k)}).
\end{aligned}
$$

Hence,

$$\underline{\lambda}_k^{\frac{1}{\delta}}(\boldsymbol{x}^{(k)}) \le \boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^{(k)} \le \bar{\lambda}_k^{\frac{1}{\delta}}(\boldsymbol{x}^{(k)}).$$

As $\boldsymbol{x}^{(k)} \to \boldsymbol{x}^*$,

$$\underline{\lambda}_k^{\frac{1}{\delta}}(\boldsymbol{x}^*) \le \boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^* \le \bar{\lambda}_k^{\frac{1}{\delta}}(\boldsymbol{x}^*).$$

To prove Statement 4, since $\boldsymbol{Q}$ is nonhomogeneous, we substitute $\boldsymbol{P}$ in $\boldsymbol{Q}$ with $\boldsymbol{F}$. Let $\bar{\boldsymbol{Q}}(\boldsymbol{x}) = (\boldsymbol{F}(\boldsymbol{x}))^{[\delta]} + \boldsymbol{x}^{[\delta]}$, where $\boldsymbol{F}(\boldsymbol{x})$ is a homogeneous monotone function as in the proof of Theorem 39. Then, the eigenvalue of $\bar{\boldsymbol{Q}}$ depending on a, p satisfying $\|\boldsymbol{x}\|_p = a$, $a, p > 0$ is also the eigenvalue of $\boldsymbol{Q}$. From Algorithm 7,

$$\underline{\lambda}_k^{\frac{1}{\delta}}\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^k\boldsymbol{x} \le \boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k+1}\boldsymbol{x}.$$

We prove Statement 4 by contradiction. Suppose that there is a positive integer k such that

$$\underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^k\boldsymbol{x}^* < \boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k+1}\boldsymbol{x}^*. \tag{4.18}$$

Then, there exists $\boldsymbol{x}^{(r)}$ close enough to $\boldsymbol{x}^*$ such that

$$\underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^k\boldsymbol{x}^{(r)} < \boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k+1}\boldsymbol{x}^{(r)}.$$

Hence, for all i, we have

$$\underline{\lambda}^{\frac{1}{\delta}} < \frac{(\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k+1}\boldsymbol{x}^{(r)})_i}{(\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^k\boldsymbol{x}^{(r)})_i}. \tag{4.19}$$

However, from Algorithm 7,

$$\begin{aligned}
\boldsymbol{x}^{(k+1)} &= \frac{(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}}{\left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}\right\|_p}, \\
\left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}\right\|_p \boldsymbol{x}^{(k+1)} &= (\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}. \qquad (4.20)
\end{aligned}$$

We put (4.20) as $\boldsymbol{x}$ in $\bar{\boldsymbol{Q}}(\boldsymbol{x})$,

$$\bar{\boldsymbol{Q}}(\left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}\right\|_p \boldsymbol{x}^{(k+1)}) = \bar{\boldsymbol{Q}}((\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}), \tag{4.21}$$

$$\left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}\right\|_p^{\delta} \bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k+1)}) = \bar{\boldsymbol{Q}}((\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}). \tag{4.22}$$

Now,

$$\begin{aligned}
\boldsymbol{T}_{\bar{\boldsymbol{Q}}}\boldsymbol{x}^{(r)} &= (\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}, \\
\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{2}\boldsymbol{x}^{(r)} &= \boldsymbol{T}_{\bar{\boldsymbol{Q}}}((\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}) \\
&= (\bar{\boldsymbol{Q}}((\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}))^{[\frac{1}{\delta}]} \\
&= \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}\right\|_p (\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+1)}))^{[\frac{1}{\delta}]}.
\end{aligned}$$

Using the same arguments,

$$\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{3}\boldsymbol{x}^{(r)} = \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}\right\|_p \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+1)}))^{[\frac{1}{\delta}]}\right\|_p (\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+2)}))^{[\frac{1}{\delta}]}$$

and

$$\begin{aligned}
\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k}\boldsymbol{x}^{(r)} = {} & \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}\right\|_p \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+1)}))^{[\frac{1}{\delta}]}\right\|_p \cdots \\
& \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+k-2)}))^{[\frac{1}{\delta}]}\right\|_p (\boldsymbol{Q}(\boldsymbol{x}^{(r+k-1)}))^{[\frac{1}{\delta}]}.
\end{aligned} \tag{4.23}$$

We shift the indices,

$$\begin{aligned}
\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k+1}\boldsymbol{x}^{(r)} = {} & \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}\right\|_p \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+1)}))^{[\frac{1}{\delta}]}\right\|_p \cdots \\
& \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+k-1)}))^{[\frac{1}{\delta}]}\right\|_p (\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+k)}))^{[\frac{1}{\delta}]}.
\end{aligned}$$

From Algorithm 7, we have

$$\boldsymbol{x}^{(k+1)} = \frac{\big(\boldsymbol{Q}(\boldsymbol{x}^{(k)})\big)^{[\frac{1}{\delta}]}}{\left\|(\boldsymbol{Q}(\boldsymbol{x}^{(k)}))^{[\frac{1}{\delta}]}\right\|_p}.$$

Shift the indices so that

$$\begin{aligned}
\boldsymbol{x}^{(r+k)} &= \frac{\big(\boldsymbol{Q}(\boldsymbol{x}^{(r+k-1)})\big)^{[\frac{1}{\delta}]}}{\left\|(\boldsymbol{Q}(\boldsymbol{x}^{(r+k-1)}))^{[\frac{1}{\delta}]}\right\|_p}, \\
\left\|\Big(\boldsymbol{Q}^{(r+k-1)}\Big)^{[\frac{1}{\delta}]}\right\|_p \boldsymbol{x}^{(r+k)} &= \big(\boldsymbol{Q}(\boldsymbol{x}^{(r+k-1)})\big)^{[\frac{1}{\delta}]}.
\end{aligned} \tag{4.24}$$

By (4.24), equation (4.23) becomes

$$\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k}\boldsymbol{x}^{(r)} = \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}\right\|_p \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+1)}))^{[\frac{1}{\delta}]}\right\|_p \cdots \tag{4.25}$$

$$\left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+k-2)}))^{[\frac{1}{\delta}]}\right\|_p \left\|\big(\boldsymbol{Q}(\boldsymbol{x}^{(r+k-1)})\big)^{[\frac{1}{\delta}]}\right\|_p \boldsymbol{x}^{(r+k)}. \tag{4.26}$$

Thus we have

$$\frac{(\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k+1}\boldsymbol{x}^{(r)})_i}{(\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k}\boldsymbol{x}^{(r)})_i}$$

$$
\begin{aligned}
&= \frac{\left(\left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}\right\|_p \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+1)}))^{[\frac{1}{\delta}]}\right\|_p \cdots \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+k-1)}))^{[\frac{1}{\delta}]}\right\|_p (\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+k)}))^{[\frac{1}{\delta}]}\right)_i}{\left(\left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r)}))^{[\frac{1}{\delta}]}\right\|_p \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+1)}))^{[\frac{1}{\delta}]}\right\|_p \cdots \left\|(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+k-1)}))^{[\frac{1}{\delta}]}\right\|_p (\boldsymbol{x}^{(r+k)})\right)_i} \\
&= \frac{(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+k)})^{[\frac{1}{\delta}]})_i}{x_i^{(r+k)}} \\
&= \left(\frac{(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(r+k)}))_i}{(x_i^{(r+k)})^\delta}\right)^{[\frac{1}{\delta}]}.
\end{aligned}
$$

From Algorithm 7,

$$
\left(\frac{(\bar{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}))_i}{(x_i^{(k)})^\delta}\right)^{[\frac{1}{\delta}]} \geq \underline{\lambda}_k^{[\frac{1}{\delta}]}.
$$

This means there exist some i_0 such that

$$
\frac{(\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k+1}\boldsymbol{x}^{(r)})_{i_0}}{(\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{k}\boldsymbol{x}^{(r)})_{i_0}} = \underline{\lambda}_{r+k}^{\frac{1}{\delta}}.
$$

This contradicts with the assumption of equation (4.18).

□

Theorem 43. If nonnegative polynomial $\boldsymbol{P}$ is irreducible, then $\underline{\lambda} = \lambda_0 = \bar{\lambda}$ and $\boldsymbol{x}^{(k)} \to \boldsymbol{x}^*$, which means $\boldsymbol{x}^{(k)}$ converges to the positive eigenvector associated with λ_0.

Proof. From Algorithm 7, let $\boldsymbol{Y} = \boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^* \geq \underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^* = \boldsymbol{X}$ and suppose $\boldsymbol{Y} = \boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^* \neq \underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^* = \boldsymbol{X}$.

By Theorem 42, there exists $r \in \mathbb{N}$ such that $\boldsymbol{T}^r_{\boldsymbol{Q}}\boldsymbol{X} < \boldsymbol{T}^r_{\boldsymbol{Q}}\boldsymbol{Y}$. Shift the index and we have

$$
\begin{aligned}
\boldsymbol{T}_{\boldsymbol{Q}}^{r-1}(\underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*) &< \boldsymbol{T}_{\boldsymbol{Q}}^{r-1}(\boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^*), \\
\boldsymbol{T}_{\boldsymbol{Q}}^{r-1}(\underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*) &< \boldsymbol{T}^r_{\boldsymbol{Q}}\boldsymbol{x}^*.
\end{aligned}
$$

Since $\boldsymbol{Q}$ is nonhomogeneous, we substitute $\boldsymbol{P}$ in $\boldsymbol{Q}$ with $\boldsymbol{F}$, $\bar{\boldsymbol{Q}} = (\boldsymbol{F}(\boldsymbol{x}))^{[\delta]} + \boldsymbol{x}^{[\delta]}$, where $\boldsymbol{F}$ is as in the proof of Theorem 39. Then, the eigenvalue of $\bar{\boldsymbol{Q}}$ depending on a, p satisfying $\|\boldsymbol{x}\|_p = a$, $a, p > 0$ is also the eigenvalue of $\boldsymbol{Q}$. Algorithm 7 satisfies this conditions. Now,

$$
\underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{T}_{\bar{\boldsymbol{Q}}}^{r-1}(\boldsymbol{x}^*) < \boldsymbol{T}^r_{\bar{\boldsymbol{Q}}}\boldsymbol{x}^*.
$$

However, this contradicts statement (4) of Proposition 5. Hence, we must have $\boldsymbol{T}_{\bar{\boldsymbol{Q}}}\boldsymbol{x}^* = \underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*$.

By using similar argument as above, we want to show $\boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^* = \bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*$. From Algorithm 7, we have $\boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^* \leq \bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*$. Now, let say $\boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^* \neq \bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*$. By Theorem 42, there exists $r \in \mathbb{N}$ such that $\boldsymbol{T}^r_{\boldsymbol{Q}}(\boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^*) < \boldsymbol{T}^r_{\boldsymbol{Q}}(\bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*)$. Shift the index and we have

$$\begin{aligned} \boldsymbol{T}^{r-1}_{\boldsymbol{Q}}(\boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{x}^*) &< \boldsymbol{T}^{r-1}_{\boldsymbol{Q}}(\bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*), \\ \boldsymbol{T}^r_{\boldsymbol{Q}}\boldsymbol{x}^* &< \boldsymbol{T}^{r-1}_{\boldsymbol{Q}}(\bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*). \end{aligned}$$

Since $\boldsymbol{Q}$ is nonhomogeneous, we use $\bar{\boldsymbol{Q}} = (\boldsymbol{F}(x))^{[\delta]} + \boldsymbol{x}^{[\delta]}$, where $\boldsymbol{F}$ is as in the proof of Theorem 39. Now,

$$\boldsymbol{T}^r_{\bar{\boldsymbol{Q}}}\boldsymbol{x}^* < \bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{T}^{r-1}_{\bar{\boldsymbol{Q}}}(\boldsymbol{x}^*).$$

However, this contradicts statement (4) of Proposition 5. Hence, we must have $\boldsymbol{T}_{\bar{\boldsymbol{Q}}}\boldsymbol{x}^* = \bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*$.

Therefore, $\boldsymbol{T}_{\bar{\boldsymbol{Q}}}\boldsymbol{x}^* = \underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^* = \bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{x}^*$. This means $\lambda_0 = \underline{\lambda} = \bar{\lambda}$.

Next, we show that $\boldsymbol{x}^{(k)}$ converges to $\boldsymbol{x}^*$. We are going to prove this by contradiction. Suppose there exists $\{r_k\}$ such that $\boldsymbol{x}^{(r_k)} \to \boldsymbol{y}^*$ where $||\boldsymbol{y}^*|| = 1$ and $\boldsymbol{y}^* \neq \boldsymbol{x}^*$. By using the same argument as in the previous steps, we can easily show $\boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{y}^* = \underline{\lambda}^{\frac{1}{\delta}}\boldsymbol{y}^*$ and $\boldsymbol{T}_{\boldsymbol{Q}}\boldsymbol{y}^* = \bar{\lambda}^{\frac{1}{\delta}}\boldsymbol{y}^*$. Hence, $\lambda_0 = \underline{\lambda} = \bar{\lambda}$ and $\boldsymbol{Q}(\boldsymbol{y}^*) = \lambda_0(\boldsymbol{y}^*)^{[\delta]}$. However, by Theorem 39, the eigenvector associated with eigenvalue λ_0 is unique, which means $\boldsymbol{y}^* = \boldsymbol{x}^*$.

□

4.7 Numerical Results

In order to show that Algorithm 7 is efficient, we use MATLAB R2010b to test it on randomly generated nonhomogeneous polynomials.

Let $\mathcal{A}$ be an m-order n-dimensional square tensor and $\mathcal{B}$ be a d-order n-dimensional square tensor.

$$\begin{aligned}\mathcal{A} &= (a_{i_1 i_2 \dots i_m}), \quad a_{i_1 i_2 \dots i_m} \in \mathbb{R}, \quad 1 \leq i_1, i_2, ..., i_m \leq n. \\ \mathcal{B} &= (b_{i_1 i_2 \dots i_d}), \quad b_{i_1 i_2 \dots i_d} \in \mathbb{R}, \quad 1 \leq i_1, i_2, ..., i_d \leq n.\end{aligned}$$

We define n-dimensional column vectors $\mathcal{A}\boldsymbol{x}^{m-1}$ and $\mathcal{B}\boldsymbol{x}^{d-1}$ as

$$\begin{aligned}\mathcal{A}\boldsymbol{x}^{m-1} &= \left(\sum_{i_2,\dots,i_m=1}^{n} a_{i\, i_2 \dots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leq i \leq n}. \\ \mathcal{B}\boldsymbol{x}^{d-1} &= \left(\sum_{i_2,\dots,i_d=1}^{n} b_{i\, i_2 \dots i_d} x_{i_2} \cdots x_{i_d} \right)_{1 \leq i \leq n}.\end{aligned}$$

Our test function $\boldsymbol{P}(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^{m-1} + \mathcal{B}\boldsymbol{x}^{d-1}$ is defined as follows.

$$P_i(\boldsymbol{x}) = \sum_{i_2,\dots,i_m=1}^{n} a_{i\, i_2 \dots i_m} x_{i_2} \cdots x_{i_m} + \sum_{i_2,\dots,i_d=1}^{n} b_{i\, i_2 \dots i_d} x_{i_2} \cdots x_{i_d}, \quad 1 \leq i \leq n,$$

Each entry of the tensors is randomly generated between 0 and 10. We choose $d > m$ so that $\boldsymbol{P}(\boldsymbol{x})$ is a nonhomogeneous polynomial with degree $d-1$. We present the numerical results in the tables below. Table 4.1, 4.2, 4.3 and 4.4 show the results when $p = 1$, $p = 2$, $p = 3$ and $p = \infty$, respectively. In the tables, n denotes the dimension of the polynomials, d denotes the order of tensor $\mathcal{B}$, *Ite* denotes the number of iterations, $\bar{\lambda} - \underline{\lambda}$ and λ denote the value of $\bar{\lambda}_k - \underline{\lambda}^k$ and $0.5(\bar{\lambda}_k - \underline{\lambda}^k)$, respectively at the final iteration. $||\boldsymbol{P}(\boldsymbol{x}) - \lambda \boldsymbol{x}^{[d-1]}||_p$ denotes the value of $||\boldsymbol{P}(\boldsymbol{x}^{(k)}) - \lambda(\boldsymbol{x}^{(k)})^{[d-1]}||_p$ at the final iteration. We use initial value $\boldsymbol{x}^{(0)} = (1, 1, ..., 1)^T$ for all the computational experiments and terminate the iteration when $|\bar{\lambda} - \underline{\lambda}| \leq 1 \times 10^{-6}$ or $||\boldsymbol{P}(\boldsymbol{x}) - \lambda \boldsymbol{x}^{[d-1]}||_p \leq 1 \times 10^{-6}$. As shown in the tables, the algorithm managed to produce the largest eigenvalue of all the randomly generated polynomials.

4.8 Conclusion

An iterative method for finding the largest eigenvalue of nonhomogenous nonnegative polynomials was proposed in this chapter. This method was proven to be convergent for irreducible nonhomogenous nonnegative polynomials. We also extended the Collatz minimax theorem to nonhomogeneous polynomials and the concept of primitivity to polynomials.

(n, d)	Ite	$\bar{\lambda} - \underline{\lambda}$	$\|\|\boldsymbol{P}(\boldsymbol{x}) - \lambda \boldsymbol{x}^{[d]}\|\|_1$
(4,3)	8	1.20e-06	1.33e-07
(5,4)	8	5.71e-06	7.55e-08
(5,5)	7	1.93e-04	4.53e-07
(5,6)	5	5.17e-05	3.40e-08
(10,3)	7	1.27e-05	4.04e-07
(10,4)	7	7.12e-05	1.87e-07
(10,5)	7	7.47e-05	2.12e-08
(15,3)	8	6.04e-06	9.60e-08
(15,4)	6	1.52e-05	1.82e-08
(15,5)	5	5.31e-05	3.48e-09
(15,6)	4	4.86e-03	2.14e-08
(20,3)	7	2.48e-05	2.48e-07
(30,3)	7	2.58e-05	1.53e-07
(40,3)	7	2.36e-05	1.07e-07
(50,3)	7	1.31e-05	4.70e-08
(60,3)	7	1.46e-05	3.97e-08
(70,3)	7	1.35e-05	3.42e-08
(80,3)	7	1.04e-05	2.35e-08
(90,3)	7	1.24e-05	2.61e-08
(100,3)	7	9.83e-06	1.41e-08

Table 4.1: Numerical results of Algorithm 7 for randomly generated polynomials when $p = 1$.

(n, d)	Ite	$\bar{\lambda} - \underline{\lambda}$	$\|\|\boldsymbol{P}(\boldsymbol{x}) - \lambda \boldsymbol{x}^{[d]}\|\|_2$
(4,3)	7	3.26e-06	5.75e-07
(5,4)	6	2.38e-06	1.82e-07
(5,5)	6	6.93e-07	2.60e-08
(5,6)	5	1.29e-05	1.76e-07
(10,3)	7	1.98e-06	2.03e-07
(10,4)	6	7.48e-07	2.44e-08
(10,5)	5	1.31e-05	1.16e-07
(15,3)	7	1.61e-06	1.39e-07
(15,4)	6	5.71e-07	1.16e-08
(15,5)	5	6.13e-06	2.71e-08
(15,6)	4	4.88e-04	6.19e-07
(20,3)	6	1.69e-05	9.19e-07
(30,3)	6	4.45e-06	2.02e-07
(40,3)	6	7.17e-06	1.96e-07
(50,3)	6	3.84e-06	1.14e-07
(60,3)	6	2.19e-06	5.17e-08
(70,3)	6	1.15e-06	2.90e-08
(80,3)	6	1.15e-06	2.53e-08
(90,3)	6	6.61e-07	1.54e-08
(100,3)	6	4.96e-07	1.30e-08

Table 4.2: Numerical results of Algorithm 7 for randomly generated polynomials when $p = 2$.

(n,d)	Ite	$\bar{\lambda} - \underline{\lambda}$	$\|\|\boldsymbol{P}(\boldsymbol{x}) - \lambda \boldsymbol{x}^{[d]}\|\|_3$
(4,3)	7	3.26e-06	5.75e-07
(5,4)	6	2.38e-06	1.82e-07
(5,5)	6	6.93e-07	2.60e-08
(5,6)	5	1.29e-05	1.76e-07
(10,3)	7	1.98e-06	2.03e-07
(10,4)	6	7.48e-07	2.44e-08
(10,5)	5	1.31e-05	1.16e-07
(15,3)	7	1.61e-06	1.39e-07
(15,4)	6	5.71e-07	1.16e-08
(15,5)	5	6.13e-06	2.71e-08
(15,6)	4	4.88e-04	6.19e-07
(20,3)	6	1.69e-05	9.19e-07
(30,3)	6	4.45e-06	2.02e-07
(40,3)	6	7.17e-06	1.96e-07
(50,3)	6	3.84e-06	1.14e-07
(60,3)	6	2.19e-06	5.17e-08
(70,3)	6	1.15e-06	2.90e-08
(80,3)	6	1.15e-06	2.53e-08
(90,3)	6	6.61e-07	1.54e-08
(100,3)	6	4.96e-07	1.30e-08

Table 4.3: Numerical results of Algorithm 7 for randomly generated polynomials when $p = 3$.

(n,d)	Ite	$\bar{\lambda}-\underline{\lambda}$	$\|\|\boldsymbol{P}(\boldsymbol{x})-\lambda\boldsymbol{x}^{[d]}\|\|_{\infty}$
(4,3)	9	2.74e-07	1.25e-07
(5,4)	6	3.39e-07	1.69e-07
(5,5)	5	1.87e-06	9.36e-07
(5,6)	5	1.79e-08	8.95e-09
(10,3)	7	3.87e-07	1.81e-07
(10,4)	6	2.77e-08	1.35e-08
(10,5)	5	3.23e-08	1.61e-08
(15,3)	6	5.53e-07	2.76e-07
(15,4)	5	3.80e-07	1.90e-07
(15,5)	5	6.81e-09	3.38e-09
(15,6)	4	3.59e-07	1.80e-07
(20,3)	6	7.54e-07	3.67e-07
(30,3)	6	2.30e-07	1.10e-07
(40,3)	6	3.71e-08	1.79e-08
(50,3)	5	1.87e-06	9.07e-07
(60,3)	5	1.20e-06	5.96e-07
(70,3)	5	1.18e-06	5.90e-07
(80,3)	5	4.88e-07	2.42e-07
(90,3)	5	5.13e-07	2.52e-07
(100,3)	5	3.33e-07	1.64e-07

Table 4.4: Numerical results of Algorithm 7 for randomly generated polynomials when $p=\infty$.

Chapter 5

Nonnegative Polynomial Optimisation

5.1 Introduction

In this chapter, we study optimisation problem where the objective function is a nonnegative polynomial and the constraint is spherical. Polynomial optimisation can be found in vast amount in literature. The application ranges are from investment science [39, 27], control theory [55] to psychometrics and chemometrics [30].

Let $\boldsymbol{P} : \mathbb{R}^n \to \mathbb{R}$,

$$\boldsymbol{P}(\boldsymbol{y}) = \sum_{\boldsymbol{\alpha} \in \mathbb{R}^n_+} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}},$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$, $a_\alpha = (a_{\alpha_1}, \cdots, a_{\alpha_n})$ and $\boldsymbol{x}^{\boldsymbol{\alpha}} = (x_1^{\alpha_1}, \cdots, x_n^{\alpha_n})$, be a generalised irreducible polynomial with nonnegative coefficients of degree h with $h \leq p$. Assume that $\boldsymbol{P}(\boldsymbol{y})$ is monotone. We are going to study the following problem.

$$\begin{aligned} &\max \boldsymbol{P}(\boldsymbol{y}) \\ s.t. \quad &\|\boldsymbol{y}\|_p = a, \qquad \boldsymbol{y} \geq \mathbf{0}, \quad a > 0, \end{aligned} \tag{5.1}$$

where $\|\boldsymbol{y}\|_p$ denotes p-norm. Recently, some kind of nonnegative polynomial optimisation models was studied in [77]. The models have the following form:

$$\max \boldsymbol{P}_{\mathcal{B}}(\boldsymbol{x}) \quad (5.2)$$
$$s.t. \quad \|\boldsymbol{x}\| = 1, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$

where the notation $\|\boldsymbol{x}\|$ denotes Euclidean norm and

$$\boldsymbol{P}_{\mathcal{B}}(\boldsymbol{x}) = \sum_{i_1, i_2, \ldots, i_d = 1}^{n} b_{i_1 i_2 \ldots i_d} x_{i_1} x_{i_2} \cdots x_{i_d}.$$

The polynomial $\boldsymbol{P}_{\mathcal{B}}(\boldsymbol{x})$ is formed from d-th order n-dimensional square tensor $\mathcal{B}$, where

$$\mathcal{B} = (b_{i_1 i_2 \ldots i_d}), \quad b_{i_1 i_2 \ldots i_d} \in \mathbb{R}, \quad 1 \leq i_1, i_2, ..., i_d \leq n.$$

This problem is a special case of Problem (5.1). The second polynomial problem that was studied in [77] is

$$\max \boldsymbol{P}_{\mathcal{C}}(\boldsymbol{x}, \boldsymbol{y}) \quad (5.3)$$
$$s.t. \quad \|\boldsymbol{x}\| = 1, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$
$$\|\boldsymbol{y}\| = 1, \qquad \boldsymbol{y} \in \mathbb{R}^m,$$

where

$$\boldsymbol{P}_{\mathcal{C}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i_1, \ldots, i_p = 1}^{n} \sum_{j_1, \ldots, j_q = 1}^{m} c_{i_1 \ldots i_p j_1 \ldots j_q} x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}.$$

The polynomial $\boldsymbol{P}_{\mathcal{C}}(\boldsymbol{x}, \boldsymbol{y})$ is formed from a (p, q)-th order (m, n)-dimensional rectangular tensor $\mathcal{C}$, where

$$\mathcal{C} = (c_{i_1 \ldots i_p j_1 \ldots j_q}), \quad c_{i_1 \ldots i_p j_1 \ldots j_q} \in \mathbb{R},$$
$$i_k = 1, ..., n, \;\; k = 1, ..., p, \;\; and \;\; j_l = 1, ..., m, \;\; l = 1, ..., q.$$

The polynomial $\boldsymbol{P}_{\mathcal{C}}(\boldsymbol{x}, \boldsymbol{y})$ has the order of $d = p+q$. Another polynomial problem which was studied in [77] is

$$\max \boldsymbol{P}_{\mathcal{A}}(\boldsymbol{x}^1, \boldsymbol{x}^2, ..., \boldsymbol{x}^d) \quad (5.4)$$
$$s.t. \quad \|\boldsymbol{x}^i\| = 1, \qquad \boldsymbol{x}^i \in \mathbb{R}^{n_i}, i = 1, 2, ..., d,$$

where

$$\boldsymbol{P}_{\mathcal{A}}(\boldsymbol{x}^1, \boldsymbol{x}^2, ..., \boldsymbol{x}^d) = \sum_{1\le i_1\le n_1, 1\le i_2\le n_2, ..., 1\le i_d\le n_d} a_{i_1 i_2 \dots i_d} x^1_{i_1} x^2_{i_2} \cdots x^d_{i_d}.$$

The polynomial $\boldsymbol{P}_{\mathcal{A}}(\boldsymbol{x}^1, \boldsymbol{x}^2, ..., \boldsymbol{x}^d)$ is formed from a d-th order tensor $\mathcal{A}$, where

$$\mathcal{A} = (a_{i_1 i_2 \dots i_d}), \quad a_{i_1 i_2 \dots i_d} \in \mathbb{R}, \quad 1 \le i_1 \le n_1, 1 \le i_2 \le n_2, ..., 1 \le i_d \le n_d.$$

In the paper, Problem (5.2), (5.3) and (5.4) were shown to be NP-hard. It is known that NP-hard problem is difficult to solve. In order to overcome the matter, the Problem (5.2), (5.3) and (5.4) were relaxed before solved using some computational methods. The followings are some results related to the relaxations.

Lemma 19. [77] Suppose that $\boldsymbol{x}^* \in \mathbb{R}^n$ is a global solution of Problem (5.2). Then, $|\boldsymbol{x}^*| = [|x_1^*|, |x_2^*|, ..., |x_n^*|]^T$ is a global solution of Problem (5.2).

Lemma 20. [77] Suppose that $\boldsymbol{x}^* \in \mathbb{R}^n$ is a global solution of the following problem:

$$\begin{aligned} &\max \boldsymbol{P}_{\mathcal{B}}(\boldsymbol{x}) \\ s.t. \quad &\|\boldsymbol{x}\| \le 1, \boldsymbol{x} \ge \boldsymbol{0}, \boldsymbol{x} \in \mathbb{R}^n. \end{aligned} \tag{5.5}$$

Then, $\boldsymbol{x}^*$ is a global solution of Problem (5.2).

Likewise, the following results are produced for Problem (5.3) and (5.4).

Lemma 21. [77] Suppose that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a global solution of the following problem:

$$\begin{aligned} &\max \boldsymbol{P}_{\mathcal{C}}(\boldsymbol{x}, \boldsymbol{y}) \\ s.t. \quad &\|\boldsymbol{x}\| \le 1, \boldsymbol{x} \ge \boldsymbol{0}, \boldsymbol{x} \in \mathbb{R}^n, \\ &\|\boldsymbol{y}\| \le 1, \boldsymbol{y} \ge \boldsymbol{0}, \boldsymbol{y} \in \mathbb{R}^m. \end{aligned} \tag{5.6}$$

Then,$(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a global solution of Problem (5.3).

Lemma 22. [77] Suppose that $((\boldsymbol{x}^1)^*, ..., (\boldsymbol{x}^d)^*)$ is a global solution of the following problem:

$$\begin{aligned} &\max \boldsymbol{P}_{\mathcal{A}}(\boldsymbol{x}^1, \boldsymbol{x}^2, ..., \boldsymbol{x}^d) \\ s.t. \quad &\|\boldsymbol{x}^i\| \le 1, \boldsymbol{x}^i \ge \boldsymbol{0}, \boldsymbol{x}^i \in \mathbb{R}^{n_i}, i = 1, 2, ..., d. \end{aligned} \tag{5.7}$$

Here, $d > 2$. Then, $((\boldsymbol{x}^1)^*, ..., (\boldsymbol{x}^d)^*)$ is a global solution of Problem (5.4).

To continue the process of relaxation, we further relax the constraint of Problem (5.5) from $\|\boldsymbol{x}\| \leq 1$ to $\|\boldsymbol{x}\|_d \leq 1$, constraints of Problem (5.6) from $\|\boldsymbol{x}\| \leq 1,\ \|\boldsymbol{y}\| \leq 1$ to $\|\boldsymbol{x}\|_d \leq 1,\ \|\boldsymbol{y}\|_d \leq 1$ and constraints of Problem (5.7) from $\|\boldsymbol{x}^i\| \leq 1,\ i = 1, 2, ..., d$ to $\|\boldsymbol{x}^i\|_d \leq 1,\ i = 1, 2, ..., d$. Now we have the relaxations of Problem (5.5), (5.6) and (5.7).

$$\begin{aligned} & \max \boldsymbol{P}_{\mathcal{B}}(\boldsymbol{x}) \\ s.t. \quad & \|\boldsymbol{x}\|_d \leq 1, \quad \boldsymbol{x} \geq \boldsymbol{0} \quad \boldsymbol{x} \in \mathbb{R}^n, \end{aligned} \tag{5.8}$$

$$\begin{aligned} & \max \boldsymbol{P}_{\mathcal{C}}(\boldsymbol{x}, \boldsymbol{y}) \\ s.t. \quad & \|\boldsymbol{x}\|_d \leq 1, \quad \boldsymbol{x} \geq 1 \quad \boldsymbol{x} \in \mathbb{R}^n \\ & \|\boldsymbol{y}\|_d \leq 1, \quad \boldsymbol{y} \geq 1 \quad \boldsymbol{y} \in \mathbb{R}^m, \end{aligned} \tag{5.9}$$

$$\begin{aligned} & \max \boldsymbol{P}_{\mathcal{A}}(\boldsymbol{x}^1, \boldsymbol{x}^2, ..., \boldsymbol{x}^d) \\ s.t. \quad & \|\boldsymbol{x}^i\|_d \leq 1, \quad \boldsymbol{x}^i \geq \boldsymbol{0} \quad \boldsymbol{x}^i \in \mathbb{R}^{n_i}, i = 1, 2, ..., d, \end{aligned} \tag{5.10}$$

Three methods were proposed in [77] to solve (5.8), (5.9) and (5.10) which were polynomial time algorithm by reformulating the relaxations into geometric programming problem, smoothing Newton methods and power methods . These power methods are a variation of the methods discussed in previous chapters. Amongst the methods proposed to solve the relaxations, the power methods have been proven the most efficient when tested numerically as reported in [77]. The quality of the approximation solutions produced by solving the relaxations (5.8), (5.9) and (5.10) using power method was also tested. The results indicate that solutions produced are of high quality.

5.2 Optimisation of Nonnegative Generalised Polynomials

We first introduce some useful notations and definitions. For $p > 0$, let $S_p^n = \{\boldsymbol{x} \in \mathbb{R}^n : \sum_i |x_i|^p = 1\}$ denote the ℓ_p unit sphere in $\mathbb{R}^n$. Let set $S_{p,+}^n = \mathbb{R}_+^n \bigcap S_p^n$

and set $S^n_{p,>0} = \mathbb{R}^n_{>0} \bigcap S^n_p$.

We can rewrite Problem (5.1) so that it is easier to solve. Observe that

$$\begin{aligned} \|\boldsymbol{y}\|_p &= a, \\ (y_1^p + y_2^p + ... + y_n^p)^{\frac{1}{p}} &= a, \\ \left(\frac{1}{a^p}\right)^{\frac{1}{p}} (y_1^p + y_2^p + ... + y_n^p)^{\frac{1}{p}} &= 1, \\ \left(\left(\frac{y_1}{a}\right)^p + \left(\frac{y_2}{a}\right)^p + ... + \left(\frac{y_n}{a}\right)^p\right)^{\frac{1}{p}} &= 1, \\ \|\boldsymbol{x}\|_p &= 1, \qquad \boldsymbol{x} = \frac{\boldsymbol{y}}{a}. \end{aligned}$$

Notice that

$$\begin{aligned} \|\boldsymbol{x}\|_p &= 1 \\ (x_1^p + x_2^p + ... + x_n^p)^{\frac{1}{p}} &= 1 \\ x_1^p + x_2^p + ... + x_n^p &= 1 \\ \sum_{i=1}^n x_i^p &= 1. \end{aligned}$$

Now Problem (5.1) becomes

$$\begin{aligned} & \max \boldsymbol{P}(\boldsymbol{x}) \\ s.t. \quad & \sum_i^n x_i^p = 1, \qquad x_i \geq 0, \qquad i = 1, ..., n, \end{aligned} \tag{5.11}$$

The problem of maximizing nonnegative generalised polynomial under ℓ_p-constraints where the degree of polynomial is at most p was discussed in [4] and we have the following result.

Theorem 44. [4] If $\boldsymbol{P}$ has no proper homogeneous irreducible component of degree p (in particular if $\boldsymbol{P}$ is irreducible), then Problem (5.11) has a unique solution $\boldsymbol{x}^*$. Moreover, $\boldsymbol{x}^* \in \mathbb{R}^n_{>0}$ in this case, and is the unique critical point of $\boldsymbol{P}$ on $S^n_{p,>0} = \{\boldsymbol{x} \in \mathbb{R}^n_{>0} : \sum_i |x_i|^p = 1\}$.

The theorem above states that if $\boldsymbol{P}(\boldsymbol{x})$ is irreducible, the unique solution of Problem (5.11) is the unique critical point of $\boldsymbol{P}$ on $S^n_{p,>0}$. We have the following results for the optimal condition for Problem (5.11).

Lemma 23. Suppose $\boldsymbol{P}(\boldsymbol{x})$ is an irreducible generalised polynomial with non-negative coefficients of degree h with $h \leq p$. The Karush-Kuhn-Tucker conditions for Problem (5.11) are

$$\begin{aligned} \nabla \boldsymbol{P}(\bar{\boldsymbol{x}}) &= \lambda_0(\bar{\boldsymbol{x}})^{[p-1]}, \\ \sum_i^n (\bar{x}_i)^p &= 1, \qquad \bar{x}_i > 0, \quad i = 1, ..., n, \qquad \lambda_0 > 0. \end{aligned}$$

Proof. The Karush-Kuhn-Tucker conditions for (5.11) are

$$\begin{aligned} -\nabla \boldsymbol{P}(\bar{\boldsymbol{x}}) + \mu p(\bar{\boldsymbol{x}})^{[p-1]} - \boldsymbol{\nu} = \mathbf{0}, \quad \mu \in \mathbb{R}, \quad \boldsymbol{\nu} \in \mathbb{R}^n \\ \sum_i^n (\bar{x})_i^p - 1 = 0, \\ -\bar{x}_i \leq 0, \quad i = 1, ..., n, \\ -\nu_i \bar{x}_i = 0, \quad \boldsymbol{\nu} \geq 0, \quad i = 1, ..., n. \end{aligned}$$

By Theorem 44, $\bar{\boldsymbol{x}} > 0$, hence, $\boldsymbol{\nu} = \mathbf{0}$. Now,

$$\nabla \boldsymbol{P}(\bar{\boldsymbol{x}}) = \lambda_0(\bar{\boldsymbol{x}})^{[p-1]}, \quad \lambda_0 = \mu p, \tag{5.12}$$

$$\sum_i^n (\bar{x}_i)^p = 1, \qquad \bar{x}_i > 0, \quad i = 1, ..., n, \qquad \lambda_0 > 0. \tag{5.13}$$

□

We know from Theorem 44 that $\bar{\boldsymbol{x}}$ is the unique positive critical point. Note that $\nabla \boldsymbol{P}(\bar{\boldsymbol{x}}) = \lambda_0(\bar{\boldsymbol{x}})^{[p-1]}$ is similar to the definition of eigenvalue and eigenvector of polynomial $\nabla \boldsymbol{P}(\boldsymbol{x})$. By Theorem 39, the Perron-Frobenius Theorem for polynomial, λ_0 is the largest eigenvalue of polynomial $\nabla \boldsymbol{P}(\boldsymbol{x})$.

Solving Problem (5.11) is now reduced to solving

$$\begin{aligned} & \nabla \boldsymbol{P}(\boldsymbol{x}) = \lambda(\boldsymbol{x})^{[p-1]} \\ s.t. \quad & \sum_i^n (x_i)^p = 1, x_i \geq 0, \quad i = 1, ..., n. \end{aligned} \tag{5.14}$$

The associated graph of $\boldsymbol{P}(\boldsymbol{x})$ is $\mathcal{G}(\boldsymbol{P})$ whose vertices are the variables of $\boldsymbol{P}(\boldsymbol{x})$. An edge of $\mathcal{G}(\boldsymbol{P})$, (i, j) exists if $a_{\alpha_1,\cdots,\alpha_n} \neq 0$ where $\alpha_i \neq 0$ and $\alpha_j \neq 0$. It means if variable x_i and x_j appears in a monomial, there exists an edge between vertices i and j. Polynomial $\boldsymbol{P}(\boldsymbol{x})$ is irreducible if and only if the associated graph of $\boldsymbol{P}(\boldsymbol{x})$, $\mathcal{G}(\boldsymbol{P})$ is connected.

Lemma 24. If polynomial $\boldsymbol{P}(\boldsymbol{x})$ is irreducible, then the gradient of $\boldsymbol{P}(\boldsymbol{x})$, polynomial $\nabla\boldsymbol{P}(\boldsymbol{x})$ is weakly irreducible.

Proof. Let us say x_{i_0} and x_{j_0} appears in a monomial of $\boldsymbol{P}(\boldsymbol{x})$. It implies there exists an edge between i_0 and j_0 in $\mathcal{G}(\boldsymbol{P})$. Now consider the gradient of $\boldsymbol{P}(\boldsymbol{x})$, $\nabla\boldsymbol{P}(\boldsymbol{x})$. Variable x_{j_0} appears in the expression of $\nabla P_{i_0}(\boldsymbol{x})$ and variable x_{i_0} appears in the expression of $\nabla P_{j_0}(\boldsymbol{x})$. Hence, there exists an edge between i_0 and j_0 for both directions in the graph of $\nabla\boldsymbol{P}(\boldsymbol{x})$, $\mathcal{G}(\nabla\boldsymbol{P})$. Now we can say that if $\mathcal{G}(\boldsymbol{P})$ is connected, then $\mathcal{G}(\nabla\boldsymbol{P})$ is strongly connected. This means $\nabla\boldsymbol{P}(\boldsymbol{x})$ is weakly irreducible. □

5.3 Algorithm

In this section, we give the algorithm for solving Problem (5.14) given that $\nabla\boldsymbol{P}(\boldsymbol{x})$ is irreducible and monotone. The following iterative process was suggested for solving Problem (5.14) in [4]. Select an initial value $\boldsymbol{x}^{(0)}$, then compute

$$\boldsymbol{x}^{(k+1)} = \alpha^{(k+1)}\Big(\ \nabla\boldsymbol{P}(\boldsymbol{x}^{(k)})\ \Big)^{[\frac{1}{p-1}]}, \tag{5.15}$$

$\alpha^{(k+1)}$ is adjusted so that

$$\alpha^{(k+1)} = \frac{1}{||(\nabla\boldsymbol{P}(\boldsymbol{x}^{(k)}))^{\frac{1}{[p-1]}}||_p}. \tag{5.16}$$

However, this method has not been proven convergent. The key to overcoming this problem lies in [9]. A convergent method was given in [9] to solve the following kind of problem.

$$\begin{aligned} \boldsymbol{P}_1(\boldsymbol{x}) &= \lambda(\boldsymbol{x})^{[p]} \qquad (5.17)\\ s.t. \qquad \|\boldsymbol{x}\|_1 &= 1, x_i \geq 0, \quad i = 1, ..., n, \end{aligned}$$

where $\boldsymbol{P}_1$ is a homogeneous and monotone polynomial of degree p. The problem above deals with the constraint $\|\boldsymbol{x}\|_1 = 1$. However, the problem we want to solve has the constraint $\|\boldsymbol{x}\|_p = 1$. Notice that if we let

$$\boldsymbol{z} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_p},$$

then

$$\begin{aligned} \|\boldsymbol{z}\|_p &= \left\| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_p} \right\|_p \\ &= \frac{\|\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p} \\ &= 1. \end{aligned}$$

It means the condition $\|\boldsymbol{x}\|_p = 1$ is satisfied. Therefore, we can conclude that solving Problem (5.17) is equivalent to solving Problem (5.14). We give the algorithm below to find the solution to Problem (5.17):

Algorithm 8.

Step 0: Choose $\boldsymbol{x}^{(1)} \in \mathbb{R}^n_{>0}$. Set $k = 1$.

Step 1: Compute

$$\boldsymbol{Q}(\boldsymbol{x}^{(k)}) = \nabla \boldsymbol{P}(\boldsymbol{x}^{(k)}) + \boldsymbol{x}^{[p-1]}$$

Step 2: Compute

$$\boldsymbol{x}^{(k+1)} = \frac{\boldsymbol{Q}(\boldsymbol{x})^{\frac{1}{[p-1]}}}{||\boldsymbol{Q}(\boldsymbol{x})^{\frac{1}{[p-1]}}||_1}$$

If $\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\| \leq \epsilon$ then stop. Replace k by $k+1$ and go to Step 1.

First, consider the case where $\boldsymbol{P}(\boldsymbol{x})$ is homogeneous.

Theorem 45. Let polynomial $\boldsymbol{P}(\boldsymbol{x})$ be as in Problem (5.17). Assume $\boldsymbol{P}(\boldsymbol{x})$ is homogeneous and irreducible. Then the sequence $\{\boldsymbol{x}^{(k)}\}$ produced by Algorithm 8 converges to the unique vector $\boldsymbol{x}^* \in \mathbb{R}^n_{>0}$ satisfying $\nabla \boldsymbol{P}(\boldsymbol{x}^*) = \lambda(\boldsymbol{x}^*)^{[p-1]}$ and $\boldsymbol{\psi}^T \boldsymbol{x}^* = 1$.

Proof. When $\boldsymbol{P}(\boldsymbol{x})$ is homogeneous, clearly $\nabla \boldsymbol{P}(\boldsymbol{x})$ is also homogeneous. The term $\boldsymbol{x}^{[p-1]}$ is obviously homogeneous, hence, the polynomial $\boldsymbol{Q}(\boldsymbol{x})$ is homogeneous. The irreducibility of polynomial $\boldsymbol{P}(\boldsymbol{x})$ implies that $\nabla \boldsymbol{P}(\boldsymbol{x})$ is weakly irreducible by Lemma 24. By Corollary 10, $\boldsymbol{Q}(\boldsymbol{x})$ is weakly primitive and by Corollary 7, this theorem holds. □

Now for the case where $\boldsymbol{P}(\boldsymbol{x})$ is nonhomogeneous, we have the following result.

Theorem 46. Let polynomial $\boldsymbol{P}(\boldsymbol{x})$ be as in Problem (5.17). Assume $\boldsymbol{P}(\boldsymbol{x})$ is irreducible. Then the sequence $\{\boldsymbol{x}^{(k)}\}$ generated by Algorithm 8 converges to the unique vector $\boldsymbol{x}^* \in \mathbb{R}^n_{>0}$ satisfying $\nabla \boldsymbol{P}(\boldsymbol{x}^*) = \lambda(\boldsymbol{x}^*)^{[p-1]}$ and $\boldsymbol{\psi}^T \boldsymbol{x}^* = 1$.

Proof. The irreducibility of polynomial $\boldsymbol{P}(\boldsymbol{x})$ implies $\nabla \boldsymbol{P}(\boldsymbol{x})$ is weakly irreducible by Lemma 24. By Corollary 10, $\boldsymbol{Q}(\boldsymbol{x})$ is weakly primitive. Since $\boldsymbol{Q}(\boldsymbol{x})$ is non-homogeneous, we substitute $\boldsymbol{P}(\boldsymbol{x})$ in $\boldsymbol{Q}(\boldsymbol{x})$ with $\boldsymbol{F}(\boldsymbol{x})$, where $\boldsymbol{F}(\boldsymbol{x})$ is as in the proof of Theorem 39. Now $\bar{\boldsymbol{Q}}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x})^{[p-1]} + \boldsymbol{x}^{[p-1]}$. $\boldsymbol{F}(\boldsymbol{x})$ is homogeneous. Since $\boldsymbol{x}^{[p-1]}$ is also homogeneous, $\bar{\boldsymbol{Q}}(\boldsymbol{x})$ is homogeneous. Now $\bar{\boldsymbol{Q}}(\boldsymbol{x})$ has satisfied the conditions for Corollary (7), which are homogeneous and weakly primitive. It follows that $\boldsymbol{x}^{(k)}$ converges to the unique vector $\boldsymbol{x}^* \in \mathbb{R}^n_{>0}$ satisfying $\nabla \boldsymbol{P}(\boldsymbol{x}) = \lambda(\boldsymbol{x}^*)^{[p-1]}$ and $\boldsymbol{\psi}^T \boldsymbol{x}^* = 1$. Algorithm 8 satisfies $\|\boldsymbol{x}\|_p = a$ where $a, p > 0$. Therefore, the solution of $\bar{\boldsymbol{Q}}(\boldsymbol{x})$ is also the solution of $\boldsymbol{Q}(\boldsymbol{x})$. □

5.4 Conclusion

In this chapter, with some modifications, we have applied the algorithm presented in Chapter 4 to the optimisation problem whereby the objective function is a nonnegative polynomial and the constraint is spherical.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

Finding the largest eigenvalue of tensors was the main focus in this thesis. The Collatz method was the base of our algorithms as discussed in the introductory chapter. The method was later extended to square tensors, rectangular tensors and the wider class; the general polynomials. We summarise this thesis in this concluding chapter.

In Chapter 2, we studied the extended Collatz method for calculating the largest eigenvalue of nonnegative tensors [43]. However the algorithm is proven to be convergent only under primitive condition. The algorithm is modified so that it is convergent under a wider class, that is irreducible. The convergence of the method was discussed at length and proven to be Q-linear convergent [78].

In Chapter 3, we considered the method for rectangular tensors given by [7, 76] which was an extension of the discussion in Chapter 2. This method was proven to be convergent for irreducible rectangular tensors [76]. We showed that the method of [76] was also convergent for weakly irreducible rectangular tensors and we proved that it had a Q-linear rate of convergence.

In Chapter 4, we took the methods presented in Chapter 2 and 3 to a wider class namely, nonhomogeneous polynomials. The methods in the previous chapters only considered homogeneous polynomials. Another method for finding the largest eigenvalue of polynomials was also proposed in [16], however that method is for homogeneous polynomials. We extended some properties of nonnegative square tensors to nonnegative polynomials and presented a convergent method for finding the largest eigenvalue of nonnegative polynomials. This method was convergent under weakly irreducible condition for both homogenous and non-homogeneous polynomials. The numerical results showed that the method was efficient.

In Chapter 5, we studied the optimisation problem of nonnegative polynomials subject to spherical constraint. This kind of problem whereby the polynomials were induced by tensors was studied in [77]. The problem was relaxed and solved by three methods: geometric programming problem, smoothing Newton method and the power method. The power method used was a variation of the methods in Chapters 2 and 3. When tested numerically, the power method was most efficient. Motivated by this, we presented a convergent algorithm for solving the optimisation problem of both homogeneous and nonhomogeneous nonnegative general polynomials subject to spherical constraint.

6.2 Suggestions

In Chapters 2 and 3, we did not discuss the effect of the choice of ρ. In Chapters 4 and 5, we only considered $\rho = 1$. For matrices, the issue of the optimal shift was discussed in [66]; however, it is unknown for the case of tensors and polynomials. It might accelerate the rate of convergence.

Regarding the rate of convergence, a weaker condition might exist. This opens a door to further research.

The methods in this thesis were proven to be convergent under irreducible conditions. The case for matrices under reducible condition was discussed in [66] and for square tensors in [25]. For general polynomials, interested readers should refer to [4]. It is unknown whether the algorithms converge under a broader class and this deserves further investigation.

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